Problems of Information Transmission, vol. 56, no. 2, pp. 64-81, 2020.

New Upper Bounds in the Hypothesis Testing Problem with Information Constraints¹

M. V. Burnashev

Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia email: burn@iitp.ru

We consider a hypothesis testing problem where a part of data cannot be observed. Our helper observes the missed data and can send us a limited amount of information about them. What kind of this limited information will allow us to make the best statistical inference? In particular, what is the minimum information sufficient to obtain the same results as if we directly observed all the data? We derive estimates for this minimum information and some other similar results.

Key words: testing of hypothesis, information constraints, error probabilities.

§ 1. Introduction and main results

1. Statement of the problem. Similarly to [1, 2], a binary symmetric channel BSC(p) on length n, with unknown crossover probability p is considered. In order to distinguish input and output alphabets $E^n = \{0, 1\}^n$, denote them E_{in}^n and E_{out}^n , respectively. Concerning the value p, there are two hypotheses (one of them is true) : $H_0 : p = p_0$ and $H_1 : p = p_1$, where $0 < p_0, p_1 \le 1/2$.

Denote by **P** and **Q** conditional output distributions on the BSC(p) output for hypotheses H_0 and H_1 , respectively. Then probabilities to get the output block $\boldsymbol{y} = (y_1, \ldots, y_n)$ provided the input block $\boldsymbol{x} = (x_1, \ldots, x_n)$ are given by

$$\mathbf{P}(\boldsymbol{y}|\boldsymbol{x}) = (1 - p_0)^{n - d(\boldsymbol{x}, \boldsymbol{y})} p_0^{d(\boldsymbol{x}, \boldsymbol{y})}, \qquad \mathbf{Q}(\boldsymbol{y}|\boldsymbol{x}) = (1 - p_1)^{n - d(\boldsymbol{x}, \boldsymbol{y})} p_1^{d(\boldsymbol{x}, \boldsymbol{y})}$$

where $d(\boldsymbol{x}, \boldsymbol{y})$ - the Hamming distance between blocks \boldsymbol{x} and \boldsymbol{y} (i.e. the number of noncoincident components of those vectors on length n).

The following problem of minimax testing of hypotheses H_0 and H_1 is considered. We (i.e. "the statistician") observe only the channel output block $\boldsymbol{y} \in E_{out}^n$, while our "helper" observes only the channel input $\boldsymbol{x} \in E_{in}^n$. It is assumed that we do not have any prior information on the input block \boldsymbol{x} . Clearly, that based only on the output block \boldsymbol{y} we are not able to make any reasonable conclusions on unknown value p.

¹The reported study was funded by RFBR according to the research project 19-01-00364.

Assume further that for a prescribed value R > 0, our helper is allowed to partition in advance the input space $E_{in}^n = \{0, 1\}^n$ on $N \leq 2^{Rn}$ arbitrary parts $\{X_1, \ldots, X_N\}$, and to inform us (in some additional way) to which part X_i belongs the input block \boldsymbol{x} . Clearly, only the case $N < 2^n$, i.e. R < 1, is interesting (otherwise, the helper can simply inform us on the block \boldsymbol{x}).

For example, the helper may transmit to the statistician exact values of the first Rn components x_1, \ldots, x_{Rn} (but inform nothing on the next values x_i). Such simple partitioning of the input space E_{in}^n (on cylinder sets $\{X_i\}$), generally speaking, is not optimal. From the statistician point of view input data (x_1, \ldots, x_n) represent very strong nuisance parameter.

We may also say that the optimal limited information on the block \boldsymbol{x} means the optimal "contraction" of full information on the block \boldsymbol{x} . Of course, such optimal "contraction" depends on prior information on transfer probability p and a quality criteria used.

Remark 1. Clearly, the problem will not be changed if the statistician observes the channel input, and the helper observes the channel output.

Based on observation \boldsymbol{y} and the index *i* of the part X_i the statistician makes a decision in favor of one of hypotheses H_0 or H_1 . In order to avoid overcomplification we consider only nonrandomized decision methods (then the problem essence and results remain the same).

We consider partitions $\{X_1, \ldots, X_N\}$ and decision methods that are asymptotically (as $n \to \infty$) optimal. Similar, but much more general problem statements were considered, for example, in [3, 4, 5, 6, 7, 8].

Remark 2. As far as we know, all results in that area (see, for example, [1, 2, 3, 4, 5, 6, 7, 8]) have the form: "it is possible to get the following testing performance ...". Our aim is to get an opposite result, i.e. to show that "it is impossible to get a better result than ...".

Below we denote $\log x = \log_2 x$. For a finite set A we denote by |A| its cardinality. Introduce balls and spheres in E^n

$$\mathbf{B}_{\boldsymbol{x}}(p) = \{ \boldsymbol{u} : d(\boldsymbol{x}, \boldsymbol{u}) \le pn \}, \qquad \boldsymbol{x}, \boldsymbol{u} \in E^n, \\ \mathbf{S}_{\boldsymbol{x}}(p) = \{ \boldsymbol{u} : d(\boldsymbol{x}, \boldsymbol{u}) = pn \}.$$
(1)

2. Error probability exponents and dual problem. Let a partition $\{X_1, \ldots, X_N\}$ of the input space $E_{in}^n = \{0, 1\}^n$ be chosen. Then general decision making can be described as follows. For each partition element X_i we choose a set $\mathcal{A}(X_i) \subset E_{out}^n$, and based on observation \boldsymbol{y} and known element X_i , make a decision $(\mathcal{A}^c = E_{out}^n \setminus \mathcal{A})$:

$$\boldsymbol{y} \in \mathcal{A}(X_i) \Longrightarrow H_0; \qquad \boldsymbol{y} \in \mathcal{A}^c(X_i) \Longrightarrow H_1.$$

Assume that we set a partition $\{X_1, \ldots, X_N\}$ of the input space $E_{in}^n = \{0, 1\}^n$. For each partition element X_i we choose a set $\mathcal{A}(X_i) \subset E_{out}^n$, and based on observation \boldsymbol{y} and known element X_i make a decision $(\mathcal{A}^c = E_{out}^n \setminus \mathcal{A})$:

$$\boldsymbol{y} \in \mathcal{A}(X_i) \Longrightarrow H_0; \quad \boldsymbol{y} \in \mathcal{A}^c(X_i) \Longrightarrow H_1.$$

Define error probabilities of the 1-kind α_n and the 2-kind β_n as

$$\alpha_n = \Pr(H_1|H_0) = \max_{i=1,\dots,N} \max_{\boldsymbol{x} \in X_i} \mathbf{P} \left(\mathcal{A}^c(X_i) | \boldsymbol{x} \right),$$

$$\beta_n = \Pr(H_0|H_1) = \max_{i=1,\dots,N} \max_{\boldsymbol{x} \in X_i} \mathbf{Q} \left(\mathcal{A}(X_i) | \boldsymbol{x} \right).$$

Let $\gamma \geq 0$ - a given constant. We demand that the 1-kind error probability α_n satisfies the condition

$$\alpha_n = \Pr(H_1|H_0) \le 2^{-\gamma n}. \tag{2}$$

We are interested in the minimal possible (over all partitions $\{X_i\}$ of the input space E_{in}^n and all decisions) 2-kind error probability inf β_n . We investigate the asymptotic case as $n \to \infty$ and $N = 2^{Rn}$, where 0 < R < 1 – a given constant. ¹ Then for the best partition $\{X_i\}$ and decision methods denote

$$e(\gamma, R) = \lim_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\inf \beta_n} > 0, \tag{3}$$

where inf is taken over all partitions $\{X_i\}$ and decision methods satisfying the condition (2).

Our main aim is upperbounds for the function $e(\gamma, R)$ (see lowerbounds in [1]). In the paper we limit ourselves to the case $\gamma \to 0$, evaluating the function e(0, R) = e(R), and the related function $r_{\text{crit}}(p_0, p_1)$ (that case sometimes is called Neiman-Pierson problem). In other paper we will consider the case $\gamma > 0$.

It will be convenient for us to consider also the equivalent dual problem (without the helper). Let a value r, 0 < r < 1 be given, and we may choose any set $\mathcal{X} \subset E_{in}^n$ of $X = 2^{rn}$ input blocks. It is known also that the input block \boldsymbol{x} belongs to the chosen set \mathcal{X} . We observe the channel output \boldsymbol{y} and, knowing the set \mathcal{X} , consider the testing of hypotheses H_0 and H_1 problem. We choose a set \mathcal{A} and depending on observation \boldsymbol{y} make the decision:

$$\boldsymbol{y} \in \mathcal{A} \Longrightarrow H_0; \quad \boldsymbol{y} \in \mathcal{A}^c \Longrightarrow H_1.$$

Define 1-kind α_n and 2-kind β_n error probabilities as

$$lpha_n = \max_{oldsymbol{x} \in \mathcal{X}} \mathbf{P}\left(\mathcal{A}^c | oldsymbol{x}
ight), \quad eta_n = \max_{oldsymbol{x} \in \mathcal{X}} \mathbf{Q}\left(\mathcal{A} | oldsymbol{x}
ight).$$

Assume that for the 1-kind error probability α_n condition (2) is fulfilled, and we want to choose the set $\mathcal{X} \subset E_{in}^n$ of cardinality $X = 2^{rn}$ and decision method in order to achieve the minimal possible 2-kind error probability inf β_n . Similarly to (3), for such dual problem define the function $e_d(\gamma, r)$

$$e_{\rm d}(\gamma, r) = \lim_{n \to \infty} \frac{1}{n} \log_2 \frac{1}{\min \beta_n} > 0, \tag{4}$$

where minimum is taken over all sets $\mathcal{X} \subset E_{in}^n$ of cardinality $X = 2^{rn}$ and all decision methods.

The following result establishes simple relation between functions $e(\gamma, R)$ and $e_d(\gamma, r)$. P r o p o s i t i o n 1 [1, Proposition 1]. The following relation holds true

$$e(\gamma, 1 - R) = e_{\mathrm{d}}(\gamma, R); \qquad 0 \le R \le 1, \quad \gamma \ge 0.$$
(5)

¹In order to simplify formulas we don't use integer part sign of value 2^{Rn}

By virtue of Proposition 1 and the formula (5) it is sufficient to investigate the function $e_{\rm d}(\gamma, r)$. In the paper we limit ourselves to the case $\gamma \to 0$, investigating the function $e_{\rm d}(0, r)$.

Remark 3. Essentially, we consider the case when distributions P(x, y) and Q(x, y) have the form: P(x, y) = p(x)P(y|x) and Q(x, y) = p(x)Q(y|x).

3. Known input block. Assume that we know the input block \boldsymbol{x} (then we may set $\boldsymbol{x} = \boldsymbol{0}$) and we observe the output block \boldsymbol{y} . If we demand only $\alpha_n \to 0, n \to \infty$ (i.e. $\gamma = 0$), and we are interested only in the exponent (on n) of 2-kind error probability β_n , then as $n \to \infty$ by Central Limit Theorem and Pearson-Neiman lemma the optimal decision set in favor of H_0 (i.e. p_0) is the spherical slice $\mathbf{B}_0(p_0 + \delta) \setminus \mathbf{B}_0(p_0 - \delta)$ in E_{out}^n (see (1)), where $\delta > 0$ - small. Then for the exponent (on n) of 2-kind error probability β_n we have

$$\frac{1}{n}\log\beta_n = \frac{1}{n}\log\left[\binom{n}{p_0n}(1-p_1)^{(1-p_0)n}p_1^{p_0n}\right] + o(1), \qquad n \to \infty,$$

and therefore we get as $n \to \infty$

$$\frac{1}{n}\log\frac{1}{\beta_n} = -(1-p_0)\log(1-p_1) - p_0\log p_1 - h(p_0) + o(1) = D(p_0||p_1) + o(1), \quad (6)$$

where

$$D(a||b) = a\log\frac{a}{b} + (1-a)\log\frac{1-a}{1-b}.$$
(7)

Remark 4. The function D(a||b) is the divergence for two binomial random variables with parameters a and b, respectively. In other words, it gives the best possible exponent for 2-kind error probability provided fixed 1-kind error probability (i.e. its exponent equals 0), when testing two simple hypotheses: $H_0: p = a$ versus $H_1: p = b$.

With $\gamma = r = 0$ for the value $e_d(\gamma, 0)$ (see (4)) we have from (6)

$$e_{\rm d}(0,0) = D(p_1||p_0).$$
 (8)

4. Unknown input block and critical rate. If we know the input block \boldsymbol{x} and $\alpha_n \to 0$, then the best exponent $e_d(0,0)$ for 2-kind error probability β_n is given by the formula (8).

If we know only that the input block \boldsymbol{x} belongs to the set \mathcal{X} of cardinality $X \sim 2^{rn}$, then for the best such set \mathcal{X} the exponent $e_{d}(0, r)$ of 2-kind error probability β_{n} is defined by the formula (4). It is clear that

$$e_{\rm d}(\gamma, r) \le e_{\rm d}(\gamma, 0), \quad \gamma \ge 0, \quad 0 \le r \le 1.$$
(9)

The function $e_d(\gamma, r)$ does not increase in r. Then the following natural question arises: does there exist $r(\gamma) > 0$ for which the equality in (9) holds, and, if so, what is the maximal rate $r_{\text{crit}}(\gamma)$? Limiting ourselves to the case $\gamma = 0$, define the critical rate $r_{\text{crit}}(p_0, p_1) =$ $r_{\text{crit}}(p_0, p_1, 0)$ as (see (8))

$$r_{\rm crit} = r_{\rm crit}(p_0, p_1) = \sup\{r : e_{\rm d}(0, r) = e_{\rm d}(0, 0) = D(p_0||p_1)\}.$$
(10)

In other words, what is the maximal cardinality 2^{rn} of the best set \mathcal{X} for which we can achieve the same asymptotic efficiency as for known input block \boldsymbol{x} (although we don't know the input block \boldsymbol{x})?

Similarly, introduce the critical rate $R_{\rm crit}$ for the original problem (see (3))

$$R_{\rm crit}(p_0, p_1) = \inf\{R : e(0, R) = e(0, 1) = D(p_0||p_1)\}.$$
(11)

By virtue of Proposition 1 and (11) we have

$$R_{\rm crit}(p_0, p_1) = 1 - r_{\rm crit}(p_0, p_1).$$
(12)

The paper main result is

T е орема 1. If $p_1 < p_0 \leq 1/2$, then there exists $p_1^*(p_0) \leq p_0$, such that for any $p_1 \leq p_1^*(p_0)$ the formula holds

$$r_{\rm crit}(p_0, p_1) = 1 - R_{\rm crit}(p_0, p_1) = 1 - h(p_0), \qquad 0 < p_1 \le p_1^* < p_0 \le 1/2.$$
 (13)

Remark 5. Although the value $r_{\rm crit}(p_0, p_1)$ in (13) coincides with the channel BSC (p_0) capacity, its origin (10) is related with the function $e_d(0, r)$, similar to the channel reliability function E(r, p) in information theory [9, 10]. Exact form of the reliability function E(r, p) is only partially known [11]. For that reason, in the proof of Theorem 1 rather recent results on spectrum of binary codes are used (as in [11, 12, 13]). Complete description of the function $e_d(\gamma, r)$ looks rather difficult problem.

In §2 the lower bound for $r_{\rm crit}$ (Proposition 2) is presented. In §3 the general formula for 2-kind error probability β_n (Lemma 1) is derived. Using the method of "two hypotheses", in §4 Theorem 1 is proved. But generally speaking, the upper bound (13) for $r_{\rm crit}$ is weaker than the corresponding lower bound from §2. In §5 using additional combinatoric arguments one more upper bound for $r_{\rm crit}$ (Proposition 3) is derived. In §6 the accuracy of the lower bound for $r_{\rm crit}$ from Proposition 2 is shown, provided some additional condition is fulfilled. In Appendix some necessary analytic results are presented.

Below in the paper $f \sim g$ means $n^{-1} \ln f = n^{-1} \ln g + o(1), n \to \infty$, and $f \lesssim g$ means $n^{-1} \ln f \leq n^{-1} \ln g + o(1), n \to \infty$.

§ 2. Lower bound for $r_{\rm crit}$

Next result follows from [1, Proposition 2]. P r o p o s i t i o n 2. For $r_{crit}(p_0, p_1)$ lower bounds hold

$$r_{\rm crit}(p_0, p_1) \ge 1 - h(p_0), \quad if \quad 0 < p_1 < p_0 \le 1/2.$$
 (14)

and

$$r_{\rm crit}(p_0, p_1) \ge 1 - h(p_0) - D(p_0||p_1), \quad \text{if} \quad 0 < p_0 < p_1 \le 1/2.$$
 (15)

Proof. For given r, 0 < r < 1, choose randomly and equiprobably a set \mathcal{X} of $X = 2^{rn}$ input blocks \boldsymbol{x} . It was shown in [1, Proposition 2] that if $p_0 < p_1 \leq 1/2$, then for any τ ,

 $p_0 \leq \tau \leq p_1$, there exist a set \mathcal{X} and a decision method for which the following inequalities hold

$$\frac{1}{n}\log\frac{1}{\alpha_n} \ge D(\tau||p_0), \qquad \frac{1}{n}\log\frac{1}{\beta_n} \ge \min\{D(\tau||p_1), 1 - h(\tau) - r\}.$$
(16)

If it is sufficient to have $\alpha_n \to 0$, $n \to \infty$, then setting in (16) $\tau = p_0$, we get (15) from (10).

If $p_1 < p_0 \leq 1/2$, then changing p_0 with p_1 and α_n with β_n in (16) then for any τ we have

$$\frac{1}{n}\log\frac{1}{\alpha_n} \ge \min\{D(\tau||p_0), 1 - h(\tau) - r\}, \qquad \frac{1}{n}\log\frac{1}{\beta_n} \ge D(\tau||p_1).$$
(17)

If it is sufficient to have $\alpha_n \to 0$, $n \to \infty$, then setting $\tau = p_0$ in (17), from (10) we get (14).

§ 3. General formula for 2-kind error probability β_n .

Let $C_n(r) = \{x_1, \ldots, x_M\}$ - a set (code) of $M = 2^{rn}$ different input codeblocks. For the code $C_n(r)$ and 1-kind error probability α_n denote by $\mathcal{D}_0 = \mathcal{D}_0(\mathcal{C}_n, \alpha_n) \subseteq E_{out}^n$ the optimal decision set in favor of H_0 , minimizing 2-kind error probability β_n . Although the set \mathcal{D}_0 has rather complicated form, it is possible to establish some its properties sufficient for proving Theorem 1.

Set a small $\delta > 0$ and for each $\boldsymbol{x}_k, k = 1, \dots, M$, introduce the spherical slide in E_{out}^n

$$SL_{\boldsymbol{x}_{k}}(p_{0},\delta) = \mathbf{B}_{\boldsymbol{x}_{k}}(p_{0}+\delta) \setminus \mathbf{B}_{\boldsymbol{x}_{k}}(p_{0}-\delta) = \{\boldsymbol{u} : |\boldsymbol{d}(\boldsymbol{x}_{k},\boldsymbol{u}) - p_{0}\boldsymbol{n}| \le \delta\boldsymbol{n}\},$$
(18)

where $B_{\boldsymbol{x}}(p)$ is defined in (1). For each \boldsymbol{x}_k introduce also the set

$$D_{\boldsymbol{x}_k}(\delta) = \mathcal{D}_0 \bigcap SL_{\boldsymbol{x}_k}(p_0, \delta).$$
(19)

Since we need $\alpha_n \to 0, n \to \infty$, the optimal set \mathcal{D}_0 contains an "essential" part of each set $SL_{\boldsymbol{x}_k}(p_0, \delta), k = 1, \ldots, M$. In order to evaluate it, note that for any \boldsymbol{x}_k and $\boldsymbol{u}, \boldsymbol{z} \in SL_{\boldsymbol{x}_k}(p_0, \delta)$ we have

$$\frac{\mathbf{P}(\boldsymbol{u}|p_0,\boldsymbol{x}_k)}{\mathbf{P}(\boldsymbol{z}|p_0,\boldsymbol{x}_k)} = \left(\frac{q_0}{p_0}\right)^{d(\boldsymbol{z},\boldsymbol{x}_k)-d(\boldsymbol{u},\boldsymbol{x}_k)} \le \left(\frac{q_0}{p_0}\right)^{2\delta n}, \qquad q_0 = 1 - p_0.$$
(20)

By Chebychev exponential inequality (Chernov bound) for any \boldsymbol{x}_k and small $\delta > 0$ we get

$$\log \mathbf{P}\{\boldsymbol{u} \notin SL_{\boldsymbol{x}_k}(p_0, \delta) | \boldsymbol{x}_k, p_0\} \le -\frac{n\delta^2}{2p_0q_0}.$$
(21)

Then by (18), (19) and (21) we have for any \boldsymbol{x}_k

$$\mathbf{P}\left\{D_{\boldsymbol{x}_{k}}(\delta)|p_{0},\boldsymbol{x}_{k}\right\} \geq 1 - \mathbf{P}\left\{\boldsymbol{u} \notin \mathcal{D}_{0}|p_{0},\boldsymbol{x}_{k}\right\} - \mathbf{P}\left\{\boldsymbol{u} \notin SL_{\boldsymbol{x}_{k}}(p_{0},\delta)|p_{0},\boldsymbol{x}_{k}\right\} \geq \\ \geq 1 - \alpha_{n} - e^{-n^{2}\delta^{2}/(2p_{0}q_{0})},$$
(22)

and by (20) also have

$$\delta_1 |SL_{\boldsymbol{x}_k}(p_0, \delta)| \le |D_{\boldsymbol{x}_k}(\delta)| \le |SL_{\boldsymbol{x}_k}(p_0, \delta)|,$$

$$\delta_1 = \left(1 - \beta_n - e^{-n^2 \delta^2 / (2p_0 q_0)}\right) \left(\frac{p_0}{q_0}\right)^{2\delta n}.$$
(23)

Since $D_{\boldsymbol{x}_k}(\delta) \subseteq \mathcal{D}_0$ for any \boldsymbol{x}_k , then by (19), (22) and (23) for the probability $\mathbf{P}(e|p_1, \boldsymbol{x}_i)$ we have

$$\mathbf{P}(e|p_1, \boldsymbol{x}_i) = \mathbf{P}\{\mathcal{D}_0|p_1, \boldsymbol{x}_i\} \sim \mathbf{P}\left\{\bigcup_{k=1}^M D_{\boldsymbol{x}_k}(\delta)|p_1, \boldsymbol{x}_k\right\} \sim \\ \geq \delta_1 \mathbf{P}\left\{\bigcup_{k=1}^M SL_{\boldsymbol{x}_k}(p_0, \delta)|p_1, \boldsymbol{x}_i\right\}.$$
(24)

For t > 0 and each \boldsymbol{x}_i introduce the set

$$D_{\boldsymbol{x}_i}(t,p) = \left\{ \boldsymbol{u} : \begin{array}{c} \text{there exists } \boldsymbol{x}_k \neq \boldsymbol{x}_i, \text{ such that} \\ d(\boldsymbol{x}_i, \boldsymbol{u}) = tn, d(\boldsymbol{x}_k, \boldsymbol{u}) = pn \end{array} \right\}.$$
(25)

L e m m a 1. For 2-kind error probability β_n of a code $C_n = \{x_1, \ldots, x_M\}$ and the optimal set \mathcal{D}_0 in favor of H_0 , the formula holds as $n \to \infty$

$$\frac{\log \beta_n}{n} \sim \max_{t>0} \left\{ \frac{1}{n} \log \left[\frac{1}{M} \sum_{i=1}^M |D_{\boldsymbol{x}_i}(t, p_0)| \right] + t \log p_1 + (1-t) \log(1-p_1) \right\}.$$
 (26)

The critical rate $r_{\text{crit}}(p_0, p_1)$ is defined by the formula $(M = 2^{rn})$

$$r_{\rm crit}(p_0, p_1) = \sup\left\{r : F(p_0, p_1, r) \le 0\right\} = \inf\left\{r : F(p_0, p_1, r) > 0\right\},\tag{27}$$

where

$$F(p_0, p_1, r) = \lim_{n \to \infty} \min_{|\mathcal{C}_n| \le M} \max_t F(p_0, p_1, r, \mathcal{C}_n, t),$$

$$F(p_0, p_1, r, \mathcal{C}_n, t) = \frac{1}{n} \log \left[\sum_{i=1}^M |D_{\boldsymbol{x}_i}(t, p_0)| \right] + (p_0 - t) \log \frac{1 - p_1}{p_1} - r - h(p_0).$$
(28)

Proof. Using (24) with $\delta = o(1)$ and $\delta_1 = e^{o(n)}$ as $n \to \infty$, we have

$$\beta_n = \max_i \mathbf{P}(e|p_1, \boldsymbol{x}_i) \sim \frac{1}{M} \sum_{i=1}^M \mathbf{P}(e|p_1, \boldsymbol{x}_i) \sim \frac{\delta_1}{M} \sum_{i=1}^M \mathbf{P}\left\{\bigcup_{k=1}^M SL_{\boldsymbol{x}_k}(p_0, \delta) \middle| p_1, \boldsymbol{x}_i\right\}.$$
 (29)

From (25) and (26) for each \boldsymbol{x}_i

$$\mathbf{P}\left\{\bigcup_{k=1}^{M} SL_{\boldsymbol{x}_{k}}(p_{0},\delta) \left| p_{1},\boldsymbol{x}_{i} \right\} \sim \mathbf{P}\left\{\bigcup_{t>0} D_{\boldsymbol{x}_{i}}(t,p_{0}) \left| p_{1},\boldsymbol{x}_{i} \right\} \sim \\ \sim \max_{t>0} \left\{ p_{1}^{tn}(1-p_{1})^{(1-t)n} \left| D_{\boldsymbol{x}_{i}}(t,p_{0}) \right| \right\}.$$

$$(30)$$

Therefore from (29) and (30) the formula (26) follows.

Since

 $\mathbf{P} \{ SL_{\boldsymbol{x}_{i}}(p_{0}, \delta) | p_{1}, \boldsymbol{x}_{i} \} \sim \mathbf{P} \{ d(\boldsymbol{x}_{i}, \boldsymbol{u}) \ge p_{0}n | p_{1}, \boldsymbol{x}_{i} \} \sim 2^{-D(p_{0}||p_{1})n},$

the right-hand side of (26) increases with r (i.e. with $M = 2^{rn}$), starting from $(-D(p_1||p_0))$. Therefore, from (6) and (26) it follows that the critical rate $r_{\rm crit}$ is the maximal rate r, such that

$$\min_{\{\boldsymbol{x}_i\}} \max_{t>0} \left\{ \frac{1}{n} \log \left[\sum_{i=1}^M |D_{\boldsymbol{x}_i}(t, p_0)| \right] + t \log p_1 + (1-t) \log(1-p_1) \right\} - r \le -D(p_0||p_1).$$
(31)

Note that

$$D(p_0||p_1) + t\log p_1 + (1-t)\log(1-p_1) = -h(p_0) + (p_0-t)\log\frac{1-p_1}{p_1}.$$
(32)

From (31) and (32) the formulas (27)-(28) follow.

In particular, from (53) with $t = p_0$ we have

$$F(p_0, p_1, r, \mathcal{C}_n, p_0) = o(1), \qquad n \to \infty.$$

The main difficulty in analysis of relations (27)-(28) constitutes estimation of cardinalities $|D_{\boldsymbol{x}_i}(t, p_0)|$ in (28), which depend on the code C_n geometry. Similar problem arose in [11, 12, 13], where the reliability function E(R, p) of the channel BSC(p) was investigated. Direct estimation of those cardinalities leads to quite bulky formulas.

§ 4. Upper bound for r_{crit} : two hypotheses.

We get a simple (but not very accurate) upper bound for $r_{\rm crit}(p_0, p_1)$, using quite popular in mathematical statistics (mainly, in estimation theory) method of "two hypotheses". Using the formula (26), choose from the code $C_n(r) = \{ \boldsymbol{x}_1, \ldots, \boldsymbol{x}_M \}$, $M = 2^{rn}$, any two codewords, say, $\boldsymbol{x}_1 \bowtie \boldsymbol{x}_2$ with $d(\boldsymbol{x}_1, \boldsymbol{x}_2) = \omega n$. We may assume that for a rate r > 0 the value ω satisfies constraints

$$0 < \omega \le \omega_{\min}(r),$$

where the value $\omega_{\min}(r)$ will be defined later. Replace the code $C_n(r)$ by the code C' of two chosen codewords $C' = \{x_1, x_2\}$. Then $\beta_n(C) \ge \beta_n(C')$. Similarly to (29)-(30) we have

$$\beta_n(\mathcal{C}') \sim 2^{-D(p_0||p_1)n} + \mathbf{P}\left\{SL_{\boldsymbol{x}_2}(p_0,\delta) \middle| p_1, \boldsymbol{x}_1\right\}$$

We are interested when for x_1, x_2 the following inequality holds

٨

$$\frac{1}{n}\log \mathbf{P}\left\{SL_{\boldsymbol{x}_{2}}(p_{0},\delta)\Big|p_{1},\boldsymbol{x}_{1}\right\} > -D(p_{0}||p_{1}).$$
(33)

Evaluate the probability in the left-hand side of (33). For $d(\boldsymbol{x}_i, \boldsymbol{x}_k) = \omega n$ denote

$$S_{\boldsymbol{x}_i,\boldsymbol{x}_k}(t,p,\omega) = \{ \boldsymbol{u} : d(\boldsymbol{x}_i,\boldsymbol{u}) = tn, d(\boldsymbol{x}_k,\boldsymbol{u}) = pn, d(\boldsymbol{x}_i,\boldsymbol{x}_k) = \omega n \}.$$
 (34)

Then (see Appendix)

$$\frac{1}{n} \log |S_{\boldsymbol{x}_{i},\boldsymbol{x}_{k}}(t,p,\omega)| = g(t,p,\omega) + o(1), \qquad n \to \infty,$$

$$\frac{1}{n} \log \mathbf{P} \left\{ S_{\boldsymbol{x}_{i},\boldsymbol{x}_{k}}(t,p,\omega) \Big| p_{1},\boldsymbol{x}_{i} \right\} = g(t,p,\omega) - t \log \frac{1-p_{1}}{p_{1}} + \log(1-p_{1}) + o(1),$$
(35)

where $g(t, p, \omega)$ is defined in (78). Therefore as $n \to \infty$ (see (76)-(77))

$$\frac{1}{n}\log \mathbf{P}\left\{SL_{\boldsymbol{x}_{2}}(p_{0},\delta)\Big|p_{1},\boldsymbol{x}_{1}\right\} = \frac{1}{n}\max_{t}\log \mathbf{P}\left\{S_{\boldsymbol{x}_{1},\boldsymbol{x}_{2}}(t,p_{0},\omega)\Big|p_{1},\boldsymbol{x}_{1}\right\} + o(1) = f(p_{0},p_{1},\omega) + o(1)$$
(36)

where

$$f(p_0, p_1, \omega) = \max_t f(p_0, p_1, \omega, t),$$

$$f(p_0, p_1, \omega, t) = g(t, p_0, \omega) - t \log \frac{1 - p_1}{p_1} + \log(1 - p_1).$$
(37)

We have

$$f'_t(p_0, p_1, \omega, t) = \log \frac{\omega - t}{t} - \log \frac{p_0 + t - \omega}{1 - p_0 - t} - 2\frac{1 - p_1}{p_1}, \qquad f''_{tt}(p_0, p_1, \omega, t) < 0.$$
(38)

By (32) and (35)-(37) the inequality (33) takes the form

$$\max_{t} F(p_0, p_1, \omega, t) > 0, \tag{39}$$

where

$$F(p_0, p_1, \omega, t) = f(p_0, p_1, \omega, t) + D(p_0||p_1) = g(t, p_0, \omega) + (p_0 - t)\log\frac{1 - p_1}{p_1} - h(p_0).$$
(40)

If for some p_0, p_1 and ω the inequality (39) holds, then the appropriate upper bound (14)-(15) is valid. Denote by $t_1^0 = t_1^0(p_0, p_1, \omega)$ the maximizing value t in (37) (it remains the maximizing one in (39) as well). Then

$$f(p_0, p_1, \omega) = f(p_0, p_1, \omega, t_1^0(p_0, p_1, \omega)).$$
(41)

From the equation $f'_t(p_0, p_1, \omega, t) = 0$ for t_1^0 from (38) we get

$$t_1^0 = t_1^0(p_0, p_1, \omega) = \frac{\sqrt{1 + (v_0 - 1)[(\omega - p_0)^2 v_0 - (1 - \omega - p_0)^2 + 1]} - 1}{v_0 - 1},$$

$$v_0(p_1) = \left(\frac{1 - p_1}{p_1}\right)^2 \ge 1.$$
(42)

Then from (40) and (42) we have

$$F(p_0, p_1, \omega, t_1^0) = g(t_1^0, p_0, \omega) + (p_0 - t_1^0) \log \frac{1 - p_1}{p_1} - h(p_0).$$
(43)

It is possible to check that for the function $F(p_0, p_1, \omega, t_1^0)$ from (43) we have

 $F(p_0, p_1, 0, t_1^0) = 0$ and $F''_{\omega\omega} < 0, \omega > 0$. Therefore, it is sufficient to check the inequality (39) with $t = t_1^0$ only for the minimal value ω for the code $C_n(r)$ (i.e. for its code distance $d(\mathcal{C})$). Let $\omega_{\min}(r)n$ - the maximal possible code distance of $C_n(r)$. For the value $\omega_{\min}(r)$ the

following bound is known [14, formula (1.5)]

$$r \le h \left[\frac{1}{2} - \sqrt{\omega_{\min}(1 - \omega_{\min})} \right], \qquad \omega_{\min} = \omega_{\min}(r).$$
(44)

Consider two possible cases 1) $p_1 < p_0 \leq 1/2$ and 2) $p_0 < p_1 \leq 1/2$.

1. Case $p_1 < p_0 \le 1/2$. Setting $r = 1 - h(p_0)$, denote by $\omega_0 = \omega_0(p_0)$ the root of the equation (see (44))

$$1 - h(p_0) = h\left[\frac{1}{2} - \sqrt{\omega(1-\omega)}\right]$$

Then the inequality (39) takes the form $(\omega_0 = \omega_0(p_0))$

$$F(p_0, p_1, \omega_0, t_1^0) = g(t_1^0, p_0, \omega_0) + (p_0 - t_1^0) \log \frac{1 - p_1}{p_1} - h(p_0) > 0.$$
(45)

It is possible to check (Maple), that the inequality (45) is satisfied, if $p_1 \leq p_1^*(p_0)$, where

$$p_0 = 0.1 = 0.12 = 0.15 = 0.2 = 0.3 = 0.4 = 0.45 = 0.49$$

 $p_1^*(p_0) = 0.0003 = 0.003 = 0.016 = 0.056 = 0.17 = 0.317 = 0.4 = 0.48$

If $p_0 \leq 0.20707$ (i.e. $\omega < 0.273$), then in [14, формула (1.4)] there is a little bit more accurate than (44) bound (but much more bulky).

2. Case $p_0 < p_1 \le 1/2$. It is possible to check (Maple), that the inequality (39) is not satisfied for any $p_0 < p_1$!

§ 5. Upper bound for r_{crit} : combinatorics.

We will get one more upper bound for $r_{\rm crit}$, based on the same formula (26), but using additional combinatorics arguments.

1. Combinatorics lemma. In the code $C_n = \{x_i\}$ we call $(x_i, x_j) \omega$ -pair, if $d(x_i, x_j) = \omega n$. The total number of ω -pairs in a code C_n equals $MB_{\omega n}$ (see (65)). We say that a point $y \in E^n$ is (ω, p, t) -covered, if there exists ω -pair (x_i, x_j) such that $d(x_i, y) = pn$, $d(x_j, y) = tn$. Denote by $K(y, \omega, p, t)$ the number of (ω, p, t) -coverings of the point y (taking into account multiplicity of coverings), i.e.

$$K(\boldsymbol{y},\omega,p,t) = |\{(\boldsymbol{x}_i,\boldsymbol{x}_j) : d(\boldsymbol{x}_i,\boldsymbol{x}_j) = \omega n, \ d(\boldsymbol{x}_i,\boldsymbol{y}) = pn, \ d(\boldsymbol{x}_j,\boldsymbol{y}) = tn\}|, \quad \omega > 0.$$
(46)

Introduce sets (see (25))

$$D_{\boldsymbol{x}_{i}}(t, p, \omega) = \bigcup_{\boldsymbol{x}_{k}} S_{\boldsymbol{x}_{i}, \boldsymbol{x}_{k}}(t, p, \omega) =$$

$$= \left\{ \boldsymbol{u}: \text{ there exists } \boldsymbol{x}_{k} \text{ such that } d(\boldsymbol{x}_{i}, \boldsymbol{x}_{k}) = \omega n, \\ d(\boldsymbol{x}_{i}, \boldsymbol{u}) = tn, d(\boldsymbol{x}_{k}, \boldsymbol{u}) = pn \right\}.$$
(47)

Then

$$D_{\boldsymbol{x}_i}(t,p) = \bigcup_{\omega>0} D_{\boldsymbol{x}_i}(t,p,\omega)$$

For t > 0 introduce the value

$$m_t(\boldsymbol{y}) = \{$$
число $\boldsymbol{x}_i \in \mathbf{S}_{\boldsymbol{y}}(t)\}.$ (48)

Then for any $\boldsymbol{y}, p, t > 0$

$$K(\boldsymbol{y},t,p) = m_t(\boldsymbol{y})m_p(\boldsymbol{y}).$$
(49)

L e m m a 2. For a code $\{x_i\}$ and $\omega, p, t > 0$ the formula holds (see (46) u (47))

$$\sum_{i=1}^{M} |D_{\boldsymbol{x}_{i}}(t, p, \omega)| \leq \sum_{\boldsymbol{y} \in E^{n}} K(\boldsymbol{y}, \omega, t, p).$$
(50)

Also, if (see (48))

$$\max_{\boldsymbol{y}} m_p(\boldsymbol{y}) = 2^{o(n)}, \qquad n \to \infty, \tag{51}$$

then for any $\omega, t > 0$

$$\sum_{i=1}^{M} |D_{\boldsymbol{x}_i}(t, p, \omega)| = 2^{o(n)} \sum_{\boldsymbol{y} \in E^n} K(\boldsymbol{y}, \omega, t, p), \qquad n \to \infty.$$
(52)

Proof. Let $\boldsymbol{y} \in E^n$ and there are m ordered pairs $(\boldsymbol{x}_i, \boldsymbol{x}_j)$ with $d(\boldsymbol{x}_i, \boldsymbol{x}_j) = \omega n$ and $d(\boldsymbol{x}_i, \boldsymbol{y}) = tn$, $d(\boldsymbol{x}_j, \boldsymbol{y}) = pn$. Those m pairs $(\boldsymbol{x}_i, \boldsymbol{x}_j)$ have $m_1 \leq m$ different first arguments $\{\boldsymbol{x}_i\}$. Then \boldsymbol{y} appears m times in the right-hand side of (50) and m_1 times in the left-hand side, what proves the formula (50). If the condition (51) is satisfied, then $m_1 = me^{o(n)}$, from where the equality (52) follows. Note also that by (49) we have

$$\sum_{i=1}^{M} |D_{\boldsymbol{x}_{i}}(t,p)| = \sum_{\boldsymbol{y}:m_{p}(\boldsymbol{y})\geq 1} \frac{K(\boldsymbol{y},t,p)}{m_{p}(\boldsymbol{y})} = \sum_{\boldsymbol{y}:m_{p}(\boldsymbol{y})\geq 1} m_{t}(\boldsymbol{y}) \sim M2^{h(t)n} - \sum_{\boldsymbol{y}:m_{p}(\boldsymbol{y})=0} m_{t}(\boldsymbol{y}).$$
(53)

From the first of the equality (53) formulas (50) and (52) follow as well. \Box

The formula (53) looks simple and attractive, but its right-hand side has the form "large minus large", what is not pleasant. Note that in (53) we can not neglect the last sum, because then we get only $r_{\text{crit}} \leq 1$, what is not interesting.

2. One more upper bound for r_{crit} . We upperbound the last sum in B (53) as follows. We have

$$\sum_{\boldsymbol{y}:m_{p_0}(\boldsymbol{y})=0} m_t(\boldsymbol{y}) \le 2^{h(t)n} |\{\boldsymbol{y}: m_{p_0}(\boldsymbol{y})=0\}|.$$
(54)

Maximum of the cardinality $|\{\boldsymbol{y} : m_{p_0}(\boldsymbol{y}) = 0\}|$ is attained when the code \mathcal{C} is the ball $\mathbf{B}_0(\tau)$ of radius τn , where $r = h(\tau)$. Therefore

$$\max_{\mathcal{C}} |\{\boldsymbol{y} : m_{p_0}(\boldsymbol{y}) = 0\}| = 2^n - |\mathbf{B}_{\mathbf{0}}(\tau + p_0)| \sim 2^{h(\tau + p_0)n}, \quad \tau + p_0 \ge 1/2; \max_{\mathcal{C}} |\{\boldsymbol{y} : m_{p_0}(\boldsymbol{y}) = 0\}| \sim 2^n, \quad \tau + p_0 \le 1/2.$$
(55)

If $\tau + p_0 \ge 1/2$, i.e. if $r \ge h(1/2 - p_0)$, then from (53), (54) and (55) we get

$$\sum_{i=1}^{M} |D_{\boldsymbol{x}_{i}}(t,p_{0})| \ge 2^{h(t)n} \left[M - 2^{h(\tau+p_{0})n} \right] = 2^{h(t)n} \left[2^{h(\tau)n} - 2^{h(1-\tau-p_{0})n} \right] \sim M 2^{h(t)n},$$

if $\tau > 1 - \tau - p_0$, i.e. if $\tau > (1 - p_0)/2$, or, equivalently, if $r > h[(1 - p_0)/2]$.

Therefore, if $r \ge \max\{h(1/2 - p_0), h[(1 - p_0)/2]\} = h[(1 - p_0)/2]$, then for any $p_0 \ne p_1$ (28) takes the form

$$F(p_0, p_1, r) = \max_{t>0} \left\{ h(t) + (p_0 - t) \log \frac{1 - p_1}{p_1} \right\} - h(p_0) =$$
$$= h(p_1) + (p_0 - p_1) \log \frac{1 - p_1}{p_1} - h(p_0) > 0, \qquad p_0 \neq p_1,$$

since maximum over t is attained for $t = p_1$. Therefore, it gives the following upper bound for r_{crit} (weaker than (13))

$$r_{\rm crit}(p_0, p_1) \le h[(1-p_0)/2], \qquad p_0 \ne p_1.$$
 (56)

Remark 6. Note that $1 - h(p_0) < h(1/2 - p_0) < h[(1 - p_0)/2], 0 < p_0 < 1/2$. We improve the bound (56). In addition to (54) we also have

$$\sum_{\boldsymbol{y}:m_{p_0}(\boldsymbol{y})=0} m_t(\boldsymbol{y}) \le M |\{ \boldsymbol{y}: m_{p_0}(\boldsymbol{y})=0\}|.$$

Therefore, if $\tau + p_0 \ge 1/2$ and $t \ge 1 - \tau - p_0$, then

$$\sum_{i=1}^{M} |D_{\boldsymbol{x}_i}(t, p_0)| \ge M \left[2^{h(t)n} - 2^{h(1-\tau-p_0)n} \right] \sim M 2^{h(t)n}.$$

By (39)-(40) it is necessary to have

$$\max_{t \ge 1 - \tau - p_0} f(t, p_0, p_1) > 0,$$

$$f(t, p_0, p_1) = h(t) + (p_0 - t) \log \frac{1 - p_1}{p_1} - h(p_0).$$
(57)

Maximum of the function $f(t, p_0, p_1)$ over $t \ge 1 - \tau - p_0$ is attained for $t = \max\{p_1, 1 - \tau - p_0\}$, since $\max_t f(t, p_0, p_1) = f(p_1, p_0, p_1) > 0, \quad p_0 \ne p_1; \qquad f(p_0, p_0, p_1) = 0,$

$$f'_{t}(t, p_{0}, p_{1}) = \log \frac{1-t}{t} - \log \frac{1-p_{1}}{p_{1}}, \qquad f''_{tt}(t, p_{0}, p_{1}) < 0,$$

$$\operatorname{sign} f'_{t}(t, p_{0}, p_{1}) = \operatorname{sign} (p_{1} - t).$$
(58)

1) Therefore, if $p_1 \ge 1 - \tau - p_0$, then from (57)-(58) for $p_0 \ne p_1$ we get

$$\max_{t \ge 1-\tau-p_0} f(t, p_0, p_1) = h(p_1) + (p_0 - p_1) \log \frac{1 - p_1}{p_1} - h(p_0) > 0.$$
(59)

Hence if $\tau \ge \max\{1/2 - p_0, 1 - p_0 - p_1\} = 1 - p_0 - p_1$, then for $p_0 \ne p_1$ the inequality (59) holds, from where the estimate follows

$$\tau_{\rm crit} \le 1 - p_0 - p_1, \qquad r_{\rm crit} = h(\tau_{\rm crit}). \tag{60}$$

2) If $p_1 < 1 - \tau - p_0$, then maximum in (57) is attained for $t = 1 - \tau - p_0$, and then

$$\max_{t \ge 1 - \tau - p_0} f(t, p_0, p_1) = f(1 - \tau - p_0, p_0, p_1).$$

Note that

$$f(p_0, p_0, p_1) = 0, \qquad f'_{t=p_0}(t, p_0, p_1) \neq 0, \quad p_0 \neq p_1;$$

$$f''_{tt}(t, p_0, p_1) < 0, \qquad \text{sign } f'_t(t, p_0, p_1) = \text{sign } (p_1 - t).$$

Let also $p_0 > 1 - \tau - p_0$ (i.e. $\tau > 1 - 2p_0$). Then $\max_{t \ge 1 - \tau - p_0} f(t, p_0, p_1) > 0$ (it is sufficient to set t, close to p_0). Therefore

$$\tau_{\rm crit} \le 1 - 2p_0, \qquad r_{\rm crit} = h(\tau_{\rm crit}).$$
(61)

As a result, from (60) and (61) we get

Propositio 3. For any $p_0, p_1 \in [0, 1/2]$ for r_{crit} the upper bound holds

$$\tau_{\rm crit}(p_0, p_1) \le \min\{1 - p_0 - p_1, 1 - 2p_0\}, \qquad r_{\rm crit} = h(\tau_{\rm crit}).$$
 (62)

C or ollary. If $p_0 = 1/2$, then from (62) it follows $\tau_{\text{crit}}(1/2, p_1)r_{\text{crit}}(1/2, p_1) = 0$.

Earlier that particular result was proved by different method in [1, предложение 3]. Also the best exponent $e_d(\gamma, r)$ for $\gamma \ge 0, 0 \le r \le 1$ from (4) was obtained there.

§ 6. "Potential" additive upper bound for r_{crit} .

Theorem 1 was proved replacing in the formula (26) the exponential number M of codewords $\{x_i\}$ by two closest codewords (x_i, x_j) . Such method gives optimal results only if it is possible to choose a pair (x_i, x_j) with $d(x_i, x_j) = \omega n$ and small $\omega > 0$. In the problem statement considered we can not do that.

In order to strengthen Theorem 1 it is necessary to consider in (26) an exponential number M of codewords $\{x_i\}$, what is much more difficult [11, 12, 13]. We strengthen Theorem 1 provided it is possible to use in the formula (26) an additive approximation.

We assume that for all $\{x_i\}$ in the formula (26) the additive approximation holds as $n \to \infty$

$$\mathbf{P}\left\{\bigcup_{k\neq i} SL_{\boldsymbol{x}_{k}}(p_{0},\delta) \middle| p_{1},\boldsymbol{x}_{i}\right\} = 2^{o(n)} \sum_{k\neq i} \mathbf{P}\left\{SL_{\boldsymbol{x}_{k}}(p_{0},\delta) \middle| p_{1},\boldsymbol{x}_{i}\right\}.$$
(63)

Then (see (36)) with $d(\boldsymbol{x}_i, \boldsymbol{x}_k) = \omega_{ik} n$

$$\mathbf{P}\left\{\bigcup_{k\neq i} SL_{\boldsymbol{x}_{k}}(p_{0},\delta) \middle| p_{1},\boldsymbol{x}_{i}\right\} = 2^{o(n)} \sum_{k\neq i} 2^{f(p_{0},p_{1},\omega_{ik})n}$$

and

$$\sum_{i=1}^{M} \mathbf{P} \left\{ \bigcup_{k \neq i} SL_{\boldsymbol{x}_{k}}(p_{0}, \delta) \Big| p_{1}, \boldsymbol{x}_{i} \right\} = 2^{o(n)} \sum_{i=1}^{M} \sum_{k \neq i} 2^{f(p_{0}, p_{1}, \omega_{ik})n}.$$
(64)

In order to develop relations (64), introduce some additional notions.

Code spectrum (distance distribution) of length $n \operatorname{code} C$ is the (n + 1)-tuple $B(C) = (B_0, B_1, \ldots, B_n)$ with components

$$B_i = |\mathcal{C}|^{-1} \left| \{ (\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}, \, d(\boldsymbol{x}, \boldsymbol{y}) = i \} \right|, \qquad i = 0, 1, \dots, n.$$
(65)

In other words, B_i is average number of codewords \boldsymbol{y} on the distance i from the codeword \boldsymbol{x} . The total number of ordered codepairs $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ with $d(\boldsymbol{x}, \boldsymbol{y}) = i$ equals $|\mathcal{C}|B_i$. Denote also $B_{\omega n} = 2^{b(\omega,r)n}$.

Then we can continue the formula (64) as follows

$$\sum_{i=1}^{M} \mathbf{P} \left\{ \bigcup_{k \neq i} SL_{\boldsymbol{x}_{k}}(p_{0}, \delta) \middle| p_{1}, \boldsymbol{x}_{i} \right\} = 2^{o(n)} M \sum_{\omega > 0} 2^{[b(\omega, r) + f(p_{0}, p_{1}, \omega)]n}$$

Therefore (see (36)-(37))

$$\frac{1}{n}\log\left[\sum_{i=1}^{M}\mathbf{P}\left\{\bigcup_{k\neq i}SL_{\boldsymbol{x}_{k}}(p_{0},\delta)\Big|p_{1},\boldsymbol{x}_{i}\right\}\right] = r + \max_{\omega,t}\left\{b(\omega,r) + f(p_{0},p_{1},\omega,t)\right\} + o(1),\quad(66)$$

where $f(p_0, p_1, \omega, t)$ is defined in (37). Then for the function $F(p_0, p_1, r)$ from (28) and (66) we have

$$F(p_0, p_1, r) = \max_{\omega, t} \left\{ b(\omega, r) + g(p_0, t, \omega) + (p_0 - t) \log \frac{1 - p_1}{p_1} - h(p_0) \right\}.$$
 (67)

As an estimate for $b(\omega, r)$ in (67) we use a function $b_{\text{low}}(\omega, r)$ with the following property: there exists a value $\omega_{\text{max}} = \omega_{\text{max}}(r) > 0$, such that

$$\max_{0<\omega\le\omega_{\max}} \left[b(\omega, r) - b_{\text{low}}(\omega, r) \right] \ge 0, \qquad r > 0.$$
(68)

Then in order the inequality $F(p_0, p_1, r) > 0$ (see (27)) be valid, it is sufficient the following condition (see (37) and (67)) be satisfied

$$\min_{0<\omega\le\omega_{\max}}\max_{t>0}\left\{b_{\text{low}}(\omega,r) + g(p_0,t,\omega) + (p_0-t)\log\frac{1-p_1}{p_1} - h(p_0)\right\} > 0.$$
(69)

We use in (69) as $b_{\text{low}}(\omega, r)$ the best of known such functions $\mu(r, \alpha, \omega)$, $h_2(\tau) = h_2(\alpha) - 1 + r$, with arbitrary $\alpha \in [\delta_{GV}(r), 1/2]$ (see (83), (84) and Theorem 2 in Appendix). The function $\mu(r, \alpha, \omega)$ satisfies the condition (68). Moreover, it monotonically increases in r and $\omega_{\text{max}} = G(\alpha, \tau)$, where $G(\alpha, \tau)$ is defined in (81). Then in order the inequality (69) be satisfied, it is sufficient the condition be fulfilled

$$\min_{0 < \omega \le \omega_{\max}} \max_{t > 0} K(p_0, p_1, r, \omega, t) > 0,$$
(70)

where

$$K(p_0, p_1, r, \omega, t) = \mu(r, p_0, \omega) + g(p_0, t, \omega) + (p_0 - t) \log \frac{1 - p_1}{p_1} - h(p_0).$$
(71)

Note that $K(p_0, p_1, r, 0, p_0) = 0$. In order to avoid bulky calculations, we set $t = p_0$. The function $K(p_0, p_1, r, \omega, p_0) = 0$ is concave in ω , i.e. $K''(p_0, p_1, r, \omega, p_0)_{\omega\omega} < 0$ (the simplest way is to check that with Maple). Therefore, minimum over ω is attained for $\omega = \omega_{\max} = G(\alpha, \tau)$ and it is sufficient to check the condition (70) for $\omega = G(\alpha, \tau)$. The following useful formula [11, Lemma 4] is known:

$$\mu(r, \alpha, G(\alpha, \tau)) = h_2(G(\alpha, \tau)) + r - 1, \qquad h_2(\alpha) - h_2(\tau) = 1 - r.$$
(72)

Consider only more simple

Case $p_1 < p_0 \leq 1/2$. Set $r = r_0 = 1 - h(p_0)$ and $\alpha = p_0$ (then $\delta_{GV}(r_0) = p_0, \tau = 0$). We have $G(\alpha, \tau) = 2p_0(1-p_0)$ and it is sufficient to check the condition (70) for $\omega = 2p_0(1-p_0)$. From (71)-(72) with $\alpha = p_0, \tau = 0, r = r_0 = 1 - h(p_0), t = p_0$ and $\omega_{\max} = G(\alpha, \tau) = 2p_0(1-p_0)$ we have

$$K(p_0, p_1, 1 - h(p_0), \omega_{\max}, p_0) = h_2(\omega_{\max}) + g(p_0, p_0, \omega_{\max}) - 2h(p_0),$$

where

$$g(p, p, 2p(1-p)) = 2p(1-p) + [1 - 2p(1-p)]h\left[\frac{p^2}{1 - 2p(1-p)}\right]$$

It is possible to check that for $\omega_0 = 2p_0(1-p_0)$ the equality holds

$$K(p_0, p_1, 1 - h(p_0), \omega_0, p_0) = h_2(\omega_0) + \omega_0 + (1 - \omega_0)h\left(\frac{p_0^2}{1 - \omega_0}\right) - 2h(p_0) = 0.$$
(73)

We also have

$$[K(p_0, p_1, 1 - h(p_0), \omega_0, t)]'_t = \frac{1}{2} \log \frac{(1 - t)^2 - (1 - \omega_0 - p_0)^2}{t^2 - (\omega_0 - p_0)^2} - \log \frac{1 - p_1}{p_1}, \qquad (74)$$
$$[K(p_0, p_1, 1 - h(p_0), \omega_0, t)]''_{tt} < 0.$$

Therefore, for $t = p_0$ we have

$$[K(p_0, p_1, 1 - h(p_0), \omega_0, t)]'_{t=p_0} = \log \frac{1 - p_0}{p_0} - \log \frac{1 - p_1}{p_1} < 0, \qquad p_1 < p_0, \tag{75}$$

It follows from (73)-(75) that

$$K(p_0, p_1, 1 - h(p_0), \omega_0, t) > 0, \qquad t < p_0.$$

Therefore, the inequality (70) holds for any $r > r_0 = 1 - h(p_0)$ if $p_1 < p_0 \le 1/2$.

As a result, we get the conditional result:

Proposition 4. If the additive approximation (63) holds, then $r_{\text{crit}}(p_0, p_1) = 1 - h(p_0)$, $0 < p_1 < p_0 \le 1/2$.

Remark 6. It is possible to show that Theorem 1 and the formula (13) hold for any $p_1 < p_0 \leq 1/2$. For that purpose we can perform similarly to [11], using Lemma 2 and considering separately the case of equality in the formula (50) (essentially, it is equivalent to the considered in §6 case), and the case of inequality in the formula (50). Proof in the second case turns out to be too bulky (and oriented only to the binary channel BSC(p)). For that reason we omit that proof. Certainly, there should be a simpler proof.

APPENDIX

1. F u n c t i o n $g(t, p, \omega)$ a n d f o r m u l a (35). Consider codewords $\boldsymbol{x} = \boldsymbol{0}$ and \boldsymbol{x}_1 with $d(\boldsymbol{x}, \boldsymbol{x}_1) = w(\boldsymbol{x}_1) = \omega n$, and the set $S_{\boldsymbol{x}, \boldsymbol{x}_1}(t, p, \omega)$ from (34). We may assume that $\boldsymbol{x}_1 = (1, \ldots, 1, 0, \ldots, 0)$ and has first ωn "ones", and then $(1 - \omega)n$ "zeros". Let also $\boldsymbol{u} \in S_{\boldsymbol{x}, \boldsymbol{x}_1}(t, p, \omega)$ has $u_1 n$ "ones" on the first ωn positions, and then $u_2 n$ "ones" on the next $(1 - \omega)n$ positions. Since $u_1 + u_2 = t$, $\omega - u_1 + u_2 = p$, then

$$u_1 = \frac{t - p + \omega}{2}, \qquad u_2 = \frac{t + p - \omega}{2},$$
 (76)

and as $n \to \infty$ we get

$$\frac{1}{n}\log|S_{\boldsymbol{x},\boldsymbol{x}_1}(t,p,\omega)| = \frac{1}{n}\log\left[\binom{\omega n}{u_1 n}\binom{(1-\omega)n}{u_2 n}\right] = \omega h\left(\frac{u_1}{\omega}\right) + (1-\omega)h\left(\frac{u_2}{1-\omega}\right) + o(1) = g(t,p,\omega) + o(1),$$
(77)

where

$$g(t, p, \omega) = \omega h\left(\frac{t + \omega - p}{2\omega}\right) + (1 - \omega)h\left(\frac{t + p - \omega}{2(1 - \omega)}\right).$$
(78)

We also have

$$2g'_{\omega}(p,t,\omega) = -2\log\frac{1-\omega}{\omega} + \log\frac{(1-\omega)^2 - (1-t-p)^2}{\omega^2 - (t-p)^2},$$

$$2g'_t(p,t,\omega) = \log\frac{(1-t)^2 - (1-\omega-p)^2}{t^2 - (\omega-p)^2}, \qquad g''_{tt}(p,t,\omega) < 0, \qquad g''_{\omega\omega}(p,t,\omega) \le 0.$$
(79)

For the root ω_0 of the equation $g'_{\omega}(t, p, \omega) = 0$ we have

$$\omega_0 = \frac{p-t}{1-2t}, \qquad g(t, p, \omega_0) = h(t).$$
(80)

2. F u n c t i o n $\mu(R, \alpha, \omega)$. Introduce the function [14] $(0 \le \tau \le \alpha \le 1/2)$

$$G(\alpha, \tau) = 2 \frac{\alpha(1-\alpha) - \tau(1-\tau)}{1 + 2\sqrt{\tau(1-\tau)}} \ge 0.$$
 (81)

For α, τ , such that $0 \le \tau \le \alpha \le 1/2$ and $h_2(\alpha) - h_2(\tau) = 1 - R$, introduce the function [16]

$$\mu(R,\alpha,\omega) = h_2(\alpha) - 2 \int_0^{\omega/2} \log \frac{P + \sqrt{P^2 - 4Qy^2}}{Q} dy - (1-\omega)h_2\left(\frac{\alpha - \omega/2}{1-\omega}\right), \qquad (82)$$
$$P = \alpha(1-\alpha) - \tau(1-\tau) - y(1-2y), \qquad Q = (\alpha - y)(1-\alpha - y).$$

Denote the function $\delta_{GV}(R) \leq 1/2$ (Varshamov - Gilbert bound) as

$$1 - R = h_2(\delta_{GV}(R)), \qquad 0 \le R \le 1.$$
 (83)

Importance of the function $\mu(R, \alpha, \omega)$ and its relation to the code spectrum $\{B_i\}$ (see (65)) is described by the following variant of Theorem 3 from [15].

T h e o r e m 2 [15, Theorem 3]. For any (R, n)-code and any $\alpha \in [\delta_{GV}(R), 1/2]$ there exist $r_1(R, \alpha) > 0$ and ω , $0 < r_1(R, \alpha) \le \omega \le G(\alpha, \tau)$, where $h_2(\tau) = h_2(\alpha) - 1 + R$, and $G(\alpha, \tau)$ is defined in (81), such that

$$n^{-1}\log B_{\omega n} \ge \mu(R, \alpha, \omega) + o(1), \qquad n \to \infty.$$
(84)

For $\mu(R, \alpha, \omega)$ from (82) the non-integral representation (85)-(87) also holds.

Remark 7. Theorem 2 makes more precise Theorem 5 from [16] (see also [12, Theorem 2]. With $r_1 = 0$ Theorem 2 turns into Theorem 5 from [16]. In [15, теорема 3] there are estimates for $r_1(R, \alpha) > 0$.

Proposition 5 [11, Proposition 3]. For the function $\mu(R, \alpha, \omega)$ the representation holds

$$\mu(R,\alpha,\omega) = (1-\omega)h_2\left(\frac{\alpha-\omega/2}{1-\omega}\right) - h_2(\alpha) + 2h_2(\omega) + \omega\log\frac{2\omega}{e} - T(A,B,\omega), \quad (85)$$

where

$$T(A, B, \omega) = \omega \log(v - 1) - (1 - \omega) \log \frac{v^2 - A^2}{v^2 - B^2} + B \log \frac{v + B}{v - B} - A \log \frac{v + A}{v - A} - \frac{(v - 1)(B^2 - A^2)}{(v^2 - B^2) \ln 2},$$

$$v = \frac{\sqrt{B^2 \omega^2 - 2a_1 \omega + a_1^2} + a_1}{\omega}, \quad a_1 = \frac{B^2 - A^2}{2}.$$
(86)

and

$$h_2(\alpha) - h_2(\tau) = 1 - R, \quad A = 1 - 2\alpha, \quad B = 1 - 2\tau, \quad 0 \le \tau \le \alpha \le 1/2.$$
 (87)

We have for any $\alpha_0(R) \leq \alpha < 1/2$ and $\omega > 0$

$$\frac{d\mu(R,\alpha,\omega)}{d\alpha} > 0, \qquad \alpha_0(R) = h_2^{-1}(1-R).$$

For any $\alpha > 0$ and R > 0 we also have $\mu(R, \alpha, 0) = 0$ and $\mu'_{\omega}(R, \alpha, \omega)\Big|_{\omega=0} > 0$. Moreover, for any $0 \le \tau \le \alpha \le 1/2$ and $0 < \omega < G(\alpha, \tau)$

$$\mu_{\omega^2}''(R,\alpha,\omega) > 0.$$

3) For any $\omega > 0$ we have $\mu(0, 1/2, \omega) = 0$.

ACKNOWLEDGEMENTS

The author appreciates Shun Watanabe and the reviewer for useful discussions and constructive critical remarks which improved the paper.

REFERENCES

- 1. Burnashev M.V., Amari S., Han T. S., On some testing of hypotheses problems with information constraints, Theory of Probab. and Its Applications, 45, no. 4, pp. 625-638, 2000.
- 2. Burnashev M.V., Han T. S., Amari S., On some estimation problems with information constraints, Theory of Probab. and Its Applications, 46, no. 2, pp. 233-246, 2001.
- Ahlswede R., Csiszár I. Hypothesis testing with communication constraints. IEEE Trans. on Inform. Theory, 1986, v. IT-32, No. 4, p. 533-542.
- 4. Han T. S., Kobayashi K. Exponential-type error probabilities for multiterminal hypothesis testing. IEEE Trans. on Inform. Theory, 1989, v. IT-35, No. 1, p. 2-14.
- Ahlswede R., Burnashev M. V. On Minimax estimation in the presence of side information about remote data. - The Annals of Statistics, 1990, v. 18, No. 1, p. 141-171.
- Han T. S., Amari S. Statistical inference under multiterminal data compression. -IEEE Trans. on Inform. Theory, 1998, v. IT-44, No. 6, p. 2300-2324.
- Han T. S., Shimokawa H., Amari S. Error bounds of hypothesis testing with data compression. Proc. IEEE Int. Symp. Information Theory. Trondheim, Norway, 1994, p. 29.
- 8. Watanabe S. Neyman-Pearson Test for Zero-Rate Multiterminal Hypothesis Testing. 2017, arXiv: 1611.08175v2.

- Elias P. Coding for noisy channels // IRE Conv. Rec. 1955. March, P. 37-46. Reprinted in D. Slepian, Ed., Key papers in the development of information theory, IEEE Press, 1974, P. 102-111.
- 10. Gallager R. G. Information theory and reliable communication. Wiley, NY, 1968.
- 11. Burnashev M.V. On the BSC Reliability Function: Expanding the Region Where It Is Known Exactly // Probl. Peredachi Inf., 2015, vol. 51, no. 4, pp. 3–22.
- 12. Burnashev M.V. Code Spectrum and the Reliability Function: Binary Symmetric Channel // Probl. Peredachi Inf., 2006, vol. 42, no. 4, pp. 3–22.
- 13. Burnashev M.V. Sharpening of an Upper Bound for the Reliability Function of a Binary Symmetric Channel // Probl. Peredachi Inf., 2005, vol. 41, no. 4, pp. 3–22.
- McEliece R. J., Rodemich E. R., Rumsey H., Jr., Welch L. R. New Upper Bounds on the Rate of a Code via the Delsarte–MacWilliams Inequalities // IEEE Trans. Inform. Theory. 1977. V. 23. № 2. P. 157–166.
- 15. Burnashev M.V. On Lower Bounds on the Spectrum of a Binary Code // Probl. Peredachi Inf., 2019, vol. 55, no. 4, pp. 76–85.
- 16. Litsyn S. New Bounds on Error Exponents // IEEE Trans. Inform. Theory. 1999. V. 45. № 2. P. 385–398.