Computing Rational Generating Function of a Solution to the Initial Value Problem of Two-dimensional Difference Equation with Constant Coefficients

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Abstract. Algorithms for computing rational generating functions of solutions of one-dimensional difference equations are well-known and easy to implement. We propose an algorithm for computing rational generating functions of solutions of two-dimensional difference equations in terms of initial data of the corresponding initial value problems. The crucial part of the algorithm is the reconstruction of infinite one-dimensional initial data on the basis of finite input data. The proposed technique can be used for the development of similar algorithms in higher dimensions. We furnish examples of implementation of the proposed algorithm.

1 Introduction

The study of linear difference equations and their solutions is an important topic of modern computer algebra (see, e.g., [1, 2]).

The method of generating functions is a powerful tool for the study of differential equations with applications in the theory of discrete dynamical systems and in the enumerative combinatorial analysis. It allows one to apply the methods of complex analysis to the problems of enumerative combinatorics.

The one-dimensional case is well-studied (see [8, 9]) due to the absence of geometric obstacles. In [16], A. Moivre considered the power series

$$f(0) + f(1)z + \ldots + f(k)z^k + \ldots$$

with the coefficients $f(0), f(1), \ldots$ satisfying the difference equation

$$c_m f(x+m) + c_{m-1} f(x+m-1) + \ldots + c_0 f(x) = 0, \quad x = 0, 1, 2 \ldots, \quad (1)$$

where $c_m \neq 0$, and $c_j \in \mathbb{C}$ are constants. He proved that this series always represents a rational function (De Moivre's Theorem, [16]).

In the multidimensional case, which is not adequately explored (see [5, 10, 11, 13]), the rational generating functions are the most useful class of generating functions according to the Stanley's hierarchy (see [19]). A broad class of two-dimensional sequences that lead to rational generating functions is well-known in the enumerative combinatorics. For example, one can consider the problem of finding a number of lattice paths, the problem on generating trees with marked labels, Bloom's strings, a number of placement of the pieces on the chessboard etc. (see [3, 4, 7]).

Generating functions for multiple sequences with elements which could be expressed in terms of rational, exponential functions or gamma function, form a wide class of hypergeometric-type functions [6, 18]. Their study leads to the problem of solving overdetermined systems of linear equations with polynomial coefficients.

A multidimensional analogue of the De Moivre Theorem was formulated and proved in [12]. We now give some definitions and notations that we will need for formulating the main result.

2 Known Results and Formulation of the Problem

Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, where $\mathbb{Z}^n = \mathbb{Z} \times \ldots \times \mathbb{Z}$ is the n-dimensional integer lattice. Let $A = \{\alpha\}$ be a finite set of points in $\mathbb{Z}^n$. Let $f(x)$ be a function of integer arguments $x = (x_1, \ldots, x_n)$ with constant coefficients c_α. By a difference equation with respect to the unknown function $f(x)$ we call the equation of the form

$$\sum_{\alpha \in A} c_\alpha f(x + \alpha) = 0. \tag{2}$$

In the present work we consider the case when the set A belongs to the positive octant $\mathbb{Z}_0^n = \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{Z}, x_i \geqslant 0, i = 1, \ldots, n\}$ of the integer lattice and satisfies to the condition:

There exists a point $m = (m_1, \ldots, m_n) \in A$ such that

for any $\alpha \in A$ the inequalities $\alpha_j \leqslant m_j,\ j = 1, 2, \ldots, n$ hold. (3)

Denote the *characteristic polynomial* of (2) by

$$P(z) = \sum_{\alpha \in A} c_\alpha z^\alpha = \sum_{\alpha \in A} c_{\alpha_1, \ldots, \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

The generating function (or the z-transformation) of the function $f(x)$ in integer variables $x \in \mathbb{Z}_0^n$ is defined as follows:

$$F(z) = \sum_{x \geqslant 0} \frac{f(x)}{z^{x+I}}, \quad \text{where } I = (1, \ldots, 1).$$

For example, a solution of the difference equation

$$f(x_1+1,x_2+1)-f(x_1+1,x_2)-f(x_1,x_2+1)+f(x_1,x_2)=0$$

is an arbitrary function of the form $f(x_1,x_2)=\varphi(x_1)+\psi(x_2)$, where φ and ψ are arbitrary univariate functions of integer arguments and the corresponding generating series is in general divergent.

Define the "initial data set" for the difference equation (2) that satisfies the condition (3), as follows:

$$X_0=\{\tau\in\mathbb{Z}^n\colon \tau\geqslant 0,\ \tau\ngeqslant m\}.$$

Here $\ngeqslant$ means that the point τ belongs to the complement of the set defined by the system of inequalities

$$\tau_j\geqslant m_j,\ j=1,\dots,n.$$

The initial value problem is set up as follows: we need to find the solution $f(x)$ of the difference equation (2), which coincides with a given function $\varphi(x)$ on X_0:

$$f(x)=\varphi(x),\ x\in X_0. \tag{4}$$

It is easy to show (see, e.g., [11]), that if the condition (3) holds, then the initial value problem (2), (4) has the unique solution. The solvability of the problem (2), (4) without the constraints (3) has been studied in [5].

We now proceed with some more notations that we will need later on.

Let $J=(j_1,\dots,j_n)$, where $j_k\in\{0,1\}$, $k=1,\dots,n$, be an ordered set. With each such set J we associate the face Γ_J of the n-dimensional integer parallelepiped

$$\Pi_m=\{x\in\mathbb{Z}^n : 0\leqslant x_k\leqslant m_k, k=1,\dots,n\}$$

as follows:

$$\Gamma_J=\{x\in\Pi_m\colon x_k=m_k,\ \text{if } j_k=1,\\ \text{and } x_k<m_k,\ \text{if } j_k=0\}.$$

For example, $\Gamma_{(1,\dots,1)}=\{m\}$ and

$$\Gamma_{(0,\dots,0)}=\{x\in\mathbb{Z}^n\colon 0\leqslant x_k<m_k,\ k=1,\dots,n\}.$$

It is easy to check that $\Pi_m=\bigcup\limits_J\Gamma_J$ and for any J,J' the corresponding faces do not intersect: $\Gamma_J\cap\Gamma_{J'}=\emptyset$.

Let $\Phi(z)=\sum\limits_{\substack{\tau\geqslant 0\\ \tau\ngeqslant m}}\dfrac{\varphi(\tau)}{z^{\tau+I}}$ be the generating function of the initial data of the solution of (2), (4). With each point $\tau\in\Gamma_J$ we associate the series

$$\Phi_{\tau,J}(z)=\sum_{y\geqslant 0}\frac{\varphi(\tau+Jy)}{z^{\tau+Jy+I}},$$

and with each face Γ_J we associate the series

$$\Phi_J(z) = \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z).$$

If we extend the domain of $\varphi(x)$ by zero on $\mathbb{Z}^n_+ \setminus X_0$, then the generating function of the initial data can be written down as the sum

$$\Phi(z) = \sum_J \Phi_J(z) = \sum_J \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z).$$

Theorem 1. *The generating function $F(z)$ of the solution of the problem (2),(4) under the assumption (3) and the generating function $\Phi(z)$ of the initial data are connected by the formula*

$$P(z)F(z) = \sum_J \sum_{\tau \in \Gamma_J} \Phi_{\tau,J}(z) P_\tau(z), \tag{5}$$

where $P_\tau(z) = \sum\limits_{\substack{\alpha \leqslant m \\ \alpha \nleqslant \tau}} c_\alpha z^\alpha$ are polynomials.

For $n = 1$ it is easy to verify the statement of the Theorem 1. Indeed, in this case the generating function

$$F(z) = \sum_{x=0}^{\infty} \frac{f(x)}{z^x}$$

of the solution $f(x)$ to the initial value problem (1) with the coefficients $(c_0, c_1, \ldots, c_m)$ and given initial data $(\varphi(0), \varphi(1), \ldots, \varphi(m-1))$ is a rational function:

$$F(z) = \frac{\sum\limits_{\alpha=1}^{m} \sum\limits_{x=0}^{\alpha-1} \frac{c_\alpha \varphi(x)}{z^{x-\alpha}}}{\sum\limits_{\alpha=0}^{m} c_\alpha z^\alpha}. \tag{6}$$

For $n = 2$ Theorem 1 was proved in [15] in connection with studying the rational Riordan arrays. For $n > 1$ the proof was given in [12]. The properties of generating function of solutions of a difference equation in rational cones of integer lattice were studied by T. Nekrasova (see, e.g., [17]).

Theorem 1 yields the following multidimensional analog of the De Moivre Theorem that is essential for the construction of algorithm:

Theorem 2. *The generating function $F(z)$ of the solution of the initial value problem (2), (4) under the assumption (3) is rational if and only if the generating function $\Phi(z)$ of the initial data is rational.*

The proof of Theorem 2 was given in [12].

3 Description of the Algorithm

For $n = 1$ the expression (6) consists of a finite number of terms, which makes the corresponding algorithm and computational procedure trivial. In this case the input data consists of two finite sets of numbers, namely: coefficients of the difference equation and the initial data. The output data of the algorithm is a rational function (6).

In the case when $n > 1$, the initial data set X_0 is infinite. For $n = 2$ the algorithm for computing the generating function $F(z)$ can be reduced to computation of a finite number of one-dimensional generating functions of sequences with elements along the coordinate axes. These elements are uniquely determined by coefficients of the corresponding one-dimensional difference equation and by finite set of the corresponding initial data (that is different for each sequence).

We now describe the structure of input data of the algorithm which consists of the matrices c, C and $InData$.

The matrix $c = (c_{\alpha_1,\alpha_2})$, where $0 \leqslant \alpha_1 \leqslant m_1$, $0 \leqslant \alpha_2 \leqslant m_2$ has size $(m_1 + 1) \times (m_2 + 1)$. It's elements c_{α_1,α_2} equal to the coefficients of two-dimensional difference equation (2) if $(\alpha_1, \alpha_2) \in A$ and equal 0 otherwise.

The coefficients of one-dimensional difference equations (that allow one to determine infinite initial data of two-dimensional initial value problem) are given by the matrix C. It consists of two columns: elements of the first column define starting point and direction of the corresponding one-dimensional initial data while elements of the second column are the coefficients of the one-dimensional difference equations, which define initial data on the corresponding direction.

We now consider the structure of the matrix C more precisely. Suppose that for each $\xi_1 = 0, \dots, m_1-1$ there exist numbers $\mu_{\xi_1} \in \mathbb{N}$ and $c^{\xi_1}_{\mu_{\xi_1}}, c^{\xi_1}_{\mu_{\xi_1}-1}, \dots, c^{\xi_1}_0 \in \mathbb{C}$ such that a subset of the initial data $\{\varphi(\xi_1, x_2)\}_{x_2=0}^{\infty}$ along the horizontal axis satisfies one-dimensional initial value problem with the coefficients $(c^{\xi_1}_0, c^{\xi_1}_1, \dots, c^{\xi_1}_{\mu_{\xi_1}})$ of the difference equation and with initial data $(\varphi(\xi_1, 0), \varphi(\xi_1, 1), \dots, \varphi(\xi_1, \mu_{\xi_1} - 1))$. Then (6) implies that for each $\xi_1 = 0, \dots, m_1 - 1$ the one-dimensional generating function of the sequence $\{\varphi(\xi_1, x_2)\}_{x_2=0}^{\infty}$ is rational.

Similarly, suppose that for each $\xi_2 = 0, \dots, m_2 - 1$ there exist numbers $\nu_{\xi_2} \in \mathbb{N}$ and $d^{\xi_2}_{\nu_{\xi_2}}, d^{\xi_2}_{\nu_{\xi_2}-1}, \dots, d^{\xi_2}_0 \in \mathbb{C}$ such that a subset of the initial data $\{\varphi(x_1, \xi_2)\}_{x_1=0}^{\infty}$ along the vertical axis are satisfies one-dimensional initial value problem with the coefficients $(d^{\xi_2}_0, d^{\xi_2}_1, \dots, d^{\xi_2}_{\nu_{\xi_2}})$ of the difference equation and with initial data $(\varphi(0, \xi_2), \varphi(1, \xi_2), \dots, \varphi(\nu_{\xi_1} - 1, \xi_2))$. Then (6) implies that for each $\xi_2 = 0, \dots, m_2 - 1$ the one-dimensional generating function of the sequence $\{\varphi(\xi_1, x_2)\}_{x_2=0}^{\infty}$ is rational function. Both these functions can be easily computed.

Consequently, the matrix C has the following structure

$$C = \begin{pmatrix} (1,0) & (c^0_{\mu_0}, c^0_{\mu_0-1}, \ldots, c^0_0) \\ (2,0) & (c^1_{\mu_1}, c^1_{\mu_1-1}, \ldots, c^1_0) \\ \ldots & \ldots \\ (m_1,0) & (c^{m_1-1}_{\mu_{m_1-1}}, c^{m_1-1}_{\mu_{m_1-1}-1}, \ldots, c^{m_1-1}_0) \\ (0,1) & (d^0_{\nu_0}, d^0_{\nu_0-1}, \ldots, d^0_0) \\ (0,2) & (d^1_{\nu_1}, d^1_{\nu_1-1}, \ldots, d^1_0) \\ \ldots & \ldots \\ (0,m_2) & (d^{m_1-1}_{\nu_{m_1-1}}, d^{m_1-1}_{\nu_{m_1-1}-1}, \ldots, d^{m_1-1}_0) \end{pmatrix}.$$

The initial data of one-dimensional difference equations are given by the matrix $InData$ of the size $M_1 \times M_2$, where $M_1 = \max\{\mu_0, \ldots, \mu_{m_1-1}\}$ and $M_2 = \max\{\nu_0, \ldots, \nu_{m_2-1}\}$.

Note that some elements of $InData$ can be dependent from each other (or, in other words, some elements can be derived from another ones), so that it is necessary to define only a part of its' elements. We illustrate this fact on the following example.

Suppose that the subsets of the initial data of two-dimensional difference equation along the horizontal axis are given by the following three one-dimensional difference equations

$$c^0_3\varphi(x+3,0) + c^0_2\varphi(x+2,0) + c^0_1\varphi(x+1,0) + c^0_0\varphi(x,0) = 0,$$

$$c^1_2\varphi(x+2,1) + c^1_1\varphi(x+1,1) + c^1_0\varphi(x,1) = 0,$$

$$c^2_2\varphi(x+2,2) + c^2_1\varphi(x+1,2) + c^2_0\varphi(x,2) = 0.$$

Similarly, let the subsets of the initial data along the vertical axis be given by the two one-dimensional difference equations

$$d^0_2\varphi(0,y+2) + d^0_1\varphi(0,y+1) + d^0_0\varphi(0,y) = 0,$$

$$d^1_3\varphi(1,y+3) + d^1_2\varphi(1,y+2) + d^1_1\varphi(1,y+1) + d^1_0\varphi(1,y) = 0.$$

Then the initial data of one-dimensional difference equations is given by a matrix

$$InData = \begin{pmatrix} * & \varphi(1,2) & * \\ \varphi(0,1) & \varphi(1,1) & * \\ \varphi(0,0) & \varphi(1,0) & \varphi(2,0) \end{pmatrix}.$$

The computational procedure does not read elements denoted by $*$, but compute them from the other entries of the matrix and corresponding difference equations.

Consequently, the input data of the algorithm is finite.

Now we rewrite (5) in the form that is convenient for the implementation of the algorithm:

$$F(z_1,z_2)=\left(\sum_{\xi_1=0}^{m_1-1}\sum_{\xi_2=0}^{m_2-1}\frac{P_{\xi_1,\xi_2}(z_1,z_2)}{z_1^{\xi_1+1}z_2^{\xi_2+1}}\varphi(\xi_1,\xi_2)+\right.$$
$$+\sum_{\xi_1=0}^{m_1-1}\frac{P_{\xi_1,m_2}(z_1,z_2)}{z_1^{\xi_1+1}}\Phi_{m_2}^{\xi_1}(z_2)+$$
$$\left.+\sum_{\xi_2=0}^{m_2-1}\frac{P_{m_1,\xi_2}(z_1,z_2)}{z_2^{\xi_2+1}}\Psi_{m_1}^{\xi_2}(z_1)\right)/P(z), \quad (7)$$

where

$$\Phi_{m_2}^{\xi_1}(z_2)=\frac{z^{m_2}\sum\limits_{\alpha=1}^{\mu_{\xi_1}}\sum\limits_{x_2=0}^{\alpha-1}\frac{c_\alpha^{\xi_1}\varphi(\xi_1,x_2+m_2)}{z_2^{x_2-\alpha}}}{\sum\limits_{\alpha=0}^{\mu_{\xi_1}}c_\alpha^{\xi_1}z_2^\alpha} \quad (8)$$

and

$$\Psi_{m_1}^{\xi_2}(z_1)=\frac{z^{m_1}\sum\limits_{\alpha=1}^{\nu_{\xi_2}}\sum\limits_{x_1=0}^{\alpha-1}\frac{d_\alpha^{\xi_2}\varphi(x_1+m_1,\xi_2)}{z_1^{x_1-\alpha}}}{\sum\limits_{\alpha=0}^{\nu_{\xi_2}}d_\alpha^{\xi_2}z_1^\alpha}. \quad (9)$$

The right hand side of (7) is split into 3 groups. The first group consists of a finite number of summands. The second and the third groups require computing of a finite number of one-dimensional generating functions (8) and (9).

The algorithm itself consists of three procedures that we present below.

The first procedure (Algorithm 1) computes the missing terms $\varphi(\xi_1,\xi_2)$ in the first sum of (7). It is also used for computing generating functions in the second and third sums of (7).

Algorithm 3 is the main procedure that assemblies the two-dimensional generating function from one-dimensional generating functions obtained by Algorithm 2.

4 Experiments

The algorithm was implemented in Maple 2015. It was tested on the following examples.

Example 1. The binomial coefficients

$$C_n^k=\frac{n!}{k!(n-k)!}$$

Algorithm 1 The algorithm for computing the initial data $\varphi(x_0, y_0)$ at arbitrary point $(x_0, y_0) \in X_0$.

Input: Matrixes c, C, $InData$ and a point $(x_0, y_0) \in X_0$.
Output: A value of function of initial data $\varphi(x_0, y_0)$.

```
procedure φ(c, C, InData, (x_0, y_0))
    m_1 := is a number of rows of the matrix c
    m_2 := is a number of columns of the matrix c
    if x_0 < m_1 then
        for i from 1 to m_1 + m_2 do
            if c[i, 1] = (x_0 + 1, 0) then
                c_0 := C[i, 2]
            end if
        end for
        φ_0 := x_0-th column of InData
        l_0 := y_0
    end if
    if y_0 < m_2 then
        for i from 1 to m_1 + m_2 do
            if c[i, 1] = (0, y_0 + 1) then
                c_0 := C[i, 2]
            end if
        end for
        φ_0 := y_0-th row of InData
        l_0 := x_0
    end if
    l := is a length of c_0
    for i from l to l_0 do
        φ_0(i) := − Σ_{j=0}^{l−1} (c_0[j] / c_0[m]) φ_0(i + j − l)
    end for
    return φ_0(l_0)
end procedure
```

Algorithm 2 The algorithm for computing one-dimensional generating function according to formulas (8) or (9).

Input: A vector M of coefficients of a difference equation (with increasing order of indices); a vector Φ of initial data (with increasing order of indices) of the length equal to the length of $M-1$; an integer $start$ (the initial value of summation); an integer $t \in \{1, 2\}$; a variable z_t.

Output: Generating function, given by formulae (8) or (9) according to t.

```
procedure GF(M, Φ, start, t, z_t)
    F := 0
    n := length of M − 1
    for α from 1 to n + 1 do
        c_{α−1} := M[α]
    end for
    for α from 1 to n do
        φ[α − 1] := Φ[α]
    end for
    for α from n to start + n − 1 do
        φ[α] := 0
        for j from 1 to n do
            φ[α] := φ[α] − (c_{j−1}/c_n) · φ[α − n + j − 1]
        end for
    end for
    for α from 0 to n − 1 do
        φ[α] := φ[start + α]
    end for
    for α from 1 to n do
        for x from 0 to α − 1 do
            F := F + (c_α · φ[x]) / z_t^{x−α+1}
        end for
    end for
    Q := 0
    for α from 0 to n do
        Q := Q + c_α · z_t^α
    end for
    F := F · z_t^{start}/Q
    return F
end procedure
```

Algorithm 3 The algorithm for computing two-dimensional generating function of the solution of the initial value problem (2), (4).

Input: Matrices c, C, $InData$,
Output: Generating function $F(z)$ of the solution to the Cauchy problem.

procedure $GenFunc(c,\ C,\ InData)$
    $m_1 :=$ is a number of rows in/of the matrix c
    $m_2 :=$ is a number of cloumns in/of the matrix c
    $F := 0$
    for x_1 from 1 to $m_1 - 1$ **do**
        for x_2 from 1 to $m_2 - 1$ **do**
            $F := F + \dfrac{P_{x_1,x_2}(z_1,z_2)}{z_1^{x_1} z_2^{x_2}} \cdot \varphi(c, C, InData, (x_0, y_0))$
        end for
    end for
    for ξ_1 from 0 to $m_1 - 1$ **do**
        $\varPhi := (\xi_1 + 1)$-th column of $InData$
        $M := C[i,2]$ such that $C[i,1] = [\xi_1 + 1, 0]$
        for i from 1 to $\xi_1 + \xi_2$ **do**
            if $C[i,1] = (\xi_1 + 1, 0)$ **then**
                $M := C[i,2]$
            end if
        end for
        $F := F + \frac{P_{\xi_1,m_2}(z_1,z_2)}{z_1^{\xi_1+1}} \cdot GF(M, \varPhi, m_2, 2)$
    end for
    for ξ_2 from 0 to $m_2 - 1$ **do**
        $\varPhi := (\xi_2 + 1)$-th row of $InData$
        for i from 1 to $\xi_1 + \xi_2$ **do**
            if $C[i,1] = (0, \xi_2 + 1)$ **then**
                $M := C[i,2]$
            end if
        end for
        $F := F + \frac{P_{m_1,\xi_2}(z_1,z_2)}{z_2^{\xi_2+1}} \cdot GF(M, \varPhi, m_1, 1)$
    end for
    $F := F / P(z_1, z_2)$
    return F
end procedure

give a solution to the Cauchy problem

$$f(x+1,y+1)-f(x,y+1)-f(x,y)=0$$

with the initial data

$$\varphi(x,y)=\begin{cases}1, & x\geqslant 0, y=0;\\ 0, & x=0, y\geqslant 0.\end{cases}$$

In the example $m_1=m_2=1$ and the set of initial data

$$X_0=\{(x_1,x_2)\in\mathbb{Z}^2_+\colon (x_1,x_2)\ngeqslant(1,1)\}.$$

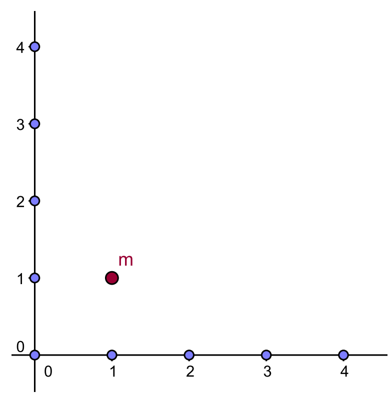

Fig. 1. The initial data set X_0

The input data for the computational procedure:

$$c=\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix},$$

$$InData=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

$$C=\begin{pmatrix}(0,1) & (-1,1)\\ (1,0) & (1,0)\end{pmatrix}.$$

The result is the generating function of the binomial coefficients:

$$F(z,w)=\frac{1}{zw-w-1}.$$

Example 2. Bloom studies the number of singles in all 2^x x-length bit strings [4], where a single is any isolated 1 or 0, i.e. any run of length 1. Let $r(x, y)$ be the number of x-length bit strings beginning with 0 and having y singles. Apparently $r(x, y) = 0$ if $x < y$. Then $r(x, y)$ is:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 & \dots \\ 0 & 0 & 0 & 1 & 0 & 4 & 4 & \dots \\ 0 & 0 & 1 & 0 & 3 & 3 & 9 & \dots \\ 0 & 1 & 0 & 2 & 2 & 5 & 8 & \dots \\ 1 & 0 & 1 & 1 & 2 & 3 & 5 & \dots \end{pmatrix},$$

where the element $r(0, 0)$ is in left lower corner.

In [4] D. Bloom proves that $r(x, y)$ is a solution to the Cauchy problem

$$r(x + 2, y + 1) - r(x + 1, y + 1) - r(x + 1, y) - r(x, y + 1) + r(x, y) = 0$$

with the initial data

$$\begin{aligned} &\varphi(0, 0) = 1, \quad \varphi(1, 0) = 0, \\ &\varphi(x, 0) = \varphi(x - 1, 0) + \varphi(x - 2, 0), \ x \geqslant 2, \\ &\varphi(0, y) = 0, \ y \geqslant 1, \\ &\varphi(1, 1) = 1, \quad \varphi(1, y) = 0, \ y \geqslant 2. \end{aligned}$$

In this case the input data of the algorithm will be:

$$c = \begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$InData = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} (0, 1) & (-1, -1, 1) \\ (1, 0) & (0, 1) \\ (2, 0) & (0, 0, 1) \end{pmatrix}.$$

The result is the generating function of the considered initial value problem:

$$F(z, w) = \frac{z - 1}{z^2 w - zw - w - z + 1}.$$

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