MATHEMATICAL MODELING OF SEMICONDUCTOR LASERS

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Abstract. In semiconductor lasers the electrostatic potential φ is harmonic function both in the p-region Ω_p and in the n-region Ω_n; however, across the photoactive layer Γ separating these regions both φ and its normal derivative experience jumps which are determined implicitly by a system of differential and functional equations on Γ. It is proved that the mathematical formulation of the problem is well posed, i.e., it has a unique solution for the range of parameters which occur in the physical problem.

Key words. semiconductor, laser, electrostatic potential.

AMS(MOS) subject classifications. 35J65, 78A60, 81G10

§1. The mathematical model. In this paper we study the mathematical aspects of a coupled electrical and optical model for conversion of electrical energy into coherent optical energy by solid state device. The model, described in [4], consists of a semiconductor diode with a thin photoactive layer separating the p- and n- regions. Simplifying to 2-dimensions, such a device is (quite crudely) described in Figure 1, where Ω_p is the p-type semiconductor and Ω_n is the n-type semiconductor.

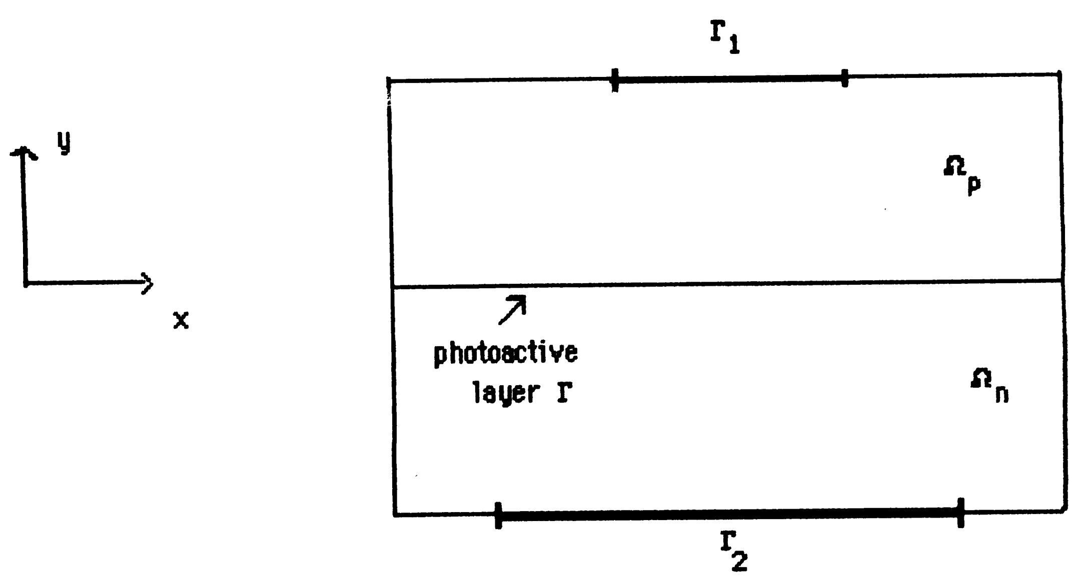

FIGURE 1

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The electrostatic potential is denoted by φ_p in Ω_p and by φ_n in Ω_n. We apply Ohmic contact: On $\Gamma_1 = \{(x,b); |x| \le d\}$ voltage V is applied ($V > 0$) and on $\Gamma_2 = \{(x,-c); |x| \le e\}$ the prescribed potential is 0. The device produces a laser beam at the active layer Γ. For simplicity it is assumed that the active layer has negligible width. Thus

$$\Omega_p = \{-a < x < a, 0 < y < b\} , \qquad \Omega_n = \{-a < x < a, -c < y < 0\} ,$$

and

$$\begin{aligned} \nabla(\sigma_p \nabla \varphi_p) &= 0 \quad \text{in} \quad \Omega_p , \\ \nabla(\sigma_n \nabla \varphi_n) &= 0 \quad \text{in} \quad \Omega_n , \end{aligned}$$

where σ_p, σ_n are the conductivities in Ω_p and Ω_n respectively; we shall henceforth assume that σ_p, σ_n are constants. The boundary conditions on $\partial\big(\overline{\Omega_p \cup \Omega_n}\big)$ are:

$$\begin{aligned} &\varphi_p = V && \text{on} \quad \Gamma_1 , \\ &\varphi_n = 0 && \text{on} \quad \Gamma_2 , \\ &\frac{\partial \varphi_p}{\partial n} = 0 && \text{on} \quad \partial\Omega_p \backslash (\Gamma \cup \Gamma_1) , \\ &\frac{\partial \varphi_n}{\partial n} = 0 && \text{on} \quad \partial\Omega_n \backslash (\Gamma \cup \Gamma_2) \end{aligned}$$

where $\Gamma = \{(x,0); -a < x < a\}$ and $\partial/\partial n$ is the normal derivative.

On the active layer Γ ,

$$\begin{aligned} &\varphi_p - \varphi_n = \Psi_n - \Psi_p , \\ &\frac{1}{q}\frac{\partial}{\partial x} J_p = G_p - U_p , \\ &\frac{1}{q}\frac{\partial}{\partial x} J_n = U_n - G_n , \end{aligned}$$

where q = unit charge per particle, Ψ_p, G_p, U_p are respectively the quasi-Fermi level, the generation, and recombination for holes, and Ψ_n, G_n, U_n are similarly defined for electrons.

Denote by p and n the hole and electron number densities. Then, on Γ,

$$p = N_v F_{1/2}\left(\frac{E_v - \Psi_p}{kT}\right) , \quad n = N_c\, F_{1/2}\left(\frac{\Psi_n - E_c}{kT}\right)$$

where

$$E_{1/2}(z) = \int_0^\infty \frac{\sqrt{\eta}}{1+e^{\eta - z}}\, d\eta \qquad \text{(Fermi function)} ,$$

$$J_p = p\mu_p \frac{\partial \Psi_p}{\partial x} , \quad J_n = n\mu_n \frac{\partial \Psi_n}{\partial x} ,$$

$$G_p = \frac{j_p}{qt} = \frac{\sigma_p}{qt} \frac{\partial \varphi_p}{\partial y} ,$$

$$G_n = \frac{j_n}{qt} = \frac{\sigma_n}{qt} \frac{\partial \varphi_n}{\partial y} \quad (t = \text{thickness of the active layer}),$$

$$U_p = U_n = U_{\text{trap}} + U_{\text{spon}} + U_{\text{stim}} = \frac{1}{\tau} f(p,n) ,$$

$$p = n + N_a^- - N_d^+ \quad (N_a^- > N_d^+) .$$

The function $F_{1/2}(z)$ satisfies:

(1.1)

$$F'_{1/2}(z) > 0 , \quad F_{1/2}\,(z) \sim \text{const}\,.e^z \quad \text{if } z < 0 \text{ and } |z| \text{ large, } F_{1/2}(z) \to \infty \text{ if } z \to \infty .$$

For simplicity we shall take

$$F_{1/2}\,(z) \equiv e^z ; \tag{1.2}$$

however, all our results extend to the case of any function $F_{1/2}$ satisfying (1.1).

Assuming (1.2) we then have

$$p = N_v e^{\frac{E_v - \Psi_p}{kT}} = \widetilde{N}_v\, e^{-\Psi_p/kT}$$

where k is the Boltzmann constant and T the absolute temperature, and similarly

$$n = \widetilde{N}_c e^{\Psi_n/kT} .$$

Eliminating Ψ_p, Ψ_n, we get the following relations on Γ:

$$\varphi_p - \varphi_n = \log \frac{pn}{\widetilde{N}_c \widetilde{N}_n} ,$$

$$p = n + N_a^- - N_d^+ ,$$

$$(kT)\, p_{xx} = -\frac{\sigma_p}{\mu_p t} \frac{\partial \varphi_p}{\partial y} + \frac{q}{\mu_p t} f(p,n) ,$$

$$(kT)\, n_{xx} = -\frac{\sigma_n}{\mu_n t} \frac{\partial \varphi_n}{\partial y} + \frac{q}{\mu_n t} f(p,n) .$$

Set

$$
\begin{aligned}
&p = \sqrt{\widetilde{N}_c \widetilde{N}_v}\ \hat{p}\ , \quad n = \sqrt{\widetilde{N}_c \widetilde{N}_v}\ \hat{n}\ , \\
&x = a\hat{x}\ ,\ y = a\hat{y}\ , \\
&\hat{f}(\hat{n}) = \frac{1}{\sqrt{\widetilde{N}_c \widetilde{N}_v}}\ f\left(\sqrt{\widetilde{N}_c \widetilde{N}_v}\ \hat{p}\ , \sqrt{\widetilde{N}_c \widetilde{N}_v \hat{n}}\right)\Big|_{\hat{p}=\hat{n}+\frac{N_a^- - N_d^+}{\sqrt{\widetilde{N}_c\ \widetilde{N}_v}}}\ .
\end{aligned}
$$

Then

$$
\begin{aligned}
\varphi_p - \varphi_n &= \log \hat{n}(\hat{n} + \beta)\ , \\
\hat{n}_{\hat{x}\hat{x}} &= -A_p\ \frac{\partial \varphi_p}{\partial \hat{y}} + B_p \hat{f}(\hat{n}), \\
\hat{n}_{\hat{x}\hat{x}} &= -A_n \frac{\partial \varphi_n}{\partial \hat{y}} + B_n \hat{f}(\hat{n})\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
&\beta = \frac{N_a^- - N_d^+}{\sqrt{\widetilde{N}_c \widetilde{N}_v}}\ , \ A_p = \frac{\sigma_p a}{\mu_p \sqrt{\widetilde{N}_c \widetilde{N}_v}\ tkT}\ , \ A_n = \frac{\sigma_n a}{\mu_n \sqrt{\widetilde{N}_c \widetilde{N}_v}\ tkT}\ , \\
&B_p = \frac{qa^2}{\mu_p \tau kT}\ , \quad B_n = \frac{qa^2}{\mu_n \tau kT}\ .
\end{aligned}
$$

Since typically (see [3])

$$
\begin{aligned}
&\sigma_p = 8\ ,\ \sigma_n = 200 - 1000, \quad \mu_p = 300, \quad \mu_n = 4000\ , \\
&q = 1.6 \times 10^{-17}\ , \quad \tau = 10^{-7}, \quad a = 10^{-2}\ , \quad t = 10^{-5}\ , \\
&N_c = 4.7 \times 10^{17}\ , \quad N_v = 7 \times 10^{18}\ , \quad N_a^- - N_d^+ = 3 \times 10^{17}\ , \quad kT = 0.26
\end{aligned}
$$

in CGS units, we find upon assuming that $\widetilde{N}_c, \widetilde{N}_v$ are of the same order of magnitude as N_c and N_v respectively, that

(1.3) $\beta \sim \frac{3}{18}$, $A_p = 6\times10^{-17}$, $A_n = 5.5\times10^{-16}$, (if $\sigma_n = 1000$) $B_p = 2\times10^{-16}$, $B_n = 1.6\times10^{-17}$.

By slight change of notation we then get, after dropping " $\hat{}$ " everywhere, denoting $b/a, c/a, d/a, e/a$ by b, c, d, e, and designating the transformed domains Ω_p, Ω_n again by Ω_p, Ω_n:

(1.4) $$\Delta \varphi_p = 0 \quad \text{in} \quad \Omega_p\ ,$$

(1.5) $$\Delta \varphi_n = 0 \quad \text{in} \quad \Omega_n\ ,$$

$$\varphi_p(x,0) - \varphi_n(x,0) = \log n(n+1) \ , \quad -1 < x < 1, \tag{1.6}$$

$$n_{xx} = -A_p \frac{\partial \varphi_p}{\partial y}(x,0) + B_p f(n) \ , \qquad -1 < x < 1, \tag{1.7}$$

$$n_{xx} = -A_n \frac{\partial \varphi_n}{\partial y} \ (x,0) + B_n f(n) \ , \qquad -1 < x < 1, \tag{1.8}$$

$$n_x(-1) = n_x(1) = 0 \ , \tag{1.9}$$

$$\varphi_p(x,b) = V \ , \qquad |x| \ \leq d \ , \tag{1.10}$$

$$\varphi_n(x,-c) = 0 \ , \qquad |x| \ \leq c \ , \tag{1.11}$$

$$\frac{\partial \varphi_p}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_p \backslash (\Gamma \cup \Gamma_1) \ , \tag{1.12}$$

$$\frac{\partial \varphi_n}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_n \backslash (\Gamma \cup \Gamma_2) \ ; \tag{1.13}$$

here we have taken for simplicity $\beta = 1$.

The following formula for $f(n)$ was given in [4]:

$$f(n) = a_0 + a_1 n + a_2 n^2 + a_3 \frac{n^2}{2n+1} \ , \qquad a_i \quad \text{positive constants.} \tag{1.14}$$

In the sequel we shall not require the special form (1.14). All we shall assume is that

$$f(s) \in C^1[0,\infty) \ , \quad f(0) > 0 \ , \quad f'(s) \geq 0 \ . \tag{1.15}$$

An important feature in (1.7), (1.8) is that

$$A_p, \ A_n \ , \ B_p \quad \text{and} \quad B_n \quad \text{are “very small”} \tag{1.16}$$

(in fact, smaller than 2×10^{-16}) and that

$$\frac{A_n}{A_p} \ , \frac{B_p}{B_n} = O(1) \ . \tag{1.17}$$

For more details on the physics of the problem we refer the reader to [4], as well as to [3] and the references in both articles.

The present authors became interested in the problem after a talk given at the IMA by John Spence from Eastman Kodak and subsequent discussion. This was reported in [1; Chap. 13] together with some computations carried out by J. Spence and Keith Kahen (from Eastman Kokak). (The numerical values of σ_p, σ_n were misquoted in [1]; this resulted in incorrect order of magnitude for the quantities in (1.3)).

§2. **A special case.** In this section we assume that

$$\Gamma_1 = \{(b,x), \quad |x| \le 1\}\ , \quad \Gamma_2 = \{(-c,x);\ |x| \le 1\}\ . \tag{2.1}$$

We then look for a solution independent of x:

$$\varphi = \varphi(y) \quad , \quad n = \text{const}\,. \tag{2.2}$$

where $\varphi = \varphi_p$ in Ω_p, $\varphi = \varphi_n$ in Ω_n.

We easily find that

$$\varphi = \begin{cases} V + K_1(y-b) & \text{if} \quad 0 < y < b\ , \\ K_2(y+c) & \text{if} \quad -c < y < 0 \end{cases} \tag{2.3}$$

where

$$K_1 = \frac{B_p}{A_p}\, f(n), \quad K_2 = \frac{B_n}{A_n}\, f(n)$$

and (1.6) becomes

$$n(n+1) = e^{V-(K_1 b + K_2 c)} = e^{V - \left(\frac{B_p b}{A_p} + \frac{B_n c}{A_n}\right) f(n)}\ . \tag{2.4}$$

THEOREM 2.1. *There exists a unique solution $n > 0$ of (2.4); it provides a unique solution to (1.4)–(1.13) of the form (2.3).*

Proof. Denoting the right-hand side of (2.4) by $g(n)$ we have, by (1.15):

$$n(n+1) - g(n) \quad \text{is monotone increasing in} \quad n\ ,$$

negative when $n \downarrow 0$ and $+\infty$ when $n \uparrow \infty$. Hence (2.4) has a unique solution.

The solution $\varphi(n)$ has a jump $\log n(n+1)$ at $y = 0$; consequently,

$$\varphi(0+) > \varphi(0-) \quad \text{if} \quad n > \frac{\sqrt{5}-1}{2}\ , \quad \text{or equivalently} \quad V > \left(\frac{B_p b}{A_p} + \frac{B_n c}{A_n}\right) f\left(\frac{\sqrt{5}-1}{2}\right),$$

$$\varphi(0+) < \varphi(0-) \quad \text{if} \quad n < \frac{\sqrt{5}-1}{2}\ , \quad \text{or} \quad V < \left(\frac{B_p b}{A_p} + \frac{B_n c}{A_n}\right) f\left(\frac{\sqrt{5}-1}{2}\right),$$

and

$$\varphi(0+) = \varphi(0-) \quad \text{if} \quad n = \frac{\sqrt{5}-1}{2}\ , \quad \text{or} \quad V = \left(\frac{B_p b}{A_p} + \frac{B_n c}{A_n}\right) f\left(\frac{\sqrt{5}-1}{2}\right).$$

§3. **The general case: reformulation.** In this section and in the following one we consider the system (1.4)–(1.13) in the general case. We shall prove existence and uniqueness by a method which is quite general and can be extended to other systems. We do not wish however to formulate here the most general conditions; instead, we shall indicate the generality of the method by setting up more general notation. In §4 we shall discuss one important generalization.

We write $\Omega_1 = \Omega_p$, $\Omega_2 = \Omega_n$ to indicate that the method can be applied to more general domains, although we shall carry out the details only for the rectangular domains Ω_p, Ω_n as above. We further set $\varphi_1 = \varphi_p$, $\varphi_2 = \varphi_n$, $L_1\varphi_1 = \Delta\varphi_1$, $L_2\varphi_2 = \Delta\varphi_n$,

$$S_1 = \partial\Omega_1 \backslash \Gamma \ , \quad S_2 = \partial\Omega_2 \backslash \Gamma,$$

and denote by B_1, B_2 the boundary operators:

$$\begin{aligned} B_1\varphi_1 &= \frac{\partial\varphi_1}{\partial n} \quad \text{on} \quad S_1\backslash\Gamma_1, \quad B_1\varphi_1 = \varphi_1 \quad \text{on} \quad \Gamma_1 \ , \\ B_2\varphi_2 &= \frac{\partial\varphi_2}{\partial n} \quad \text{on} \quad S_2\backslash\Gamma_2, \quad B_2\varphi_2 = \varphi_2 \quad \text{on} \quad \Gamma_2 \ , \end{aligned}$$

$$\begin{aligned} & u = n, \ \varepsilon = A_p, \ \alpha = \frac{A_n}{A_p} \ , \\ & h_1(u) = \frac{B_p}{A_p} \ f(u) \ , \\ & h_2(u) = \frac{B_n}{A_p} \ f(u) \ , \\ & h_3(u) = \log u(u+1) \ ; \\ & Lu = u''(x) \ , \ -1 < x < 1 \ , \\ & Bu = u' \quad \text{at} \quad x = \pm 1 \ . \end{aligned}$$

Then the system (1.4)–(1.13) can be written in the following form:

(3.1) $$\varphi_1 \in H^2(\Omega_1) \cap H^1(\Gamma) \ , \ \varphi_2 \in H^2(\Omega_2) \cap H^1(\Gamma), \ u \in H^2(\Gamma)$$

(3.2) $$L_i\varphi_i = 0 \quad \text{in} \quad \Omega_i \ ,$$

(3.3) $$Lu = \varepsilon\frac{\partial\varphi_1}{\partial n} + \varepsilon h_1(u) \quad \text{on} \quad \Gamma \ ,$$

(3.4) $$B_i\varphi_i = g_i(x) \quad \text{on} \quad S_i \ ,$$

(3.5) $$\varphi_1 - \varphi_2 = h_3(u) \quad \text{on} \quad \Gamma \ ,$$

(3.6) $$\frac{\partial\varphi_1}{\partial n} - \alpha\frac{\partial\varphi_2}{\partial n} = -h_1(u) + \alpha h_2(u) \quad \text{on} \quad \Gamma \ ,$$

(3.7) $$Bu = 0 \quad \text{on} \quad \partial\Gamma$$

where the normal n on Γ points into Ω_2, and

$$(3.8) \qquad \begin{aligned} &g_1 = V \quad \text{on} \quad \Gamma_1, \quad g_1 = 0 \quad \text{on} \quad S_1 \backslash \Gamma_1 , \\ &g_2 = 0 \quad \text{on} \quad S_2 . \end{aligned}$$

Here ε is a very small number ($\leq 6 \times 10^{-17}$) and $\alpha = O(1)$. The heights b, c, of Ω_1, Ω_2, as well as b/c, c/b are also $O(1)$, and so are the lengths $2d, 2e$ of Γ_1 and Γ_2 respectively.

In the sequel we shall use the fact that the boundary conditions (3.4) satisfy the "consistency" condition at the endpoints $(\pm 1, 0)$, i.e., by reflection one can deduce regularity of solutions in Ω_1 (or Ω_2) at these corner points.

Our main result (to be proved in §4) is:

THEOREM 3.1. *For any large positive number M there exists an $\varepsilon_0 > 0$ depending on M such that if $-\varepsilon_0 < \varepsilon < \varepsilon_0$ then the systems (3.1)–(3.8) has a unique solution $(\varphi_1, \varphi_2, u)$ with*

$$(3.9) \qquad \frac{1}{M} \leq u(x) \leq M .$$

The proof consists of several steps. We begin by introducing the solution φ_i^0 of

$$(3.10) \qquad \begin{aligned} &L_i \varphi_i^0 = 0 \quad \text{in} \quad \Omega_i , \\ &B_i \varphi_i^0 = g_i \quad \text{on} \quad S_i , \\ &\frac{\partial \varphi_i^0}{\partial n} = 0 \quad \text{in} \quad \Gamma , \qquad i = 1, 2 . \end{aligned}$$

Due to the consistency of the boundary conditions $(\pm 1, 0)$, the solution φ_i^0 satisfies:

$$(3.11) \qquad \varphi_i^0 \in H^2(\Omega_i) .$$

Actually, from the special assumptions (3.8) we see that

$$(3.12) \qquad \varphi_1^0 \equiv V , \qquad \varphi_2^0 \equiv 0 .$$

For any $u \in H^1(\Gamma)$, $u > 0$ on $\overline{\Gamma}$ we can solve the system

$$(3.13) \qquad \begin{aligned} &L_i w_i = 0 \quad \text{in} \quad \Omega_i, , \\ &B_i w_i = 0 \quad \text{on} \quad S_i , \\ &\frac{\partial w_1}{\partial n} = h_1(u), \quad \frac{\partial w_2}{\partial n} = -h_2(u) \quad \text{on} \quad \Gamma . \end{aligned}$$

We note by (1.15), that

$$(3.14) \qquad h_i(u) > 0 , \ h_i'(u) > 0 \quad \text{for} \quad u \geq 0 \quad (i = 1, 2) .$$

Since $u \in H^1(\Gamma)$, $u(x)$ is bounded. By the maximum principle

$$w_i > 0 \quad \text{in} \quad \Omega_i \ , \tag{3.15}$$

and, by elliptic regularity,

$$\begin{aligned} &w_i \in H^1(\Omega_i) \quad \text{and} \quad w_i \in H^2(\Omega_i \backslash N_i) \\ &\qquad \text{where} \quad N_i \quad \text{is any neighborhood of } \partial\Gamma_i \ . \end{aligned} \tag{3.16}$$

Writing $w_i = w_i(u)$, we set

$$\varphi_i = \varphi_i^0 + (-1)^{i+1} w_i(u) + \psi_i \qquad (i = 1, 2) \ . \tag{3.17}$$

Then the system (3.1)–(3.7) can be rewritten as a system for (ψ_1, ψ_2, u):

$$\begin{aligned} &L_i \psi_i = 0 \quad \text{in} \quad \Omega_i \ , \\ &Lu = \varepsilon \frac{\partial \psi_i}{\partial n} \quad \text{on} \quad \Gamma \ , \\ &B_i \psi_i = 0 \quad \text{on} \quad S_i \ , \\ &\frac{\partial \psi_1}{\partial n} = \alpha \frac{\partial \psi_2}{\partial n} \quad \text{on} \quad \Gamma \ , \\ &Bu = 0 \quad \text{on} \quad \partial\Gamma \ , \end{aligned} \tag{3.18}$$

and

$$\psi_1 - \psi_2 = \varphi_2^0 - \varphi_1^0 + (w_2(u) + w_1(u)) + h_3(u) \equiv F(u) \ ; \tag{3.19}$$

$F(u)$ is a function of x and depends on u in a nonlocal way. Note that $F(u) \in H^1(\Gamma)$.

Let (v_1, v_2) be the solution to

$$\begin{aligned} &L_i v_i = 0 \quad \text{in} \quad \Omega_i \ , \\ &B_i v_i = 0 \quad \text{on} \quad S_i \ , \\ &\frac{\partial v_i}{\partial n} = \alpha \frac{\partial v_2}{\partial n} \quad \text{on} \quad \Gamma \ , \\ &v_1 - v_2 = F(u) \quad \text{on} \quad \Gamma \ . \end{aligned} \tag{3.20}$$

We can construct (v_1, v_2) uniquely as a solution to a variational problem. Further, by working with v_1 and with $\tilde{v}_2(x, y) = v_2(x, -y)$ in Ω_1-neighborhood of Γ and applying H^s elliptic estimates to $v_1 - v_2$ and $v_1 + \alpha \tilde{v}_2$ [2; p. 201 and p. 202], we get

$$\begin{aligned} &v_1, v_2 \in H^1(\Gamma) \ , \\ &v_1, \tilde{v}_2 \in H^{3/2} \quad \text{in } \overline{\Omega}_1\text{- neighborhood of } \Gamma \ . \end{aligned}$$

It follows that

$$\frac{\partial v_1}{\partial n} \in L^2(\Gamma) \ .$$

We define the operator A from $H^2(\Gamma)$ into $L^2(\Gamma)$ by

$$Au = \frac{\partial v_1}{\partial n} \ . \tag{3.21}$$

Comparing (3.20), (3.21) with (3.18), (3.19) we see that (ψ_1, ψ_2, u) is a solution of (3.18), (3.19) if and only if u is such that

$$\begin{aligned} Lu &= \varepsilon A(u) \quad \text{in} \quad \Gamma \ , \\ Bu &= 0 \quad \text{on} \quad \partial\Gamma \ . \end{aligned} \tag{3.22}$$

When $\varepsilon = 0$ we get

$$\begin{aligned} Lu &= 0 \quad \text{in} \quad \Gamma \ , \\ Bu &= 0 \quad \text{on} \quad \partial\Gamma \ , \end{aligned} \tag{3.23}$$

and

(3.24) the only solutions of (3.23) are multiples of $u_0 \equiv 1$.

Any function $u \in H^1(\Gamma)$ can be written in the form

$$u = mu_0 + \varepsilon v$$

where m is constant and $(v, u_0)_{L^2(\Gamma)} = 0$. The problem (3.22) (for $u > 0$) can be written in the form

$$\begin{aligned} & mu_0 + \varepsilon v > 0 \quad \text{on} \quad \overline{\Gamma}, \quad v \in H^1(\Gamma), \ (v, u_0)_{L^2(\Gamma)} = 0 \ , \\ & Lv = A(mu_0 + \varepsilon v) \quad \text{on} \quad \Gamma \ , \\ & Bv = 0 \quad \text{on} \quad \partial\Gamma \ . \end{aligned} \tag{3.25}$$

In the next section we prove that this problem has a unique solution.

§4. **Proof of Theorem 3.1.** We begin with several lemmas.

LEMMA 4.1. *Let (U_1^0, U_2^0) be the solution to*

$$\begin{aligned} & L_i U_i^0 = 0 \quad \textit{in} \quad \Omega_i \ , \\ & B_i U_i^0 = 0 \quad \textit{on} \quad S_i \ , \\ & \frac{\partial U_1^0}{\partial n} = \alpha \frac{\partial U_2^0}{\partial n} \quad \textit{on} \quad \Gamma \ , \\ & U_1^0 - U_2^0 = u_0 \ . \end{aligned} \tag{4.1}$$

Then, for any $v \in H^1(\Gamma)$,

$$(A(mu_0 + \varepsilon v)\ ,\ u_0) = -\left(F(mu_0 + \varepsilon v),\ \frac{\partial U_1^0}{\partial n}\right) \tag{4.2}$$

where $(\ ,\ ) = (\ ,\ )_{L^2(\Gamma)}$.

Proof. By (3.21), the left-hand side of (4.2) is equal to

$$\begin{aligned}
\left(\frac{\partial v_1}{\partial n}\ , u_0\right) &= \left(\frac{\partial v_1}{\partial n}\ ,\ U_1^0 - U_2^0\right) \\
&= \left(\frac{\partial v_1}{\partial n}\ ,\ U_1^0\right) - \alpha\left(\frac{\partial v_2}{\partial n}\ , U_2^0\right) \\
&= -\left(v_1, \frac{\partial U_1^0}{\partial n}\right) + \alpha\left(v_2,\ \frac{\partial U_2^0}{\partial n}\right) \quad \text{(by Green's formula)} \\
&= -\left(v_1 - v_2, \frac{\partial U_1^0}{\partial n}\right) = -\left(F(mu_0 + \varepsilon v),\ \frac{\partial U_1^0}{\partial n}\right)
\end{aligned}$$

by the last equation in (3.20).

Notice that the functions U_i^0 are in $C^1(\overline{\Omega}_j \backslash \overline{\Gamma}_j)$.

LEMMA 4.2. *The functions U_j^0 satisfy:*

$$\frac{\partial U_j^0}{\partial n} > 0 \quad \textit{on} \quad \overline{\Gamma}. \tag{4.3}$$

Proof. Set

$$W = \begin{cases} U_1^0 - u_0 & \text{in} \quad \Omega_1 \\ U_2^0 & \text{in} \quad \Gamma_2\ . \end{cases}$$

Since $u_0 \equiv 1$,

$$\Delta W = 0 \quad \text{in} \quad \Omega_1 \cup \Omega_2\ ; \tag{4.4}$$

further

$$W \text{ is continuous across } \Gamma\ , \tag{4.5}$$

$$\frac{\partial W^+}{\partial n} = \alpha\,\frac{\partial W^-}{\partial n} \quad \text{on} \quad \Gamma \tag{4.6}$$

where $W^+ = W|_{\Omega_1}$, $W^- = W|_{\Omega_2}$. If W takes minimum at a point of Γ, then, by the maximum principle (noting that n points from Ω_1 into Ω_2),

$$\frac{\partial W^+}{\partial n} < 0 , \quad \frac{\partial W^-}{\partial n} > 0$$

at that point, a contradiction to (4.6). We conclude that W must take its minimum in $\overline{\Omega_1 \cup \Omega_2}$ at the boundary, and since $W = 0$ on Γ_2, $W = -1$ on Γ_1 whereas $\partial W/\partial n = 0$ elsewhere in $\partial\,(\overline{\Omega, \cup \Omega_2})$, it follows that the minimum is attained on Γ_1. Further,

$$\frac{\partial W}{\partial y} < 0 \quad \text{on} \quad \Gamma_1 \ .$$

Similarly W takes its maximum (=zero) on Γ_2 and

$$\frac{\partial W}{\partial y} < 0 \quad \text{on} \quad \Gamma_2 \ .$$

Further, $\partial W/\partial y = 0$ on the horizontal part of $\partial(\overline{\Omega_1 \cup \Omega_2})\backslash(\overline{\Gamma}_1 \cup \overline{\Gamma}_2)$ and

$$\frac{\partial}{\partial n}\left(\frac{\partial W}{\partial y}\right) = 0 \quad \text{on the vertical part of} \quad \partial(\overline{\Omega_1 \cup \Omega_2})\backslash(\overline{\Gamma}_1 \cup \overline{\Gamma}_2) \ .$$

We claim that

$$\frac{\partial W^\pm}{\partial y} < 0 \quad \text{on} \quad \overline{\Gamma} \ , \tag{4.7}$$

and this of course establishes the assertion (4.3). Indeed, if (4.7) is not true then $\partial W^+/\partial y$ (or $\partial W^-/\partial y$) must take nonnegative values on $\overline{\Gamma}$, and in fact it attains its nonnegative maximum in $\overline{\Omega}_1$ (or $\overline{\Omega}_2$) at a point $(x_0, 0)$ of $\overline{\Gamma}$. By (4.6), the same is true of $\partial W^-/\partial y$ (or $\partial W^+/\partial y$). Hence, by the maximum principle,

$$\frac{\partial^2 W^+}{\partial y^2} < 0 \ , \ \frac{\partial^2 W^-}{\partial y^2} > 0 \quad \text{at} \quad (x_0, 0) \ ,$$

and consequently $\partial^2 W^+/\partial x^2 > 0$, $\partial^2 W^-/\partial x^2 < 0$ at $(x_0, 0)$. (If $x_0 = \pm 1$ then since we can extend W by reflection across $x = \pm 1$, the maximum principle can still be applied). This contradicts the fact that

$$W^+(x, 0) \equiv W^-(x, 0) \ .$$

LEMMA 4.3. *There exists a unique positive constant* m_0 *such that*

$$(A(m_0 u_0), u_0) = 0 \ . \tag{4.8}$$

Proof. In view of Lemma 4.1, (4.8) is equivalent to

$$\left(F(m_0u_0),\ \frac{\partial U_1^0}{\partial n}\right) = 0\ . \tag{4.9}$$

From (3.13), (3.14) it is clear that

$$w_i(mu_0) \quad \text{is strictly increasing in} \quad m\ .$$

Hence, by (3.19),

$$\frac{\partial}{\partial m}\ F(mu_0) > \frac{1}{m} + \frac{1}{m+1} > 0\ . \tag{4.10}$$

Also, $F(mu_0) \to -\infty$ if $m \to 0$ and $F(mu_0) \to \infty$ if $m \to \infty$. Recalling also (4.3), the assertion (4.9) follows.

LEMMA 4.4. *There exists a positive constant M_0 such that*

$$\textit{if} \qquad \|u_1\|_{L^\infty(\Gamma)} \le \frac{1}{M_0}\ ,\ \|u_2\|_{L^\infty(\Gamma)} \le \frac{1}{M_0}$$

then $m_0u_0 + u_j > 0$ in $\overline{\Gamma}$ $(j = 1, 2)$ and

$$\|A(m_0u_0 + u_1) - A(m_0u_0 + u_2)\|_{L^2(\Gamma)} \le CM_0\|u_1 - u_2\|_{H^1(\Gamma)} \tag{4.11}$$

where C is a constant independent of M_0.

The proof follows from the dependence of the $w_i(u)$ on u (cf. (3.16)):

$$\|w_i(m_0u_0 + u_1) - w_i(m_0u_0 + u_2)\|_{H^2(\Omega_i \setminus N_i)} \le C\|u_1 - u_2\|_{H^1(\Gamma)}\ .$$

LEMMA 4.5. *If M_0 is sufficiently large then*

$$\frac{d}{dm}(A(mu_0 + u), u_0) \le -\frac{1}{M_0} \tag{4.12}$$

provided $mu_0 + u > 0$ on $\overline{\Gamma}$ and $\|u\|_{L^\infty(\Gamma)} \le \frac{1}{M_0}$.

The proof for $u = 0$ was already established above (following (4.9)). The proof for $u \not\equiv 0$ is the same.

THEOREM 4.6. *For any large positive number M $(M \geq M_0)$ there exists an $\varepsilon_0 > 0$ depending on M such that if $-\varepsilon_0 < \varepsilon < \varepsilon_0$ then (3.25) has a unique solution satisfying*

$$\|v\|_{L^\infty(\Gamma)} \leq M \ .$$

Since (3.25) is equivalent to (3.1)–(3.7), the assertion of Theorem 3.1 follows from Theorem 4.6

Proof. Introduce the set

$$X = \{u \in H^1(\Gamma) \ , \ (u, u_0) = 0 \ , \ \|u\|_{L^\infty(\Gamma)} < M \ , \ \|u\|_{H^1(\Gamma)} < M\}$$

in $H^1(\Gamma)$. We define a map $G : X \to H^2(\Gamma)$ as follows: $\tilde{v} = G(v)$ if $\tilde{v}$ is the solution of

$$\begin{aligned} &L\tilde{v} = A(mu_0 + \varepsilon v) \quad \text{in} \quad \Gamma \ , \\ &B\tilde{v} = 0 \quad \text{on} \quad \partial\Gamma \ , \\ &(\tilde{v}, u_0) = 0 \end{aligned} \tag{4.13}$$

where $m = m(v)$ is defined as the solution of

$$(A(mu_0 + \varepsilon v) \ , \ u_0) = 0 \ . \tag{4.14}$$

In order to justify this definition we note, by Lemmas 4.3, 4.4, that

$$|(A(m_0 u_0 + \varepsilon v), \ u_0)| \ \leq C_0 \varepsilon \|v\|_{H^1(\Gamma)} \ .$$

Recalling Lemma 4.5 we deduce that there exists a solution to (4.14) with

$$|m - m_0| \ < c\varepsilon \ .$$

The solution is unique in the set of all positive number m such that $mu_0 + \varepsilon u > 0$ on $\overline{\Gamma}$.

From (4.14) it follows that the right-hand side of the elliptic equation in (4.13) is orthogonal to the eigenfunction u_0 of the homogeneous problem. Hence, by Fredholdm alternative, (4.13) has a unique solution v. Clearly

$$\begin{aligned} \|v\|_{H^2(\Gamma)} &\leq C\|A(mu_0 + \varepsilon v)\|_{L^2(\Gamma)} \\ &\leq C(m_0 + |m - m_0| + \|\varepsilon v\|_{L^2}) < M \end{aligned}$$

if M is large enough. Thus G maps X into a compact subset.

We next show that G is a contraction. We begin with

$$\begin{aligned} \|Gv_1 - Gv_2\|_{H^1} &\leq C\|A(m_1 u_0 + \varepsilon v_1) - A(m_2 u_0 + \varepsilon v_2)\|_{L^2} \\ &\leq C(|m_1 - m_2| + \varepsilon\|v_1 - v_2\|_{H^1(\Gamma)}) \ . \end{aligned} \tag{4.15}$$

Since

$$(A(m_1u_0 + \varepsilon v_1), u_0) = (A(m_2u_0 + \varepsilon v_2), u_0) = 0 \ ,$$

we have

$$\begin{aligned}
&|(A(m_2u_0 + \varepsilon v_1), u_0)| \\
&\qquad = |(A(m_2u_0 + \varepsilon v_1), u_0) - (A(m_2u_0 + \varepsilon v_2), u_0)| \\
&\qquad \le CM_0 \|\varepsilon v_1 - \varepsilon v_2\|_{H^1}
\end{aligned}$$

by Lemmas 4.4, so that by Lemma 4.5

$$|m_1 - m_2| \ \le CM_0^2 \|\varepsilon v_1 - \varepsilon v_2\|_{H^1} \ .$$

Using this in (4.15) we conclude that

$$\|Gv_1 - Gv_2\|_{H^1} \le C_1 \varepsilon \|v_1 - v_2\|_{H^1}$$

for some constant C_1 independent of ε. Hence, if ε is small enough G maps X into X and is a contraction. It follows that G has a unique fixed point in X, and the proof of Theorem 4.6 is complete.

Generalizations. In the actual model of the semiconductor laser, the conductivity σ_p is not a uniform constant (see [4]). Indeed, there is some $\overline{\Omega}_p$-neighborhood R of Γ_1, such that

$$\sigma_p = \begin{cases} \sigma_p^1 & \text{in} \quad R \\ \sigma_p^2 & \text{in} \quad \Omega_p \backslash R \end{cases} \tag{4.15}$$

where σ_p^i are constants and $\sigma_p^1 > \sigma_p^2$. The potential φ_1 satisfies

$$\nabla \cdot (\sigma_p \nabla \varphi_p) = 0 \quad \text{in} \quad \Omega_p \ . \tag{4.16}$$

We can now proceed similarly to the proof of Theorem 3.1. The only difficulty arises in the proof of Lemma 4.2; if the assertion (4.3) is valid then the rest of the proof extends without changes, provided we replace $\Delta\varphi_p = 0$ by (4.16).

By simple continuity argument, (4.3) remains valid if R lies in a "small" neighborhood of Γ_1. Otherwise, the proof cannot be extended; one can however proceed to verify (4.3) numerically

REMARK 4.1. The shape of the function $u(x)$ (or $n(x)$ is the notation of §1) is important for determining the intensity of the laser. Some numerical work was carried out by J. Spence and K. Kahen (see [1; Chap. 13]) in the case (4.15). Our results for the case where $\sigma_p = \text{const.}$ in Ω_p show that

$$n(x) = \text{const.} + \varepsilon v(x) .$$

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