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COMPUTATION OF MATRIX CHAIN PRODUCTS. PART I*

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Abstract. This paper considers the computation of matrix chain products of the form $M_1 \times M_2 \times \cdots \times M_{n-1}$. If the matrices are of different dimensions, the order in which the product is computed affects the number of operations. An optimum order is an order which minimizes the total number of operations. We present some theorems about an optimum order of computing the matrices. Based on these theorems, an $O(n \log n)$ algorithm for finding an optimum order will be presented in Part II.

Key words. matrix multiplication, polygon partition, dynamic programming

1. Introduction. Consider the evaluation of the product of $n-1$ matrices

$$M = M_1 \times M_2 \times \cdots \times M_{n-1}, \tag{1}$$

where M_i is a $w_i \times w_{i+1}$ matrix. Since matrix multiplication satisfies the associative law, the final result M in (1) is the same for all orders of multiplying the matrices. However, the order of multiplication greatly affects the total number of operations to evaluate M. The problem is to find an optimum order of multiplying the matrices such that the total number of operations is minimized. Here, we assume that the number of operations to multiply a $p \times q$ matrix by a $q \times r$ matrix is pqr.

In [1], [7], a dynamic programming algorithm is used to find an optimum order. The algorithm needs $O(n^3)$ time and $O(n^2)$ space. In [2], Chandra proposed a heuristic algorithm to find an order of computation which requires no more than $2T_o$ operations where T_o is the total number of operations to evaluate (1) in an optimum order. This heuristic algorithm needs only $O(n)$ time. Chin [3] proposed an improved heuristic algorithm to give an order of computation which requires no more than $1.25T_o$. This improved heuristic algorithm also needs only $O(n)$ time.

In this paper we first transform the matrix chain product problem into a problem in graph theory—the problem of partitioning a convex polygon into nonintersecting triangles, see [9], [10], [11], [12]; then we state several theorems about the optimum partitioning problem. Based on these theorems, an $O(n \log n)$ algorithm for finding an optimum partition is developed.

2. Partitioning a convex polygon. Given an n-sided convex polygon, such as the hexagon shown in Fig. 1, the number of ways to partition the polygon into $(n-2)$ triangles by nonintersecting diagonals is the Catalan number (see for example, Gould [8]). Thus, there are 2 ways to partition a convex quadrilateral, 5 ways to partition a convex pentagon, and 14 ways to partition a convex hexagon.

Let every vertex V_i of the polygon have a positive weight w_i. We can define the cost of a given partition as follows: The cost of a triangle is the product of the weights of the three vertices, and the cost of partitioning a polygon is the sum of the costs of all its triangles. For example, the cost of the partition of the hexagon in Fig. 1 is

$$w_1w_2w_3 + w_1w_3w_6 + w_3w_4w_6 + w_4w_5w_6. \tag{2}$$

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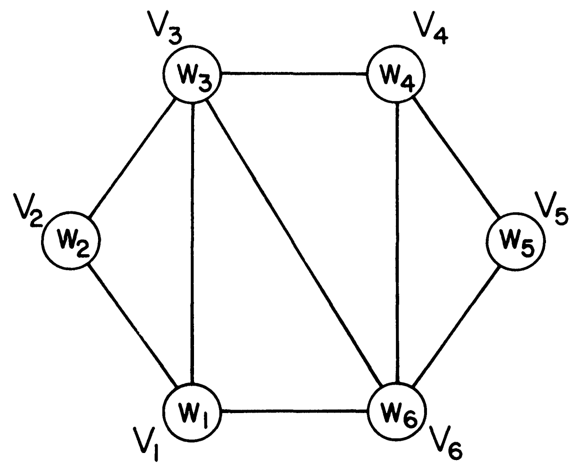


FIG. 1

If we erase the diagonal from V_3 to V_6 and replace it by the diagonal from V_1 to V_4, then the cost of the new partition will be

$$w_1w_2w_3 + w_1w_3w_4 + w_1w_4w_6 + w_4w_5w_6. \tag{3}$$

We will prove that an order of multiplying $n-1$ matrices corresponds to a partition of a convex polygon with n sides. The cost of the partition is the total number of operations needed in multiplying the matrices. For brevity, we shall use n-gon to mean a convex polygon with n sides, and the partition of an n-gon to mean the partitioning of an n-gon into $n-2$ nonintersecting triangles.

For any n-gon, one side of the n-gon will be considered to be its base, and will usually be drawn horizontally at the bottom such as the side V_1-V_6 in Fig. 1. This side will be called the base; all other sides are considered in a clockwise way. Thus, V_1-V_2 is the first side, V_2-V_3 the second side, $\cdots$ and V_5-V_6 the fifth side.

The first side represents the first matrix in the matrix chain and the base represents the final result M in (1). The dimensions of a matrix are the two weights associated with the two end vertices of the side. Since the adjacent matrices are compatible, the dimensions $w_1 \times w_2$, $w_2 \times w_3, \cdots, w_{n-1} \times w_n$ can be written inside the vertices as $w_1, w_2, \cdots, w_n$. The diagonals are the partial products. A partition of an n-gon corresponds to an alphabetic tree of $n-1$ leaves or the parenthesis problem of $n-1$ symbols (see, for example, Gardner [6]). It is easy to see the one-to-one correspondence between the multiplication of $n-1$ matrices to either the alphabetic binary tree or the parenthesis problem of $n-1$ symbols. Here, we establish the correspondence between the matrix-chain product and the partition of a convex polygon directly.

LEMMA 1. *Any order of multiplying $n-1$ matrices corresponds to a partition of an n-gon.*

Proof. We shall use induction on the number of matrices. For two matrices of dimensions $w_1 \times w_2$, $w_2 \times w_3$, there is only one way of multiplication; this corresponds to a triangle where no further partition is required. The total number of operations in multiplication is $w_1w_2w_3$, the product of the three weights of the vertices. The resulting matrix has dimension $w_1 \times w_3$. For three matrices, the two orders of multiplication $(M_1 \times M_2) \times M_3$ and $M_1 \times (M_2 \times M_3)$ correspond to the two ways of partitioning a 4-gon. Assume that this lemma is true for k matrices where $k \leqq n-2$, and we now consider $n-1$ matrices. The n-gon is shown in Fig. 2.

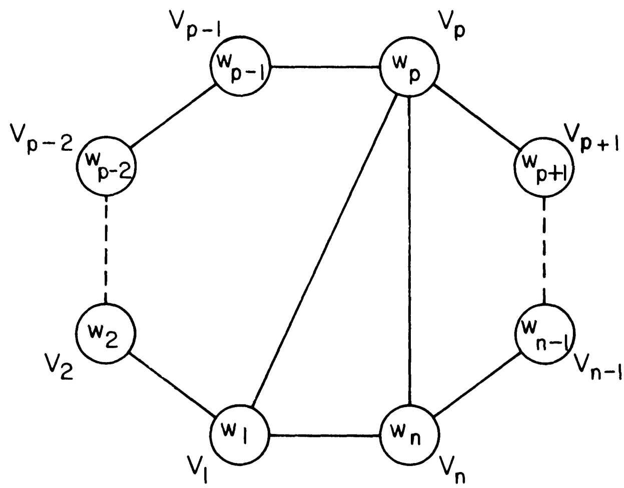


FIG. 2

Let the order of multiplication be represented by

$$M=(M_1\times M_2\times\cdots\times M_{p-1})\times(M_p\times\cdots\times M_{n-1});$$

i.e., the final matrix is obtained by multiplying a matrix of dimension $(w_1\times w_p)$ and a matrix of dimension $(w_p\times w_n)$. Then in the partition of the n-gon, we let the triangle with vertices V_1 and V_n have the third vertex V_p. The polygon $V_1-V_2-\cdots-V_p$ is a convex polygon of p sides with base V_1-V_p and its partition corresponds to an order of multiplying matrices $M_1,\cdots,M_{p-1}$, giving a matrix of dimension $w_1\times w_p$. Similarly, the partition of the polygon $V_p-V_{p+1}-\cdots-V_n$ with base V_p-V_n corresponds to an order of multiplying matrices $M_p,\cdots,M_{n-1}$, giving a matrix of dimension $w_p\times w_n$. Hence the triangle $V_1V_pV_n$ with base V_1-V_n represents the multiplication of the two partial products, giving the final matrix of dimension $w_1\times w_n$. □

LEMMA 2. *The minimum numbers of operations needed to evaluate the following matrix chain products are identical.*

$$M_1\times M_2\times\cdots\times M_{n-2}\times M_{n-1},$$
$$M_n\times M_1\times\cdots\times M_{n-3}\times M_{n-2},$$
$$\vdots$$
$$M_2\times M_3\times\cdots\times M_{n-1}\times M_n,$$

where M_i has dimension $w_i\times w_{i+1}$ and $w_{n+1}\equiv w_1$. Note that in the first matrix chain, the resulting matrix is of dimension $w_1\times w_n$. In the last matrix chain, the resulting matrix is of dimension $w_2\times w_1$. But in all the cases, the total number of operations in the optimum orders of multiplication is the same.

Proof. The cyclic permutations of the $n-1$ matrices all correspond to the same n-gon and thus have the same optimum partitions. □
(This lemma was obtained independently in [4] with a long proof.)

From now on, we shall concentrate only on the partitioning problem.

The diagonals inside the polygon are called arcs. Thus, one easily verifies inductively that every partition consists of $n-2$ triangles formed by $n-3$ arcs and n sides.

In a partition of an n-gon, the degree of a vertex is the number of arcs incident on the vertex plus two (since there are two sides incident on every vertex).

LEMMA 3. *In any partition of an n-gon, $n \geqq 4$, there are at least two triangles, each having a vertex of degree two.* (*For example, in Fig.* 1, *the triangle $V_1V_2V_3$ has vertex V_2 with degree* 2 *and the triangle $V_4V_5V_6$ has vertex V_5 with degree* 2.) (*See also* [5].)

Proof. In any partition of an n-gon, there are $n-2$ nonintersecting triangles formed by $n-3$ arcs and n sides. And for any $n \geqq 4$, no triangle can be formed by 3 sides. Let x be the number of triangles with two sides and one arc, y be the number of triangles with one side and two arcs, and z be the number of triangles with three arcs. Since an arc is used in two triangles, we have

$$x+2y+3z=2(n-3). \tag{4}$$

Since the polygon has n sides, we have

$$2x+y=n. \tag{5}$$

From (4) and (5), we get

$$3x=3z+6.$$

Since $z \geqq 0$, we have $x \geqq 2$. □

LEMMA 4. *Let P and P' both be n-gons where the corresponding weights of the vertices satisfy $w_i \leqq w'_i$. Then the cost of an optimum partition of P is less than or equal to the cost of an optimum partition of P'.*

Proof. Omitted. □

If we use $C(w_1, w_2, w_3, \cdots, w_k)$ to mean the minimum cost of partitioning the k-gon with weights w_i optimally, Lemma 4 can be stated as

$$C(w_1, w_2, \cdots, w_k) \leqq C(w'_1, w'_2, \cdots, w'_k) \quad \text{if } w_i \leqq w'_i.$$

We say that two vertices are *connected* in an optimum partition if the two vertices are connected by an arc or if the two vertices are adjacent to the same side.

In the rest of the paper, we shall use $V_1, V_2, \cdots, V_n$ to denote vertices which are ordered according to their weights, i.e., $w_1 \leqq w_2 \leqq \cdots \leqq w_n$. To facilitate the presentation, we introduce a tie-breaking rule for vertices of equal weights.

If there are two or more vertices with weights equal to the smallest weight w_1, we can arbitrarily choose one of these vertices to be the vertex V_1. Once the vertex V_1 is chosen, further ties in equal weights are resolved by regarding the vertex which is closer to V_1 in the clockwise direction to be of less weight. With this tie-breaking rule, we can unambiguously label the vertices $V_1, V_2, \cdots, V_n$ for each choice of V_1. A vertex V_i is said to be *smaller than* another vertex V_j, denoted by $V_i < V_j$, either if $w_i < w_j$ or if $w_i = w_j$ and $i < j$. We say that V_i is the *smallest* vertex in a subpolygon if it is smaller than any other vertices in the subpolygon.

After the vertices are labeled, we define an arc $V_i - V_j$ to be *less than* another arc $V_p - V_q$ if

$$\min(i, j) < \min(p, q) \quad \text{or} \quad \begin{cases} \min(i, j) = \min(p, q), \\ \max(i, j) < \max(p, q). \end{cases}$$

(For example, the arc $V_3 - V_9$ is less than the arc $V_4 - V_5$.) Every partition of an n-gon has $n-3$ arcs which can be sorted from the smallest to the largest into an ordered sequence of arcs, i.e., each partition is associated with a unique ordered sequence of arcs. We define a partition P to be *lexicographically less than* a partition Q if the ordered sequence of arcs associated with P is lexicographically less than that associated with Q.

When there is more than one optimum partition, we use the *l-optimum partition* (i.e., lexicographically-optimum partition) to mean the lexicographically smallest optimum partition, and use *an optimum partition* to mean some partition of minimum cost.

We shall use V_a, V_b, $\cdots$ to denote vertices which are unordered in weights, and T_{ijk} to denote the product of the weights of any three vertices V_i, V_j and V_k.

THEOREM 1. *For every way of choosing* V_1, V_2, $\cdots$ *(as prescribed), there is always an optimum partition containing* $V_1 - V_2$ *and* $V_1 - V_3$. *(Here,* $V_1 - V_2$ *and* $V_1 - V_3$ *may be either arcs or sides.)*

Proof. The proof is by induction, For the optimum partitions of a triangle and a 4-gon, the theorem is true. Assume that the theorem is true for all k-gons $(3 \leqq k \leqq n-1)$ and consider the optimum partitions of an n-gon.

From Lemma 3, in any optimum partition, we can find at least two vertices having degree two. Call these two vertices V_i and V_j. We can divide this into two cases.

(i) One of the two vertices V_i (or V_j) is not V_1, V_2 or V_3 in some optimum partition of the n-gon. In this case, we can remove the vertex V_i with its two sides and obtain an $(n-1)$-gon. In this $(n-1)$-gon, V_1, V_2, V_3 are the three vertices with smallest weights. By the induction assumption, V_1 is connected to both V_2 and V_3 in an optimum partition.

(ii) Consider the complementary case of (i), in all the optimum partitions of the n-gon, all the vertices with degree two are from the set $\{V_1, V_2, V_3\}$. (In this case, there will be at most three vertices with degree two in every optimum partition.) We have the following three subcases:

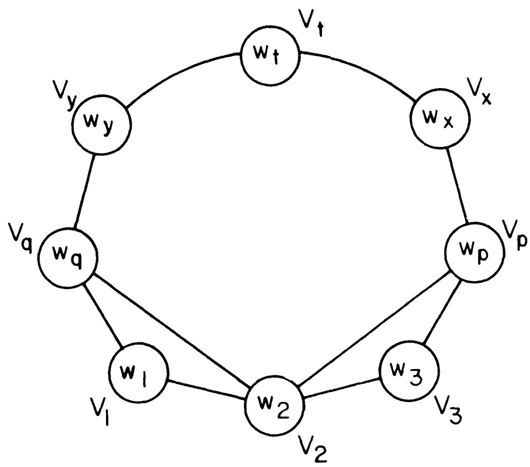
(a) $V_i = V_2$ and $V_j = V_3$ in some optimum partition of the n-gon, i.e., both V_2 and V_3 have degree two simultaneously. In this case, we first remove V_2 with its two sides and form an $(n-1)$-gon. By the induction assumption, V_1, V_3 must be connected in some optimum partition. If $V_1 - V_3$ appears as an arc, it reduces to (i). So $V_1 - V_3$ must appear as a side of the $(n-1)$-gon, and reattaching V_2 to the $(n-1)$-gon shows that either V_1, V_2 and V_3 are mutually adjacent or $V_1 - V_3$ is a side of the n-gon. In the former case, the proof is complete, so we assume that $V_1 - V_3$ is a side of the n-gon. Similarly, we can remove V_3 with its two sides and show that V_1, V_2 are connected by a side of the n-gon.

(b) $V_i = V_1$ and $V_j = V_2$ in some optimum partition of the n-gon, i.e., V_1 and V_2 both have degree two simultaneously. In this case, we can first remove V_1 and form an $(n-1)$-gon where V_2, V_3, V_4 are the three vertices with smallest weights. By the induction assumption, V_2 is connected to both V_3 and V_4 in an optimum partition. If $V_2 - V_3$ or $V_2 - V_4$ appears as an arc, it reduces to (i). Hence, $V_2 - V_3$ and $V_2 - V_4$ must both be sides of the n-gon. Similarly, we can remove V_2 with its two sides and form an $(n-1)$-gon where V_1, V_3, V_4 are the three vertices with smallest weights. Again, V_1 must be connected to V_3 and V_4 by sides of the n-gon. But for any n-gon with $n \geqq 5$, it is impossible to have V_3 and V_4 both adjacent to V_1 and V_2 at the same time, i.e., V_1 and V_2 cannot both have degree two in an optimum partition of any n-gon with $n \geqq 5$.

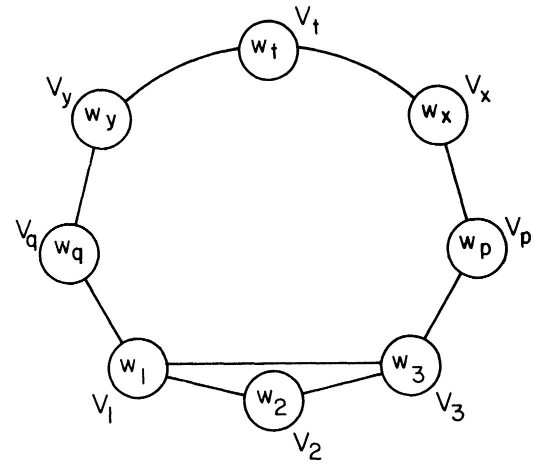
(c) $V_i = V_1$, $V_j = V_3$ in some optimum partition of the n-gon. By argument similar to (b), we can show that V_2 must be adjacent to V_1 and V_3 in the n-gon. The situation is as shown in Fig. 3(a). Then the partition in Fig. 3(b) is cheaper because

$$T_{123} \leq T_{12q}$$

and $C(w_1, w_q, w_y, w_t, w_x, w_p, w_3) \leqq C(w_2, w_q, w_y, w_t, w_x, w_p, w_3)$ according to Lemma 4. □



(a)



(b)

FIG. 3

COROLLARY 1. *For every way of choosing $V_1, V_2, \cdots$ (as prescribed), the l-optimum partition always contains $V_1 - V_2$ and $V_1 - V_3$.*

Proof. It follows from Theorem 1 and the definition of the l-optimum partition. □

Once we know $V_1 - V_2$ and $V_1 - V_3$ always exist in the l-optimum partition, we can use this fact recursively. Hence, in finding the l-optimum partition of a given polygon, we can decompose it into subpolygons by joining the smallest vertex with the second smallest and third smallest vertices repeatedly, until each of these subpolygons has the property that its smallest vertex is adjacent to both its second smallest and third smallest vertices.

A polygon having V_1 adjacent to V_2 and V_3 by sides will be called a *basic* polygon.

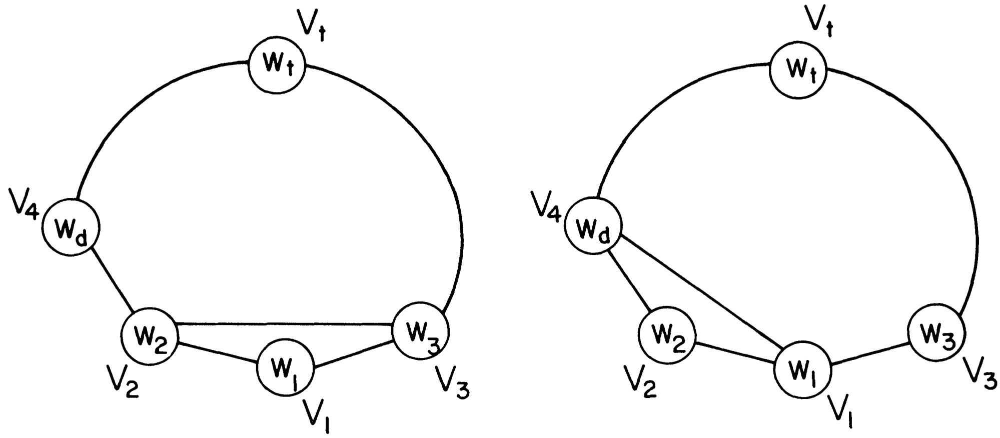
THEOREM 2. *A necessary but not sufficient condition for V_2–V_3 to exist in an optimum partition of a basic polygon is*

$$\frac{1}{w_1}+\frac{1}{w_4} \leqq \frac{1}{w_2}+\frac{1}{w_3}. \tag{6}$$

Furthermore, if $V_2 - V_3$ is not present in the l-optimum partition, then V_1, V_4 are always connected in the l-optimum partition.

Proof. If V_2, V_3 are not connected in the l-optimum partition of a basic polygon, the degree of V_1 is greater than or equal to 3. Let V_p be a vertex in the polygon and V_1, V_p be connected in the l-optimum partition. V_4 is either in the subpolygon containing V_1, V_2 and V_p or in the subpolygon containing V_1, V_3 and V_p. In either case, V_4 will be the third smallest vertex in the subpolygon. From Corollary 1, V_1, V_4 are connected in the l-optimum partition of the subpolygon and it also follows that V_1, V_4 are connected in the l-optimum partition of the basic polygon.

If V_2, V_3 are connected in an optimum partition, then we have an $(n-1)$-gon where V_2 is the smallest vertex and V_4 is the third smallest vertex. By Theorem 1, there exists an optimum partition of the $(n-1)$-gon in which V_2, V_4 are connected. Thus by induction on n, we can assume that V_4 is adjacent to V_2 in the basic polygon as shown in Fig. 4(a).



(a) (b)

FIG. 4

The cost of the partition in Fig. 4(a) is

$$T_{123}+C(w_2, w_4, \cdots, w_t, \cdots, w_3), \tag{7}$$

and the cost of the partition in Fig. 4(b) is

$$T_{124}+C(w_1, w_4, \cdots, w_t, \cdots, w_3).$$

According to Lemma 4,

$$C(w_1, w_4, \cdots, w_t, \cdots, w_3) \leqq C(w_2, w_4, \cdots, w_t, \cdots, w_3). \tag{9}$$

Since the weights of the vertices between V_4 and V_3 in the clockwise direction are all greater than or equal to w_4, the difference between the right-hand side and the left-hand side of (9) is at least

$$T_{243}-T_{143}.$$

So the necessary condition for (7) to be no greater than (8) is

$$T_{123}+T_{243} \leqq T_{124}+T_{134}$$

or

$$\frac{1}{w_1}+\frac{1}{w_4} \leqq \frac{1}{w_2}+\frac{1}{w_3}. \qquad \square$$

LEMMA 5. *In an optimum partition of an n-gon, let V_x, V_y, V_z and V_w be four vertices of an inscribed quadrilateral (V_x and V_z are not adjacent in the quadrilateral). A necessary condition for V_x–V_z to exist is*

$$\frac{1}{w_x}+\frac{1}{w_z} \geqq \frac{1}{w_y}+\frac{1}{w_w}. \tag{10}$$

Proof. The cost of partitioning the quadrilateral by the arc V_x–V_z is

$$T_{xyz}+T_{xzw}, \tag{11}$$

and the cost partitioning the quadrilateral by the arc V_y-V_w is

$$T_{xyw}+T_{yzw}. \tag{12}$$

For optimality, we have (11)$\leqq$(12) which is (10). $\square$

Note that if strict inequality holds in (10), the necessary condition is also sufficient. If equality holds in (10), the condition is sufficient for $V_x - V_z$ to exist in the l-optimum partition provided $\min(x, z) < \min(y, w)$. This lemma is a generalization of [3, Lemma 1] where V_y is the vertex with the smallest weight and V_x, V_w, V_z are three consecutive vertices with w_w greater than both w_x and w_z.

A partition is called *stable* if every quadrilateral in the partition satisfies (10).

COROLLARY 2. *An optimum partition is stable but a stable partition may not be optimum.*

Proof. The fact that an optimum partition has to be stable follows from Lemma 5. Figure 5 gives an example that a stable partition may not be optimum. □

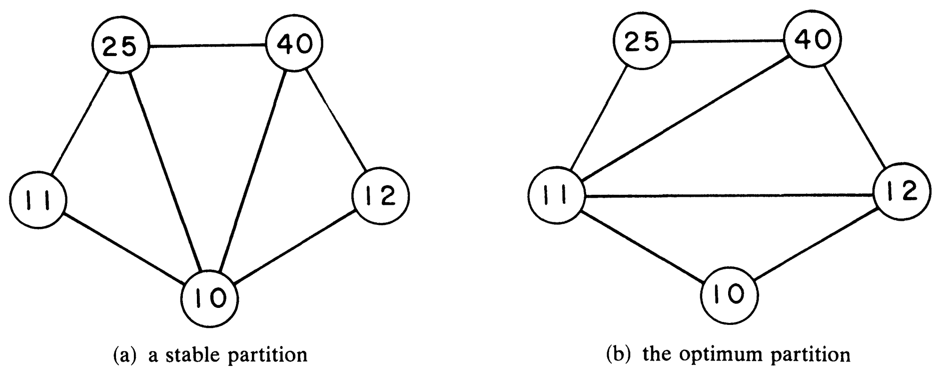


FIG. 5

In any partition of an n-gon, every arc dissects a unique quadrilateral. Let V_x, V_y, V_z, V_w be the four vectices of an inscribed quadrilateral and $V_x - V_z$ be the arc which dissects the quadrilateral. We define $V_x - V_z$ to be a *vertical* arc if (13) or (14) is satisfied.

$$\min(w_x, w_z) < \min(w_y, w_w), \tag{13}$$

$$\min(w_x, w_z) = \min(w_y, w_w), \qquad \max(w_x, w_z) \leqq \max(w_y, w_w). \tag{14}$$

We define $V_x - V_z$ to be a *horizontal* arc if (15) is satisfied

$$\min(w_x, w_z) > \min(w_y, w_w), \qquad \max(w_x, w_z) < \max(w_y, w_w). \tag{15}$$

For brevity, we shall use *h-arcs* and *v-arcs* to denote horizontal arcs and vertical arcs from now on.

COROLLARY 3. *All arcs in an optimum partition must be either vertical arcs or horizontal arcs.*

Proof. Let $V_x - V_z$ be an arc which is neither vertical nor horizontal. There are two cases:

Case 1. $\min(w_x, w_z) = \min(w_y, w_w)$ and $\max(w_x, w_z) > \max(w_y, w_w)$;

Case 2. $\min(w_x, w_z) > \min(w_y, w_w)$ and $\max(w_x, w_z) \geqq \max(w_y, w_w)$.

In both cases, the inequality (10) in Lemma 5 cannot be satisfied. This implies that the partition is not stable and hence cannot be optimum. □

THEOREM 3. *Let V_x and V_z be two arbitrary vertices which are not adjacent in a polygon, and V_w be the smallest vertex from V_x to V_z in the clockwise manner ($V_w \neq V_x$, $V_w \neq V_z$), and V_y be the smallest vertex from V_z to V_x in the clockwise manner*

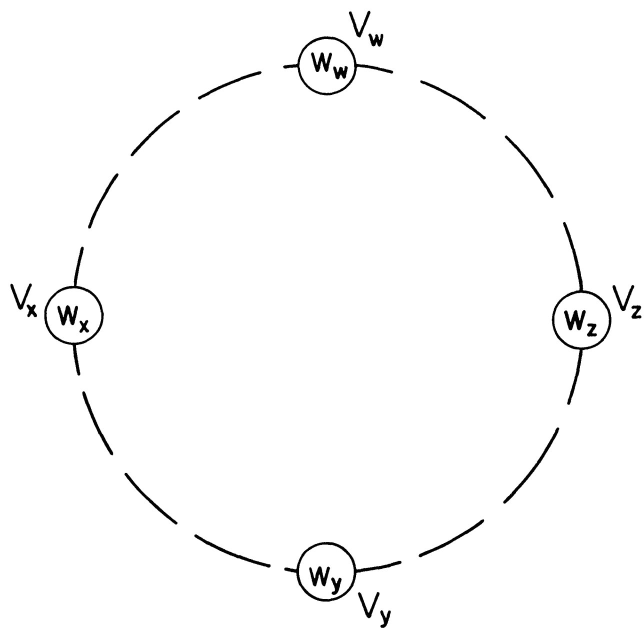


FIG. 6

($V_y \neq V_x$, $V_y \neq V_z$). This is shown in Fig. 6, *where without loss of generality we assume that $V_x < V_z$ and $V_y < V_w$. A necessary condition for $V_x - V_z$ to exist as an h-arc in the l-optimum partition is that*

$$w_y < w_x \leqq w_z < w_w.$$

(Note that the necessary condition still holds when the positions of V_y and V_w are interchanged.)

Proof. The proof is by contradiction. If $w_x \leqq w_y$, w_x must be equal to the smallest weight w_1, and $V_x - V_z$ can never satisfy (15). Hence, in order that $V_x - V_z$ exist as an h-arc in the l-optimum partition, we must have $w_y < w_x \leqq w_z$. Since V_y is the smallest vertex from V_z to V_x in the clockwise manner and $V_x < V_w$, we must have $V_y = V_1$.

Assume for the moment that $V_3 < V_x < V_z$. From Corollary 1, both $V_1 - V_2$ and $V_1 - V_3$ exist in the l-optimum partition, and the two arcs would divide the polygon into subpolygons. If V_x and V_z are in different subpolygons, then they cannot be connected in the l-optimum partition. Without loss of generality, we can assume that the polygon is a basic polygon. In this basic polygon, either $V_2 - V_3$ or $V_1 - V_4$ exists in the l-optimum partition (Theorem 2).

If V_2, V_3 are connected, then V_x and V_z are both in a smaller polygon in which we can treat V_2 as the smallest vertex and repeat the argument. If V_1, V_4 are connected, the basic polygon is again divided into two subpolygons and V_x and V_z both have to be in one of the subpolygons and the subpolygon has at most $n-1$ sides. (Otherwise $V_x - V_z$ can never exist in the l-optimum partition.) The successive reduction in the size of the polygon will either make the connection $V_x - V_z$ impossible, or force V_x and V_z to become the second smallest and the third smallest vertices in a basic subpolygon. Let V_m be the smallest vertex in this basic subpolygon. In order that $V_x - V_z$ appear as an h-arc, we must have $w_x > w_m$. From Theorem 2, the necessary condition for $V_x - V_z$ (i.e., $V_2 - V_3$) to exist in an optimum partition of the subpolygon is

$$\frac{1}{w_x}+\frac{1}{w_z} \geqq \frac{1}{w_m}+\frac{1}{w_w}.$$

Since $w_x > w_m$, the inequality is valid only if $w_z < w_w$. □

COROLLARY 4. *A weaker necessary condition for $V_x - V_z$ to exist as an h-arc in the l-optimum partition is that*

$$V_y < V_x < V_z < V_w.$$

Proof. This follows from Theorem 3. □

We call any arc which satisfies this weaker necessary condition a *potential h-arc*. Let P be the set of potential h-arcs in the n-gon and H be the set of h-arcs in the l-optimum partition, we have $P \supseteq H$ where the inclusion could be proper.

COROLLARY 5. *Let V_w be the largest vertex in the polygon and V_x and V_z be its two neighboring vertices. If there exists a vertex V_y such that $V_y < V_x$ and $V_y < V_z$, then $V_x - V_z$ is a potential h-arc.*

Proof. This follows directly from Corollary 4 where there is only one vertex between V_x and V_z. □

Two arcs are called *compatible* if both arcs can exist simultaneously in a partition. Assume that all weights of the vertices are distinct, then there are $(n-1)!$ distinct permutations of the weights around an n-gon. For example, the weights 10, 11, 25, 40, 12 in Fig. 5(a) correspond to the permutation w_1, w_2, w_4, w_5, w_3 (where $w_1 < w_2 < w_3 < w_4 < w_5$). There are infinitely many values of weights which correspond to the same permutation. For example, 1, 16, 34, 77, 29 also corresponds to w_1, w_2, w_4, w_5, w_3 but its optimum partition is different from that of 10, 11, 25, 40, 12. However, all the potential h-arcs in all the n-gons with the same permutation of weights are compatible. We state this remarkable fact as Theorem 4.

THEOREM 4. *All potential h-arcs are compatible.*

Proof. The proof is by contradiction. Let V_x, V_y, V_z and V_w be the four vertices described in Theorem 3. Hence, we have $V_y < V_x < V_z < V_w$ and $V_x - V_z$ is a potential h-arc. Let $V_p - V_q$ be a potential h-arc which is not compatible to $V_x - V_z$, as shown in Fig. 7. Without loss of generality, we can assume $V_p < V_q$. (The proof for the case $V_q < V_p$ is similar to that which follows.)

Since V_w is the smallest vertex between V_x and V_z in the clockwise manner, we have $V_z < V_w < V_q$. Hence, we have either $V_y < V_p < V_z < V_q$ or $V_y < V_z < V_p < V_q$. Both cases violate Corollary 4 and $V_p - V_q$ cannot be a potential h-arc. □

Note that the potential h-arc $V_x - V_z$ always dissects the n-gon into two subpolygons and one of these subpolygons has the property that all its vertices except V_x and V_z have weights no smaller than max (w_x, w_z). We shall call this subpolygon the *upper* subpolygon of $V_x - V_z$. For example, the subpolygon $V_x - \cdots - V_w - \cdots - V_z$ in Fig. 7 is the upper subpolygon of $V_x - V_z$.

Using Corollary 4 and Theorem 4, we can generate all the potential h-arcs of a polygon.

Let $V_x - V_z$ be the arc defined in Corollary 5, i.e., $V_1 < V_x < V_z < V_w$. The arc $V_x - V_z$ is a potential h-arc compatible with all other potential h-arcs in the n-gon. Furthermore, there is no other potential h-arc in its upper subpolygon. Now consider the $(n-1)$-gon obtained by cutting out V_w. In this $(n-1)$-gon, let $V_{w'}$ be the largest vertex and $V_{x'}$ and $V_{z'}$ be the two neighbors of $V_{w'}$ where $V_1 < V_{x'} < V_{z'} < V_{w'}$. Then $V_{x'} - V_{z'}$ is again a potential h-arc compatible with all other potential h-arcs in the

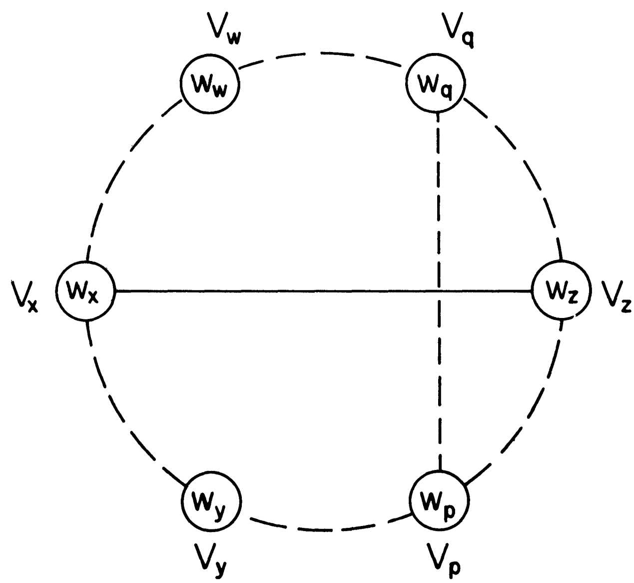


FIG. 7

n-gon and there is no other potential h-arc in its upper subpolygon which has not been generated. This is true even if V_w is in the upper subpolygon of $V_{x'} - V_{z'}$. If we repeat the process of cutting out the largest vertex, we get a set P of arcs, all of which satisfy Corollary 4. The h-arcs of the l-optimum partition must be a subset of these arcs.

The process of cutting out the largest vertex can be made into an algorithm which is $O(n)$. We shall call this algorithm the *one-sweep algorithm*. The output of the one-sweep algorithm is a set S of n-3 arcs. S is empty initially.

The one-sweep algorithm. Starting from the smallest vertex, say V_1, we travel in the clockwise direction around the polygon and push the weights of the vertices successively onto the stack as follows (w_1 will be at the bottom of the stack).

(a) Let V_t be the top element on the stack, V_{t-1} be the element immediately below V_t, and V_c be the element to be pushed onto the stack. If there are two or more vertices on the stack and $w_t > w_c$, add $V_{t-1} - V_c$ to S, pop V_t off the stack; if there is only one vertex on the stack or $w_t \leqq w_c$, *push* w_c onto the stack. Repeat this step until the nth vertex has been pushed onto the stack.

(b) If there are more than three vertices on the stack, add $V_{t-1} - V_1$ to S, pop V_t off the stack and repeat this step, else stop.

Since we do not check for the existence of a smallest vertex whose weight is no larger than those of the two neighbors of the largest vertex, i.e., the existence of the vertex V_y in Corollary 4, not all the $n-3$ arcs generated by the algorithm are potential h-arcs. However, it is not difficult to verify that the one-sweep algorithm always generates a set S of $n-3$ arcs which contains the set P of all potential h-arcs which contains the set H of all h-arcs in the l-optimum partition of the n-gon, i.e.,

$$S \supseteq P \supseteq H,$$

where each inclusion could be proper. For example, if the weights of the vertices around the n-gon in the clockwise direction are $w_1, w_2, \cdots, w_n$ where $w_1 \leqq w_2 \leqq \cdots \leqq w_n$, none of the arcs in the n-gon can satisfy Corollary 4 and hence there are no potential h-arcs in the n-gon. The one-sweep algorithm would still generate $n-3$ arcs for the n-gon but none of the arcs generated is a potential h-arc.

3. Conclusion. In this paper, we have presented several theorems on the polygon partitioning problem. Some of these theorems are characterizations of the optimum partitions of any n-sided convex polygon, while the others apply to the unique lexicographically smallest optimum partition. Based on these theorems an $O(n)$ algorithm for finding a near-optimum partition can be developed [12]. The cost of the partition produced by the heuristic algorithm never exceeds $1 \cdot 155\, C_{\text{opt}}$, where C_{opt} is the optimum cost of partitioning the polygon. An $O(n \log n)$ algorithm for finding the unique lexicographically smallest optimum partition will be presented in Part II [13].

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