Research report 74

UNIVERSAL CHAINS AND WIRING LAYOUTS

by

M.S. Paterson

(RR74)

Abstract

A universal chain can be considered as a fixed schedule for a traveller's visits such that any set of m locations to be visited can be sequenced so that travelling times are accommodated within this schedule. Upper and lower bounds are proved for the lengths of universal chains in the unit interval, unit square and in higher dimensions. Applications to wiring layouts in circuit boards are presented.

Department of Computer Science
University of Warwick
Coventry, CV4 7AL, England

April 1986

(Revised August 1986)

UNIVERSAL CHAINS AND WIRING LAYOUTS

by

M.S.Paterson

Department of Computer Science
University of Warwick
Coventry, CV4 7AL, England

Abstract

A universal chain can be considered as a fixed schedule for a traveller's visits such that any set of m locations to be visited can be sequenced so that travelling times are accommodated within this schedule. Upper and lower bounds are proved for the lengths of universal chains in the unit interval, unit square and in higher dimensions. Applications to wiring layouts in circuit boards are presented.

1. Definitions

A finite sequence of non-negative real numbers is a **chain.** The **length** of a chain is the sum of the numbers.

A chain $a_1, \ldots, a_{k-1}$ **covers** a multiset of k points in d-dimensional space if there is an ordering $x_1, \ldots, x_k$ of these points such that for $i = 1, \ldots, k-1$ the distance $d(x_i, x_{i+1})$ is less than or equal to a_i. Here we might take any metric d. For the sake of our main application we choose the L_∞ metric, $d(\langle u_1, u_2, \ldots \rangle, \langle v_1, v_2, \ldots \rangle) = \max\{ |u_1 - v_1|, |u_2 - v_2|, \ldots \}$. Other L-metrics merely yield different constants in some of our theorems.

A **universal chain for** a family F of multisets is a chain which covers each multiset in F.

We use log to mean $\log_2$ throughout.

2. Main results

We present our combinatorial results in order of increasing dimension: one, two, many. The applications appear in Section 3. The upper bound in Theorem 1 was proved by Tom Leighton, improving the result of an earlier version of this paper.

Theorem 1 (Leighton and Paterson)

For the family of all multisets of size m in the unit interval there is a universal chain which is of length $O((\log m)^3)$. Any universal chain for this family has length $\Omega((\log m)^2)$.

Proof

For the upper bound we construct a universal chain recursively. The resulting path visiting the elements of a multiset may be described roughly as follows. A rather compact multiset of elements with a binary tree-like structure is identified and these elements are visited in sequence using a suitable chain. Then the remaining multiset is visited recursively. The details of this construction are given below.

We shall prove the bound on $\mathbf{u_1(m)}$, the minimum size of a universal chain for multisets of m points in the unit interval, of:

$$u_1(m) \leq c.(\log m)^3 + 1 \quad \text{where } c = (\ln 2)/3.$$

Since there must exist some pair of points in the interval with distance apart no more than $1/(m-1)$, the chain $1/(m-1), 1, 1, \ldots, 1$ of total length $(m-2) + 1/(m-1)$ is universal. This suffices to prove the bound for all $m \leq 31$, and yields a base for our induction. The inductive step which follows actually holds for all $m \geq 10$.

Let the integer d be such that:

$$d.2^d < m \leq (d+1).2^{d+1}.$$

Working through the unit interval from left to right we identify the sequence of disjoint pairs of elements whose distance apart is at most 2^{-d}. Since each element which is unpaired by this process, except perhaps the rightmost, is followed by an empty interval of size greater than 2^{-d}, the number of such elements is less than 2^d. Next we identify in a similar way disjoint pairs of these pairs whose distance apart is at most 2^{-d+1}. The number of unpaired pairs is less than 2^{d-1}. Continuing in this way we identify disjoint pairs of (2^{r-1})-tuples with separating distance at most 2^{-d+r-1}, for $r = 1,2,\ldots,d$. The total number of elements which are excluded at some stage of this pairing process is at most:

$$2^d + 2^d + \ldots + 2^d = d.2^d < m.$$

Hence the process generates at least one 2^d-tuple satisfying the given distance constraints. The elements of this 2^d-tuple can be visited in left-to-right order using the following chain:

$$2^{-d}, 2^{-d+1}, 2^{-d}, 2^{-d+2}, 2^{-d}, 2^{-d+1}, 2^{-d}, 2^{-d+3}, \ldots.$$

Half the links have size 2^{-d}, a quarter have size 2^{-d+1}, an eighth have size 2^{-d+2}, and so on. The total length is therefore exactly $d/2$. Our universal sequence begins in this way, then has a unit link and continues recursively with a universal sequence for the remaining $m - 2^d$ elements. We have therefore shown the following inequality:

$$u_1(m) \leq d/2 + 1 + u_1(m - 2^d).$$

Now,

$$u_1(m) \leq d/2 + 1 + c.(\log(m - 2^d))^3 + 1$$

by the inductive hypothesis,

$$\leq d/2 + 1 + c.(\log m)^3 + 1 - 2^d.c.3(\log m)^2 / (m.\ln 2)$$

since the derivative of $(\log m)^3$ is a decreasing function of m,

$$\leq c.(\log m)^3 + 1 + (d+2)/2 - (\log m)^2 / (2(d+1))$$

using $c = (\ln 2)/3$ and $m \leq (d+1).2^{d+1}$,

$$\leq c.(\log m)^3 + 1$$

since $(d+1)(d+2) \leq (\log m)^2$ for $m \leq 10$.

This establishes the upper bound claimed.

For the lower bound, suppose C is a minimal length universal chain. We can assume that no element of C has length greater than 1, and for $i \geq 0$ we define k_i to be the number of elements of C with length in the range $[2^{-i}, 2^{-i+1})$. Let F_0 be the family of m+1 multisets $s_0, \ldots, s_m$ where s_j has j of its elements at 0 and $m - j$ at 1. When using C to cover members of F_0 the only choices which can be made are at which end of the interval to start and then for each member of C whether to remain at the current end or move to the other, a total of $2^{(k_0 + 1)}$ possibilities. Since C is universal for F_0 we must have

$$2^{(k_0 + 1)} \geq m+1$$

This gives already a lower bound of $\Omega(\log m)$. Now we define F_t for all $t \geq 0$ to be the family of multisets of size m where each element is at one of the points $j \cdot 2^{-t}$ for $j = 0, \ldots, 2^t$. In using C to cover members of F_t we have for each element of length less than 2^{-i+1}, a choice of at most $2^{t-i+2} - 1$ points to visit, i.e. at most $2^{t-i+1} - 1$ points on either side of the current point. Arguing as above that the total number of choices in using C for F_t must be at least as great as the cardinality of F_t we obtain the following inequality:

$$(2^t + 1) \prod_{0 \leq i \leq t} (2^{t-i+2} - 1)^{k_i} \geq \binom{m+2^t}{2^t}$$

Recognising that $(T + 1)! < T^T$ for $T \geq 3$, and taking logarithms we derive:

$$2^{-t} \sum_{0 \leq i \leq t} (t - i + 2) k_i \geq \log m - t$$

for $t \geq 2$. The same inequality may be verified explicitly for $t = 0$ and 1. Summing each side of these inequalities for $t = 0, \ldots, \log m$, we may deduce:

$$(2 + 3.2^{-1} + 4.2^{-2} + 5.2^{-3} + \ldots) \sum_{0 \leq i \leq \log m} (2^{-t}.k_i) \geq (\log m)^2 / 2$$

The series on the left has a limit of 6, so we have a lower bound on the total length of C given by $(\log m)^2/12$. ♦

For two dimensions our upper and lower bounds are closer.

Theorem 2

For the family of all sets of size m in the unit square there is a universal chain of length $O(\sqrt{m})$. Any universal chain for this family has length $\Omega(\sqrt{m})$.

Proof We shall show a bound of the form $a\sqrt{m} - 1$. If $m \leq a^2$ then a chain consisting just of unit links suffices (in the L_∞ metric). Assume therefore that $m > a^2$ and that there is a chain of length at most $a\sqrt{r} - 1$ for r points in the square where $r < m$. Let $n = \lceil \sqrt{m}/7 \rceil$ and $p = 1/n$. Divide the unit square into an $n \times n$ array of $p \times p$ squares. A **cluster** is defined to be any multiset of 10 points lying within one such subsquare. Identify a maximal number of disjoint clusters. The number of points not included within some cluster is at most $9n^2$. Select $q = \lceil (m - 9n^2)/10) \rceil$ clusters and consider the multiset Q consisting of the centres of the subsquares containing each cluster. Given a covering chain for Q we can modify this by increasing each link by an amount p and inserting 9 links of size p for each cluster. The resulting chain can cover all points in the clusters. Our universal chain is got by appending one further link of size 1 and then a universal chain for the remaining points. Thus

$$\begin{aligned} \text{total length} &\leq a\sqrt{q} - 1 + p(q-1) + 9pq + 1 + a\sqrt{(m - 10q)} - 1 \\ &< a\sqrt{q'} + 10q'/n + a\sqrt{(m - 10q')} - 1 \end{aligned}$$

for $a \geq 10$, where $q' = (m - 9n^2)/10$. To show this result to be less than or equal to $a\sqrt{m} - 1$ as required, we shall verify that:

$$a \cdot (1 - \sqrt{((1 - 9v^2)/10)} - 3v) \geq 1/v - 9v$$

where $v = n/\sqrt{m}$. We now select the value $a = 21$ and observe that since $\sqrt{m} > a$ the choice of n ensures that v lies in the interval $[1/7, 4/21]$. For this range of r the above inequality holds and the result is proved.

The lower bound is obvious from consideration of a set of m points which is distributed as a $\sqrt{m} \times \sqrt{m}$ square array with distance $1/\sqrt{m}$ between adjacent points. ♦

Finally in this section we consider the natural extension to higher dimensions and derive bounds which tighten asymptotically as the dimension tends to infinity.

Theorem 3

For all $d \geq 2$, if $u_d(m)$ is the minimal length of a universal chain for multisets of size m in the d-dimensional unit hypercube then there is a constant c(d) such that:

$$m^{1-1/d} \lesssim u_d(m) \lesssim c(d) \cdot m^{1-1/d}$$

Furthermore $c(d) \rightarrow 1$ as $d \rightarrow \infty$.

Proof

The lower bound comes from consideration of an $m^{1/d} \times \ldots \times m^{1/d}$ array of points with a spacing of $m^{-1/d}$.

For the upper bound we follow the pattern in the proof of Theorem 2, but now take **clusters** to be multisets of $k = \lceil b^d/2 \rceil$ points lying within one $b \cdot m^{-1/d} \times \ldots \times b \cdot m^{-1/d}$ subhypercube in a regular $m^{1/d}/b \times \ldots \times m^{1/d}/b$ array. The number excluded from a maximal set of clusters is at most $(k-1)m/b^d$ which is less than $m/2$, so for our construction we select $q = \lceil m/(2k) \rceil$ clusters. The resulting recurrence, following the pattern from Theorem 2, is:

$$u_d(m) \lesssim f(q) + kqb \cdot m^{-1/d} + u_d(m - qk)$$

The claimed upper bound is shown by the following inequalities:

$$c \cdot q^{1-1/d} + kqb \cdot m^{-1/d} + c \cdot (m-qk)^{1-1/d} \lesssim m^{1-1/d} (c/b^{(d-1)} + b/2 + c/2^{1-1/d})$$
$$\lesssim c \cdot m^{1-1/d}$$

provided that

$$c = c(d) \geq b/(2 - 2/b^{d-1} - 2^{-1/d}).$$

Any large enough constant b is adequate here. For the claim that $c(d) \to 1$ we choose $b = b(d)$ such that $b \to 1$ but $b^{d-1} \to \infty$ as $d \to \infty$. ♦

3. Applications

Although the above results are offered as being of purely combinatorial interest, the motivation for our research comes from the design of "semi-custom" integrated circuits. Here the main components of a layout are designed and positioned in advance leaving various wiring details and local choices to be made for particular applications. As an example of a fragment of such a design consider figure 1.

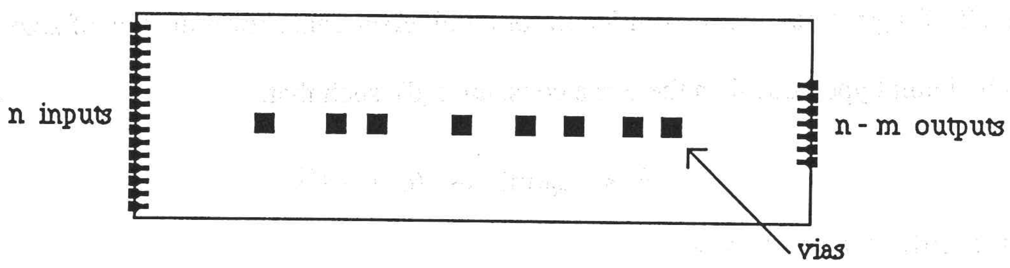

Figure 1. Input /output board with drop-offs

On the circuit board shown above there are contacts for n wires to enter from the left edge and for $n - m$ of these to leave at the right edge. Some m of these, which will be specified in the particular application are to be connected in an arbitrary order to the m "vias" (connections between the front and back faces) which are shown as black squares. Furthermore no wires are allowed to touch or cross. It should be clear that if two inputs separated by r inputs on the left edge are chosen to connect to adjacent vias then there must be a sufficient spacing between these vias to allow r wires to pass through. Figure 2 shows the form of a simple **partial weave** layout method which can always be used if the above spacing condition between adjacent vias is met. The asterisks show which subset of the inputs is to terminate at vias in this instance. The wires that pass above a via run in a contiguous parallel group at the top of the rectangle, and similarly for those passing below. Between each pair of vias there is room for all the required transfers between top and bottom. Extra spaces of size n and $n - m$ are allowed at the left and right ends respectively. As a final design stage the height of the layout is reduced as much as possible.

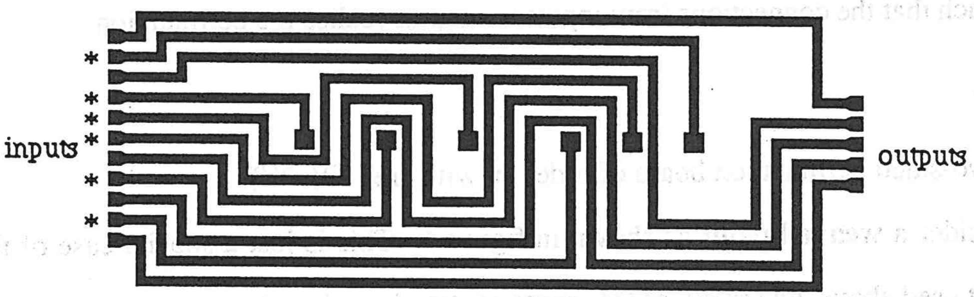

Figure 2 Partial weave layout

Theorem 4

The layout problem posed above can be solved within a rectangle as shown in figure 1 of height $O(n)$ and width $O(n(\log m)^3)$. If the vias are constrained to lie in a horizontal line, the layout requires height $\Omega(n)$ and width $\Omega((n - m)(\log m)^2)$.

Proof

The spacing of vias used corresponds to a universal chain for sets of size m within an interval of size either n or $n - m$ for the upper or lower bound respectively. ♦

A **two-sided permutation board of order** n is a square with two faces described as follows. Along one edge of the front face are n input terminals labelled $x_1, \ldots, x_n$, while along one edge of the back face are n output terminals labelled $y_1, \ldots, y_n$. In addition there are n vias arranged in the interior of the square. (See figure 3.)

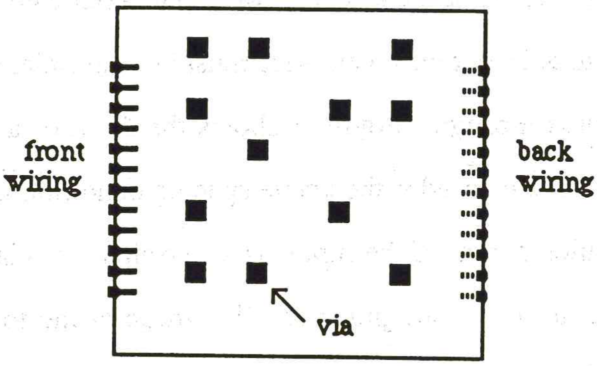

Figure 3. Two-sided permutation board

The board is to have the property that, for any n-permutation, there is a planar layout on the front face of n disjoint wires from the inputs to the vias and a similar layout from vias to outputs on the back face, such that the connections from inputs to outputs realise the permutation.

Theorem 5

There is a two-sided permutation board of order n with area $O(n^{5/2})$.

Proof Consider a **weave layout** as shown in figure 4. This is just a special case of the partial weave layout used above, where *all* of the inputs are terminated at vias.

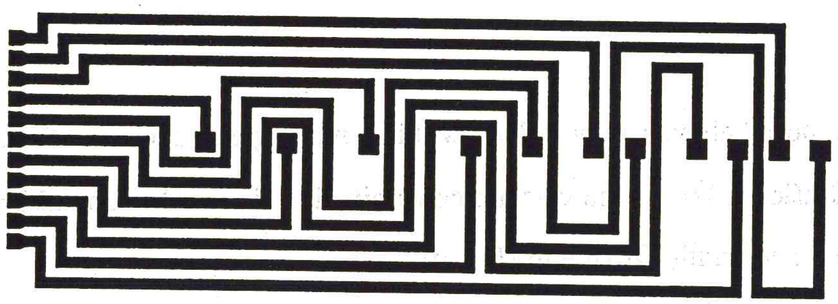

Figure 4. Weave layout

Labelling the vias $v_1, \ldots, v_n$ from left to right, we see that it is enough for the horizontal spacing between vias to satisfy:

$$d(v_i, v_{i+1}) \geq |\pi_i - \pi_{i+1}| \quad \text{for } i = 1, \ldots, n-1$$

where input π_j is connected to via j for all j.

Represent some permutation to be realised on the permutation board as an $n \times n$ permutation matrix. Embed this matrix in a unit square and consider the set of n points defined by the 1's in the matrix. By Theorem 2 there is a fixed universal chain $a_1, \ldots, a_{n-1}$ of length $O(\sqrt{m})$ which covers any such set, visiting it in some order. Using this order for the ones of the matrix let the corresponding order for the rows and the columns be $\pi_1, \ldots, \pi_n$ and $\pi_1', \ldots, \pi_n'$ respectively. A weave layout with horizontal spacings $n \cdot a_1, \ldots, n \cdot a_{n-1}$ can be made corresponding to the connection pattern given by the row order or the column order. A rectangular permutation board can therefore be made with the input and output terminals on the two faces of the left-hand edge and the vias arranged in a horizontal row with the spacings given above. This would have height $O(n)$ and width $O(n^{3/2})$. This strip can be "folded" as indicated in figure 5, and the terminals set in whichever edges are desired within a final square permutation board of area $O(n^{5/2})$. This folding technique is described in [4]. ♦

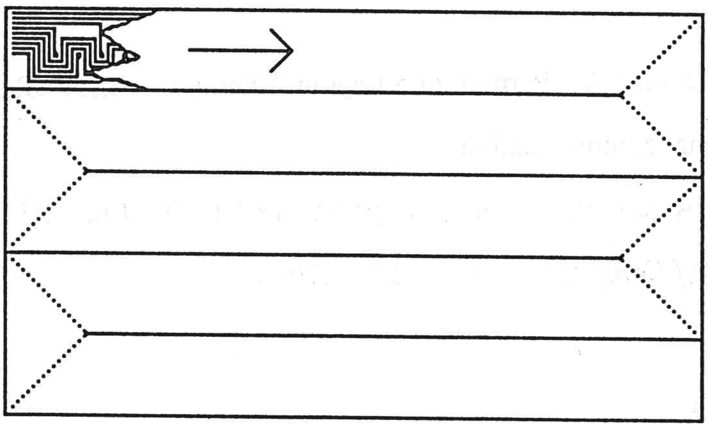

Figure 5. Folded weave layout

4. Related work and Conclusions.

Cutler and Shiloach [2] showed that $O(n^3)$ area was sufficient for the layout of all permutations between n input and n output terminals in fixed locations. Their construction is like a single weave layout as in fig. 4, but with the vias replaced by the output terminals spaced regularly with an interval of $O(n)$. In [1] a matching lower bound of $\Omega(n^3)$ is stated for this problem. This $\Omega(n^3)$ lower bound result has been extended [3] to permit the use of both sides of a circuit board and cn fixed vias, where $c < 1$. Our Theorem 5 shows that the area can be reduced to $O(n^{2.5})$ when $c = 1$. Of course if the locations of the vias are not fixed then a straightforward crossbar layout achieves an area of $O(n^2)$ using n vias.

It is an open problem to improve the bound in Theorem 5, though the lower bound from Theorem 2 suggests that this may be optimal. For the unit interval, the upper and lower bounds given in Theorem 1 leave in question the true value of u_1.

References

[1] A.Aggarwal, M.Klawe, D.Lichtenstein, N.Linial and A.Wigderson, "Layer Changes in VLSI," *Proc. 26th Ann. IEEE Symp. on Foundations of Computer Science, 1985*, 186-196.

[2] M.Cutler and Y.Shiloach, "Permutation Layout," *Networks*, **8**,(1978),253-278.

[3] M.Klawe, personal communication.

[4] C.E.Leiserson, "Area Efficient Graph Layouts (for VLSI)," *Proc. 21st Ann. IEEE Symp. on Foundations of Computer Science, 1980*, 270-281.