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009

DISJOINT PATHS IN A PLANAR GRAPH—A GENERAL THEOREM*

GUOLI DING†, A. SCHRIJVER‡, AND P. D. SEYMOUR§

Abstract. Let $D = (V, A)$ be a directed planar graph, let $(r_1, s_1), \cdots, (r_k, s_k)$ be pairs of vertices on the boundary of the unbounded face, let $A_1, \cdots, A_k$ be subsets of A, and let H be a collection of unordered pairs from $\{1, \cdots, k\}$. Given are necessary and sufficient conditions for the existence of a directed $r_i - s_i$ path P_i in (V, A_i) (for $i = 1, \cdots, k$), such that P_i and P_j are vertex-disjoint whenever $\{i, j\} \in H$.

Key words. disjoint paths, trees, planar graph

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1. Introduction. Let $D = (V, A)$ be a directed graph, let $(r_1, s_1), \ldots, (r_k, s_k)$ be pairs of vertices of D, let $A_1, \cdots, A_k$ be subsets of A, and let H be a collection of unordered pairs from $\{1, \cdots, k\}$. We are interested in the conditions under which there exist directed paths $P_1, \cdots, P_k$ so that

(1)
(i) P_i is a directed $r_i - s_i$ path in (V, A_i) $(i = 1, \cdots, k)$;

(ii) P_i and P_j are vertex-disjoint for each $\{i, j\} \in H$.

In §3 we will discuss some special cases of this problem.

Since the problem is NP-complete, we may not expect a nice set of necessary and sufficient conditions characterizing the existence of paths satisfying (1). The problem is NP-complete even if we restrict the problem to instances with $k = 2, A_1 = A_2 = A$, and $H = \{\{1, 2\}\}$. Moreover, it is NP-complete when restricted to $A_1 = \cdots = A_k = A$, H is the collection of all pairs from $\{1, \cdots, k\}$, and D arises from an undirected planar graph by replacing each edge by two opposite arcs.

In this paper we give necessary and sufficient conditions for the problem when

(2) D is planar and the vertices $r_1, s_1, \cdots, r_k, s_k$ all belong to the boundary of one fixed face I.

The characterization extends the one given by Robertson and Seymour [1]. In fact, if (2) holds, there is an easy, greedy-type algorithm for finding the path P_i, as we discuss below.

Let D be embedded in the plane $\mathbb{R}^2$. We identify D with its image in the plane. Without loss of generality, we may assume I to be the unbounded face. (Each face is considered as an *open* region.) Moreover, we may assume that the boundary $\mathrm{bd}(I)$ of I is a simple closed curve. This is no restriction, since we can extend D by new arcs as long as we do not include them in any A_i and as long as we keep $r_1, s_1, \cdots, r_k, s_k$ on $\mathrm{bd}(I)$.

We say that two pairs (r, s) and (r', s') of vertices on $\mathrm{bd}(I)$ *cross* if each $r - s$ curve in $\mathbb{R}^2 \setminus I$ intersects each $r' - s'$ curve in $\mathbb{R}^2 \setminus I$. Clearly, the following is a

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† Rutgers Center for Operations Research, Rutgers University, New Brunswick, New Jersey 08903.

‡ Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, the Netherlands.

§ Bellcore, 445 South Street, Morristown, New Jersey 07960.

necessary condition for the existence of paths satisfying (1):

(3) *cross-freeness condition*: if $\{i,j\} \in H$ then (r_i, s_i) and (r_j, s_j) do not cross.

Now the following algorithm finds paths as in (1) if (2) holds. First, check if the cross-freeness condition is satisfied. If not, our problem has no solution. If the cross-freeness condition is satisfied, choose a pair (r_i, s_i) so that the shortest of the two $r_i - s_i$ paths along $\mathrm{bd}(I)$ is as short as possible (over all $i = 1, \cdots, k$). Without loss of generality, $i = k$. Let Q be this shortest $r_k - s_k$ path along $\mathrm{bd}(I)$. If (V, A_k) does not contain any $r_k - s_k$ path, then there are no paths satisfying (1). If (V, A_k) does contain an $r_k - s_k$ path, let P_k be the (unique) directed $r_k - s_k$ path in (V, A_k) that is nearest to Q. Next, repeat the algorithm for $D, (r_1, s_1), \cdots, (r_{k-1}, s_{k-1})$, removing from any A_i with $\{i,k\} \in H$ all those arcs incident with some vertex in P_k. After at most k iterations we either find paths as required, or we find that no such paths exist.

The correctness of the algorithm follows from the following observation. Suppose that there exist paths $Q_1, \cdots, Q_k$ as required. Then, if k is as above, we may assume without loss of generality that Q_k is equal to P_k. Indeed, $Q_1, \cdots, Q_{k-1}, P_k$ also form a solution, since if P_k intersects some Q_i, then also Q_k intersects Q_i.

We describe a second necessary condition. Let C be some curve in $\mathbb{R}^2$, starting in I and ending in some face F. Let $f(C)$ and $l(C)$ denote the first and last point of intersection of C with D. Let $i_1, \cdots, i_n$ be indices from $\{1, \cdots, k\}$ such that

(4)

(i) $f(C), r_{i_1}, s_{i_1}, \ldots, r_{i_n}, s_{i_n}$ are all distinct;

(ii) The $r_{i_j} - s_{i_j}$ part of $\mathrm{bd}(I)$ containing $f(C)$ is contained in the $r_{i_{j+1}} - s_{i_{j+1}}$ part of $\mathrm{bd}(I)$ containing $f(C)$, for $j = 1, \cdots, n-1$;

(iii) $\{i_j, i_{j+1}\} \in H$ for $j = 1, \cdots, n-1$.

For each $j = 1, \cdots, n$ we define a set W_j as follows. If $f(C), r_{i_j}, s_{i_j}$ occur clockwise around $\mathrm{bd}(I)$, W_j is the set of points p on D traversed by C such that some arc in A_{i_j} is entering C at p from the left and some arc in A_{i_j} is leaving C at p from the right. Similarly, if $f(C), r_{i_j}, s_{i_j}$ occur counterclockwise around $\mathrm{bd}(I)$, W_j is the set of points p on D traversed by C such that some arc in A_{i_j} is entering C at p from the right, and some arc in A_{i_j} is leaving C at p from the left.

We say that C *fits* $i_1, \cdots, i_n$ if there exist distinct points $p_1, \cdots, p_n$ so that $p_j \in W_j$ for $j = 1, \cdots, n$ and so that C traverses $p_1, \cdots, p_n$ in this order. Now we have the following condition:

(5) *cut condition*: each curve C starting and ending in I fits each choice of $i_1, \cdots, i_n$ satisfying (4), whenever $(f(C), l(C))$ crosses each (r_{i_j}, s_{i_j}) $(j = 1, \cdots, n)$.

2. The theorem.

We now prove the following theorem.

THEOREM. *Let $D = (V, A)$ be a directed planar graph, embedded in the plane $\mathbb{R}^2$, let $(r_1, s_1), \cdots, (r_k, s_k)$ be pairs of vertices of D on $\mathrm{bd}(I)$, with $r_i \neq s_i$ for $i = 1, \cdots, n$, let $A_1, \cdots, A_k$ be subsets of A, and let H be a set of unordered pairs from $\{1, \cdots, k\}$.*

Then there exist paths $P_1, \cdots, P_k$ satisfying (1) *if and only if the cross-freeness condition* (3) *and the cut condition* (5) *hold.*

Proof. Necessity of the conditions is trivial. To see sufficiency, we assume without loss of generality that the arcs on $\mathrm{bd}(I)$ do not belong to any A_i. (We can add new arcs to D (but not to any A_i), without violating the cross-freeness and cut conditions.)

Choose an arbitrary point p_0 on $\mathrm{bd}(I)$, not being a vertex of D. For each $i = 1, \cdots, k$, let Q_i be that of the two $r_i - s_i$ parts of $\mathrm{bd}(I)$ that does not contain p_0. For each $i = 1, \cdots, k$, let $\mathcal{F}_i$ be the set of faces $F \neq I$ of D for which there exists a curve C starting in I and ending in F, such that $f(C) \in Q_i$, and such that C does not fit some choice of $i_1, \cdots, i_n$ satisfying (4) with $i_n = i$.

Note that, since no arc on $\mathrm{bd}(I)$ belongs to A_i, each arc in Q_i is on the boundary of $\bigcup \mathcal{F}_i$. Let B_i be the set of arcs on the boundary of $\bigcup \mathcal{F}_i$ but not in Q_i. We show that

(6) B_i is contained in A_i and contains a directed $r_i - s_i$ path.

Assume without loss of generality that r_i, p_0, s_i occur in this order clockwise around $\mathrm{bd}(I)$. Let a be an arc on the boundary of $\bigcup \mathcal{F}_i$ and not in Q_i. We show that a belongs to A_i and that a is oriented clockwise with respect to $\bigcup \mathcal{F}_i$.

Let a separate faces $F \in \mathcal{F}_i$ and $F' \notin \mathcal{F}_i$. By definition of $\mathcal{F}_i$, there exists a curve C starting in I and ending in F, such that $f(C) \in Q_i$ and such that C does not fit some choice $i_1, \cdots, i_n$ satisfying (4) with $i_n = i$. Now extend C to F' by crossing a, obtaining a curve C'.

If C' does not fit $i_1, \cdots, i_n$, then $F' = I$ (as $F' \notin \mathcal{F}_i$). Then, however, C' violates the cut condition.

So C' does fit $i_1, \cdots, i_n$. Since C itself does not fit $i_1, \cdots, i_n$, this implies that a belongs to A_i and that a is oriented clockwise with respect to $\bigcup \mathcal{F}_i$. This proves (6).

Choose for each $i = 1, \cdots, k$ a directed $r_i - s_i$ path P_i in B_i. We finally show that if $\{i, j\} \in H$, then P_i and P_j are vertex-disjoint. Assume without loss of generality that $i = 1, j = 2$, and let $\{1, 2\} \in H$. Suppose some vertex v is traversed both by P_1 and P_2. Hence v is incident with some face F_1 in $\mathcal{F}_1$ and with some face F_2 in $\mathcal{F}_2$. It follows that there exists a curve C from I to F_1 such that $f(C) \in Q_i$ and such that C does not fit indices $i_1, \cdots, i_n$ satisfying (4) with $i_n = 1$.

By the cross-freeness condition, we know that parts Q_1 and Q_2 of $\mathrm{bd}(I)$ are either contained in each other or are disjoint.

First, assume that they are contained in each other, say $Q_1 \subseteq Q_2$. Then each face $F' \neq I$ incident with v is contained in $\mathcal{F}_2$. To see this, we can extend curve C via v to F', yielding curve C'. As C does not fit $i_1, \ldots, i_n = 1$, it follows that C' does not fit $i_1, \cdots, i_n = 1, i_{n+1} = 2$. So $F' \in \mathcal{F}_2$. As this holds for each face $F' \neq I$ incident with v, no arc incident with v belongs to B_2, and hence P_2 does not traverse v.

Next, assume that Q_1 and Q_2 are disjoint. (So p_0 is in between Q_1 and Q_2.) Since F_2 belongs to $\mathcal{F}_2$, there exists a curve C' from I to F_2 not fitting indices $i'_1, \cdots, i'_{n'}$ satisfying (4) (adapted to $C', i'_1, \cdots, i'_{n'}$), such that $f(C') \in Q_2$ and such that $i'_{n'} = 2$.

Connect the curves C and C' by a $F_1 - F_2$ curve via v, yielding a curve C'' from I to I. Then C'' does not fit $i_1, \cdots, i_n, i'_{n'}, \cdots, i'_1$, as we can easily check. This violates the cut condition. □

The theorem can be seen to give a "good characterization."

3. Special cases. In this section we describe some special cases of the problem and the theorem.

First, let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $\{r_1, s_1\}, \cdots, \{r_k, s_k\}$ be pairs of vertices of G, each on the boundary of the unbounded face I of G. Robertson and Seymour [1] proved that there exist pairwise vertex-disjoint paths $P_1, \cdots, P_k$ in G where P_i connects r_i and s_i for $i = 1, \cdots, k$, if and only if no two of the pairs $\{r_i, s_i\}$ cross and each vertex cut of G contains at least as many vertices as it separates pairs from $\{r_1, s_1\}, \cdots, \{r_k, s_k\}$.

This follows trivially from our theorem by replacing each arc by two opposite arcs, and taking for H the collection of all pairs from $\{1, \cdots, k\}$.

The second special case generalizes the first. Let $G = (V, E)$ be an undirected planar graph, embedded in $\mathbb{R}^2$. Let $R_1, \cdots, R_t$ be pairwise disjoint sets of vertices of G, all on the boundary of the unbounded face I of G.

We say that two sets R and R' of vertices on the boundary of I *cross* if some pair of vertices in R crosses some pair of vertices in R'. We say that a cut *separates* a set R of vertices if the cut separates $\{r, s\}$ for some r, s in R.

Robertson and Seymour [1] proved more generally that there exist pairwise vertex-disjoint trees $T_1, \cdots, T_t$ in G such that T_i covers R_i $(i = 1, \cdots, t)$ if and only if no two of the R_i cross, and each vertex cut of G contains at least as many vertices as it separates sets from $R_1, \cdots, R_t$.

This follows from the theorem by replacing each edge of G by two opposite edges, by taking as pairs $(r_1, s_1), \cdots, (r_k, s_k)$ all pairs (r, s) for which there exists an $i \in \{1, \cdots, t\}$ such that $r, s \in R_i$, and by taking for H all pairs $\{j, j'\}$ from $\{1, \cdots, k\}$ for which $r_j, s_j, r_{j'}$, and $s_{j'}$ do not all belong to the same set among $R_1, \cdots, R_t$. (We take each A_j to be equal to the full arc set.)

As a third special case, consider a planar directed graph $D = (V, A)$ and a collection of ordered pairs $(r_1, s_1), \cdots, (r_k, s_k)$ on the boundary of the unbounded face I (with $r_i \neq s_i$ for $i = 1, \cdots, k$). Then the theorem implies that there exists a directed $r_i - s_i$ path P_i for $i = 1, \cdots, k$ so that $P_1, \cdots, P_k$ are pairwise vertex-disjoint if and only if no two of the (r_i, s_i) cross, and for each cut C not intersecting any of $r_1, s_1, \cdots, r_k, s_k$, the following *cut condition* holds:

(7) If C separates $(r_{i_1}, s_{i_1}), \cdots, (r_{i_n}, s_{i_n})$, in this order, then C contains vertices $p_1, \cdots, p_n$, in this order so that for each $j = 1, \cdots, n$:

- if r_{i_j} is at the left-hand side of C, then at least one arc of D is entering C at p_j from the left and at least one arc of D is leaving C at p_j from the right;
- if r_{i_j} is at the right-hand side of C, then at least one arc of D is entering C at p_j from the right and at least one arc of D is leaving C at p_j from the left.

This follows by taking for H the set of all pairs from $\{1, \cdots, k\}$ and taking each A_i equal to A.

More generally, let $D = (V, A)$ be a planar directed graph, let $R_1, \cdots, R_t$ be sets of vertices on the boundary of the unbounded face I of D, and let, for each $i = 1, \cdots, k$, r_i be some vertex from R_i. The theorem gives necessary and sufficient conditions for the existence of pairwise vertex-disjoint rooted trees $T_1, \cdots, T_k$ in D, where T_i has root r_i and covers R_i $(i = 1, \cdots, k)$. Again this follows straightforwardly with reductions like the above.

Finally, let $D = (V, A)$ be a planar directed graph and let $R_1, \cdots, R_k$ be sets of vertices on the boundary of the unbounded face I of G. Again, it is straightforward to derive necessary and sufficient conditions for the existence of pairwise vertex-disjoint strongly connected subgraphs $D_1, \cdots, D_k$ such that D_i covers R_i (for $i = 1, \cdots, k$).

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