# MULTIADAPTIVE GALERKIN METHODS FOR ODES III: A PRIORI ERROR ESTIMATES* 

ANDERS LOGG ${ }^{\dagger}$


#### Abstract

The multiadaptive continuous/discontinuous Galerkin methods mcG $(q)$ and mdG(q) for the numerical solution of initial value problems for ordinary differential equations are based on piecewise polynomial approximation of degree $q$ on partitions in time with time steps which may vary for different components of the computed solution. In this paper, we prove general order a priori error estimates for the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods. To prove the error estimates, we represent the error in terms of a discrete dual solution and the residual of an interpolant of the exact solution. The estimates then follow from interpolation estimates, together with stability estimates for the discrete dual solution.


Key words. multiadaptivity, individual time steps, local time steps, ODE, continuous Galerkin, discontinuous Galerkin, $\operatorname{mcG}(q), \operatorname{mdG}(q)$, a priori error estimates, existence, stability, Peano kernel theorem, interpolation estimates, piecewise smooth

AMS subject classifications. 65L05, 65L07, 65L20, 65L50, 65L60, 65L70
DOI. 10.1137/040604133

1. Introduction. This is part 3 in a sequence of papers [32, 33] on multiadaptive Galerkin methods, $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$, for approximate (numerical) solution of ODEs of the form

$$
\begin{align*}
\dot{u}(t) & =f(u(t), t), \quad t \in(0, T],  \tag{1.1}\\
u(0) & =u_{0}
\end{align*}
$$

where $u:[0, T] \rightarrow \mathbb{R}^{N}$ is the solution to be computed, $u_{0} \in \mathbb{R}^{N}$ a given initial condition, $T>0$ a given final time, and $f: \mathbb{R}^{N} \times(0, T] \rightarrow \mathbb{R}^{N}$ a given function that is Lipschitz-continuous in $u$ and bounded.

In the previous two parts of our series on multiadaptive Galerkin methods, we proved a posteriori error estimates, through which the time steps are adaptively determined from residual feedback and stability information, obtained by solving a dual linearized problem. In this paper, we prove a priori error estimates for $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$. We also prove the stability estimates and interpolation estimates which are essential to the a priori error analysis.

Standard methods for the time-discretization of (1.1) require that the resolution is equal for all components $U_{i}(t)$ of the computed approximate solution $U(t)$ of (1.1). This includes all standard Galerkin or Runge-Kutta methods; see [9, 4, 23, 24, 41, 2]. Using the same time step sequence $k=k(t)$ for all components could become very costly if the different components of the solution exhibit multiple time scales of different magnitudes. We therefore propose a new representation of the solution in which the difference in time scales is reflected in the componentwise time-discretization of (1.1), that is, each component $U_{i}(t)$ is computed using an individual time step sequence $k_{i}=k_{i}(t)$.

[^0]The multiadaptive Galerkin methods $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ first presented in [32] are formulated as extensions of the standard continuous and discontinuous Galerkin methods $\mathrm{cG}(q)$ and $\mathrm{dG}(q)$, studied earlier in detail by Hulme [28, 27, Jamet 29], Delfour, Hager, and Trochu [7, Eriksson, Johnson, and Thomée 16, 30, 11, 12, 10, 13, 14, 15, 8, , and Estep et al. [17, 18, 19, 21, 20. As such, the analysis of the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods can be carried out within the existing framework, but the extension to multiadaptive time-stepping leads to some technical challenges, in particular, proving the appropriate interpolation estimates.

Local (multiadaptive) time-stepping has been explored before to some extent for specific applications, including specialized integrators for the $n$-body problem [37, 5, 1] and low-order methods for conservation laws [39, 22, 6]. Early attempts at local time-stepping include [25, 26]. Recently, a new class of related methods, known as asynchronous variational integrators (AVI) with local time steps, has been proposed 31.
1.1. Main results. The main results of this paper are a priori error estimates for the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods, respectively, of the form

$$
\begin{equation*}
\|e(T)\|_{l_{p}} \leq C S(T)\left\|k^{2 q} u^{(2 q)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\|e(T)\|_{l_{p}} \leq C S(T)\left\|k^{2 q+1} u^{(2 q+1)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{1.3}
\end{equation*}
$$

for $p=2$ or $p=\infty$, where $C$ is an interpolation constant, $S(T)$ is a (computable) stability factor, and $k^{2 q} u^{(2 q)}\left(\right.$ or $\left.k^{2 q+1} u^{(2 q+1)}\right)$ combines local time steps $k_{i}=k_{i}(t)$ with derivatives of the exact solution $u$. The norm $L_{\infty}(I,\|\cdot\|)$ is defined by $\|v\|_{L_{\infty}(I,\|\cdot\|)}=$ $\sup _{t \in I}\|v(t)\|$. These estimates state that the $\operatorname{mcG}(q)$ method is of order $2 q$ and that the $\operatorname{mdG}(q)$ method is of order $2 q+1$ in the local time step. We refer to section 6.2 for the exact results. It should be noted that superconvergence is obtained only at synchronized time levels, such as the end-point $t=T$. For the general nonlinear problem, we obtain exponential estimates for the stability factor $S(T)$. In 34, we prove that for a parabolic model problem, the stability factor remains bounded and of unit size, independent of $T$ (up to a logarithmic factor).
1.2. Notation. The following notation is used throughout this paper. Each component $U_{i}(t), i=1, \ldots, N$, of the approximate $\mathrm{m}(\mathrm{c} / \mathrm{d}) \mathrm{G}(q)$ solution $U(t)$ of (1.1) is a piecewise polynomial on a partition of $(0, T]$ into $M_{i}$ subintervals. Subinterval $j$ for component $i$ is denoted by $I_{i j}=\left(t_{i, j-1}, t_{i j}\right]$, and the length of the subinterval is given by the local time step $k_{i j}=t_{i j}-t_{i, j-1}$. This is illustrated in Figure 1 On each subinterval $I_{i j},\left.U_{i}\right|_{I_{i j}}$ is a polynomial of degree $q_{i j}$ and we refer to $\left(I_{i j},\left.U_{i}\right|_{I_{i j}}\right)$ as an element.

Furthermore, we shall assume that the interval $(0, T]$ is partitioned into blocks between certain synchronized time levels $0=T_{0}<T_{1}<\cdots<T_{M}=T$. We refer to the set of intervals $\mathcal{T}_{n}$ between two synchronized time levels $T_{n-1}$ and $T_{n}$ as a time slab:

$$
\mathcal{T}_{n}=\left\{I_{i j}: T_{n-1} \leq t_{i, j-1}<t_{i j} \leq T_{n}\right\}
$$

We denote the length of a time slab by $K_{n}=T_{n}-T_{n-1}$. We also refer to the entire collection of intervals $I_{i j}$ as the partition $\mathcal{T}$.

Since different components use different time steps, a local interval $I_{i j}$ may contain nodal points for other components, that is, some $t_{i^{\prime} j^{\prime}} \in\left(t_{i, j-1}, t_{i j}\right)$. We denote the set of such internal nodes on a local interval $I_{i j}$ by $\mathcal{N}_{i j}$.


Fig. 1. Individual partitions of the interval ( $0, T]$ for different components. Elements between common synchronized time levels are organized in time slabs. In this example, we have $N=6$ and $M=4$.
1.3. Outline of the paper. The outline of this paper is as follows. In section 2 we give the full definition of the multiadaptive Galerkin methods $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$. We also introduce the dual methods $\operatorname{mcG}(q)^{*}$ and $\operatorname{mdG}(q)^{*}$, which are of importance to the a priori error analysis. In sections 3and 4, respectively, we then prove existence and stability of the discrete solutions as defined in section 2

In section 5, we prove the interpolation estimates that we later use to prove the a priori error estimates in section 6. Proving the interpolation estimates is technically challenging, since the function to be interpolated may be discontinuous within the interval of interpolation. To measure the regularity of the interpolated function, it is then necessary to take into consideration the size of the jump in function value and derivatives at each point of discontinuity.

Finally, in section 7 we present some numerical evidence for the a priori error estimates by solving a simple model problem and showing that we obtain the predicted convergence rates, $k^{2 q}$ and $k^{2 q+1}$, respectively, for the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods.
2. Definition of methods. In this section, we give the definitions of the multiadaptive Galerkin methods $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$. The multiadaptive methods are obtained as extensions of the standard (monoadaptive) Galerkin methods $\mathrm{cG}(q)$ and $\mathrm{dG}(q)$ by extending the trial and test spaces to allow individual time step sequences for different components.

As an important tool for the a priori error analysis in section 6e also introduce the discrete dual problem and the discrete dual methods $\operatorname{mcG}(q)^{*}$ and $\operatorname{mdG}(q)^{*}$.
2.1. Multiadaptive continuous Galerkin, $\operatorname{mcG}(\boldsymbol{q})$. To formulate the $\operatorname{mcG}(q)$ method, we define the trial space $V$ and the test space $V$ as

$$
\begin{align*}
V & =\left\{v \in[\mathcal{C}([0, T])]^{N}:\left.v_{i}\right|_{I_{i j}} \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right), j=1, \ldots, M_{i}, i=1, \ldots, N\right\},  \tag{2.1}\\
\hat{V} & =\left\{v:\left.v_{i}\right|_{I_{i j}} \in \mathcal{P}^{q_{i j}-1}\left(I_{i j}\right), j=1, \ldots, M_{i}, i=1, \ldots, N\right\},
\end{align*}
$$

where $\mathcal{P}^{q}(I)$ denotes the linear space of polynomials of degree $q$ on an interval $I \subset \mathbb{R}$. In other words, $V$ is the space of vector-valued continuous piecewise polynomials of degree $q=\left(q_{i}(t)\right)$ with $q_{i}(t) \geq 1$ on the partition $\mathcal{T}$, and $\hat{V}$ is the space of vectorvalued (possibly discontinuous) piecewise polynomials of degree $q-1=\left(q_{i}(t)-1\right)$ on the same partition.

We now define the mcG $(q)$ method for (1.1) as follows: Find $U \in V$ with $U(0)=$ $u_{0}$ such that

$$
\begin{equation*}
\int_{0}^{T}(\dot{U}, v) d t=\int_{0}^{T}(f(U, \cdot), v) d t \quad \forall v \in \hat{V} \tag{2.2}
\end{equation*}
$$

where $(\cdot, \cdot)$ denotes the $\mathbb{R}^{N}$ inner product. With a suitable choice of test function $v$, it follows that the global problem (2.2) can be restated as a sequence of successive local problems for each component: For $i=1, \ldots, N, j=1, \ldots, M_{i}$, find $\left.U_{i}\right|_{I_{i j}} \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right)$ with $U_{i}\left(t_{i, j-1}\right)$ given such that

$$
\begin{equation*}
\int_{I_{i j}} \dot{U}_{i} v d t=\int_{I_{i j}} f_{i}(U, \cdot) v d t \quad \forall v \in \mathcal{P}^{q_{i j}-1}\left(I_{i j}\right) \tag{2.3}
\end{equation*}
$$

where the initial condition is specified for $i=1, \ldots, N$ by $U_{i}(0)=u_{i}(0)$.
We define the residual $R$ of the approximate solution $U$ by $R_{i}(U, t)=\dot{U}_{i}(t)-$ $f_{i}(U(t), t)$. In terms of the residual, we can rewrite (2.3) in the form

$$
\begin{equation*}
\int_{I_{i j}} R_{i}(U, \cdot) v d t=0 \quad \forall v \in \mathcal{P}^{q_{i j}-1}\left(I_{i j}\right), \quad j=1, \ldots, M_{i}, \quad i=1, \ldots, N \tag{2.4}
\end{equation*}
$$

that is, the residual is orthogonal to the test space on each local interval. We refer to (2.4) as the Galerkin orthogonality of the $\operatorname{mcG}(q)$ method.
2.2. Multiadaptive discontinuous Galerkin, $\operatorname{mdG}(\boldsymbol{q})$. For $\operatorname{mdG}(q)$, we define the trial and test spaces by

$$
\begin{equation*}
V=\hat{V}=\left\{v:\left.v_{i}\right|_{I_{i j}} \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right), j=1, \ldots, M_{i}, i=1, \ldots, N\right\} \tag{2.5}
\end{equation*}
$$

that is, both trial and test functions are vector-valued (possibly discontinuous) piecewise polynomials of degree $q=\left(q_{i}(t)\right)$ with $q_{i}(t) \geq 0$ on the partition $\mathcal{T}$. By definition, the $\operatorname{mdG}(q)$ solution $U \in V$ is left-continuous.

We now define the $\operatorname{mdG}(q)$ method for (1.1) as follows: Find $U \in V$ with $U\left(0^{-}\right)=$ $u_{0}$ such that

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{j=1}^{M_{i}}\left[\left[U_{i}\right]_{i, j-1} v_{i}\left(t_{i, j-1}^{+}\right)+\int_{I_{i j}} \dot{U}_{i} v_{i} d t\right]=\int_{0}^{T}(f(U, \cdot), v) d t \quad \forall v \in \hat{V} \tag{2.6}
\end{equation*}
$$

where $\left[U_{i}\right]_{i, j-1}=U_{i}\left(t_{i, j-1}^{+}\right)-U_{i}\left(t_{i, j-1}^{-}\right)$denotes the jump in $U_{i}(t)$ across the node $t=t_{i, j-1}$, and where $v\left(t^{+}\right)=\lim _{s \rightarrow t^{+}} v(s)$.

The $\operatorname{mdG}(q)$ method in local form, corresponding to (2.3), reads as follows: For $i=1, \ldots, N, j=1, \ldots, M_{i}$, find $\left.U_{i}\right|_{I_{i j}} \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right)$ such that

$$
\begin{equation*}
\left[U_{i}\right]_{i, j-1} v\left(t_{i, j-1}\right)+\int_{I_{i j}} \dot{U}_{i} v d t=\int_{I_{i j}} f_{i}(U, \cdot) v d t \quad \forall v \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right) \tag{2.7}
\end{equation*}
$$

where the initial condition is specified for $i=1, \ldots, N$ by $U_{i}\left(0^{-}\right)=u_{i}(0)$.
The residual $R$ is defined on the inner of each local interval $I_{i j}$ by $R_{i}(U, t)=$ $\dot{U}_{i}(t)-f_{i}(U(t), t)$. In terms of the residual, (2.7) can be restated in the form

$$
\begin{equation*}
\left[U_{i}\right]_{i, j-1} v\left(t_{i, j-1}^{+}\right)+\int_{I_{i j}} R_{i}(U, \cdot) v d t=0 \quad \forall v \in \mathcal{P}^{q_{i j}}\left(I_{i j}\right) \tag{2.8}
\end{equation*}
$$

for $j=1, \ldots, M_{i}, i=1, \ldots, N$. We refer to (2.8) as the Galerkin orthogonality of the $\operatorname{mdG}(q)$ method.
2.3. The dual problem. The dual problem is the standard tool for error analysis, a priori or a posteriori, of Galerkin finite element methods for the numerical solution of differential equations; see [8, 3]. For the a posteriori error analysis of the multiadaptive Galerkin methods $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ in [32, we formulate a continuous dual problem. For the a priori error analysis of this paper, we formulate instead a discrete dual problem. The discrete dual problem was first introduced for the family of discontinuous Galerkin methods $\mathrm{dG}(q)$ in [16]. As we shall see, the discrete dual problem can be expressed as a Galerkin method for a continuous problem.

The discrete dual solution $\Phi:[0, T] \rightarrow \mathbb{R}^{N}$ is a Galerkin approximation of the exact solution $\phi:[0, T] \rightarrow \mathbb{R}^{N}$ of the continuous dual problem

$$
\begin{align*}
-\dot{\phi}(t) & =J^{\top}(\pi u, U, t) \phi(t)+g(t), \quad t \in[0, T)  \tag{2.9}\\
\phi(T) & =\psi
\end{align*}
$$

where $\pi u$ is an interpolant or a projection of the exact solution $u$ of (1.1), $g:[0, T] \rightarrow$ $\mathbb{R}^{N}$ is a given function, $\psi \in \mathbb{R}^{N}$ is a given initial condition, and

$$
\begin{equation*}
J^{\top}(\pi u, U, t)=\left(\int_{0}^{1} \frac{\partial f}{\partial u}(s \pi u(t)+(1-s) U(t), t) d s\right)^{\top} \tag{2.10}
\end{equation*}
$$

that is, an appropriate mean value of the transpose of the Jacobian of the right-hand side $f(\cdot, t)$ evaluated at $\pi u(t)$ and $U(t)$. Note that by the chain rule, we have

$$
\begin{equation*}
J(\pi u, U, \cdot)(U-\pi u)=f(U, \cdot)-f(\pi u, \cdot) \tag{2.11}
\end{equation*}
$$

The data $(\psi, g)$ of the dual problem allow us to obtain error estimates for different functionals $L_{\psi, g}$ of the error $e=U-u$.

We define below two new Galerkin methods for the dual problem (2.9): the dual methods $\operatorname{mcG}(q)^{*}$ and $\operatorname{mdG}(q)^{*}$. We will later use the $\operatorname{mcG}(q)^{*}$ method to express the error of the $\operatorname{mcG}(q)$ solution of (1.1) in terms of the $\operatorname{mcG}(q)^{*}$ solution of (2.9). Similarly, we will express the error of the $\operatorname{mdG}(q)$ solution of (1.1) in terms of the $\operatorname{mdG}(q)^{*}$ solution of (2.9).
2.4. Multiadaptive dual continuous Galerkin, $\mathbf{m c G}(\boldsymbol{q})^{*}$. In the formulation of the dual method of $\operatorname{mcG}(q)$, we interchange the trial and test spaces of $\mathrm{mcG}(q)$. With the same definitions of $V$ and $\hat{V}$ as in (2.1), we thus define the $\operatorname{mcG}(q)^{*}$ method for (2.9) as follows: Find $\Phi \in \hat{V}$ with $\Phi\left(T^{+}\right)=\psi$ such that

$$
\begin{equation*}
\int_{0}^{T}(\dot{v}, \Phi) d t=\int_{0}^{T}(J(\pi u, U, \cdot) v, \Phi)+L_{\psi, g}(v) \tag{2.12}
\end{equation*}
$$

for all $v \in V$ with $v(0)=0$, where

$$
\begin{equation*}
L_{\psi, g}(v) \equiv(v(T), \psi)+\int_{0}^{T}(v, g) d t \tag{2.13}
\end{equation*}
$$

Notice the extra condition that the test functions should vanish at $t=0$, which is introduced to make the dimension of the test space equal to the dimension of the trial space. Integrating by parts, (2.12) can alternatively be expressed in the form

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{j=1}^{M_{i}}\left[-\left[\Phi_{i}\right]_{i j} v_{i}\left(t_{i j}\right)-\int_{I_{i j}} \dot{\Phi}_{i} v_{i} d t\right]=\int_{0}^{T}\left(J^{\top}(\pi u, U, \cdot) \Phi+g, v\right) d t \tag{2.14}
\end{equation*}
$$

2.5. Multiadaptive dual discontinuous Galerkin, $\operatorname{mdG}(\boldsymbol{q})^{*}$. With the same definitions of $V$ and $\hat{V}$ as in (2.5), we define the $\operatorname{mdG}(q)^{*}$ method for (2.9) as follows: Find $\Phi \in \hat{V}$ with $\Phi\left(T^{+}\right)=\psi$ such that

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{j=1}^{M_{i}}\left[\left[v_{i}\right]_{i, j-1} \Phi_{i}\left(t_{i, j-1}^{+}\right)+\int_{I_{i j}} \dot{v}_{i} \Phi_{i} d t\right]=\int_{0}^{T}(J(\pi u, U, \cdot) v, \Phi) d t+L_{\psi, g}(v) \tag{2.15}
\end{equation*}
$$

for all $v \in V$ with $v\left(0^{-}\right)=0$. Integrating by parts, (2.15) can alternatively be expressed in the form

$$
\begin{equation*}
\sum_{i=1}^{N} \sum_{j=1}^{M_{i}}\left[-\left[\Phi_{i}\right]_{i j} v_{i}\left(t_{i j}^{-}\right)-\int_{I_{i j}} \dot{\Phi}_{i} v_{i} d t\right]=\int_{0}^{T}\left(J^{\top}(\pi u, U, \cdot) \Phi+g, v\right) d t \tag{2.16}
\end{equation*}
$$

3. Existence of solutions. To prove existence of the discrete $\operatorname{mcG}(q), \operatorname{mdG}(q)$, $\operatorname{mcG}(q)^{*}$, and $\operatorname{mdG}(q)^{*}$ solutions defined in the previous section, we formulate fixed point iterations for the construction of solutions. Existence then follows from the Banach fixed point theorem if the time steps are sufficiently small.

Lemma 3.1 (fixed point iteration). Let $\mathcal{T}_{n}$ be a time slab with synchronized time levels $T_{n-1}$ and $T_{n}$. With time reversed for the dual methods (to simplify the notation), the $\operatorname{mcG}(q), \operatorname{mdG}(q), \operatorname{mcG}(q)^{*}$, and $\operatorname{mdG}(q)^{*}$ methods can all be expressed in the following form: For all $I_{i j} \in \mathcal{T}_{n}$, find $\left\{\xi_{i j n}\right\}$ (the degrees of freedom for $U_{i}$ on $I_{i j}$ ) such that

$$
\begin{equation*}
\xi_{i j n}=u_{i}(0)+\int_{0}^{t_{i, j-1}} f_{i}(U, \cdot) d t+\int_{I_{i j}} w_{n}^{\left[q_{i j}\right]}\left(\tau_{i j}(t)\right) f_{i}(U, \cdot) d t \tag{3.1}
\end{equation*}
$$

where $\tau_{i j}(t)=\left(t-t_{i, j-1}\right) /\left(t_{i j}-t_{i, j-1}\right)$ and $\left\{w_{n}^{\left[q_{i j}\right]}\right\}$ is a set of polynomial weight functions on $[0,1]$.

Proof. The result follows from the definitions of the $\operatorname{mcG}(q), \operatorname{mdG}(q), \operatorname{mcG}(q)^{*}$, and $\operatorname{mdG}(q)^{*}$ methods, using an appropriate basis for the trial and test spaces. See [34] for details.

Theorem 3.2 (existence of solutions). Let $K=\max K_{n}$ be the maximum time slab length and define the Lipschitz constant $L_{f}>0$ by

$$
\begin{equation*}
\|f(x, t)-f(y, t)\|_{l_{\infty}} \leq L_{f}\|x-y\|_{l_{\infty}} \quad \forall t \in[0, T] \forall x, y \in \mathbb{R}^{N} \tag{3.2}
\end{equation*}
$$

If now

$$
\begin{equation*}
K C L_{f}<1 \tag{3.3}
\end{equation*}
$$

where $C=C(q)>0$ is a constant depending only on the order and method, the fixed point iteration (3.1) converges to the unique solution of (2.2), (2.6), (2.12), and (2.15), respectively.

Proof. The result follows by Lemma 3.1 and an application of the Banach fixed point theorem. See [34] for details.
4. Stability of solutions. Write the dual problem (2.9) for $\phi=\phi(t)$ in the form

$$
\begin{align*}
-\dot{\phi}(t)+A^{\top}(t) \phi(t) & =g, \quad t \in[0, T)  \tag{4.1}\\
\phi(T) & =\psi
\end{align*}
$$

For simplicity, we consider only the case $g=0$. With $w(t)=\phi(T-t)$, we have $\dot{w}(t)=-\dot{\phi}(T-t)=-A^{\top}(T-t) w(t)$, and so (4.1) can be written as a forward problem for $w$ in the form

$$
\begin{align*}
\dot{w}(t)+B(t) w(t) & =0, \quad t \in(0, T] \\
w(0) & =w_{0} \tag{4.2}
\end{align*}
$$

where $w_{0}=\psi$ and $B(t)=A^{\top}(T-t)$. Below, $w$ represents either $u$ or $\phi(T-\cdot)$ and, correspondingly, $W$ represents either the discrete $\mathrm{mc} / \mathrm{dG}(q)$ approximation $U$ of $u$ or the discrete $\mathrm{mc} / \mathrm{dG}(q)^{*}$ approximation $\Phi$ of $\phi$.
4.1. A general exponential estimate. The general exponential stability estimate is based on the following version of the discrete Gronwall inequality.

LEmma 4.1 (discrete Gronwall inequality). Assume that $z, a: \mathbb{N} \rightarrow \mathbb{R}$ are nonnegative, $a(m) \leq 1 / 2$ for all $m$, and $z(n) \leq C+\sum_{m=1}^{n} a(m) z(m)$ for all $n$. Then $z(n) \leq 2 C \exp \left(\sum_{m=1}^{n-1} 2 a(m)\right)$ for $n=1,2, \ldots$.

Proof. By a standard discrete Gronwall inequality 38, $z(n) \leq C \exp \left(\sum_{m=0}^{n-1} a(m)\right)$ if $z(n) \leq C+\sum_{m=0}^{n-1} a(m) z(m)$ for $n \geq 1$ and $z(0) \leq C$. Here, $(1-a(n)) z(n) \leq$ $C+\sum_{m=1}^{n-1} a(m) z(m)$, and so $z(n) \leq 2 C+\sum_{m=1}^{n-1} 2 a(m) z(m)$, since $1-a(n) \geq 1 / 2$. The result now follows if we take $a(0)=z(0)=0$.

Theorem 4.2 (stability estimate). Let $W$ be the $\operatorname{mcG}(q), \operatorname{mdG}(q), \operatorname{mcG}(q)^{*}$, or $\operatorname{mdG}(q)^{*}$ solution of (4.2). Then there is a constant $C=C(q)$, depending only on the highest order max $q_{i j}$, such that if $K_{n} C\|B\|_{L_{\infty}\left(\left[T_{n-1}, T_{n}\right], l_{p}\right)} \leq 1$ for $n=1, \ldots, M$, then

$$
\begin{equation*}
\|W\|_{L_{\infty}\left(\left[T_{n-1}, T_{n}\right], l_{p}\right)} \leq C\left\|w_{0}\right\|_{l_{p}} \exp \left(\sum_{m=1}^{n-1} K_{m} C\|B\|_{L_{\infty}\left(\left[T_{m-1}, T_{m}\right], l_{p}\right)}\right) \tag{4.3}
\end{equation*}
$$

for $n=1, \ldots, M, 1 \leq p \leq \infty$.
Proof. By Lemma 3.1, we can write the $\operatorname{mcG}(q), \operatorname{mdG}(q), \operatorname{mcG}(q)^{*}$, and $\operatorname{mdG}(q)^{*}$ methods in the form $\xi_{i j n^{\prime}}=w_{i}(0)+\int_{0}^{t_{i, j-1}} f_{i}(W, \cdot) d t+\int_{I_{i j}} w_{n^{\prime}}^{\left[q_{i j}\right]}\left(\tau_{i j}(t)\right) f_{i}(W, \cdot) d t$. Applied to the linear model problem (4.2), we have $\xi_{i j n^{\prime}}=w_{i}(0)-\int_{0}^{t_{i, j-1}}(B W)_{i} d t-$ $\int_{I_{i j}} w_{n^{\prime}}^{\left[q_{i j}\right]}\left(\tau_{i j}(t)\right)(B W)_{i} d t$, and so

$$
\begin{aligned}
\left|\xi_{i j n^{\prime}}\right| & \leq\left|w_{i}(0)\right|+\left|\int_{0}^{t_{i, j-1}}(B W)_{i} d t\right|+\left|\int_{I_{i j}} w_{n^{\prime}}^{\left[q_{i j}\right]}\left(\tau_{i j}(t)\right)(B W)_{i} d t\right| \\
& \leq\left|w_{i}(0)\right|+C \int_{0}^{t_{i j}}\left|(B W)_{i}\right| d t \leq\left|w_{i}(0)\right|+C \int_{0}^{T_{n}}\left|(B W)_{i}\right| d t
\end{aligned}
$$

where $T_{n}$ is smallest synchronized time level for which $t_{i j} \leq T_{n}$. It now follows that for all $t \in\left[T_{n-1}, T_{n}\right]$, we have $\left|W_{i}(t)\right| \leq C\left|w_{i}(0)\right|+C \int_{0}^{T_{n}}\left|(B W)_{i}\right| d t$, and so

$$
\|W(t)\|_{l_{p}} \leq C\left\|w_{0}\right\|_{l_{p}}+C \int_{0}^{T_{n}}\|B W\|_{l_{p}} d t=C\left\|w_{0}\right\|_{l_{p}}+C \sum_{m=1}^{n} \int_{T_{m-1}}^{T_{m}}\|B W\|_{l_{p}} d t
$$

The result now follows by letting $\bar{W}_{n}=\|W\|_{L_{\infty}\left(\left[T_{n-1}, T_{n}\right], l_{p}\right)}$.
REmARK 4.1. See [34] for an extension to multiadaptive time-stepping of the strong stability estimate Lemma 6.1 for parabolic problems in [11].
5. Interpolation estimates. In this section, we introduce a pair of carefully chosen interpolants, $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{dG}}^{[q]}$, which are central to the a priori error analysis of the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods. The interpolants are defined in section 5.1, In section 5.2 we discuss the interpolation of piecewise smooth functions, that is, the interpolation of functions which may be discontinuous within the interval of interpolation, and then present the basic general interpolation estimates for the two interpolants $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{dG}}^{[q]}$.

For the a priori error analysis of the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods, we will also need a special interpolation estimate for the function $\varphi=J^{\top} \Phi$, where $J$ is the Jacobian of the right-hand side $f$ of (1.1) and $\Phi$ is the discrete dual solution as defined in section 2, including estimates for the size of the jump in function value and derivatives for the function $\varphi$ at points of discontinuity. These estimates are proved in section 5.3, based on a representation formula for the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ solutions of (1.1).
5.1. Interpolants. The interpolant $\pi_{\mathrm{cG}}^{[q]}: V \rightarrow \mathcal{P}^{q}([a, b])$ is defined by the following conditions:

$$
\begin{array}{ll}
\pi_{\mathrm{cG}}^{[q]} v(a)=v(a) \quad \text { and } \quad \pi_{\mathrm{cG}}^{[q]} v(b)=v(b) \\
\int_{a}^{b}\left(v-\pi_{\mathrm{cG}}^{[q]} v\right) w d x=0 & \forall w \in \mathcal{P}^{q-2}([a, b]), \tag{5.1}
\end{array}
$$

where $V$ denotes the set of functions that are piecewise $\mathcal{C}^{q+1}$ and bounded on $[a, b]$. In other words, $\pi_{\mathrm{cG}}^{[q]} v$ is the polynomial of degree $q$ that interpolates $v$ at the endpoints of the interval $[a, b]$ and additionally satisfies $q-1$ projection conditions. This is illustrated in Figure 2, We also define the dual interpolant $\pi_{\mathrm{cG}}{ }^{[q]}$ as the standard $L_{2}$-projection onto $\mathcal{P}^{q-1}([a, b])$.


FIg. 2. The interpolant $\pi_{\mathrm{cG}}^{[q]} v$ (dashed) of the function $v(x)=x \sin (7 x)$ (solid) on $[0,1]$ for $q=1$ (left) and $q=3$ (right).

The interpolant $\pi_{\mathrm{dG}}^{[q]}: V \rightarrow \mathcal{P}^{q}([a, b])$ is defined by the following conditions:

$$
\begin{align*}
& \pi_{\mathrm{dG}}^{[q]} v(b)=v(b), \\
& \int_{a}^{b}\left(v-\pi_{\mathrm{dG}}^{[q]} v\right) w d x=0 \quad \forall w \in \mathcal{P}^{q-1}([a, b]), \tag{5.2}
\end{align*}
$$

that is, $\pi_{\mathrm{dG}}^{[q]} v$ is the polynomial of degree $q$ that interpolates $v$ at the right end-point of the interval $[a, b]$ and additionally satisfies $q$ projection conditions. This is illustrated



FIG. 3. The interpolant $\pi_{\mathrm{dG}}^{[q]} v$ (dashed) of the function $v(x)=x \sin (7 x)$ (solid) on $[0,1]$ for $q=0$ (left) and $q=3$ (right).


FIG. 4. A piecewise smooth function $v$ and its interpolant $\pi v$.
in Figure 3. The dual interpolant $\pi_{\mathrm{dG}^{*}}^{[q]}$ is defined similarly, with the difference being that the left end-point $x=a$ is used for interpolation.
5.2. Basic interpolation estimates. To estimate the size of the interpolation error $\pi v-v$ for a given function $v$, we express the interpolation error in terms of the regularity of $v$ and the length of the interpolation interval, $k=b-a$. Specifically, when $v \in \mathcal{C}^{q+1}([a, b]) \subset V$ for some $q \geq 0$, we obtain estimates of the form

$$
\begin{equation*}
\left\|(\pi v)^{(p)}-v^{(p)}\right\| \leq C k^{q+1-p}\left\|v^{(q+1)}\right\|, \quad p=0, \ldots, q+1 \tag{5.3}
\end{equation*}
$$

where $\|\cdot\|=\|\cdot\|_{L_{\infty}([a, b])}$ denotes the maximum norm on $[a, b]$. This estimate is a simple consequence of the Peano kernel theorem 40] if one can show that the interpolant $\pi: V \rightarrow \mathcal{P}^{q}([a, b]) \subset V$ is linear and bounded on $V$ and that $\pi$ is exact on $\mathcal{P}^{q}([a, b]) \subset V$, that is, $\pi v=v$ for all $v \in \mathcal{P}^{q}([a, b])$.

In the general case, where the interpolated function $v$ is only piecewise smooth (see Figure 4), we also need to include the size of the jump $\left[v^{(p)}\right]_{x}$ in function value and derivatives at each point $x$ of discontinuity within $(a, b)$ to measure the regularity of the interpolated function $v$. In [34], we prove the following extensions of the basic estimate (5.3).

Lemma 5.1. If $\pi$ is linear and bounded on $V$ and is exact on $\mathcal{P}^{q}([a, b]) \subset V$, then there is a constant $C=C(q)>0$ such that for all $v$ piecewise $\mathcal{C}^{q+1}$ on $[a, b]$ with


FIG. 5. If some other component $l \neq i$ has a node within $I_{i j}$, then $\Phi_{l}$ may be discontinuous within $I_{i j}$, causing $\varphi_{i}$ to be discontinuous within $I_{i j}$.
discontinuities at $a<x_{1}<\cdots<x_{n}<b$,

$$
\begin{equation*}
\left\|(\pi v)^{(p)}-v^{(p)}\right\| \leq C k^{r+1-p}\left\|v^{(r+1)}\right\|+C \sum_{j=1}^{n} \sum_{m=0}^{r} k^{m-p}\left|\left[v^{(m)}\right]_{x_{j}}\right| \tag{5.4}
\end{equation*}
$$

for $p=0, \ldots, r+1, r=0, \ldots, q$.
Lemma 5.2. If $\pi$ is linear and bounded on $V$ and is exact on $\mathcal{P}^{q}([a, b]) \subset V$, then there is a constant $C=C(q)>0$ such that for all $v$ piecewise $\mathcal{C}^{q+1}$ on $[a, b]$ with discontinuities at $a<x_{1}<\cdots<x_{n}<b$,

$$
\begin{equation*}
\left\|(\pi v)^{(p)}\right\| \leq C\left\|v^{(p)}\right\|+C \sum_{j=1}^{n} \sum_{m=0}^{p-1} k^{m-p}\left|\left[v^{(m)}\right]_{x_{j}}\right| \tag{5.5}
\end{equation*}
$$

for $p=0, \ldots, q$.
Lemmas 5.1 and 5.2 apply to both the $\pi_{\mathrm{cG}}^{[q]}$ interpolant (for $q \geq 1$ ) and the $\pi_{\mathrm{dG}}^{[q]}$ interpolant (for $q \geq 0$ ) defined in section5.1. The linearity of both interpolants follows directly from the definition of the interpolants. The proofs that both interpolants are bounded and exact on $\mathcal{P}^{q}([a, b])$ are given in detail in [34] and [35.
5.3. A special interpolation estimate. To prove a priori error estimates for $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ in section 6, we need to estimate the interpolation error $\pi \varphi-\varphi$ for the function $\varphi$ defined by

$$
\begin{equation*}
\varphi_{i}=\left(J^{\top}(\pi u, u, \cdot) \Phi\right)_{i}=\sum_{l=1}^{N} J_{l i}(\pi u, u, \cdot) \Phi_{l}, \quad i=1, \ldots, N \tag{5.6}
\end{equation*}
$$

We note that $\varphi_{i}$ may be discontinuous within $I_{i j}$ if $I_{i j}$ contains a node for some other component, which is generally the case. This is illustrated in Figure 5. Note that on the right-hand side $f$ is linearized around a mean value of $\pi u$ and $u$.

An interpolation estimate for $\pi \varphi-\varphi$ follows directly from Lemma 5.1. To use this estimate, we need to estimate the size of the jump in function value and derivatives at
each internal node $t_{i j}$ of the partition $\mathcal{T}$. To obtain this estimate, we need to make a number of additional assumptions on the right-hand side $f$ of (1.1) and the partition $\mathcal{T}$. These assumptions are discussed in section 5.3.2 Based on the assumptions and the representation formula presented in section 5.3.1, we obtain the jump estimates in section 5.3.3 and, finally, in section 5.3.4 the interpolation estimate for $\varphi$.
5.3.1. A representation formula. The proof of jump estimates for the multiadaptive Galerkin methods $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ is based on expressing the solutions as certain interpolants. These representations are obtained as follows. Let $U$ be the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution of (1.1) and define, for $i=1, \ldots, N$,

$$
\begin{equation*}
\tilde{U}_{i}(t)=u_{i}(0)+\int_{0}^{t} f_{i}(U(s), s) d s \tag{5.7}
\end{equation*}
$$

Similarly, for $\Phi$ the $\operatorname{mcG}(q)^{*}$ or $\operatorname{mdG}(q)^{*}$ solution of (2.9), we define, for $i=1, \ldots, N$,

$$
\begin{equation*}
\tilde{\Phi}_{i}(t)=\psi_{i}+\int_{t}^{T} f_{i}^{*}(\Phi(s), s) d s \tag{5.8}
\end{equation*}
$$

where $f^{*}(\Phi, \cdot)=J^{\top}(\pi u, U, \cdot) \Phi+g$. We note that $\dot{\tilde{U}}=f(U, \cdot)$ and $-\dot{\tilde{\Phi}}=f^{*}(\Phi, \cdot)$.
It now turns out that $U$ can be expressed as an interpolant of $\tilde{U}$. Similarly, $\Phi$ can be expressed as an interpolant of $\tilde{\Phi}$. We present these representations in Lemmas 5.3 and 5.4. We remind the reader about the interpolants $\pi_{\mathrm{cG}}^{[q]}, \pi_{\mathrm{cG}}{ }^{[q]}, \pi_{\mathrm{dG}}^{[q]}$, and $\pi_{\mathrm{dG}^{*}}^{[q]}$, defined in section 5.1.

Lemma 5.3. The $\operatorname{mcG}(q)$ solution $U$ of (1.1) can expressed in the form $U=\pi_{\mathrm{cG}}^{[q]} \tilde{U}$. Similarly, the $\operatorname{mcG}(q)^{*}$ solution $\Phi$ of (2.9) can be expressed in the form $\Phi=\pi_{\mathrm{cG}}\left[\frac{[q]}{}{ }^{(2)}\right.$, that is, $U_{i}=\pi_{\mathrm{cG}}^{\left[q_{i}\right]} \tilde{U}_{i}$ and $\Phi_{i}=\pi_{\mathrm{cG}}{ }^{\left[q_{i}\right]} \tilde{\Phi}_{i}$ on each local interval $I_{i j}$.

Proof. The representation formulas follow by the definitions of the $\operatorname{mcG}(q)$ and $\operatorname{mcG}(q)^{*}$ methods and the interpolants $\pi_{\mathrm{cG}}^{[q]}$ and $\pi_{\mathrm{cG}}[q]$. See 34 for details. $\quad$

LEMmA 5.4. The $\operatorname{mdG}(q)$ solution $U$ of (1.1) can expressed in the form $U=\pi_{\mathrm{dG}}^{[q]} \tilde{U}$. Similarly, the $\operatorname{mdG}(q)^{*}$ solution $\Phi$ of (2.9) can be expressed in the form $\Phi=\pi_{\mathrm{dG}}{ }^{[q]} \tilde{\Phi}$, that is, $U_{i}=\pi_{\mathrm{dG}}^{\left[q_{i j}\right]} \tilde{U}_{i}$ and $\Phi_{i}=\pi_{\mathrm{dG}^{*}}^{\left[q_{i j}\right.} \tilde{\Phi}_{i}$ on each local interval $I_{i j}$.

Proof. The representation formulas follow by the definitions of the $\operatorname{mdG}(q)$ and $\operatorname{mdG}(q)^{*}$ methods and the interpolants $\pi_{\mathrm{dG}}^{[q]}$ and $\pi_{\mathrm{dG}^{*}}^{[q]}$. See [34] for details.
5.3.2. Assumptions. To estimate the size of the jump in function value and derivatives for the function $\varphi$ defined in (5.6), we make the following assumptions. Given a time slab $\mathcal{T}$, assume that for each pair of local intervals $I_{i j}$ and $I_{m n}$ within the time slab, we have

$$
\begin{equation*}
q_{i j}=q_{m n}=\bar{q} \tag{A1}
\end{equation*}
$$

and

$$
\begin{equation*}
k_{i j}>\alpha k_{m n} \tag{A2}
\end{equation*}
$$

for some $\bar{q} \geq 0$ and some $\alpha \in(0,1)$. The dependence on $\alpha$ in the error estimates is weak (see Remark 5.1), so assumption (A2) does not prevent multiadaptivity.

We also assume that the problem (1.1) is autonomous,

$$
\begin{equation*}
\partial f_{i} / \partial t=0, \quad i=1, \ldots, N \tag{A3}
\end{equation*}
$$

noting that the dual problem nevertheless will be nonautonomous in general. Furthermore, we assume that

$$
\begin{equation*}
\left\|f_{i}\right\|_{D^{\bar{q}+1}(\mathcal{T})}<\infty, \quad i=1, \ldots, N \tag{A4}
\end{equation*}
$$

where $\|\cdot\|_{D^{p}(\mathcal{T})}$ is defined for $v: \mathbb{R}^{N} \rightarrow \mathbb{R}$ and $p \geq 0$ by $\|v\|_{D^{p}(\mathcal{T})}=\max _{n=0, \ldots, p}$ $\left\|D^{n} v\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}$, with the norm $\left\|D^{n} v\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}$ defined by $\left\|D^{n} v w^{1} \cdots w^{n}\right\|_{L_{\infty}(\mathcal{T})} \leq$ $\left\|D^{n} v\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}\left\|w^{1}\right\|_{l_{\infty}} \cdots\left\|w^{n}\right\|_{l_{\infty}}$ for all $w^{1}, \ldots, w^{n} \in \mathbb{R}^{N}$, and $D^{n} v$ the $n$ th-order tensor given by

$$
D^{n} v w^{1} \cdots w^{n}=\sum_{i_{1}=1}^{N} \cdots \sum_{i_{n}=1}^{N} \frac{\partial^{n} v}{\partial x_{i_{1}} \cdots \partial x_{i_{n}}} w_{i_{1}}^{1} \cdots w_{i_{n}}^{n}
$$

Furthermore, we choose $C_{f} \geq \max _{i=1, \ldots, N}\left\|f_{i}\right\|_{D^{\bar{q}+1}(\mathcal{T})}$ such that

$$
\begin{equation*}
\left\|d^{p} / d t^{p}(\partial f / \partial u)^{\top}(x(t))\right\|_{l_{\infty}} \leq C_{f} C_{x}^{p} \tag{5.9}
\end{equation*}
$$

for $p=0, \ldots, \bar{q}$, and

$$
\begin{equation*}
\left\|\left[d^{p} / d t^{p}(\partial f / \partial u)^{\top}(x(t))\right]_{t}\right\|_{l_{\infty}} \leq C_{f} \sum_{n=0}^{p} C_{x}^{p-n}\left\|\left[x^{(n)}\right]_{t}\right\|_{l_{\infty}} \tag{5.10}
\end{equation*}
$$

for $p=0, \ldots, \bar{q}-1$ and any given $x: \mathbb{R} \rightarrow \mathbb{R}^{N}$, where $C_{x}>0$ denotes a constant such that $\left\|x^{(n)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C_{x}^{n}$ for $n=1, \ldots, p$. Note that $C_{f}=C_{f}(t)$ defines a piecewise constant function on the partition $0=T_{0}<T_{1}<\cdots<T_{M}=T$. Note also that assumption (A4) implies that each $f_{i}$ is bounded by $C_{f}$.

We further assume that there is a constant $c_{k}>0$ such that

$$
\begin{equation*}
k_{i j} C_{f} \leq c_{k} \tag{A5}
\end{equation*}
$$

for each local interval $I_{i j}$. We summarize the list of assumptions as follows:
(A1) the local orders $q_{i j}$ are equal within each time slab;
(A2) the local time steps $k_{i j}$ are semiuniform within each time slab;
(A3) $f$ is autonomous;
(A4) $f$ and its derivatives are bounded;
(A5) the local time steps $k_{i j}$ are small.
5.3.3. Estimates of derivatives and jumps. To estimate higher-order derivatives, we face the problem of taking higher-order derivatives of $f(U(t), t)$ with respect to $t$. In Lemmas 5.5 and 5.6. we present basic estimates for composite functions $v \circ x$ with $v: \mathbb{R}^{N} \rightarrow \mathbb{R}$ and $x: \mathbb{R} \rightarrow \mathbb{R}^{N}$. The proofs are based on a straightforward application of the chain rule and Leibniz rule and are given in full detail in 34.

Lemma 5.5. Let $v: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be $p \geq 0$ times differentiable in all its variables, let $x: \mathbb{R} \rightarrow \mathbb{R}^{N}$ be $p$ times differentiable, and let $C_{x}>0$ be a constant such that $\left\|x^{(n)}\right\|_{L_{\infty}\left(\mathbb{R}, l_{\infty}\right)} \leq C_{x}^{n}$ for $n=1, \ldots, p$. Then there is a constant $C=C(p)>0$ such that

$$
\begin{equation*}
\left\|\frac{d^{p}(v \circ x)}{d t^{p}}\right\|_{L_{\infty}(\mathbb{R})} \leq C\|v\|_{D^{p}(\mathbb{R})} C_{x}^{p} \tag{5.11}
\end{equation*}
$$

Lemma 5.6. Let $v: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be $p+1 \geq 1$ times differentiable in all its variables, let $x: \mathbb{R} \rightarrow \mathbb{R}^{N}$ be $p$ times differentiable, except possibly at some $t \in \mathbb{R}$, and let
$C_{x}>0$ be a constant such that $\left\|x^{(n)}\right\|_{L_{\infty}\left(\mathbb{R}, l_{\infty}\right)} \leq C_{x}^{n}$ for $n=1, \ldots, p$. Then there is a constant $C=C(p)>0$ such that

$$
\begin{equation*}
\left|\left[\frac{d^{p}(v \circ x)}{d t^{p}}\right]_{t}\right| \leq C\|v\|_{D^{p+1}(\mathbb{R})} \sum_{n=0}^{p} C_{x}^{p-n}\left\|\left[x^{(n)}\right]_{t}\right\|_{l_{\infty}} . \tag{5.12}
\end{equation*}
$$

We now prove estimates for derivatives and jumps of the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution $U$ of the general nonlinear problem (1.1), under the assumptions listed in section 5.3.2. Similarly, one can obtain estimates for the discrete dual solution $\Phi$ and the function $\varphi$ defined in (5.6), from which the desired interpolation estimates follow.

To obtain estimates for the multiadaptive solution $U$, we first prove estimates for the function $\tilde{U}$ defined in section 5.3.1. The estimates for $U$ then follow by induction.

To simplify the estimates, we introduce the following notation. For given $p>0$, let $C_{U, p} \geq C_{f}$ be a constant such that

$$
\begin{equation*}
\left\|U^{(n)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C_{U, p}^{n}, \quad n=1, \ldots, p \tag{5.13}
\end{equation*}
$$

For $p=0$, we define $C_{U, 0}=C_{f}$. Temporarily, we assume that there is a constant $c_{k}^{\prime}>0$ such that for each $p$,

$$
k_{i j} C_{U, p} \leq c_{k}^{\prime}
$$

This assumption will be removed in Lemma 5.9. In the following lemma, we use assumptions (A1), (A3), and (A4) to derive estimates for $\tilde{U}$ in terms of $C_{U, p}$ and $C_{f}$.

Lemma 5.7 (derivative and jump estimates for $\tilde{U})$. Let $U$ be the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution of (1.1) and define $\tilde{U}$ as in (5.7). If assumptions (A1), (A3), and (A4) hold, then there is a constant $C=C(\bar{q})>0$ such that

$$
\begin{equation*}
\left\|\tilde{U}^{(p)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C C_{U, p-1}^{p}, \quad p=1, \ldots, \bar{q}+1 \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left[\tilde{U}^{(p)}\right]_{t_{i, j-1}}\right\|_{l_{\infty}} \leq C \sum_{n=0}^{p-1} C_{U, p-1}^{p-n}\left\|\left[U^{(n)}\right]_{t_{i, j-1}}\right\|_{l_{\infty}}, \quad p=1, \ldots, \bar{q}+1 \tag{5.15}
\end{equation*}
$$

for each local interval $I_{i j}$, where $t_{i, j-1}$ is an internal node of the time slab $\mathcal{T}$.
Proof. By definition, $\tilde{U}_{i}^{(p)}=\frac{d^{p-1}}{d t^{p-1}} f_{i}(U)$, and so the results follow directly by Lemmas 5.5 and 5.6, noting that $C_{f} \leq C_{U, p-1}$.

By Lemma 5.7, we now obtain the following estimate for the size of the jump in function value and derivatives for $U$.

Lemma 5.8 (jump estimates for $U$ ). Let $U$ be the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution of (1.1). If assumptions (A1)-(A5) and (A5') hold, then there is a constant $C=$ $C\left(\bar{q}, c_{k}, c_{k}^{\prime}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left\|\left[U^{(p)}\right]_{t_{i, j-1}}\right\|_{l_{\infty}} \leq C k_{i j}^{r+1-p} C_{U, r}^{r+1}, \quad p=0, \ldots, r+1, \quad r=0, \ldots, \bar{q} \tag{5.16}
\end{equation*}
$$

for each local interval $I_{i j}$, where $t_{i, j-1}$ is an internal node of the time slab $\mathcal{T}$.
Proof. The proof is by induction. We first note that at $t=t_{i, j-1}$, we have

$$
\begin{aligned}
{\left[U_{i}^{(p)}\right]_{t} } & =U_{i}^{(p)}\left(t^{+}\right)-\tilde{U}_{i}^{(p)}\left(t^{+}\right)+\tilde{U}_{i}^{(p)}\left(t^{+}\right)-\tilde{U}_{i}^{(p)}\left(t^{-}\right)+\tilde{U}_{i}^{(p)}\left(t^{-}\right)-U_{i}^{(p)}\left(t^{-}\right) \\
& \equiv e_{+}+e_{0}+e_{-}
\end{aligned}
$$

By Lemma 5.3 (or Lemma 5.4), $U$ is an interpolant of $\tilde{U}$ and so, by Lemma 5.1, we have

$$
\left|e_{+}\right| \leq C k_{i j}^{r+1-p}\left\|\tilde{U}_{i}^{(r+1)}\right\|_{L_{\infty}\left(I_{i j}\right)}+C \sum_{x \in \mathcal{N}_{i j}} \sum_{m=1}^{r} k_{i j}^{m-p}\left|\left[\tilde{U}_{i}^{(m)}\right]_{x}\right|
$$

for $p=0, \ldots, r+1$ and $r=0, \ldots, \bar{q}$. Note that the second sum starts at $m=1$ rather than at $m=0$, since $\tilde{U}$ is continuous. Similarly, we have

$$
\left|e_{-}\right| \leq C k_{i, j-1}^{r+1-p}\left\|\tilde{U}_{i}^{(r+1)}\right\|_{L_{\infty}\left(I_{i, j-1}\right)}+C \sum_{x \in \mathcal{N}_{i, j-1}} \sum_{m=1}^{r} k_{i, j-1}^{m-p}\left|\left[\tilde{U}_{i}^{(m)}\right]_{x}\right|
$$

To estimate $e_{0}$, we note that $e_{0}=0$ for $p=0$, since $\tilde{U}$ is continuous. For $p=1, \ldots, \bar{q}+$ 1, Lemma5.7gives $\left|e_{0}\right|=\left|\left[\tilde{U}_{i}^{(p)}\right]_{t}\right| \leq C \sum_{n=0}^{p-1} C_{U, p-1}^{p-n}\left\|\left[U^{(n)}\right]_{t}\right\|_{l_{\infty}}$. By assumption (A2), it then follows that (5.16) holds for $r=0$.

Assume now that (5.16) holds for $r=\bar{r}-1 \geq 0$. Then, by Lemma 5.7 and assumption ( $\mathrm{A} 5^{\prime}$ ), it follows that

$$
\begin{aligned}
\left|e_{+}\right| & \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}+C \sum_{x \in \mathcal{N}_{i j}} \sum_{m=1}^{\bar{r}} k_{i j}^{m-p} \sum_{n=0}^{m-1} C_{U, m-1}^{m-n}\left\|\left[U^{n}\right]_{x}\right\|_{l_{\infty}} \\
& \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}+C \sum k_{i j}^{m-p} C_{U, m-1}^{m-n} k_{i j}^{(\bar{r}-1)+1-n} C_{U, \bar{r}-1}^{(\bar{r}-1)+1} \\
& \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}\left(1+\sum\left(k_{i j} C_{U, \bar{r}-1}\right)^{m-1-n}\right) \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}
\end{aligned}
$$

Similarly, we obtain the estimate $\left|e_{-}\right| \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}$. Finally, we use Lemma 5.7 and assumption (A5') to obtain the estimate

$$
\begin{aligned}
\left|e_{0}\right| & \leq C \sum_{n=0}^{p-1} C_{U, p-1}^{p-n}\left\|\left[U^{n}\right]_{t}\right\|_{l_{\infty}} \leq C \sum_{n=0}^{p-1} C_{U, p-1}^{p-n} k_{i j}^{(\bar{r}-1)+1-n} C_{U, \bar{r}-1}^{(\bar{r}-1)+1} \\
& =C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1} \sum_{n=0}^{p-1}\left(k_{i j} C_{U, \bar{r}}\right)^{p-1-n} \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}
\end{aligned}
$$

Summing up, we thus obtain $\left|\left[U_{i}^{(p)}\right]_{t}\right| \leq\left|e_{+}\right|+\left|e_{0}\right|+\left|e_{-}\right| \leq C k_{i j}^{\bar{r}+1-p} C_{U, \bar{r}}^{\bar{r}+1}$, and so (5.16) follows by induction.

By Lemmas 5.7 and 5.8, we now obtain the following estimate for derivatives of the solution $U$.

Lemma 5.9 (derivative estimates for $U$ ). Let $U$ be the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution of (1.1). If assumptions (A1)-(A5) hold, then there is a constant $C=C\left(\bar{q}, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left\|U^{(p)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C C_{f}^{p}, \quad p=1, \ldots, \bar{q} \tag{5.17}
\end{equation*}
$$

Proof. By Lemma5.3(or Lemma 5.4), $U$ is an interpolant of $\tilde{U}$ and so, by Lemma 5.1. we have

$$
\left\|U_{i}^{(p)}\right\|_{L_{\infty}\left(I_{i j}\right)}=\left\|\left(\pi \tilde{U}_{i}\right)^{(p)}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C^{\prime}\left\|\tilde{U}_{i}^{(p)}\right\|_{L_{\infty}\left(I_{i j}\right)}+C^{\prime} \sum_{x \in \mathcal{N}_{i j}} \sum_{m=1}^{p-1} k_{i j}^{m-p}\left|\left[\tilde{U}_{i}^{(m)}\right]_{x}\right|
$$

for some constant $C^{\prime}=C^{\prime}(\bar{q})$. For $p=1$, we thus obtain the estimate

$$
\left\|\dot{U}_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C^{\prime}\left\|\dot{\tilde{U}}_{i}\right\|_{L_{\infty}\left(I_{i j}\right)}=C^{\prime}\left\|f_{i}(U)\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C^{\prime} C_{f}
$$

by assumption (A4), and so (5.17) holds for $p=1$.
For $p=2, \ldots, \bar{q}$, assuming that (A5') holds for $C_{U, p-1}$, we use Lemmas 5.7 and 5.8 (with $r=p-1$ ) and assumption (A2) to obtain

$$
\begin{aligned}
\left\|U_{i}^{(p)}\right\|_{L_{\infty}\left(I_{i j}\right)} & \leq C C_{U, p-1}^{p}+C \sum_{x \in \mathcal{N}_{i j}} \sum_{m=1}^{p-1} k_{i j}^{m-p} \sum_{n=0}^{m-1} C_{U, m-1}^{m-n}\left\|\left[U^{(n)}\right]_{x}\right\|_{l_{\infty}} \\
& \leq C C_{U, p-1}^{p}+C \sum k_{i j}^{m-p} C_{U, m-1}^{m-n} k_{i j}^{(p-1)+1-n} C_{U, p-1}^{(p-1)+1} \\
& \leq C C_{U, p-1}^{p}\left(1+\sum\left(k_{i j} C_{U, m-1}\right)^{m-n}\right) \leq C C_{U, p-1}^{p}
\end{aligned}
$$

where $C=C\left(\bar{q}, c_{k}, c_{k}^{\prime}, \alpha\right)$. This holds for all components $i$ and all local intervals $I_{i j}$ within the time slab $\mathcal{T}$, and so

$$
\left\|U^{(p)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C C_{U, p-1}^{p}, \quad p=1, \ldots, \bar{q}
$$

where by definition $C_{U, p-1}$ is a constant such that $\left\|U^{(n)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C_{U, p-1}^{n}$ for $n=$ $1, \ldots, p-1$. Starting at $p=1$, we now define $C_{U, 1}=C_{1} C_{f}$ with $C_{1}=C^{\prime}=C^{\prime}(\bar{q})$. It then follows that ( $\mathrm{A} 5^{\prime}$ ) holds for $C_{U, 1}$ with $c_{k}^{\prime}=C^{\prime} c_{k}$, and thus

$$
\left\|U^{(2)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C C_{U, 2-1}^{2}=C C_{U, 1}^{2} \equiv C_{2} C_{f}^{2}
$$

where $C_{2}=C_{2}\left(\bar{q}, c_{k}, \alpha\right)$. We may thus define $C_{U, 2}=\max \left(C_{1} C_{f}, \sqrt{C_{2}} C_{f}\right)$. Continuing, we note that ( $\mathrm{A} 5^{\prime}$ ) holds for $C_{U, 2}$, and thus

$$
\left\|U^{(3)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C C_{U, 3-1}^{3}=C C_{U, 2}^{3} \equiv C_{3} C_{f}^{3}
$$

where $C_{3}=C_{3}\left(\bar{q}, c_{k}, \alpha\right)$. In this way, we obtain a sequence of constants $C_{1}, \ldots, C_{\bar{q}}$, depending only on $\bar{q}, c_{k}$, and $\alpha$, such that $\left\|U^{(p)}\right\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \leq C_{p} C_{f}^{p}$ for $p=1, \ldots, \bar{q}$, and so (5.17) follows if we take $C=\max _{i=1, \ldots, \bar{q}} C_{i}$.

Having now removed the additional assumption (A5'), we obtain the following version of Lemma 5.8.

Lemma 5.10 (jump estimates for $U$ ). Let $U$ be the $\operatorname{mcG}(q)$ or $\operatorname{mdG}(q)$ solution of (1.1). If assumptions (A1)-(A5) hold, then there is a constant $C=C\left(\bar{q}, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left\|\left[U^{(p)}\right]_{t_{i, j-1}}\right\|_{l_{\infty}} \leq C k_{i j}^{\bar{q}+1-p} C_{f}^{\bar{q}+1}, \quad p=0, \ldots, \bar{q} \tag{5.18}
\end{equation*}
$$

for each local interval $I_{i j}$, where $t_{i, j-1}$ is an internal node of the time slab $\mathcal{T}$.
Similarly, we obtain estimates for the discrete dual solution $\Phi$ and the function $\varphi$. In Lemma 5.11, we present the estimates for the function $\varphi$.

Lemma 5.11 (estimates for $\varphi$ ). Let $\varphi$ be defined as in (5.6). If assumptions (A1)-(A5) hold, then there is a constant $C=C\left(\bar{q}, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left\|\varphi_{i}^{(p)}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C C_{f}^{p+1}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}, \quad p=0, \ldots, q_{i j} \tag{5.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\left[\varphi_{i}^{(p)}\right]_{x}\right| \leq C k_{i j}^{r_{i j}-p} C_{f}^{r_{i j}+1}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \quad \forall x \in \mathcal{N}_{i j}, \quad p=0, \ldots, q_{i j}-1 \tag{5.20}
\end{equation*}
$$

with $r_{i j}=q_{i j}$ for the $\operatorname{mcG}(q)$ method and $r_{i j}=q_{i j}+1$ for the $\operatorname{mdG}(q)$ method. This holds for each local interval $I_{i j}$ within the time slab $\mathcal{T}$.
5.3.4. Interpolation estimates. Using the basic interpolation estimate of section 5.2, we now obtain the following important interpolation estimates for the function $\varphi$.

Lemma 5.12 (interpolation estimates for $\varphi$ ). Let $\varphi$ be defined as in (5.6). If assumptions (A1)-(A5) hold, then there is a constant $C=C\left(\bar{q}, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left\|\pi_{\mathrm{cG}}^{\left[q_{i j}-2\right]} \varphi_{i}-\varphi_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C k_{i j}^{q_{i j}-1} C_{f}^{q_{i j}}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}, \quad q_{i j}=\bar{q} \geq 2 \tag{5.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\pi_{\mathrm{dG}}^{\left[q_{i j}-1\right]} \varphi_{i}-\varphi_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C k_{i j}^{q_{i j}} C_{f}^{q_{i j}+1}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}, \quad q_{i j}=\bar{q} \geq 1 \tag{5.22}
\end{equation*}
$$

for each local interval $I_{i j}$ within the time slab $\mathcal{T}$.
Proof. To prove (5.21), we use Lemma 5.1, with $r=q_{i j}-2$ and $p=0$, together with Lemma 5.11, to obtain

$$
\begin{aligned}
& \left\|\pi_{\mathrm{cG}}^{\left[q_{i j}-2\right]} \varphi_{i}-\varphi_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C k_{i j}^{q_{i j}-1}\left\|\varphi_{i}^{\left(q_{i j}-1\right)}\right\|_{L_{\infty}\left(I_{i j}\right)}+C \sum_{x \in \mathcal{N}_{i j}} \sum_{m=0}^{q_{i j}-2} k_{i j}^{m}\left|\left[\varphi_{i}^{(m)}\right]_{x}\right| \\
& \quad \leq C k_{i j}^{q_{i j}-1} C_{f}^{q_{i j}}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}+C \sum_{x \in \mathcal{N}_{i j}} \sum_{m=0}^{q_{i j}-2} k_{i j}^{m} k_{i j}^{q_{i j}-m} C_{f}^{q_{i j}+1}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} \\
& \quad=C k_{i j}^{q_{i j}-1} C_{f}^{q_{i j}}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)}+C k_{i j}^{q_{i j}} C_{f}^{q_{i j}+1}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)},
\end{aligned}
$$

from which the estimate follows. The estimate for $\pi_{d G}^{\left[q_{i j}-1\right]} \varphi_{i}-\varphi_{i}$ is obtained similarly.

REMARK 5.1. Note that there is only a weak dependence on $c_{k}$ and $\alpha$, since the jump term contains an extra factor $k_{i j}$. If higher-order terms can be ignored, then the dependence on $c_{k}$ and $\alpha$ can be removed.
6. A priori error estimates. To prove a priori error estimates for the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods, we derive error representations in section 6.1 and then obtain the a priori error estimates in section 6.2 for the general nonlinear case. We refer to [34] for a sharp a priori error estimate in the case of a parabolic model problem.
6.1. Error representation. For each of the two methods, $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$, we represent the error in terms of the discrete dual solution $\Phi$ and an interpolant $\pi u$ of the exact solution $u$ of (1.1), using the special interpolants $\pi u=\pi_{\mathrm{cG}}^{[q]} u$ or $\pi u=\pi_{\mathrm{dG}}^{[q]} u$ defined in section 5

We write the error $e=U-u$ in the form

$$
\begin{equation*}
e=\bar{e}+(\pi u-u) \tag{6.1}
\end{equation*}
$$

where $\bar{e} \equiv U-\pi u$ is represented in terms of the discrete dual solution and the residual of the interpolant. An estimate for the second part of the error, $\pi u-u$, follows directly from an interpolation estimate.

In Lemma 6.1, we present the error representation for the $\operatorname{mcG}(q)$ method, and then present the corresponding representation for the $\operatorname{mdG}(q)$ method in Lemma 6.2, The error representations are obtained directly by choosing $\bar{e}$ as a test function for the discrete dual problems (2.12) and (2.15).

Lemma 6.1 (error representation for $\operatorname{mcG}(q)$ ). Let $U$ be the $\operatorname{mcG}(q)$ solution of (1.1), let $\Phi$ be the corresponding $\operatorname{mcG}(q)^{*}$ solution of the dual problem (2.9), and let
$\pi u$ be any trial space approximation of the exact solution $u$ of (1.1) that interpolates $u$ at the end-points of every local interval. Then

$$
L_{\psi, g}(\bar{e}) \equiv(\bar{e}(T), \psi)+\int_{0}^{T}(\bar{e}, g) d t=-\int_{0}^{T}(R(\pi u, \cdot), \Phi) d t
$$

where $\bar{e} \equiv U-\pi u$.
Lemma 6.2 (error representation for $\operatorname{mdG}(q))$. Let $U$ be the $\operatorname{mdG}(q)$ solution of (1.1), let $\Phi$ be the corresponding $\operatorname{mdG}(q)^{*}$ solution of the dual problem (2.9), and let $\pi u$ be any trial space approximation of the exact solution $u$ of (1.1) that interpolates $u$ at the right end-point of every local interval. Then

$$
L_{\psi, g}(\bar{e})=-\sum_{i=1}^{N} \sum_{j=1}^{M_{i}}\left[\left[\pi u_{i}\right]_{i, j-1} \Phi_{i}\left(t_{i, j-1}^{+}\right)+\int_{I_{i j}} R_{i}(\pi u, \cdot) \Phi_{i} d t\right]
$$

where $\bar{e} \equiv U-\pi u$.
With a special choice of interpolant, $\pi u=\pi_{\mathrm{cG}}^{[q]} u$ and $\pi u=\pi_{\mathrm{dG}}^{[q]} u$, respectively, we obtain the following versions of the error representations.

Corollary 6.3 (error representation for $\operatorname{mcG}(q))$. Let $U$ be the $\operatorname{mcG}(q)$ solution of (1.1) and let $\Phi$ be the corresponding $\operatorname{mcG}(q)^{*}$ solution of the dual problem (2.9). Then

$$
L_{\psi, g}(\bar{e})=\int_{0}^{T}\left(f\left(\pi_{\mathrm{cG}}^{[q]} u, \cdot\right)-f(u, \cdot), \Phi\right) d t
$$

Proof. Integrate by parts and use the definition of the interpolant $\pi_{\mathrm{cG}}^{[q]}$.
Corollary 6.4 (error representation for $\operatorname{mdG}(q)$ ). Let $U$ be the $\operatorname{mdG}(q)$ solution of (1.1) and let $\Phi$ be the corresponding $\operatorname{mdG}(q)^{*}$ solution of the dual problem (2.9). Then

$$
L_{\psi, g}(\bar{e})=\int_{0}^{T}\left(f\left(\pi_{\mathrm{dG}}^{[q]} u, \cdot\right)-f(u, \cdot), \Phi\right) d t
$$

Proof. Integrate by parts and use the definition of the interpolant $\pi_{\mathrm{dG}}^{[q]}$.
6.2. A priori error estimates for the general nonlinear problem. Using the error representations of section 6.1, the stability estimates of section 4, and the interpolation estimates of section 5, we now prove our main results: a priori error estimates for general order $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$.

THEOREM 6.5 (a priori error estimate for $\operatorname{mcG}(q))$. Let $U$ be the $\operatorname{mcG}(q)$ solution of (1.1) and let $\Phi$ be the corresponding $\operatorname{mcG}(q)^{*}$ solution of the dual problem (2.9). Then there is a constant $C=C(q)>0$ such that

$$
\begin{equation*}
\left|L_{\psi, g}(\bar{e})\right| \leq C S(T)\left\|k^{q+1} \bar{u}^{(q+1)}\right\|_{L_{\infty}\left([0, T], l_{2}\right)} \tag{6.2}
\end{equation*}
$$

where $\left(k^{q+1} \bar{u}^{(q+1)}\right)_{i}(t)=k_{i j}^{q_{i j}+1}\left\|u_{i}^{\left(q_{i j}+1\right)}\right\|_{L_{\infty}\left(I_{i j}\right)}$ for $t \in I_{i j}$, and where the stability factor $S(T)$ is given by $S(T)=\int_{0}^{T}\left\|J^{\top}\left(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot\right) \Phi\right\|_{l_{2}} d t$. Furthermore, if assumptions (A1)-(A5) hold, then there is a constant $C=C\left(q, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left|L_{\psi, g}(\bar{e})\right| \leq C \bar{S}(T)\left\|k^{2 q} \overline{\bar{u}}^{(2 q)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{6.3}
\end{equation*}
$$

where $\left(k^{2 q} \overline{\bar{u}}^{(2 q)}\right)_{i}(t)=k_{i j}^{2 q_{i j}} C_{f}^{q_{i j}-1}\left\|u_{i}^{\left(q_{i j}+1\right)}\right\|_{L_{\infty}\left(I_{i j}\right)}$ for $t \in I_{i j}$, and where the stability factor $\bar{S}(T)$ is given by

$$
\bar{S}(T)=\int_{0}^{T} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} d t=\sum_{n=1}^{M} K_{n} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}_{n}, l_{\infty}\right)}
$$

Proof. By Corollary 6.3, we obtain

$$
L_{\psi, g}(\bar{e})=\int_{0}^{T}\left(f\left(\pi_{\mathrm{cG}}^{[q]} u, \cdot\right)-f(u, \cdot), \Phi\right) d t=\int_{0}^{T}\left(\pi_{\mathrm{cG}}^{[q]} u-u, J^{\top}\left(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot\right) \Phi\right) d t
$$

By Lemma 5.1, it now follows that

$$
\left|L_{\psi, g}(\bar{e})\right| \leq C\left\|k^{q+1} \bar{u}^{q+1}\right\|_{L_{\infty}\left([0, T], l_{2}\right)} \int_{0}^{T}\left\|J^{\top}\left(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot\right) \Phi\right\|_{l_{2}} d t
$$

which proves (6.2). To prove (6.3), we note that by definition, $\pi_{\mathrm{cG}}^{\left[q_{i j}\right]} u_{i}-u_{i}$ is orthogonal to $\mathcal{P}^{q_{i j}-2}\left(I_{i j}\right)$ for each local interval $I_{i j}$, and so, recalling that $\varphi=J^{\top}\left(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot\right) \Phi$,

$$
L_{\psi, g}(\bar{e})=\sum_{i, j} \int_{I_{i j}}\left(\pi_{\mathrm{cG}}^{\left[q_{i j}\right]} u_{i}-u_{i}\right) \varphi_{i} d t=\sum_{i, j} \int_{I_{i j}}\left(\pi_{\mathrm{cG}}^{\left[q_{i j}\right]} u_{i}-u_{i}\right)\left(\varphi_{i}-\pi_{\mathrm{cG}}^{\left[q_{i j}-2\right]} \varphi_{i}\right) d t
$$

where we take $\pi_{\mathrm{cG}}^{\left[q_{i j}-2\right]} \varphi_{i} \equiv 0$ for $q_{i j}=1$. By Lemmas 5.1 and 5.12, it now follows that

$$
\begin{aligned}
\left|L_{\psi, g}(\bar{e})\right| & \leq \int_{0}^{T}\left|\left(\pi_{\mathrm{cG}}^{[q]} u-u, \varphi-\pi_{\mathrm{cG}}^{[q-2]} \varphi\right)\right| d t \\
& =\int_{0}^{T}\left|\left(k^{q-1} C_{f}^{q-1}\left(\pi_{\mathrm{cG}}^{[q]} u-u\right), k^{-(q-1)} C_{f}^{-(q-1)}\left(\varphi-\pi_{\mathrm{cG}}^{[q-2]} \varphi\right)\right)\right| d t \\
& \leq C\left\|k^{2 q} \overline{\bar{u}}^{(2 q)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \int_{0}^{T} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} d t \\
& =C \bar{S}(T)\left\|k^{2 q} \overline{\bar{u}}^{(2 q)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)}
\end{aligned}
$$

where $\bar{S}(T)=\int_{0}^{T} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} d t=\sum_{n=1}^{M} K_{n} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}_{n}, l_{\infty}\right)}$.
Similarly, we obtain the following a priori error estimate for the $\operatorname{mdG}(q)$ method.
THEOREM 6.6 (a priori error estimate for $\operatorname{mdG}(q))$. Let $U$ be the $\operatorname{mdG}(q)$ solution of (1.1) and let $\Phi$ be the corresponding $\operatorname{mdG}(q)^{*}$ solution of the dual problem (2.9). Then there is a constant $C=C(q)>0$ such that

$$
\begin{equation*}
\left|L_{\psi, g}(\bar{e})\right| \leq C S(T)\left\|k^{q+1} \bar{u}^{(q+1)}\right\|_{L_{\infty}\left([0, T], l_{2}\right)} \tag{6.4}
\end{equation*}
$$

where $\left(k^{q+1} \bar{u}^{(q+1)}\right)_{i}(t)=k_{i j}^{q_{i j}+1}\left\|u_{i}^{\left(q_{i j}+1\right)}\right\|_{L_{\infty}\left(I_{i j}\right)}$ for $t \in I_{i j}$, and where the stability factor $S(T)$ is given by $S(T)=\int_{0}^{T}\left\|J^{\top}\left(\pi_{\mathrm{dG}}^{[q]} u, u, \cdot\right) \Phi\right\|_{l_{2}} d t$. Furthermore, if assumptions (A1)-(A5) hold, then there is a constant $C=C\left(q, c_{k}, \alpha\right)>0$ such that

$$
\begin{equation*}
\left|L_{\psi, g}(\bar{e})\right| \leq C \bar{S}(T)\left\|k^{2 q+1} \overline{\bar{u}}^{(2 q+1)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{6.5}
\end{equation*}
$$

where $\left(k^{2 q+1} \overline{\bar{u}}^{(2 q+1)}\right)_{i}(t)=k_{i j}^{2 q_{i j}+1} C_{f}^{q_{i j}}\left\|u_{i}^{\left(q_{i j}+1\right)}\right\|_{L_{\infty}\left(I_{i j}\right)}$ for $t \in I_{i j}$, and where the stability factor $\bar{S}(T)$ is given by

$$
\bar{S}(T)=\int_{0}^{T} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}, l_{\infty}\right)} d t=\sum_{n=1}^{M} K_{n} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}_{n}, l_{\infty}\right)}
$$

Using the stability estimate proved in section 4, we obtain the following bound for the stability factor $\bar{S}(T)$.

Lemma 6.7. Assume that $K_{n} C_{q} C_{f} \leq 1$ for all time slabs $\mathcal{T}_{n}$, with $C_{q}>0$ the constant in Theorem 4.2, and take $g=0$ in (2.9). Then

$$
\begin{equation*}
\bar{S}(T) \leq\|\psi\|_{l_{\infty}} e^{C_{q} \bar{C}_{f} T} \tag{6.6}
\end{equation*}
$$

where $\bar{C}_{f}=\max _{[0, T]} C_{f}$.
Proof. By Theorem4.2, we obtain

$$
\|\Phi\|_{L_{\infty}\left(\mathcal{T}_{n}, l_{\infty}\right)} \leq C_{q}\|\psi\|_{l_{\infty}} \exp \left(\sum_{m=n+1}^{M} K_{m} C_{q} C_{f}\right) \leq C_{q}\|\psi\|_{l_{\infty}} e^{C_{q} \bar{C}_{f}\left(T-T_{n}\right)}
$$

and so

$$
\begin{aligned}
\bar{S}(T) & =\sum_{n=1}^{M} K_{n} C_{f}\|\Phi\|_{L_{\infty}\left(\mathcal{T}_{n}, l_{\infty}\right)} d t \leq\|\psi\|_{l_{\infty}} \sum_{n=1}^{M} K_{n} C_{q} \bar{C}_{f} e^{C_{q} \bar{C}_{f}\left(T-T_{n}\right)} \\
& \leq\|\psi\|_{l_{\infty}} \int_{0}^{T} C_{q} \bar{C}_{f} e^{C_{q} \bar{C}_{f} t} d t \leq\|\psi\|_{l_{\infty}} e^{C_{q} \bar{C}_{f} T}
\end{aligned}
$$

Finally, we rewrite the estimates of Theorems 6.5 and 6.6 for special choices of data $\psi$ and $g$. We first take $\psi=0$. With $g_{n}=0$ for $n \neq i, g_{i}(t)=0$ for $t \notin I_{i j}$, and

$$
g_{i}(t)=\operatorname{sgn}\left(\bar{e}_{i}(t)\right) / k_{i j}, \quad t \in I_{i j}
$$

we obtain $L_{\psi, g}(\bar{e})=\frac{1}{k_{i j}} \int_{I_{i j}}\left|\bar{e}_{i}(t)\right| d t$ and so $\left\|\bar{e}_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C L_{\psi, g}(\bar{e})$ by an inverse estimate. By definition, it follows that $\left\|e_{i}\right\|_{L_{\infty}\left(I_{i j}\right)} \leq C L_{\psi, g}(\bar{e})+C k_{i j}^{q_{i j}+1}\left\|u_{i}^{q_{i j}+1}\right\|_{L_{\infty}\left(I_{i j}\right)}$. Note that for this choice of $g$, we have $\|g\|_{L_{1}\left([0, T], l_{2}\right)}=\|g\|_{L_{1}\left([0, T], l_{\infty}\right)}=1$.

We also make the choice $g=0$. Noting that $\bar{e}(T)=e(T)$, since $\pi u(T)=u(T)$, we obtain

$$
L_{\psi, g}(\bar{e})=(e(T), \psi)=\left|e_{i}(T)\right|
$$

for $\psi_{i}=\operatorname{sgn}\left(e_{i}(T)\right)$ and $\psi_{n}=0$ for $n \neq i$, and

$$
L_{\psi, g}(\bar{e})=(e(T), \psi)=\|e(T)\|_{l_{2}}
$$

for $\psi=e(T) /\|e(T)\|_{l_{2}}$. Note that for both choices of $\psi$, we have $\|\psi\|_{l_{\infty}} \leq 1$.
With these choices of data, we obtain the following versions of the a priori error estimates.

Corollary 6.8 (a priori error estimate for $\operatorname{mcG}(q))$. Let $U$ be the $\operatorname{mcG}(q)$ solution of (1.1). Then there is a constant $C=C(q)>0$ such that

$$
\begin{equation*}
\|e\|_{L_{\infty}\left([0, T], l_{\infty}\right)} \leq C S(T)\left\|k^{q+1} \bar{u}^{(q+1)}\right\|_{L_{\infty}\left([0, T], l_{2}\right)} \tag{6.7}
\end{equation*}
$$

where the stability factor $S(T)=\int_{0}^{T}\left\|J^{\top}\left(\pi_{\mathrm{cG}}^{[q]} u, u, \cdot\right) \Phi\right\|_{l_{2}} d t$ is taken as the maximum over $\psi=0$ and $\|g\|_{L_{1}\left([0, T], l_{\infty}\right)}=1$. Furthermore, if assumptions (A1)-(A5) and the assumptions of Lemma 6.7 hold, then there is a constant $C=C\left(q, c_{k}, \alpha\right)$ such that

$$
\begin{equation*}
\|e(T)\|_{l_{p}} \leq C \bar{S}(T)\left\|k^{2 q} \overline{\bar{u}}^{(2 q)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{6.8}
\end{equation*}
$$

for $p=2, \infty$, where the stability factor $\bar{S}(T)$ is given by $\bar{S}(T)=e^{C_{q} \bar{C}_{f} T}$.
Corollary 6.9 (a priori error estimate for $\operatorname{mdG}(q))$. Let $U$ be the $\operatorname{mdG}(q)$ solution of (1.1). Then there is a constant $C=C(q)>0$ such that

$$
\begin{equation*}
\|e\|_{L_{\infty}\left([0, T], l_{\infty}\right)} \leq C S(T)\left\|k^{q+1} \bar{u}^{(q+1)}\right\|_{L_{\infty}\left([0, T], l_{2}\right)} \tag{6.9}
\end{equation*}
$$

where the stability factor $S(T)=\int_{0}^{T}\left\|J^{\top}\left(\pi_{\mathrm{dG}}^{[q]} u, u, \cdot\right) \Phi\right\|_{l_{2}} d t$ is taken as the maximum over $\psi=0$ and $\|g\|_{L_{1}\left([0, T], l_{\infty}\right)}=1$. Furthermore, if assumptions (A1)-(A5) and the assumptions of Lemma 6.7 hold, then there is a constant $C=C\left(q, c_{k}, \alpha\right)$ such that

$$
\begin{equation*}
\|e(T)\|_{l_{p}} \leq C \bar{S}(T)\left\|k^{2 q+1} \overline{\bar{u}}^{(2 q+1)}\right\|_{L_{\infty}\left([0, T], l_{1}\right)} \tag{6.10}
\end{equation*}
$$

for $p=2, \infty$, where the stability factor $\bar{S}(T)$ is given by $\bar{S}(T)=e^{C_{q} \bar{C}_{f} T}$.
The stability factor $S(T)$ that appears in the a priori error estimates is obtained from the discrete solution $\Phi$ of the dual problem (4.1), and can thus be computed by solving the discrete dual problem. Numerical computation of the stability factor reveals the exact nature of the problem, in particular, whether or not the problem is parabolic; if the stability factor is of unit size and does not grow, then the problem is parabolic by definition; see 36.
6.3. A note on quadrature errors. The error representations presented in section 6.1 are based on the Galerkin orthogonalities of the $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q)$ methods. In particular, for the $\operatorname{mcG}(q)$ method, we assume that

$$
\int_{0}^{T}(R(U, \cdot), \Phi) d t=0
$$

In the presence of quadrature errors, this term is nonzero. As a result, we obtain an additional term of the form

$$
\int_{0}^{T}(\tilde{f}(U, \cdot)-f(U, \cdot), \Phi) d t
$$

where $\tilde{f}$ is the interpolant of $f$ corresponding the quadrature rule that is used. A convenient choice of quadrature for the $\operatorname{mcG}(q)$ method is Lobatto quadrature with $q+1$ nodal points [32], which means that the quadrature error is of order $2(q+1)-2=$ $2 q$ and so (super)convergence of order $2 q$ is obtained also in the presence of quadrature errors. Similarly for the $\operatorname{mdG}(q)$ method, we use Radau quadrature with $q+1$ nodal points, which means that the quadrature error is of order $2(q+1)-1=2 q+1$, and so the $2 q+1$ convergence order of $\operatorname{mdG}(q)$ is also maintained under quadrature.
7. A numerical example. We conclude by demonstrating the convergence of the multiadaptive methods in the case of a simple test problem.

Consider the problem

$$
\begin{align*}
& \dot{u}_{1}=u_{2} \\
& \dot{u}_{2}=-u_{1} \\
& \dot{u}_{3}=-u_{2}+2 u_{4} \\
& \dot{u}_{4}=u_{1}-2 u_{3}  \tag{7.1}\\
& \dot{u}_{5}=-u_{2}-2 u_{4}+4 u_{6} \\
& \dot{u}_{6}=u_{1}+2 u_{3}-4 u_{5}
\end{align*}
$$



Fig. 6. Convergence of the error at final time for the solution of the test problem (7.1) with $\operatorname{mcG}(q)$ and $\operatorname{mdG}(q), q \leq 5$.

Table 1
Order of convergence $p$ for $\operatorname{mcG}(q)$.

| $\operatorname{mcG}(q)$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $p$ | 1.99 | 3.96 | 5.92 | 7.82 | 9.67 |
| $2 q$ | 2 | 4 | 6 | 8 | 10 |

TABLE 2
Order of convergence $p$ for $\operatorname{mdG}(q)$.

| $\operatorname{mdG}(q)$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $p$ | 0.92 | 2.96 | 4.94 | 6.87 | 9.10 | - |
| $2 q+1$ | 1 | 3 | 5 | 7 | 9 | 11 |

on [ 0,1 ] with initial condition $u(0)=(0,1,0,2,0,3)$. The solution is given by $u(t)=$ $(\sin t, \cos t, \sin t+\sin 2 t, \cos t+\cos 2 t, \sin t+\sin 2 t+\sin 4 t, \cos t+\cos 2 t+\cos 4 t)$. For given $k_{0}>0$, we take $k_{i}(t)=k_{0}$ for $i=1,2, k_{i}(t)=k_{0} / 2$ for $i=3,4$, and $k_{i}(t)=k_{0} / 4$ for $i=5,6$, and study the convergence of the error $\|e(T)\|_{l_{2}}$ with decreasing $k_{0}$. From the results presented in Figure 6 and Tables 1 and 2 it is clear that the predicted order of convergence is obtained.

## REFERENCES

[1] S. G. Alexander and C. B. Agnor, n-body simulations of late stage planetary formation with a simple fragmentation model, ICARUS, 132 (1998), pp. 113-124.
[2] U. M. Ascher and L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, Philadelphia, 1998.
[3] R. Becker and R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods, Acta Numer., 10 (2001), pp. 1-102.
[4] J. C. Butcher, The Numerical Analysis of Ordinary Differential Equations-Runge-Kutta and General Linear Methods, Wiley, New York, 1987.
[5] R. Davé, J. Dubinski, and L. Hernquist, Parallel treeSPH, New Astron., 2 (1997), pp. 277297.
[6] C. Dawson and R. C. Kirby, High resolution schemes for conservation laws with locally varying time steps, SIAM J. Sci. Comput., 22 (2001), pp. 2256-2281.
[7] M. Delfour, W. Hager, and F. Trochu, Discontinuous Galerkin methods for ordinary differential equations, Math. Comp., 36 (1981), pp. 455-473.
[8] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, Introduction to adaptive methods for differential equations, Acta Numer., 4 (1995), pp. 105-158.
[9] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, Computational Differential Equations, Cambridge University Press, London, 1996.
[10] K. Eriksson and C. Johnson, Adaptive Finite Element Methods for Parabolic Problems III: Time Steps Variable in Space, in preparation.
[11] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems I: A linear model problem, SIAM J. Numer. Anal., 28 (1991), pp. 43-77.
[12] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems II: Optimal order error estimates in $l_{\infty} l_{2}$ and $l_{\infty} l_{\infty}$, SIAM J. Numer. Anal., 32 (1995), pp. 706-740.
[13] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems IV: Nonlinear problems, SIAM J. Numer. Anal., 32 (1995), pp. 1729-1749.
[14] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems V: Long-time integration, SIAM J. Numer. Anal., 32 (1995), pp. 1750-1763.
[15] K. Eriksson, C. Johnson, and S. Larsson, Adaptive finite element methods for parabolic problems VI: Analytic semigroups, SIAM J. Numer. Anal., 35 (1998), pp. 1315-1325.
[16] K. Eriksson, C. Johnson, and V. Thomée, Time discretization of parabolic problems by the discontinuous Galerkin method, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 611643.
[17] D. Estep, A posteriori error bounds and global error control for approximations of ordinary differential equations, SIAM J. Numer. Anal., 32 (1995), pp. 1-48.
[18] D. Estep and D. French, Global error control for the continuous Galerkin finite element method for ordinary differential equations, M2AN Math. Model. Numer. Anal., 28 (1994), pp. 815-852.
[19] D. Estep, M. Larson, and R. Williams, Estimating the error of numerical solutions of systems of nonlinear reaction-diffusion equations, Mem. Amer. Math. Soc., 696 (2000), pp. 1-109.
[20] D. Estep and A. Stuart, The dynamical behavior of the discontinuous Galerkin method and related difference schemes, Math. Comp., 71 (2002), pp. 1075-1103.
[21] D. Estep and R. Williams, Accurate parallel integration of large sparse systems of differential equations, Math. Models Methods Appl. Sci., 6 (1996), pp. 535-568.
[22] J. E. Flaherty, R. M. Loy, M. S. Shephard, B. K. Szymanski, J. D. Teresco, and L. H. Ziantz, Adaptive local refinement with octree load balancing for the parallel solution of three-dimensional conservation laws, J. Parallel Distrib. Comput., 47 (1997), pp. 139-152.
[23] E. Hairer and G. Wanner, Solving Ordinary Differential Equations I—Nonstiff Problems, Springer Ser. Comput. Math. 8, Springer, New York, 1991.
[24] E. Hairer and G. Wanner, Solving Ordinary Differential Equations II—Stiff and DifferentialAlgebraic Problems, Springer Ser. Comput. Math. 14, Springer, New York, 1991.
[25] T. J. R. Hughes, I. Levit, and J. Winget, Element-by-element implicit algorithms for heatconduction, J. Engrg. Mech.-ASCE, 109 (1983), pp. 576-585.
[26] T. J. R. Hughes, I. Levit, and J. Winget, An element-by-element solution algorithm for problems of structural and solid mechanics, Comput. Methods Appl. Mech. Engrg., 36 (1983), pp. 241-254.
[27] B. L. Hulme, Discrete Galerkin and related one-step methods for ordinary differential equations, Math. Comp., 26 (1972), pp. 881-891.
[28] B. L. Hulme, One-step piecewise polynomial Galerkin methods for initial value problems, Math. Comp., 26 (1972), pp. 415-426.
[29] P. Jamet, Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain, SIAM J. Numer. Anal., 15 (1978), pp. 912-928.
[30] C. Johnson, Error estimates and adaptive time-step control for a class of one-step methods for stiff ordinary differential equations, SIAM J. Numer. Anal., 25 (1988), pp. 908-926.
[31] A. Lew, J. E. Marsden, M. Ortiz, and M. West, Asynchronous variational integrators, Arch. Ration. Mech. Anal., 167 (2003), pp. 85-146.
[32] A. Logg, Multi-adaptive Galerkin methods for ODEs I, SIAM J. Sci. Comput., 24 (2003), pp. 1879-1902.
[33] A. Logg, Multi-adaptive Galerkin methods for ODEs II: Implementation and applications, SIAM J. Sci. Comput., 25 (2003), pp. 1119-1141.
[34] A. Logg, Automation of Computational Mathematical Modeling, Ph.D. thesis, Chalmers University of Technology, Sweden, 2004.
[35] A. Logg, Interpolation Estimates for Piecewise Smooth Functions in One Dimension, Technical report 2004-02, Chalmers Finite Element Center Preprint Series, 2004.
[36] A. Logg, Multi-adaptive time-integration, Appl. Numer. Math., 48 (2004), pp. 339-354.
[37] J. Makino and S. Aarseth, On a Hermite integrator with Ahmad-Cohen scheme for gravitational many-body problems, Publ. Astron. Soc. Japan, 44 (1992), pp. 141-151.
[38] P. Niamsup and V. N. Phat, Asymptotic stability of nonlinear control systems described by difference equations with multiple delays, Electron. J. Differential Equations, 11 (2000), pp. 1-17.
[39] S. Osher and R. Sanders, Numerical approximations to nonlinear conservation laws with locally varying time and space grids, Math. Comp., 41 (1983), pp. 321-336.
[40] M. J. D. Powell, Approximation Theory and Methods, Cambridge University Press, Cambridge, UK, 1988.
[41] L. Shampine, Numerical Solution of Ordinary Differential Equations, Chapman \& Hall, London, 1994.


[^0]:    *Received by the editors February 12, 2004; accepted for publication (in revised form) May 4, 2005; published electronically January 27, 2006.
    http://www.siam.org/journals/sinum/43-6/60413.html
    ${ }^{\dagger}$ Toyota Technological Institute at Chicago, 1427 East 60th Street, Chicago, IL 60637 (logg@ tti-c.org).

