

# Numerical Periodic Normalization for Codim 2 Bifurcations of Limit Cycles

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## Abstract

Periodic normal forms for the codim 2 bifurcations of limit cycles up to a 3-dimensional center manifold in generic autonomous ODEs and computational formulas for their coefficients are derived. The formulas are independent of the dimension of the phase space and involve solutions of certain boundary-value problems on the interval  $[0, T]$ , where  $T$  is the period of the critical cycle, as well as multilinear functions from the Taylor expansion of the right-hand sides near the cycle. The formulas allow us to distinguish between various bifurcation scenarios near codim 2 bifurcations. Our formulation makes it possible to use robust numerical boundary-value algorithms based on orthogonal collocation, rather than shooting techniques, which greatly expands its applicability. The actual implementation is described in detail with numerical examples.

*Keywords: normal forms, limit cycles, bifurcations, codimension two*

## 1 Introduction

Isolated periodic orbits (limit cycles) of smooth differential equations

$$(1) \quad \dot{u} = f(u, p), \quad u \in \mathbb{R}^n, \quad p \in \mathbb{R}^m,$$

play an important role in applications. In generic systems of the form (1) depending on one control parameter (i.e. with  $m = 1$ ) a hyperbolic limit cycle exists for an open interval of parameter values  $p$ . At a boundary of such an interval, the limit cycle may become non-hyperbolic, so that either a cycle *limit point* (*saddle-node*), or a *period-doubling* (*flip*), or a *torus* (*Neimark-Sacker*) bifurcation occurs. In two-parameter generic systems (1) (i.e. with  $m = 2$ ) these bifurcations happen at certain curves in the parameter plane. These curves of codim 1 bifurcations can meet tangentially or intersect transversally at some codim 2 points characterized by a double degeneracy of the limit cycle, which play the role of organizing centers for local dynamics, i.e. near the critical cycle and for nearby parameter values. In some cases, such codim 2 bifurcations imply the appearance of “chaotic motions”.

The codim 2 bifurcations of limit cycles in generic systems (1) are well understood with the help of the corresponding Poincaré maps and their normal forms (see for example [22, 20, 3, 29, 18]). However, applications of these results to the analysis of concrete systems (1) are exceptional, since they require accurate higher-order derivatives of the Poincaré map which are hardly available numerically [42, 19, 21, 32].

We note that there exists software, e.g. CAPD [1], TIDES [2, 5] that allows to compute up to any precision level the solution of an ODE using a Taylor series method in a variable stepsize - variable order

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formulation. It can also compute, up to any order, the partial derivatives of the solution with respect to the initial conditions. When applied to compute a periodic orbit by a shooting method, this will also provide the derivatives of the Poincaré map.

Though this is a valuable approach in the case of a single periodic orbit, it is neither practical nor efficient in a continuation context since the periodic orbits are not obtained as the solution to a set of equations. Also, the shooting method does not have the high order convergence properties of the method of approximation by piecewise polynomials with collocation in the Gauss points that is routinely used in standard software such as AUTO [14], CONTENT [31], and MATCONT [11, 10]. Moreover, the number of derivatives of the Poincaré map to be computed is  $O(n^k)$  if derivatives up to order  $k$  are needed (in several cases  $k = 5$ ). Even for moderate values of  $n$  this involves a great deal of unnecessary work since in our situation the normal form itself is known in advance and we need only compute its coefficients. We will show that this can be done without computing the derivatives of the Poincaré map.

Indeed, recently an alternative numerical method to analyse codim 1 limit cycle bifurcations has been developed and implemented in [30]. It is based on the periodic normalization proposed in [25, 23, 24] and completely avoids the numerical computation of Poincaré maps and their derivatives. Instead, the computation of the normal form coefficients is reduced to solving certain linear boundary value problems (BVP), where only the partial derivatives of the RHS of (1) are used [17, 16, 28]. In our implementation in MATCONT, we discretize these BVPs by orthogonal collocation with piecewise-polynomial functions.

In the present paper, we apply the approach developed in [30] to codim 2 bifurcations of limit cycles. It should be noted that already in [7] normal forms for some codim 2 bifurcations of cycles in (1) were derived, while [23] contains the periodic normal forms for many codim 2 bifurcations of cycles, as well as a general normalization technique applicable at any codimension. However, in neither of these publications explicit formulas for the normal form coefficients were given. The derivation of such formulas is the primary contribution of this paper.

The paper is organized as follows. In Section 2 we fix notation and list the periodic normal forms for codim 2 bifurcations of limit cycles. Then we derive explicit formulas to compute the critical normal form coefficients for these bifurcations, which we order by the dimension  $n_c$  of the cycle center manifold (i.e. the total number of *critical multipliers* with  $|\mu| = 1$ ). We restrict to the cases  $n_c = 2$  and 3. The formulas are independent of the dimension of the phase space and involve solutions of certain BVPs on the interval  $[0, T]$ , where  $T$  is the period of the critical cycle, as well as multilinear functions from the Taylor expansion of the right-hand sides of (1) near the cycle. A derivation of the critical periodic normal forms based on [23] is given in Appendix A, while their relationships with the normal forms of the Poincaré maps are discussed in Appendix B.

## 2 Periodic normal forms on the center manifold

Write (1) at the critical parameter values as

$$(2) \quad \dot{u} = F(u),$$

and suppose that there is a limit cycle  $\Gamma$  corresponding to a periodic solution  $u_0(t) = u_0(t+T)$ , where  $T > 0$  is its (minimal) period. Develop  $F(u_0(t) + v)$  into the Taylor series

$$(3) \quad \begin{aligned} F(u_0(t) + v) = & F(u_0(t)) + \\ & A(t)v + \frac{1}{2}B(t; v, v) + \frac{1}{3!}C(t; v, v, v) + \\ & \frac{1}{4!}D(t; v, v, v, v) + \frac{1}{5!}E(t; v, v, v, v, v) + O(\|v\|^6), \end{aligned}$$

where

$$A(t)v = F_u(u_0(t))v, \quad B(t; v_1, v_2) = F_{uu}(u_0(t))[v_1, v_2], \quad C(t; v_1, v_2, v_3) = F_{uuu}(u_0(t))[v_1, v_2, v_3],$$

etc. The multilinear forms  $A, B, C, D$ , and  $E$  are periodic in  $t$  with period  $T$  but this dependence will often not be indicated explicitly.

Consider the initial-value problem for the fundamental matrix solution  $Y(t)$ , namely,

$$(4) \quad \frac{dY}{dt} = A(t)Y, \quad Y(0) = I_n,$$

where  $I_n$  is the  $n \times n$  identity matrix. The eigenvalues of the monodromy matrix  $M = Y(T)$  are called (*Floquet*) *multipliers* of the limit cycle. The multipliers with  $|\mu| = 1$  are called *critical*. There is always a “trivial” critical multiplier  $\mu_n = 1$ . We denote the total number of critical multipliers by  $n_c$  and assume that the limit cycle is non-hyperbolic, i.e.  $n_c > 1$ . In this case, there exists an invariant  $n_c$ -dimensional *critical center manifold*  $W^c(\Gamma) \subset \mathbb{R}^n$  near  $\Gamma^1$ .

It is well known [3, 29], that in generic two-parameter systems (1) only eleven codim 2 bifurcations occur. To describe the normal forms of (2) on the critical center manifold  $W^c(\Gamma)$  for these codim 2 cases, we parameterize  $W^c(\Gamma)$  near  $\Gamma$  by  $(\tau, \xi)$ , where  $\tau \in [0, kT]$  for  $k \in \{1, 2, 3, 4\}$  and  $\xi$  is a real or complex vector, depending on the bifurcation. It follows from [23] that it is possible to select the  $\tau$ - and  $\xi$ -coordinates so that the restriction of (2) to the corresponding critical center manifold  $W^c(\Gamma)$  with  $n_c = 2$  or  $n_c = 3$  will take one of the following *periodic normal forms* (for derivation see Appendix A). In Section 3 we will see that in two cases, namely the cusp of cycles bifurcation and the fold-flip bifurcation, a further simplification is possible in the normal form, and more specifically in the transformation of time.

## 2.1 Bifurcations with a 2D center manifold

We list here the critical periodic normal forms with  $n_c = 2$  and briefly describe bifurcations in their generic unfoldings (see [3, 29] for more details).

### 2.1.1 Cusp of cycles bifurcation

The cycle has a *cusp of cycles* bifurcation (CPC) if the eigenvalue  $\mu_1 = \mu_n = 1$  of  $Y(T)$  corresponds to a two-dimensional Jordan block and the monodromy matrix has no other critical multipliers. The two-dimensional periodic normal form at the CPC bifurcation is

$$(5) \quad \begin{cases} \frac{d\tau}{dt} = 1 - \xi + \alpha_1 \xi^2 + \alpha_2 \xi^3 + \dots, \\ \frac{d\xi}{dt} = c\xi^3 + \dots, \end{cases}$$

where  $\tau \in [0, T]$ ,  $\xi$  is a real coordinate on  $W^c(\Gamma)$  that is transverse to  $\Gamma$ ,  $\alpha_1, \alpha_2, c \in \mathbb{R}$  and the dots denote the  $O(\xi^4)$ -terms, which are  $T$ -periodic in  $\tau$ . If  $c \neq 0$ , the limit cycle  $\Gamma$  is triple. In generic two-parameter systems (1), three hyperbolic limit cycles exist in a cuspidal wedge approaching the codim 2 point and delimited by two bifurcations curves, where two cycles collide and disappear via the saddle-node bifurcation.

### 2.1.2 Generalized period-doubling bifurcation

The cycle has a *generalized period-doubling bifurcation* (GPD) if the trivial eigenvalue  $\mu_n = 1$  of the monodromy matrix  $Y(T)$  is simple and there is only another critical simple eigenvalue  $\mu_1 = -1$ . The two-dimensional periodic normal form at the GPD bifurcation is

$$(6) \quad \begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1 \xi^2 + \alpha_2 \xi^4 + \dots, \\ \frac{d\xi}{dt} = c\xi^5 + \dots, \end{cases}$$

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<sup>1</sup>This manifold should not be confused with the  $(n_c - 1)$ -dimensional center manifold of the corresponding Poincaré map.

where  $\tau \in [0, 2T]$ ,  $\xi$  is a real coordinate on  $W^c(\Gamma)$  that is transverse to  $\Gamma$ ,  $\alpha_1, \alpha_2, e \in \mathbb{R}$  and the dots denote the  $O(\xi^6)$ -terms, which are  $2T$ -periodic in  $\tau$ . If  $e \neq 0$ , at most two period doubled limit cycles can bifurcate from the critical limit cycle  $\Gamma$ . In generic two-parameter systems (1), the GPD-point in the period-doubling bifurcation curve separates its sub- and super-critical branches and is the origin of a unique saddle-node bifurcation curve, where two period doubled cycles collide and disappear.

## 2.2 Bifurcations with a 3D center manifold

We now list the critical periodic normal forms with  $n_c = 3$  and briefly describe bifurcations in their generic unfoldings (see [3, 29]). In all cases, “chaotic motions” are possible.

### 2.2.1 Chenciner bifurcation

The cycle has a *Chenciner* bifurcation (CH) if the trivial critical eigenvalue  $\mu_n = 1$  is simple and there are only two more critical simple multipliers  $\mu_{1,2} = e^{\pm i\theta}$  with  $\theta \neq \frac{2\pi}{j}$ , for  $j = 1, 2, 3, 4, 5, 6$ . The three-dimensional periodic normal form at the CH bifurcation can be written as

$$(7) \quad \begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1|\xi|^2 + \alpha_2|\xi|^4 + \dots, \\ \frac{d\xi}{dt} = i\omega\xi + ic\xi|\xi|^2 + e\xi|\xi|^4 + \dots, \end{cases}$$

where  $\tau \in [0, T]$ ,  $\omega = \theta/T$ ,  $\xi$  is a complex coordinate on  $W^c(\Gamma)$  transverse to  $\Gamma$ ,  $\alpha_1, \alpha_2, c \in \mathbb{R}$ ,  $e \in \mathbb{C}$  and the dots denote the  $O(\|\xi^6\|)$ -terms, which are  $T$ -periodic in  $\tau$ . In generic two-parameter systems (1), at the CH-point the Neimark-Sacker bifurcation changes its criticality (i.e. the bifurcating invariant torus changes its stability). A complicated bifurcation set responsible for “collision” and destruction of two tori of opposite stability is rooted at this codim 2 point.

### 2.2.2 Strong resonance 1:1 bifurcation

The cycle has a *strong resonance* 1 : 1 bifurcation (R1) if the trivial critical eigenvalue  $\mu_n = 1$  corresponds to a three-dimensional Jordan block. The three-dimensional periodic normal form at the R1 bifurcation is

$$(8) \quad \begin{cases} \frac{d\tau}{dt} = 1 - \xi_1 + \alpha\xi_1^2 + \dots, \\ \frac{d\xi_1}{dt} = \xi_2 + \xi_1\xi_2 + \dots, \\ \frac{d\xi_2}{dt} = a\xi_1^2 + b\xi_1\xi_2 + \dots, \end{cases}$$

where  $\tau \in [0, T]$ ,  $(\xi_1, \xi_2)$  are real coordinates on  $W^c(\Gamma)$  transverse to  $\Gamma$ ,  $\alpha, a, b \in \mathbb{R}$  and the dots denote the  $O(\|\xi^3\|)$ -terms, which are  $T$ -periodic in  $\tau$ . In generic two-parameter systems (1), in the R1-point is located on the saddle-node of cycles curve. At this point, a torus bifurcation curve is rooted together with global homoclinic bifurcation curves, along which the stable and the unstable invariant manifolds of a saddle cycle are tangent. The intersection of the invariant manifolds generates a Poincaré homoclinic structure with the associated periodic and “chaotic motions”.

### 2.2.3 Strong resonance 1:2 bifurcation

The cycle has a *strong resonance* 1 : 2 bifurcation (R2) if the trivial critical eigenvalue  $\mu_n = 1$  is simple and there are only two more critical multipliers  $\mu_1 = \mu_2 = -1$  corresponding to a two-dimensional Jordan block.

The three-dimensional periodic normal form at the R2 bifurcation is

$$(9) \quad \begin{cases} \frac{d\tau}{dt} = 1 + \alpha\xi_1^2 + \dots, \\ \frac{d\xi_1}{dt} = \xi_2 + \alpha\xi_1^2\xi_2 + \dots, \\ \frac{d\xi_2}{dt} = a\xi_1^3 + b\xi_1^2\xi_2 + \dots, \end{cases}$$

where  $\tau \in [0, 2T]$ ,  $(\xi_1, \xi_2)$  are real coordinates on  $W^c(\Gamma)$  transverse to  $\Gamma$ ,  $\alpha, a, b \in \mathbb{R}$  and the dots denote the  $O(\|\xi^4\|)$ -terms, which are  $2T$ -periodic in  $\tau$ . In generic two-parameter systems (1), the R2-point is the end-point of a torus bifurcation curve. The period-doubling bifurcation curve passes through this point, and (depending on the normal form coefficients) a torus bifurcation curve of the period doubled limit cycle can originate there. As in the R1-case, global bifurcation curves related to homoclinic tangencies can be present.

### 2.2.4 Strong resonance 1:3 bifurcation

The cycle has a *strong resonance* 1 : 3 bifurcation (R3) if the trivial critical eigenvalue  $\mu_n = 1$  is simple and there are only two more critical simple multipliers  $\mu_{1,2} = e^{\pm i\frac{2\pi}{3}}$ . The three-dimensional periodic normal form at the R3 bifurcation can be written as

$$(10) \quad \begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1|\xi|^2 + \alpha_2\xi^3 + \alpha_3\bar{\xi}^3 + \dots, \\ \frac{d\xi}{dt} = b\bar{\xi}^2 + c\xi|\xi|^2 + \dots, \end{cases}$$

where  $\tau \in [0, 3T]$ ,  $\xi$  is a complex coordinate on  $W^c(\Gamma)$  transverse to  $\Gamma$ ,  $\alpha_1 \in \mathbb{R}$ ,  $\alpha_2, \alpha_3, b, c \in \mathbb{C}$  with  $\alpha_3 = \bar{\alpha}_2$  and the dots denote the  $O(\|\xi^4\|)$ -terms, which are  $3T$ -periodic in  $\tau$ . In generic two-parameter systems (1), near the R3-point a homoclinic Poincaré structure of the  $3T$ -periodic limit cycle destroys the torus that is born at the Neimark-Sacker bifurcation curve passing through this point. Homoclinic tangencies are rooted there.

### 2.2.5 Strong resonance 1:4 bifurcation

The cycle has a *strong resonance* 1 : 4 bifurcation (R4) if the trivial critical eigenvalue  $\mu_n = 1$  is simple and there are only two more critical simple multipliers  $\mu_{1,2} = e^{\pm i\frac{\pi}{2}}$ . The three-dimensional periodic normal form at the R4 bifurcation can be written as

$$(11) \quad \begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1|\xi|^2 + \alpha_2\xi^4 + \alpha_3\bar{\xi}^4 + \dots, \\ \frac{d\xi}{dt} = c\xi|\xi|^2 + d\bar{\xi}^3 + \dots, \end{cases}$$

where  $\tau \in [0, 4T]$ ,  $\xi$  is a complex coordinate on  $W^c(\Gamma)$  transverse to  $\Gamma$ ,  $\alpha_1 \in \mathbb{R}$ ,  $\alpha_2, \alpha_3, c, d \in \mathbb{C}$  with  $\alpha_3 = \bar{\alpha}_2$  and the dots denote the  $O(\|\xi^5\|)$ -terms, which are  $4T$ -periodic in  $\tau$ . In generic two-parameter systems (1), at the R4-point there can be eight different situations, depending upon the values taken by the parameter  $c$  and  $d$ . In the simplest case a homoclinic structure associated to a  $4T$ -periodic cycle destroys an invariant torus that is born at the Neimark-Sacker bifurcation curve that passes through this point.

### 2.2.6 Fold-flip bifurcation

The cycle has a *fold flip* bifurcation (LPPD) if the trivial critical eigenvalue  $\mu_1 = \mu_n = 1$  is double non semi-simple and there is only one more critical multiplier  $\mu_2 = -1$ . The three-dimensional periodic normal

form at the LPPD bifurcation is

$$(12) \quad \begin{cases} \frac{d\tau}{dt} = 1 - \xi_1 + \alpha_{20}\xi_1^2 + \alpha_{02}\xi_2^2 + \alpha_{30}\xi_1^3 + \alpha_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_1}{dt} = a_{20}\xi_1^2 + a_{02}\xi_2^2 + a_{30}\xi_1^3 + a_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_2}{dt} = b_{11}\xi_1\xi_2 + b_{21}\xi_1^2\xi_2 + b_{03}\xi_2^3 + \dots, \end{cases}$$

where  $\tau \in [0, 2T]$ ,  $(\xi_1, \xi_2)$  are real coordinates on  $W^c(\Gamma)$  transverse to  $\Gamma$ , all the coefficients are real and the dots denote the  $O(\|\xi^4\|)$ -terms, which are  $2T$ -periodic in  $\tau$ . In generic two-parameter systems (1), the period-doubling and saddle-node of cycles bifurcation curves are tangent at the LPPD-point, where (depending on the normal form coefficients) a Neimark-Sacker bifurcation curve of the  $2T$ -periodic cycle can be rooted. Global bifurcations of heteroclinic structures and invariant tori are also possible.

### 3 Computation of critical coefficients

In view of the above, we can assume that a parametrization of the center manifold  $W^c(\Gamma)$  has been selected so that the restriction of (2) to this manifold has one of the normal forms given in Section 2. We then apply the so-called *homological equation approach* [6]: the Taylor expansions of  $T$ -,  $2T$ -,  $3T$ - or  $4T$ -periodic unknown functions involved in these parametrizations can be found by solving appropriate BVPs on  $[0, T]$  so that (2) restricted to  $W^c(\Gamma)$  has the corresponding periodic normal form. The coefficients of the normal forms arise from the solvability conditions for the BVPs as integrals of scalar products over  $[0, T]$ , involving nonlinear terms of (2) near the periodic solution  $u_0$ , as well as the critical eigenfunctions and higher order expansion terms of the center manifold. The Taylor expansion coefficient functions are usually unique up to the addition of a multiple of a known eigenfunction. This has to be fixed by adding an integral condition. Among other things this leads to the fact that normal form coefficients are not unique but implications for the underlying dynamical systems are independent of this. We also remark that the solvability of all the equations up to the maximal order of the normal form has to be checked. Finally, we note that the coefficients related to the transformation of time will only be computed when needed in the computation of the critical coefficients in the normal form for the state variables.

#### 3.1 Cusp of cycles bifurcation

The two-dimensional critical center manifold  $W^c(\Gamma)$  at the CPC bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(13) \quad u = u_0 + \xi v + H(\tau, \xi), \quad \tau \in [0, T], \quad \xi \in \mathbb{R},$$

where  $H$  satisfies  $H(T, \xi) = H(0, \xi)$  and has the Taylor expansion

$$(14) \quad H(\tau, \xi) = \frac{1}{2}h_2\xi^2 + \frac{1}{6}h_3\xi^3 + O(\xi^4),$$

where  $u_0$  and all  $h_j$  are functions of  $\tau$ , with  $h_j(T) = h_j(0)$ , for  $j = 2, 3$ , while the generalized eigenfunction  $v$  is given by

$$(15) \quad \begin{cases} \dot{v} - A(\tau)v - F(u_0) &= 0, \quad \tau \in [0, T], \\ v(T) - v(0) &= 0, \\ \int_0^T \langle v, F(u_0) \rangle d\tau &= 0. \end{cases}$$

The function  $v$  exists due to Lemma 2 of [23]. Let  $\varphi^*$  be a nontrivial solution of the adjoint eigenvalue problem

$$(16) \quad \begin{cases} \dot{\varphi}^* + A^T(\tau)\varphi^* &= 0, \quad \tau \in [0, T], \\ \varphi^*(T) - \varphi^*(0) &= 0, \end{cases}$$

and the generalized adjoint eigenfunction  $v^*$  a solution of

$$(17) \quad \begin{cases} \dot{v}^* + A^T(\tau)v^* + \varphi^* &= 0, \tau \in [0, T], \\ v^*(T) - v^*(0) &= 0, \end{cases}$$

which is now defined up to the addition of a multiple of  $\varphi^*$ . Note that the first equation of (15) implies

$$(18) \quad \int_0^T \langle \varphi^*, F(u_0) \rangle d\tau = 0$$

for  $\varphi^*$  satisfying (16). Indeed,

$$\int_0^T \langle \varphi^*, F(u_0) \rangle d\tau = \int_0^T \langle \varphi^*, \left( \frac{d}{d\tau} - A(\tau) \right) v \rangle d\tau = - \int_0^T \langle \left( \frac{d}{d\tau} + A^T(\tau) \right) \varphi^*, v \rangle d\tau = 0$$

due to (16).

Moreover, due to spectral assumptions at the CPC-point, we can also assume

$$(19) \quad \int_0^T \langle \varphi^*, v \rangle d\tau = 1.$$

Notice that this assumption gives us another normalization for free, since because of (15) and (17) we have

$$(20) \quad \begin{aligned} \int_0^T \langle v^*, F(u_0) \rangle d\tau &= \int_0^T \langle v^*, \left( \frac{d}{d\tau} - A(\tau) \right) v \rangle d\tau \\ &= - \int_0^T \langle \left( \frac{d}{d\tau} + A^T(\tau) \right) v^*, v \rangle d\tau = \int_0^T \langle \varphi^*, v \rangle d\tau = 1, \end{aligned}$$

i.e. we have normalized the eigenfunction of the adjoint problem with the generalized one of the original problem and the generalized eigenfunction of the adjoint problem with the eigenfunction of the original problem. So  $\varphi^*$  is the unique solution of the BVP

$$(21) \quad \begin{cases} \dot{\varphi}^* + A^T(\tau)\varphi^* &= 0, \tau \in [0, T], \\ \varphi^*(T) - \varphi^*(0) &= 0, \\ \int_0^T \langle \varphi^*, v \rangle d\tau - 1 &= 0. \end{cases}$$

Now, we still need an integral condition for the adjoint generalized eigenfunction  $v^*$ . In all cases, for the computation of an adjoint generalized eigenfunction we will require the inproduct with an original eigenfunction to be zero. Here, the inproduct with  $v$  is appropriate. Therefore we obtain

$$(22) \quad \begin{cases} \dot{v}^* + A^T(\tau)v^* + \varphi^* &= 0, \tau \in [0, T], \\ v^*(T) - v^*(0) &= 0, \\ \int_0^T \langle v^*, v \rangle d\tau &= 0. \end{cases}$$

Now, we substitute (13) into (2), using (3), (5) and (14), as well as

$$\frac{du}{dt} = \frac{\partial u}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial u}{\partial \tau} \frac{d\tau}{dt}.$$

This gives

$$\begin{aligned} \dot{u}_0 + \xi (\dot{v} - \dot{u}_0) + \xi^2 \left( \alpha_1 \dot{u}_0 - \dot{v} + \frac{1}{2} h_2 \right) + \xi^3 \left( \alpha_2 \dot{u}_0 + \alpha_1 \dot{v} - \frac{1}{2} h_2 + \frac{1}{6} h_3 + cv \right) + O(\xi^4) \\ = F(u_0) + \xi A(\tau)v + \frac{1}{2} \xi^2 (A(\tau)h_2 + B(\tau; v, v)) + \frac{1}{6} \xi^3 (A(\tau)h_3 + 3B(\tau; h_2, v) + C(\tau; v, v, v)) + O(\xi^4), \end{aligned}$$

where dots denote the derivatives with respect to  $\tau$ .

Collecting the  $\xi^0$ -terms we get the identity

$$\dot{u}_0 = F(u_0),$$

since  $u_0$  is the periodic solution of (2).

The  $\xi^1$ -terms provide another identity, namely

$$\dot{v} - \dot{u}_0 = A(\tau)v,$$

due to (15).

From collecting the  $\xi^2$ -terms we obtain an equation for  $h_2$

$$(23) \quad \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) + 2\dot{v} - 2\alpha_1\dot{u}_0.$$

Now, we project the left-hand side of this equation on the adjoint null-eigenfunction and obtain

$$\int_0^T \langle \varphi^*, \left( \frac{d}{d\tau} - A(\tau) \right) h_2 \rangle d\tau = - \int_0^T \langle \left( \frac{d}{d\tau} + A^T(\tau) \right) \varphi^*, h_2 \rangle d\tau = 0$$

due to (16). We can use this result to impose the so-called *Fredholm solvability condition*, i.e. also project the right-hand side of (23) on  $\varphi^*$  which then has to be equal to 0

$$\int_0^T \langle \varphi^*, B(\tau; v, v) + 2\dot{v} - 2\alpha_1\dot{u}_0 \rangle d\tau = \int_0^T \langle \varphi^*, B(\tau; v, v) + 2A(\tau)v \rangle d\tau = 0.$$

Notice that this condition is actually trivially satisfied, due to the fact that we are at a cusp of cycles point, so that the second order normal form coefficient

$$b = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v, v) + 2A(\tau)v \rangle d\tau$$

(see [30]) vanishes. Hence equation (23) is solvable, independent of the value of  $\alpha_1$ . So, for any value of  $\alpha_1$  we get an equation for  $h_2$  to be solved in the space of vector-functions on  $[0, T]$  satisfying  $h_2(T) = h_2(0)$ . Notice that if  $h_2$  satisfies (23),  $h_2 + \varepsilon F(u_0)$  also satisfies (23), due to the fact that  $F(u_0)$  is in the kernel of the operator  $\frac{d}{d\tau} - A(\tau)$  and to the linearity of this operator. Now, the orthogonality condition with  $v^*$  determines the value of  $\varepsilon$  such that we can define  $h_2$  as the unique solution of

$$(24) \quad \begin{cases} \dot{h}_2 - A(\tau)h_2 - B(\tau; v, v) - 2Av - 2F(u_0) + 2\alpha_1 F(u_0) &= 0, \quad \tau \in [0, T], \\ h_2(T) - h_2(0) &= 0, \\ \int_0^T \langle v^*, h_2 \rangle d\tau &= 0. \end{cases}$$

Collecting the  $\xi^3$ -terms we finally obtain an equation in  $h_3$  which allows us to determine the normal form coefficient  $c$  of (5)

$$\dot{h}_3 - A(\tau)h_3 = -6\alpha_2\dot{u}_0 - 6\alpha_1\dot{v} + 3\dot{h}_2 - 6cv + 3B(\tau; h_2, v) + C(\tau; v, v, v).$$

As before, the null-eigenfunction of the adjoint operator  $-\frac{d}{d\tau} - A^T(\tau)$  is  $\varphi^*$ . Thus, the Fredholm solvability condition implies that

$$\int_0^T \langle \varphi^*, -6\alpha_2\dot{u}_0 - 6\alpha_1\dot{v} + 3\dot{h}_2 - 6cv + 3B(\tau; h_2, v) + C(\tau; v, v, v) \rangle d\tau = 0.$$

Using (15), (19) and (18), we get the expression

$$(25) \quad c = \frac{1}{6} \int_0^T \langle \varphi^*, -6\alpha_1 A(\tau)v + 3A(\tau)h_2 + 3B(\tau; v, v) + 6A(\tau)v + 3B(\tau; h_2, v) + C(\tau; v, v, v) \rangle d\tau,$$



where  $v$  and  $\varphi^*$  are defined by (15) and (21), while  $h_2$  satisfies (24).

In what follows we will prove that the choice of  $\alpha_1$  does not influence the value of the critical normal form coefficient  $c$ . Two solutions  $h_2$  differ by  $h_2^{(2)} - h_2^{(1)} = -2(\alpha_1^{(2)} - \alpha_1^{(1)})v$ , from which

$$\begin{aligned} c^{(2)} - c^{(1)} &= \frac{1}{6} \int_0^T \langle \varphi^*, -6(\alpha_1^{(2)} - \alpha_1^{(1)})A(\tau)v - 6A(\tau)(\alpha_1^{(2)} - \alpha_1^{(1)})v - 6(\alpha_1^{(2)} - \alpha_1^{(1)})B(\tau; v, v) \rangle d\tau \\ &= (\alpha_1^{(2)} - \alpha_1^{(1)}) \int_0^T \langle \varphi^*, -2A(\tau)v - B(\tau; v, v) \rangle d\tau \\ &= (\alpha_1^{(2)} - \alpha_1^{(1)}) \int_0^T \langle \varphi^*, -2A(\tau)v + 2A(\tau)v \rangle d\tau \\ &= 0, \end{aligned}$$

since  $b$  vanishes. So, for simplicity we take  $\alpha_1 = 0$  which further simplifies normal form (5).

Therefore, the critical coefficient  $c$  in the periodic normal form (5) has been computed. The bifurcation is nondegenerate if  $c \neq 0$ .

### 3.2 Generalized period-doubling bifurcation

The two-dimensional critical center manifold  $W^c(\Gamma)$  at the GPD bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(26) \quad u = u_0 + \xi v + H(\tau, \xi), \quad \tau \in [0, 2T], \quad \xi \in \mathbb{R},$$

where the function  $H$  satisfies  $H(2T, \xi) = H(0, \xi)$ . It has the Taylor expansion

$$(27) \quad H(\tau, \xi) = \frac{1}{2}h_2\xi^2 + \frac{1}{6}h_3\xi^3 + \frac{1}{24}h_4\xi^4 + \frac{1}{120}h_5\xi^5 + O(\xi^6),$$

with  $h_j(2T) = h_j(0)$ , with

$$(28) \quad \begin{cases} \dot{v} - A(\tau)v &= 0, \quad \tau \in [0, T], \\ v(T) + v(0) &= 0, \\ \int_0^T \langle v, v \rangle d\tau - 1 &= 0, \end{cases}$$

and

$$v(\tau + T) = -v(\tau) \text{ for } \tau \in [0, T].$$

The function  $v$  exists due to Lemma 5 of [23].

The functions  $h_i$ ,  $i = 1 \dots 5$ , can be found by solving appropriate BVPs, assuming that (2) restricted to  $W^c(\Gamma)$  has the periodic normal form (6). From (26) and (27) it follows that  $h_i(\tau + T) = h_i(\tau)$  for  $i$  even and  $h_i(\tau + T) = -h_i(\tau)$  for  $i$  odd, for  $\tau \in [0, T]$ . Indeed, since we are at a generalized period-doubling point  $u(\tau, \xi) = u(\tau + T, -\xi)$ , so

$$\sum_i \frac{1}{i!} h_i(\tau) \xi^i = \sum_i \frac{1}{i!} h_i(\tau + T) (-1)^i \xi^i,$$

and thus

$$h_i(\tau) = (-1)^i h_i(\tau + T),$$

from which the stated follows. This makes it possible to restrict our considerations to the interval  $[0, T]$  instead of  $[0, 2T]$ .

The coefficients  $\alpha_1$ ,  $\alpha_2$  and  $e$  arise from the solvability conditions for the BVPs as integrals of scalar products over the interval  $[0, T]$ . Specifically, these scalar products involve among other things the terms

up to the fifth order of (1) near the periodic solution  $u_0$ , the eigenfunction  $v$ , the adjoint eigenfunction  $\varphi^*$  satisfying

$$(29) \quad \begin{cases} \dot{\varphi}^* + A^T(\tau)\varphi^* &= 0, \quad \tau \in [0, T], \\ \varphi^*(T) - \varphi^*(0) &= 0, \\ \int_0^T \langle \varphi^*, F(u_0) \rangle d\tau - 1 &= 0, \end{cases}$$

and a similar adjoint eigenfunction  $v^*$  satisfying

$$(30) \quad \begin{cases} \dot{v}^* + A^T(\tau)v^* &= 0, \quad \tau \in [0, T], \\ v^*(T) - v^*(0) &= 0, \\ \int_0^T \langle v^*, v \rangle d\tau - 1 &= 0. \end{cases}$$

To derive the normal form coefficient, we proceed as in Section 3.1, namely, we substitute (26) into (2) and use (3), as well as (6) and (27).

Collecting the  $\xi^0$ -terms in the resulting equation gives us the identity

$$\dot{u}_0 = F(u_0),$$

since  $u_0$  is the  $T$ -periodic solution of (2).

The  $\xi^1$ -terms provide another identity

$$\dot{v} = A(\tau)v,$$

due to (28) and (3.2).

By collecting the  $\xi^2$ -terms, we obtain the equation for  $h_2$ ,

$$(31) \quad \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2\alpha_1 \dot{u}_0,$$

to be solved in the space of functions satisfying  $h_2(T) = h_2(0)$ . In this space, the differential operator  $\frac{d}{d\tau} - A(\tau)$  is singular with null-function  $\dot{u}_0$ . Thus, the following Fredholm solvability condition is involved

$$\int_0^T \langle \varphi^*, B(\tau; v, v) - 2\alpha_1 \dot{u}_0 \rangle d\tau = 0,$$

which leads to the expression

$$(32) \quad \alpha_1 = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v, v) \rangle d\tau,$$

where  $v$  and  $\varphi^*$  are defined by (28) and (29), respectively.

With  $\alpha_1$  defined in this way, let  $h_2$  be a solution of (31) in the space of functions satisfying  $h_2(0) = h_2(T)$ . Notice also here that if  $h_2$  is a solution of (31), then also  $h_2 + \varepsilon_1 F(u_0)$  satisfies (31), since  $F(u_0)$  is in the kernel of the operator  $\frac{d}{d\tau} - A(\tau)$ . In order to obtain a unique solution (without projection on the null eigenspace) we impose the following orthogonality condition which determines the value of  $\varepsilon_1$

$$\int_0^T \langle \varphi^*, h_2 \rangle d\tau = 0.$$

Thus  $h_2$  is the unique solution of the BVP

$$(33) \quad \begin{cases} \dot{h}_2 - A(\tau)h_2 - B(\tau; v, v) + 2\alpha_1 F(u_0) &= 0, \quad \tau \in [0, T], \\ h_2(T) - h_2(0) &= 0, \\ \int_0^T \langle \varphi^*, h_2 \rangle d\tau &= 0. \end{cases}$$

Collecting the  $\xi^3$ -terms, we get the equation for  $h_3$ ,

$$(34) \quad \dot{h}_3 - A(\tau)h_3 = C(\tau; v, v, v) + 3B(\tau; v, h_2) - 6\alpha_1 \dot{v},$$

to be solved in the space of functions satisfying  $h_3(T) = -h_3(0)$ . In this space the differential operator  $\frac{d}{d\tau} - A(\tau)$  has a one-dimensional nullspace, spanned by  $v$ , and (34) is solvable only if the RHS of this equation lies in the reachable space of that operator. Using (28), we can rewrite the right-hand side as

$$C(\tau; v, v, v) + 3B(\tau; v, h_2) - 6\alpha_1 A(\tau)v.$$

But the Fredholm solvability condition

$$(35) \quad \int_0^T \langle v^*, C(\tau; v, v, v) + 3B(\tau; v, h_2) - 6\alpha_1 A(\tau)v \rangle d\tau = 0$$

is trivially satisfied due to the fact that we are in a generalized period-doubling point and so the cubic coefficient of the normal form

$$c = \frac{1}{6} \int_0^T \langle v^*, C(\tau; v, v, v) + 3B(\tau; v, h_2) - 6\alpha_1 A(\tau)v \rangle d\tau$$

(for it's definition see [30]) vanishes. Since the RHS of (34) is in the range space of the operator  $\frac{d}{d\tau} - A(\tau)$ , we can solve the equation in order to find  $h_3$  as the unique solution of the BVP

$$(36) \quad \begin{cases} \dot{h}_3 - A(\tau)h_3 - C(\tau; v, v, v) - 3B(\tau; v, h_2) + 6\alpha_1 A(\tau)v &= 0, \tau \in [0, T], \\ h_3(T) + h_3(0) &= 0, \\ \int_0^T \langle v^*, h_3 \rangle d\tau &= 0. \end{cases}$$

By collecting the  $\xi^4$ -terms, we get the equation for  $h_4$ ,

$$\begin{aligned} \dot{h}_4 - A(\tau)h_4 &= D(\tau; v, v, v, v) + 6C(\tau; v, v, h_2) + 3B(\tau; h_2, h_2) \\ &\quad + 4B(\tau; v, h_3) - 12\alpha_1 \dot{h}_2 - 24\alpha_2 \dot{u}_0, \end{aligned}$$

to be solved in the space of functions satisfying  $h_4(T) = h_4(0)$ . Formulation of the Fredholm solvability condition

$$\int_0^T \langle \varphi^*, D(\tau; v, v, v, v) + 6C(\tau; v, v, h_2) + 3B(\tau; h_2, h_2) + 4B(\tau; v, h_3) - 12\alpha_1 \dot{h}_2 - 24\alpha_2 \dot{u}_0 \rangle d\tau = 0$$

gives us an equation for the parameter  $\alpha_2$

$$\alpha_2 = \frac{1}{24} \int_0^T \langle \varphi^*, D(\tau; v, v, v, v) + 6C(\tau; v, v, h_2) + 3B(\tau; h_2, h_2) + 4B(\tau; v, h_3) - 12\alpha_1 \dot{h}_2 \rangle d\tau$$

which can be simplified considering (31) into

$$\begin{aligned} \alpha_2 = \frac{1}{24} \int_0^T \langle \varphi^*, D(\tau; v, v, v, v) + 6C(\tau; v, v, h_2) + 3B(\tau; h_2, h_2) + \\ 4B(\tau; v, h_3) - 12\alpha_1 (A(\tau)h_2 + B(\tau; v, v)) \rangle d\tau + \alpha_1^2, \end{aligned}$$

where  $\alpha_1$  is given by (32), and  $h_2$ ,  $h_3$ ,  $v$  and  $\varphi^*$  are the solutions of the BVPs (33), (36), (28) and (29), respectively.

Using this value of  $\alpha_2$  we can find  $h_4$  by

$$(37) \quad \begin{cases} \dot{h}_4 - A(\tau)h_4 - D(\tau; v, v, v, v) - 6C(\tau; v, v, h_2) - 3B(\tau; h_2, h_2) - \\ 4B(\tau; v, h_3) + 12\alpha_1 (A(\tau)h_2 + B(\tau; v, v) - 2\alpha_1 F(u_0)) + 24\alpha_2 F(u_0) &= 0, \tau \in [0, T], \\ h_4(T) - h_4(0) &= 0, \\ \int_0^T \langle \varphi^*, h_4 \rangle d\tau &= 0. \end{cases}$$

Finally, by collecting the  $\xi^5$ -terms, we get the equation for  $h_5$ ,

$$\begin{aligned} \dot{h}_5 - A(\tau)h_5 = & E(\tau; v, v, v, v, v) + 10D(\tau; v, v, v, h_2) + 15C(\tau; v, h_2, h_2) \\ & + 10C(\tau; v, v, h_3) + 10B(\tau; h_2, h_3) + 5B(\tau; v, h_4) - 120\alpha_2\dot{v} - 20\alpha_1\dot{h}_3 - 120ev, \end{aligned}$$

which has to be solved in the space of functions satisfying  $h_5(T) = -h_5(0)$ . Since the operator  $\frac{d}{d\tau} - A(\tau)$  has a one-dimensional null-space, we can write

$$\begin{aligned} \int_0^T \langle v^*, E(\tau; v, v, v, v, v) + 10D(\tau; v, v, v, h_2) + 15C(\tau; v, h_2, h_2) + 10C(\tau; v, v, h_3) + \\ 10B(\tau; h_2, h_3) + 5B(\tau; v, h_4) - 120\alpha_2\dot{v} - 20\alpha_1\dot{h}_3 - 120ev \rangle d\tau = 0, \end{aligned}$$

which makes it possible to compute the parameter  $e$  of the normal form (6). Using the normalization of (30), (36) and (35) gives

$$(38) \quad e = \frac{1}{120} \int_0^T \langle v^*, E(\tau; v, v, v, v, v) + 10D(\tau; v, v, v, h_2) + 15C(\tau; v, h_2, h_2) + \\ 10C(\tau; v, v, h_3) + 10B(\tau; h_2, h_3) + 5B(\tau; v, h_4) - 120\alpha_2A(\tau)v - 20\alpha_1A(\tau)h_3 \rangle d\tau.$$

A check that this quantity doesn't vanish, guarantees us that the codim-2 bifurcation is non-degenerate.

### 3.3 Chenciner bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the CH bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(39) \quad u = u_0 + \xi v + \bar{\xi} \bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad \tau \in [0, T], \quad \xi \in \mathbb{C},$$

where the real function  $H$  satisfies  $H(T, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi})$ , and has the Taylor expansion

$$(40) \quad \begin{aligned} H(\tau, \xi, \bar{\xi}) = & \frac{1}{2}h_{20}\xi^2 + h_{11}\xi\bar{\xi} + \frac{1}{2}h_{02}\bar{\xi}^2 \\ & + \frac{1}{6}h_{30}\xi^3 + \frac{1}{2}h_{21}\xi^2\bar{\xi} + \frac{1}{2}h_{12}\xi\bar{\xi}^2 + \frac{1}{6}h_{03}\bar{\xi}^3 \\ & + \frac{1}{24}h_{40}(\tau)\xi^4 + \frac{1}{6}h_{31}(\tau)\xi^3\bar{\xi} + \frac{1}{4}h_{22}(\tau)\xi^2\bar{\xi}^2 + \frac{1}{6}h_{13}(\tau)\xi\bar{\xi}^3 + \frac{1}{24}h_{04}(\tau)\bar{\xi}^4 \\ & + \frac{1}{120}h_{50}(\tau)\xi^5 + \frac{1}{24}h_{41}(\tau)\xi^4\bar{\xi} + \frac{1}{12}h_{32}(\tau)\xi^3\bar{\xi}^2 + \frac{1}{12}h_{23}(\tau)\xi^2\bar{\xi}^3 + \frac{1}{24}h_{14}(\tau)\xi\bar{\xi}^4 \\ & + \frac{1}{120}h_{05}(\tau)\bar{\xi}^5 + O(|\xi|^6), \end{aligned}$$

with  $h_{ij}(T) = h_{ij}(0)$  and  $h_{ij} = \bar{h}_{ji}$  so that  $h_{ii}$  is real, while  $v$  and its conjugate  $\bar{v}$  are defined as

$$(41) \quad \begin{cases} \dot{v}(\tau) - A(\tau)v + i\omega v = 0, & \tau \in [0, T], \\ v(T) - v(0) = 0, \\ \int_0^T \langle v, v \rangle d\tau - 1 = 0. \end{cases}$$

These functions exist due to Lemma 2 of [23].

If we assume that (2) restricted to  $W^c(\Gamma)$  has the periodic normal form (7), as in the previous cases, we can find the functions  $h_{ij}(\tau)$  by solving appropriate BVPs.

First we introduce the two needed adjoint eigenfunctions. The first one, namely  $\varphi^*$ , satisfies (29), and the second one, namely  $v^*$ , satisfies

$$(42) \quad \begin{cases} \dot{v}^*(\tau) + A^T(\tau)v^* + i\omega v^* = 0, & \tau \in [0, T], \\ v^*(T) - v^*(0) = 0, \\ \int_0^T \langle v^*, v \rangle d\tau - 1 = 0. \end{cases}$$

As usual, we substitute (39) into (2), use (3), (7), and (40), as well as the homological equation

$$\frac{du}{dt} = \frac{\partial u}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial u}{\partial \bar{\xi}} \frac{d\bar{\xi}}{dt} + \frac{\partial u}{\partial \tau} \frac{d\tau}{dt},$$

and collect the corresponding terms in order to find the needed coefficients of (7).

The  $\xi$ -independent and the linear terms give rise to the usual identities

$$\dot{u}_0 = F(u_0), \quad \dot{v} - A(\tau)v + i\omega v = 0, \quad \dot{\bar{v}} - A(\tau)\bar{v} - i\omega\bar{v} = 0.$$

Collecting the coefficients of the  $\xi^2$ - or  $\bar{\xi}^2$ -terms leads to the equation

$$\dot{h}_{20} - A(\tau)h_{20} + 2i\omega h_{20} = B(\tau; v, v)$$

or its complex-conjugate. This equation has a unique solution  $h_{20}$  satisfying  $h_{20}(T) = h_{20}(0)$ , since due to the spectral assumptions  $e^{2i\omega T}$  is not a multiplier of the critical cycle. Thus,  $h_{20}$  can be found by solving

$$(43) \quad \begin{cases} \dot{h}_{20} - A(\tau)h_{20} + 2i\omega h_{20} - B(\tau; v, v) &= 0, \quad \tau \in [0, T], \\ h_{20}(T) - h_{20}(0) &= 0. \end{cases}$$

By collecting the  $\xi\bar{\xi}$ -terms we obtain an equation for  $h_{11}$ , namely

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - \alpha_1 \dot{u}_0,$$

to be solved in the space of the functions satisfying  $h_{11}(T) = h_{11}(0)$ . In this space the operator  $\frac{d}{d\tau} - A(\tau)$  has a range space with codimension one. As before, the null-eigenfunction of the adjoint operator  $-\frac{d}{d\tau} - A^T(\tau)$  is  $\varphi^*$ , given by (29), and thus because of the Fredholm solvability condition, we can easily obtain the needed value for  $\alpha_1$

$$(44) \quad \alpha_1 = \int_0^T \langle \varphi^*, B(\tau; v, \bar{v}) \rangle d\tau.$$

With  $\alpha_1$  defined in this way, let  $h_{11}$  be the unique solution of the BVP

$$(45) \quad \begin{cases} \dot{h}_{11} - A(\tau)h_{11} - B(\tau; v, \bar{v}) + \alpha_1 \dot{u}_0 &= 0, \quad \tau \in [0, T], \\ h_{11}(T) - h_{11}(0) &= 0, \\ \int_0^T \langle \varphi^*, h_{11} \rangle d\tau &= 0. \end{cases}$$

The coefficient in front of the third order terms in (7) is purely imaginary since the first Lyapunov coefficient vanishes at a Chenciner point. We are now ready to compute this coefficient. In fact, if we collect the  $\xi^2\bar{\xi}$ -terms we obtain

$$\dot{h}_{21} - A(\tau)h_{21} + i\omega h_{21} = C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2icv - 2\alpha_1 \dot{v},$$

to be solved in the space of functions satisfying  $h_{21}(T) = h_{21}(0)$ . In this space the operator  $\frac{d}{d\tau} - A(\tau) + i\omega$  is singular, since  $e^{i\omega T}$  is a multiplier of the critical cycle. So we can impose the usual Fredholm solvability condition, taking (42) into account

$$(46) \quad \int_0^T \langle v^*, C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2icv - 2\alpha_1 \dot{v} \rangle d\tau = 0.$$

From this equation we can find the value of the coefficient  $c$  of the normal form (7)

$$(47) \quad c = -\frac{i}{2} \int_0^T \langle v^*, C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2\alpha_1 A(\tau)v \rangle d\tau + \alpha_1 \omega$$

and, with  $c$  defined in this way, we can find  $h_{21}$  as the unique solution of the BVP

$$(48) \quad \begin{cases} \dot{h}_{21} - A(\tau)h_{21} + i\omega h_{21} - C(\tau; v, v, \bar{v}) - 2B(\tau; v, h_{11}) \\ \quad - B(\tau; \bar{v}, h_{20}) + 2icv + 2\alpha_1(A(\tau)v - i\omega v) = 0, \tau \in [0, T], \\ h_{21}(T) - h_{21}(0) = 0, \\ \int_0^T \langle v^*, h_{21} \rangle d\tau = 0. \end{cases}$$

Collecting the  $\xi^3$ -terms gives us an equation for  $h_{30}$

$$\dot{h}_{30} - A(\tau)h_{30} + 3i\omega h_{30} = C(\tau; v, v, v) + 3B(\tau; v, h_{20}),$$

which has a unique solution  $h_{30}$  satisfying  $h_{30}(T) = h_{30}(0)$ , since  $e^{3i\omega T}$  is not a multiplier of the critical cycle by the spectral assumptions. Thus,  $h_{30}$  is the unique solution of the BVP

$$(49) \quad \begin{cases} \dot{h}_{30} - A(\tau)h_{30} + 3i\omega h_{30} - C(\tau; v, v, v) - 3B(\tau; v, h_{20}) = 0, \tau \in [0, T], \\ h_{30}(T) - h_{30}(0) = 0. \end{cases}$$

By collecting the  $\xi^3\bar{\xi}$ -terms we obtain an equation for  $h_{31}$

$$\begin{aligned} \dot{h}_{31} - A(\tau)h_{31} + 2i\omega h_{31} &= D(\tau; v, v, v, \bar{v}) + 3C(\tau; v, v, h_{11}) + 3C(\tau; v, \bar{v}, h_{20}) + 3B(\tau; h_{11}, h_{20}) \\ &\quad + 3B(\tau; v, h_{21}) + B(\tau; \bar{v}, h_{30}) - 6ich_{20} - 3\alpha_1\dot{h}_{20} \end{aligned}$$

which has a unique solution  $h_{31}$  satisfying  $h_{31}(T) = h_{31}(0)$ , since  $e^{2i\omega T}$  is not a multiplier of the critical cycle by the spectral assumptions. Thus,  $h_{31}$  is the unique solution of the BVP

$$(50) \quad \begin{cases} \dot{h}_{31} - A(\tau)h_{31} + 2i\omega h_{31} - D(\tau; v, v, v, \bar{v}) - 3C(\tau; v, v, h_{11}) \\ \quad - 3C(\tau; v, \bar{v}, h_{20}) - 3B(\tau; h_{11}, h_{20}) - 3B(\tau; v, h_{21}) \\ \quad - B(\tau; \bar{v}, h_{30}) + 6ich_{20} + 3\alpha_1(A(\tau)h_{20} - 2i\omega h_{20} + B(\tau; v, v)) = 0, \tau \in [0, T], \\ h_{31}(T) - h_{31}(0) = 0. \end{cases}$$

Taking into account the  $|\xi|^4$ -terms gives an equation for  $h_{22}$

$$\begin{aligned} \dot{h}_{22} - A(\tau)h_{22} &= D(\tau; v, v, \bar{v}, \bar{v}) + C(\tau; v, v, h_{02}) \\ &\quad + 4C(\tau; v, \bar{v}, h_{11}) + C(\tau; \bar{v}, \bar{v}, h_{20}) + 2B(\tau; h_{11}, h_{11}) + 2B(\tau; v, h_{12}) \\ &\quad + B(\tau; h_{02}, h_{20}) + 2B(\tau; \bar{v}, h_{21}) - 4\alpha_1\dot{h}_{11} - 4\alpha_2\dot{u}_0, \end{aligned}$$

to be solved in the space of functions satisfying  $h_{22}(T) = h_{22}(0)$ . In this space the operator  $\frac{d}{d\tau} - A(\tau)$  has a range space with codimension one which is orthogonal to  $\varphi^*$ . So one Fredholm solvability condition is involved, namely

$$\begin{aligned} \int_0^T \langle \varphi^*, D(\tau; v, v, \bar{v}, \bar{v}) + C(\tau; v, v, h_{02}) + 4C(\tau; v, \bar{v}, h_{11}) + C(\tau; \bar{v}, \bar{v}, h_{20}) + 2B(\tau; h_{11}, h_{11}) \\ + 2B(\tau; v, h_{12}) + B(\tau; h_{02}, h_{20}) + 2B(\tau; \bar{v}, h_{21}) - 4\alpha_1\dot{h}_{11} - 4\alpha_2\dot{u}_0 \rangle d\tau = 0, \end{aligned}$$

which allows us to compute the value of the coefficient  $\alpha_2$  of our normal form

$$(51) \quad \begin{aligned} \alpha_2 = \frac{1}{4} \int_0^T \langle \varphi^*, D(\tau; v, v, \bar{v}, \bar{v}) + C(\tau; v, v, h_{02}) + 4C(\tau; v, \bar{v}, h_{11}) \\ + C(\tau; \bar{v}, \bar{v}, h_{20}) + 2B(\tau; h_{11}, h_{11}) + 2B(\tau; v, h_{12}) + B(\tau; h_{02}, h_{20}) \\ + 2B(\tau; \bar{v}, h_{21}) - 4\alpha_1(A(\tau)h_{11} + B(\tau; v, \bar{v})) \rangle d\tau + \alpha_1^2. \end{aligned}$$

Using this value for  $\alpha_2$  we can find  $h_{22}$  as the unique solution of the BVP

$$(52) \quad \begin{cases} \dot{h}_{22} - A(\tau)h_{22} - D(\tau; v, v, \bar{v}, \bar{v}) - C(\tau; v, v, h_{02}) \\ -4C(\tau; v, \bar{v}, h_{11}) - C(\tau; \bar{v}, \bar{v}, h_{20}) - 2B(\tau; h_{11}, h_{11}) - 2B(\tau; v, h_{12}) \\ -B(\tau; h_{02}, h_{20}) - 2B(\tau; \bar{v}, h_{21}) \\ +4\alpha_1(A(\tau)h_{11} + B(\tau; v, \bar{v}) - \alpha_1 F(u_0)) + 4\alpha_2 F(u_0) = 0, \quad \tau \in [0, T], \\ h_{22}(T) - h_{22}(0) = 0, \\ \int_0^T \langle \varphi^*, h_{22} \rangle d\tau = 0. \end{cases}$$

Finally, by collecting the  $\xi^3 \bar{\xi}^2$ -terms we get an equation for  $h_{32}$

$$\begin{aligned} \dot{h}_{32} - A(\tau)h_{32} + i\omega h_{32} = & E(\tau; v, v, v, \bar{v}, \bar{v}) + D(\tau; v, v, v, h_{02}) + 6D(\tau; v, v, \bar{v}, h_{11}) \\ & + 3D(\tau; v, \bar{v}, \bar{v}, h_{20}) + 6C(\tau; v, h_{11}, h_{11}) + 3C(\tau; v, v, h_{12}) + 3C(\tau; v, h_{02}, h_{20}) \\ & + 6C(\tau; \bar{v}, h_{11}, h_{20}) + 6C(\tau; v, \bar{v}, h_{21}) + C(\tau; \bar{v}, \bar{v}, h_{30}) + 3B(\tau; h_{12}, h_{20}) + 6B(\tau; h_{11}, h_{21}) \\ & + 3B(\tau; v, h_{22}) + B(\tau; h_{02}, h_{30}) + 2B(\tau; \bar{v}, h_{31}) \\ & - 12ev - 6ich_{21} - 12\alpha_2 \dot{v} - 6\alpha_1 \dot{h}_{31} \end{aligned}$$

that, since the operator is singular, allows us, using the first of (41) as well as the first and the last of (48) and (46), to compute the critical coefficient  $e$  of (7) imposing the Fredholm solvability condition, obtaining

$$(53) \quad e = \frac{1}{12} \int_0^T \langle v^*, \begin{aligned} & E(\tau; v, v, v, \bar{v}, \bar{v}) + D(\tau; v, v, v, h_{02}) + 6D(\tau; v, v, \bar{v}, h_{11}) \\ & + 3D(\tau; v, \bar{v}, \bar{v}, h_{20}) + 6C(\tau; v, h_{11}, h_{11}) + 3C(\tau; v, v, h_{12}) \\ & + 3C(\tau; v, h_{02}, h_{20}) + 6C(\tau; \bar{v}, h_{11}, h_{20}) + 6C(\tau; v, \bar{v}, h_{21}) + C(\tau; \bar{v}, \bar{v}, h_{30}) \\ & + 3B(\tau; h_{12}, h_{20}) + 6B(\tau; h_{11}, h_{21}) + 3B(\tau; v, h_{22}) + B(\tau; h_{02}, h_{30}) \\ & + 2B(\tau; \bar{v}, h_{31}) - 12\alpha_2 A(\tau)v - 6\alpha_1(A(\tau)h_{21} + 2B(\tau; v, h_{11}) + C(\tau; v, v, \bar{v}) \\ & + B(\tau; \bar{v}, h_{20}) - 2\alpha_1 Av) \rangle d\tau + i\omega\alpha_2 + ic\alpha_1 - \alpha_1^2 i\omega. \end{aligned} \rangle$$

As stated in Appendix B, we define the second Lyapunov coefficient as

$$L_2(0) = \Re(e).$$

If this coefficient does not vanish, no more degeneracies happen at this codim 2 point.

Since we have to check all equations up to the fifth order, we still have to look at the  $\xi^4$ -terms, the  $\xi^5$ -terms and the  $\xi^4 \bar{\xi}$ -terms, which give respectively

$$\begin{aligned} \dot{h}_{40} - A(\tau)h_{40} + 4i\omega h_{40} = & D(\tau; v, v, v, v) + 6C(\tau; v, v, h_{20}) + 3B(\tau; h_{20}, h_{20}) \\ & B(\tau; v, h_{30}), \end{aligned}$$

$$\begin{aligned} \dot{h}_{50} - A(\tau)h_{50} + 5i\omega h_{50} = & E(\tau; v, v, v, v, v) + 10D(\tau; v, v, v, h_{20}) + 10C(\tau; v, v, h_{30}) \\ & + 15C(\tau; v, h_{20}, h_{20}) + 10B(\tau; h_{20}, h_{30}) + 5B(\tau; v, h_{40}) \end{aligned}$$

and

$$\begin{aligned} \dot{h}_{41} - A(\tau)h_{41} + 3i\omega h_{41} = & E(\tau; v, v, v, v, \bar{v}) + 6D(\tau; v, v, \bar{v}, h_{20}) + 4D(\tau; v, v, v, h_{11}) \\ & + 4C(\tau; v, \bar{v}, h_{30}) + 8C(\tau; v, h_{20}, h_{11}) + 6C(\tau; v, v, h_{21}) \\ & + C(\tau; \bar{v}, h_{20}, h_{20}) + 6B(\tau; h_{20}, h_{21}) + B(\tau; \bar{v}, h_{40}) \\ & + 4B(\tau; v, h_{31}) + 4B(\tau; h_{30}, h_{11}) - 3\alpha_1 \dot{h}_{30} - 12ich_{30}. \end{aligned}$$

No solvability conditions have to be satisfied.

Since we are in a complex eigenvalues case,  $v$  is determined up to a factor  $\gamma$ , for which  $\gamma^H \gamma = 1$ . Then  $v^*, h_{20}, h_{21}, h_{30}, h_{31}$  are replaced by  $\gamma v^*, \gamma^2 h_{20}, \gamma h_{21}, \gamma^3 h_{30}, \gamma^2 h_{31}$  respectively, but  $\alpha_1, \alpha_2, c$  and  $e$  are not affected by this factor.

### 3.4 Strong resonance 1:1 bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the R1 bifurcation can be parameterized locally by  $(\tau, \xi)$  as

$$(54) \quad u = u_0 + \xi_1 v_1 + \xi_2 v_2 + H(\tau, \xi), \quad \tau \in [0, T], \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where  $H$  satisfies  $H(T, \xi) = H(0, \xi)$  and has the Taylor expansion

$$(55) \quad H(\tau, \xi) = \frac{1}{2} h_{20} \xi_1^2 + h_{11} \xi_1 \xi_2 + \frac{1}{2} h_{02} \xi_2^2 + O(\|\xi\|^3),$$

where the functions  $h_{20}$ ,  $h_{11}$  and  $h_{02}$  are  $T$ -periodic in  $\tau$ , where  $v_1$  and  $v_2$  are the generalized eigenfunctions associated with the trivial multiplier and defined as the unique solutions of the BVP

$$(56) \quad \begin{cases} \dot{v}_1 - A(\tau)v_1 - F(u_0) &= 0, \quad \tau \in [0, T], \\ v_1(T) - v_1(0) &= 0, \\ \int_0^T \langle v_1, F(u_0) \rangle d\tau &= 0, \end{cases}$$

and

$$(57) \quad \begin{cases} \dot{v}_2 - A(\tau)v_2 + v_1 &= 0, \quad \tau \in [0, T], \\ v_2(T) - v_2(0) &= 0, \\ \int_0^T \langle v_2, F(u_0) \rangle d\tau &= 0, \end{cases}$$

respectively. The functions  $v_1$  and  $v_2$  exist and are different due to Lemma 2 of [23]. Following our approach to find the value of the normal form constants, we define  $\varphi^*$  as a solution of the adjoint eigenfunction problem (16),  $v_1^*$  as a solution of the adjoint generalized eigenfunction problem (17) and  $v_2^*$  as a solution of

$$\begin{cases} \dot{v}_2^*(\tau) + A^T(\tau)v_2^* + v_1^* &= 0, \quad \tau \in [0, T], \\ v_2^*(T) - v_2^*(0) &= 0. \end{cases}$$

First, notice that the Fredholm solvability condition gives us immediately the following scalar products

$$(58) \quad \int_0^T \langle \varphi^*, F(u_0) \rangle d\tau = \int_0^T \langle \varphi^*, v_1 \rangle d\tau = \int_0^T \langle F(u_0), v_1^* \rangle d\tau = 0.$$

Due to the spectral assumptions at the R1 point we are free to assume that

$$(59) \quad \int_0^T \langle \varphi^*, v_2 \rangle d\tau = 1.$$

Appending this condition to the eigenproblem, we can find the eigenfunctions  $\varphi^*$  as the unique solutions of the BVP

$$(60) \quad \begin{cases} \dot{\varphi}^* + A^T(\tau)\varphi^* &= 0, \quad \tau \in [0, T], \\ \varphi^*(T) - \varphi^*(0) &= 0, \\ \int_0^T \langle \varphi^*, v_2 \rangle d\tau - 1 &= 0. \end{cases}$$

As already mentioned in the cusp of cycles case, we will choose adjoint generalized eigenfunctions orthogonal to an original eigenfunction. Therefore,  $v_1^*$  and  $v_2^*$  are obtained as the solution of

$$(61) \quad \begin{cases} \dot{v}_1^* + A^T(\tau)v_1^* - \varphi^* &= 0, \quad \tau \in [0, T], \\ v_1^*(T) - v_1^*(0) &= 0, \\ \int_0^T \langle v_1^*, v_2 \rangle d\tau &= 0, \end{cases}$$



and

$$(62) \quad \begin{cases} \dot{v}_2^*(\tau) + A^T(\tau)v_2^* + v_1^* &= 0, \tau \in [0, T], \\ v_2^*(T) - v_2^*(0) &= 0, \\ \int_0^T \langle v_2^*, v_2 \rangle d\tau &= 0, \end{cases}$$

respectively. Notice that, as in the cusp of cycles case, we have normalized in (59) the adjoint eigenfunction with the last generalized eigenfunction, which gives us in addition

$$\int_0^T \langle v_1^*, v_1 \rangle d\tau = \int_0^T \langle v_2^*, F(u_0) \rangle d\tau = 1.$$

As usual, to derive the value of the normal form coefficients we substitute (54) into (2), we use (3) as well as (8) and (55) and get different equalities for every degree of  $\xi$ . Remark that in fact the solvability of all the equations up to the maximal order of the normal form has to be checked. We will pay extra attention to it in this section.

By collecting the  $\xi^0$ -terms we get the identity

$$\dot{u}_0 = F(u_0).$$

The linear terms provide two other identities, namely

$$\dot{v}_1 - A(\tau)v_1 - F(u_0) = 0, \quad \dot{v}_2 - Av_2 + v_1 = 0,$$

cf. (56) and (57).

By collecting the  $\xi_1^2$ -terms we find an equation for  $h_{20}$

$$(63) \quad \dot{h}_{20} - A(\tau)h_{20} = -2\alpha\dot{u}_0 + 2\dot{v}_1 + B(\tau; v_1, v_1) - 2av_2,$$

to be solved in the space of periodic functions on  $[0, T]$ . In this space, the differential operator  $\frac{d}{d\tau} - A(\tau)$  is singular with a range orthogonal to  $\varphi^*$ . Thus a Fredholm solvability condition is involved, namely

$$\int_0^T \langle \varphi^*, -2\alpha\dot{u}_0 + 2\dot{v}_1 + B(\tau; v_1, v_1) - 2av_2 \rangle d\tau = 0.$$

The equations (58), (59), and (56) let us obtain the following value for  $a$

$$(64) \quad a = \frac{1}{2} \int_0^T \langle \varphi^*, 2A(\tau)v_1 + B(\tau; v_1, v_1) \rangle d\tau.$$

Notice that in the RHS of (63) we have no freedom which could change the value of the coefficient  $a$ . This confirms the theoretically proved fact that the  $\xi_1^2$ -term of normal form (8) is resonant. Notice moreover that parameter  $\alpha$  is undetermined, which gives us two degrees of freedom for  $h_{20}$ . In fact, if  $h_{20}$  is a solution of (63), then also  $\tilde{h}_{20} = h_{20} + \varepsilon_{20}^I F(u_0) + \varepsilon_{20}^{II} v_1$  is a solution, due to the fact that  $F(u_0)$  spans the nullspace of the operator  $\frac{d}{dt} - A(\tau)$  and that we can tune  $\alpha$  as desirable:

$$(65) \quad \frac{d\tilde{h}_{20}}{dt} - A(\tau)\tilde{h}_{20} = \frac{dh_{20}}{dt} - A(\tau)h_{20} + \varepsilon_{20}^{II} \left( \frac{dv_1}{dt} - A(\tau)v_1 \right) = \frac{dh_{20}}{dt} - A(\tau)h_{20} + \varepsilon_{20}^{II} \dot{u}_0.$$

By collecting the  $\xi_1\xi_2$ -terms we find an equation for  $h_{11}$

$$(66) \quad \dot{h}_{11} - A(\tau)h_{11} = B(\tau; v_1, v_2) + \dot{v}_2 - h_{20} - bv_2 - v_1,$$

to be solved in the space of  $T$ -periodic functions. As in the previous case, a solvability condition is involved

$$\int_0^T \langle \varphi^*, B(\tau; v_1, v_2) + \dot{v}_2 - h_{20} - bv_2 - v_1 \rangle d\tau = 0.$$

Equation (59) as well as (57) and (58) let us rewrite this condition as

$$b = \int_0^T \langle \varphi^*, B(\tau; v_1, v_2) + A(\tau)v_2 \rangle d\tau - \int_0^T \langle \varphi^*, h_{20} \rangle d\tau.$$

Using (61), (63), (58) and (61) we can rewrite the second term of the right-hand side as

$$\begin{aligned} \int_0^T \langle \left( \frac{d}{d\tau} + A^T(\tau) \right) v_1^*, h_{20} \rangle d\tau &= \int_0^T \langle v_1^*, \left( -\frac{d}{d\tau} + A(\tau) \right) h_{20} \rangle d\tau \\ &= - \int_0^T \langle v_1^*, -2\alpha \dot{u}_0 + 2\dot{v}_1 + B(\tau; v_1, v_1) - 2av_2 \rangle d\tau = - \int_0^T \langle v_1^*, 2Av_1 + B(\tau; v_1, v_1) \rangle d\tau \end{aligned}$$

obtaining an equation for  $b$  which involves only the original and adjoint eigenfunctions

$$(67) \quad b = \int_0^T \langle \varphi^*, B(\tau; v_1, v_2) + A(\tau)v_2 \rangle d\tau + \int_0^T \langle v_1^*, 2Av_1 + B(\tau; v_1, v_1) \rangle d\tau.$$

Notice that the freedom that we have on  $h_{20}$  can not be used to change the value to coefficient  $b$  (and so the  $\xi_1\xi_2$ -term of the normal form (8) is resonant). Indeed,  $h_{20}$  is defined up to a multiple of  $F(u_0)$  and  $v_1$ , but both vectors are orthogonal to  $\varphi^*$ , see the first two orthogonality conditions in (58). However the presence of  $h_{20}$  in the RHS gives us three degrees of freedom for  $h_{11}$ . In fact, if  $h_{11}$  is a solution of (66), also  $\tilde{h}_{11} = h_{11} + \varepsilon_{11}^I F(u_0) - \varepsilon_{20}^I v_1 + \varepsilon_{20}^{II} v_2$  is a solution, since

$$\begin{aligned} \frac{d\tilde{h}_{11}}{dt} - A(\tau)\tilde{h}_{11} &= \frac{dh_{11}}{dt} - A(\tau)h_{11} - \varepsilon_{20}^I \left( \frac{dv_1}{dt} - A(\tau)v_1 \right) + \varepsilon_{20}^{II} \left( \frac{dv_2}{dt} - A(\tau)v_2 \right) \\ &= \frac{dh_{11}}{dt} - A(\tau)h_{11} - \varepsilon_{20}^I F(u_0) - \varepsilon_{20}^{II} v_1. \end{aligned}$$

Collecting the  $\xi_2^2$ -terms gives us the following equation for  $h_{02}$

$$\dot{h}_{02} - A(\tau)h_{02} = B(\tau, v_2, v_2) - 2h_{11},$$

to be solved in the space of  $T$ -periodic functions. This equation should be solvable, so the RHS should lay in the reachable space of the operator  $\frac{d}{dt} - A(\tau)$ :

$$\int_0^T \langle \varphi^*, B(\tau, v_2, v_2) - 2h_{11} \rangle d\tau = 0.$$

This condition can be satisfied by correctly tuning  $h_{11}$ . In fact,  $\varepsilon_{20}^{II}$  is not yet determined, so  $h_{11}$  can have a projection on  $v_2$ . Due to (59)  $v_2$  does not lay in the reachable space of the  $\frac{d}{dt} - A(\tau)$  operator, and therefore we can impose that

$$\int_0^T \langle \varphi^*, h_{11} \rangle d\tau = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau, v_2, v_2) \rangle d\tau.$$

This last solvability condition determines  $\varepsilon_{20}^{II}$  uniquely, and since  $\varepsilon_{20}^{II}$  determines the value of  $\alpha$ , see (63) and (65), also  $\alpha$  is now uniquely determined. So the center manifold expansion (8) has now become unique. Note that in fact the value of  $\alpha$  is not needed since, as shown in Appendix B, it does not affect the bifurcation scenario. Remark also that in order to compute the necessary coefficients  $a$  and  $b$  by equations (64) and (67), the second order expansion of the center manifold is not needed. Indeed, we have rewritten the formulas of the normal form coefficients in terms of the original and adjoint eigenfunctions.  $h_{20}$  or  $h_{11}$  are not needed, therefore we don't write down the BVPs for their unique solutions.

### 3.5 Strong resonance 1:2 bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the R2 bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(68) \quad u = u_0 + \xi_1 v_1 + \xi_2 v_2 + H(\tau, \xi), \quad \tau \in [0, 2T], \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where  $H$  satisfies  $H(2T, \xi) = H(0, \xi)$  and has the Taylor expansion

$$(69) \quad H(\tau, \xi) = \frac{1}{2}h_{20}\xi_1^2 + h_{11}\xi_1\xi_2 + \frac{1}{2}h_{02}\xi_2^2 + \frac{1}{6}h_{30}\xi_1^3 + \frac{1}{2}h_{21}\xi_1^2\xi_2 + \frac{1}{2}h_{12}\xi_1\xi_2^2 + \frac{1}{6}h_{03}\xi_2^3 + O(\|\xi\|^4),$$

where all functions  $h_{ij}$  are  $2T$ -periodic, the eigenfunction corresponding to eigenvalue  $-1$  is given by

$$(70) \quad \begin{cases} \dot{v}_1 - A(\tau)v_1 &= 0, \quad \tau \in [0, T], \\ v_1(T) + v_1(0) &= 0, \\ \int_0^T \langle v_1, v_1 \rangle d\tau - 1 &= 0, \end{cases}$$

and the generalized eigenfunction by

$$(71) \quad \begin{cases} \dot{v}_2 - A(\tau)v_2 + v_1 &= 0, \quad \tau \in [0, T], \\ v_2(T) + v_2(0) &= 0, \\ \int_0^T \langle v_2, v_1 \rangle d\tau &= 0, \end{cases}$$

with

$$(72) \quad v_1(\tau + T) := -v_1(\tau) \text{ and } v_2(\tau + T) := -v_2(\tau) \text{ for } \tau \in [0, T].$$

The functions  $v_1$  and  $v_2$  exist due to Lemma 5 of [23]. The functions  $h_{ij}$  of (69) can be found by solving appropriate BVPs, assuming that (2) restricted to  $W^c(\Gamma)$  has normal form (9). As in the generalized period-doubling case, we first deduce a property for these functions  $h_{ij}$ . More general than in the GPD case, we here have that  $u(\tau, \xi_1, \xi_2) = u(\tau + T, -\xi_1, -\xi_2)$ . This implies that

$$\sum_{i,j} \frac{1}{i!j!} h_{ij}(\tau) \xi_1^i \xi_2^j = \sum_{i,j} \frac{1}{i!j!} h_{ij}(\tau + T) (-1)^{i+j} \xi_1^i \xi_2^j,$$

and thus

$$h_{ij}(\tau) = (-1)^{i+j} h_{ij}(\tau + T),$$

from which follows that  $h_{ij}(\tau + T) = h_{ij}(\tau)$  for  $i + j$  even and  $h_{ij}(\tau + T) = -h_{ij}(\tau)$  for  $i + j$  odd, for  $\tau \in [0, T]$ . Taking these periodicity properties into account, we can reduce our observations to the discussion of the interval  $[0, T]$  instead of  $[0, 2T]$ .

The coefficients  $\alpha$ ,  $a$  and  $b$  arise from the solvability conditions for the BVPs as integrals of scalar products over the interval  $[0, T]$ . Specifically, those scalar products involve among other things the quadratic and cubic terms of (3) near the periodic solution  $u_0$ , the eigenfunction  $v_1$ . The adjoint eigenfunction  $\varphi^*$  associated to the trivial multiplier is the  $T$ -periodic solution of (29). The adjoint eigenfunction  $v_1^*$  is the unique solution of the problem

$$(73) \quad \begin{cases} \dot{v}_1^*(\tau) + A^T(\tau)v_1^* &= 0, \quad \tau \in [0, T], \\ v_1^*(T) + v_1^*(0) &= 0 \\ \int_0^T \langle v_1^*, v_2 \rangle d\tau - 1 &= 0. \end{cases}$$

Note that we can indeed require this normalization since  $v_2$  is the last generalized eigenfunction of the original problem and therefore not orthogonal to all the eigenfunctions of the adjoint problem. We further define the generalized adjoint eigenfunction  $v_2^*$  as the unique solution of

$$(74) \quad \begin{cases} \dot{v}_2^*(\tau) + A^T(\tau)v_2^* - v_1^* &= 0, \quad \tau \in [0, T], \\ v_2^*(T) + v_2^*(0) &= 0 \\ \int_0^T \langle v_2, v_2^* \rangle d\tau &= 0, \end{cases}$$

since, as above,  $v_1^*$  is not orthogonal to  $v_2$ . Moreover, we have

$$\begin{aligned} \int_0^T \langle v_2^*, v_1 \rangle d\tau &= - \int_0^T \langle v_2^*, \left( \frac{d}{d\tau} - A(\tau) \right) v_2 \rangle d\tau \\ &= \int_0^T \langle \left( \frac{d}{d\tau} + A^T(\tau) \right) v_2^*, v_2 \rangle d\tau = \int_0^T \langle v_1^*, v_2 \rangle d\tau = 1. \end{aligned}$$

Note that

$$(75) \quad \int_0^T \langle v_2, v_1 \rangle d\tau = \int_0^T \langle v_1^*, v_1 \rangle d\tau = \int_0^T \langle v_2^*, v_2 \rangle d\tau = 0.$$

To derive the normal form coefficients, we proceed as in the previous sections, namely, we substitute (68) into (2), and use (3) as well as (9) and (69).

By collecting the  $\xi^0$ -terms we get the identity

$$\dot{u}_0 = F(u_0).$$

The linear terms provide two other identities, namely

$$\dot{v}_1 = A(\tau)v_1, \quad v_1 + \dot{v}_2 = A(\tau)v_2,$$

in correspondance with (70) and (71).

Collecting the  $\xi_2^2$ -terms gives us an equation for  $h_{02}$

$$\dot{h}_{02} - A(\tau)h_{02} = B(\tau; v_2, v_2) - 2h_{11},$$

to be solved in the space of functions satisfying  $h_{02}(T) = h_{02}(0)$ . In this space, the differential operator  $\frac{d}{d\tau} - A(\tau)$  is singular and its null-space is spanned by  $\dot{u}_0$ . The Fredholm solvability condition

$$\int_0^T \langle \varphi^*, B(\tau; v_2, v_2) - 2h_{11} \rangle d\tau = 0$$

gives us a normalization condition for function  $h_{11}$ , i.e.

$$\int_0^T \langle \varphi^*, h_{11} \rangle d\tau = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v_2, v_2) \rangle d\tau.$$

By collecting the  $\xi_1\xi_2$ -terms we obtain the differential equation for  $h_{11}$

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v_1, v_2) - h_{20},$$

which must be solved in the space of functions satisfying  $h_{11}(T) = h_{11}(0)$ . The orthogonality condition

$$\int_0^T \langle \varphi^*, B(\tau; v_1, v_2) - h_{20} \rangle d\tau = 0$$

gives us a normalization condition for  $h_{20}$ , i.e.

$$(76) \quad \int_0^T \langle \varphi^*, h_{20} \rangle d\tau = \int_0^T \langle \varphi^*, B(\tau; v_1, v_2) \rangle d\tau.$$

By collecting the  $\xi_1^2$ -terms we find an equation for  $h_{20}$

$$(77) \quad \dot{h}_{20} - A(\tau)h_{20} = B(\tau; v_1, v_1) - 2\alpha\dot{u}_0,$$

to be solved in the space of functions satisfying  $h_{20}(T) = h_{20}(0)$ . In this space, the differential operator  $\frac{d}{d\tau} - A(\tau)$  is singular and its null-space is spanned by  $\dot{u}_0$ . The Fredholm solvability condition

$$\int_0^T \langle \varphi^*, B(\tau; v_1, v_1) - 2\alpha\dot{u}_0 \rangle d\tau = 0$$

leads to the expression

$$(78) \quad \alpha = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v_1, v_1) \rangle d\tau,$$

where  $v_1$  is defined in (70).

With  $\alpha$  defined in this way we have to find a normalization condition which makes the solution of (77) unique. Indeed, if  $h_{20}$  is a solution of (77) with  $h_{20}(T) = h_{20}(0)$ , also  $\tilde{h}_{20} = h_{20} + \varepsilon_1 \dot{u}_0$  is a solution, since  $\dot{u}_0$  spans the kernel of the operator  $\frac{d}{dt} - A(\tau)$  in the space of  $T$ -periodic functions. The projection along the space generated by  $\dot{u}_0$  is fixed by solvability condition (76). So  $h_{20}$  can be found as the unique solution of the BVP

$$(79) \quad \begin{cases} \dot{h}_{20} - A(\tau)h_{20} - B(\tau; v_1, v_1) + 2\alpha F(u_0) &= 0, \tau \in [0, T], \\ h_{20}(T) - h_{20}(0) &= 0, \\ \int_0^T \langle \varphi^*, h_{20} \rangle d\tau &= \int_0^T \langle \varphi^*, B(\tau; v_1, v_2) \rangle d\tau. \end{cases}$$

In the line of the previous observations, we can define  $h_{11}$  as the unique solution of the BVP

$$(80) \quad \begin{cases} \dot{h}_{11} - A(\tau)h_{11} - B(\tau; v_1, v_2) + h_{20} &= 0, \tau \in [0, T] \\ h_{11}(T) - h_{11}(0) &= 0, \\ \int_0^T \langle \varphi^*, h_{11} \rangle d\tau &= \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v_2, v_2) \rangle d\tau, \end{cases}$$

with  $h_{20}$  defined in (79).

By collecting the  $\xi_1^3$ -terms we get an equation for  $h_{30}$

$$(81) \quad \dot{h}_{30} - A(\tau)h_{30} = C(\tau; v_1, v_1, v_1) + 3B(\tau; v_1, h_{20}) - 6\alpha v_2 - 6\alpha \dot{v}_1,$$

which again must be solved in the space of functions satisfying  $h_{30}(T) = -h_{30}(0)$ . Taking the integral condition of (73) into account, we obtain

$$(82) \quad a = \frac{1}{6} \int_0^T \langle v_1^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; v_1, h_{20}) - 6\alpha A(\tau)v_1 \rangle d\tau,$$

where  $\alpha$  is defined by (78),  $h_{20}$  is the solution of (79) and  $v_1$  and  $v_1^*$  are defined in (70) and (71), respectively. As remarked before, it is important to note that if  $h_{30}$  is a solution of (81) with  $h_{30}(T) = h_{30}(0)$ , also  $\tilde{h}_{30} = h_{30} + \epsilon_{30}^I v_1$  is a solution, since  $v_1$  spans the nullspace of the operator  $\frac{d}{dt} - A(\tau)$ .

Collecting the  $\xi_1^2 \xi_2$ -terms we get the equation for  $h_{21}$

$$(83) \quad \dot{h}_{21} - A(\tau)h_{21} = -h_{30} - 2bv_2 - 2\alpha \dot{v}_2 - 2\alpha v_1 + C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1),$$

to be solved in the space of functions satisfying  $h_{21}(T) = -h_{21}(0)$ . The solvability of this equation implies

$$\int_0^T \langle v_1^*, -h_{30} - 2bv_2 - 2\alpha v_2 - 2\alpha v_1 + C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) \rangle d\tau = 0.$$

Notice that the  $b\xi_1^2\xi_2$ -term in normal form (9) is resonant: in fact we cannot use the freedom on  $h_{30}$  to make the normal form parameter  $b$  zero since

$$\int_0^T \langle v_1^*, \tilde{h}_{30} \rangle d\tau = \int_0^T \langle v_1^*, h_{30} + \varepsilon_{30}^I v_1 \rangle d\tau = \int_0^T \langle v_1^*, h_{30} \rangle d\tau.$$

Using the normalization from (73) and (75) gives us the following expression for  $b$

$$b = \frac{1}{2} \int_0^T \langle v_1^*, -2\alpha A(\tau)v_2 + C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) \rangle d\tau - \frac{1}{2} \int_0^T \langle v_1^*, h_{30} \rangle d\tau.$$

There is no need to compute explicitly the cubic expansion of the center manifold since the last term of this sum can be rewritten as

$$\begin{aligned} \int_0^T \left\langle \left( \frac{d}{d\tau} + A^T(\tau) \right) v_2^*, h_{30} \right\rangle d\tau &= \int_0^T \langle v_2^*, \left( -\frac{d}{d\tau} + A(\tau) \right) h_{30} \rangle d\tau \\ &= - \int_0^T \langle v_2^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; v_1, h_{20}) - 6\alpha v_2 - 6\alpha A v_1 \rangle d\tau \\ &= - \int_0^T \langle v_2^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; v_1, h_{20}) - 6\alpha A v_1 \rangle d\tau, \end{aligned}$$

obtaining

$$\begin{aligned} (84) \quad b &= \frac{1}{2} \int_0^T \langle v_1^*, -2\alpha A(\tau)v_2 + C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) \rangle d\tau \\ &\quad + \frac{1}{2} \int_0^T \langle v_2^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; v_1, h_{20}) - 6\alpha A v_1 \rangle d\tau, \end{aligned}$$

where  $h_{20}$  is defined in (79) and  $\alpha$  calculated in (78). Notice that, since  $h_{30}$  appears on the RHS of equation (83), we have two degrees of freedom on  $h_{21}$ . In fact, if  $h_{21}$  is a solution of (83), also  $\tilde{h}_{21} = h_{21} + \varepsilon_{21}^I v_1 + \varepsilon_{30}^I v_2$  is a solution since

$$\frac{d\tilde{h}_{21}}{dt} - A(\tau)\tilde{h}_{21} = \frac{dh_{21}}{dt} - A(\tau)h_{21} + \varepsilon_{30}^I \left( \frac{dv_2}{dt} - A(\tau)v_2 \right) = \frac{dh_{21}}{dt} - A(\tau)h_{21} - \varepsilon_{30}^I v_1.$$

By collecting the  $\xi_1\xi_2^2$ -terms we get the equation for  $h_{12}$

$$\dot{h}_{12} - A(\tau)h_{12} = C(\tau, v_1, v_2, v_2) + B(\tau, v_1, h_{02}) + 2B(\tau, v_2, h_{11}) - 2h_{21},$$

to be solved in the space of functions satisfying  $h_{12}(T) = -h_{12}(0)$ . The Fredholm solvability condition implies that

$$\int_0^T \langle v_1^*, C(\tau, v_1, v_2, v_2) + B(\tau, v_1, h_{02}) + 2B(\tau, v_2, h_{11}) - 2h_{21} \rangle d\tau = 0.$$

As mentioned before,  $h_{21}$  has a component in the direction of  $v_2$ , which is not orthogonal to the adjoint eigenfunction  $v_1^*$ , so it is possible to impose

$$\int_0^T \langle v_1^*, h_{21} \rangle d\tau = \frac{1}{2} \int_0^T \langle v_1^*, C(\tau, v_1, v_2, v_2) + B(\tau, v_1, h_{02}) + 2B(\tau, v_2, h_{11}) \rangle d\tau.$$

This condition defines  $\varepsilon_{30}^I$  uniquely; the freedom of  $\varepsilon_{21}^I$  gives us as usual another freedom on  $h_{12}$  in the direction of  $v_2$ .

Finally, collecting the  $\xi_2^3$ -terms gives

$$\dot{h}_{03} - A(\tau)h_{03} = C(\tau, v_2, v_2, v_2) + 3B(v_2, h_{02}) - 3h_{12},$$

to be solved in the space of functions satisfying  $h_{03}(T) = -h_{03}(0)$ . The Fredholm solvability condition is

$$\int_0^T \langle v_1^*, C(\tau, v_2, v_2, v_2) + 3B(v_2, h_{02}) - 3h_{12} \rangle d\tau = 0,$$

which can be satisfied imposing

$$\int_0^T \langle v_1^*, h_{12} \rangle d\tau = \frac{1}{3} \int_0^T \langle v_1^*, C(\tau, v_2, v_2, v_2) + 3B(v_2, h_{02}) \rangle d\tau.$$

This last condition determines the value of  $\varepsilon_{21}^I$  and thus the third order center manifold expansion is uniquely determined. However, since this third order expansion of the center manifold is not needed for the computation of the critical coefficients, we don't write down those conditions.

### 3.6 Strong resonance 1:3 bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the R3 bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(85) \quad u = u_0 + \xi v + \bar{\xi} \bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad \tau \in [0, 3T], \quad \xi \in \mathbb{C},$$

where the real function  $H$  satisfies  $H(3T, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi})$  and has the Taylor expansion

$$(86) \quad \begin{aligned} H(\tau, \xi, \bar{\xi}) &= \frac{1}{2}h_{20}\xi^2 + h_{11}\xi\bar{\xi} + \frac{1}{2}h_{02}\bar{\xi}^2 + \frac{1}{6}h_{30}\xi^3 + \frac{1}{2}h_{21}\xi^2\bar{\xi} \\ &+ \frac{1}{2}h_{12}\xi\bar{\xi}^2 + \frac{1}{6}h_{03}\bar{\xi}^3 + O(|\xi|^4), \end{aligned}$$

with  $h_{ij}(3T) = h_{ij}(0)$  and  $h_{ij} = \bar{h}_{ji}$  so that  $h_{ii}$  is real. The eigenfunction  $v$  is defined as the unique solution of the BVP

$$(87) \quad \begin{cases} \dot{v}(\tau) - A(\tau)v &= 0, \quad \tau \in [0, T], \\ v(T) - e^{i\frac{2\pi}{3}}v(0) &= 0, \\ \int_0^T \langle v, v \rangle d\tau - 1 &= 0, \end{cases}$$

and extended on the interval  $[0, 3T]$  using the equivariance property of the normal form, i.e.

$$v(\tau + T) := e^{i\frac{2\pi}{3}}v(\tau) \text{ and } v(\tau + 2T) := e^{i\frac{4\pi}{3}}v(\tau) \text{ for } \tau \in [0, T].$$

The definition of the conjugate eigenfunction  $\bar{v}$  follows immediately. These functions exist due to Lemma 2 of [23].

As usual the functions  $h_{ij}$  can be found by solving appropriate BVPs, assuming that (2) restricted to  $W^c(\Gamma)$  has the periodic normal form (10). Also here we can deduce a property for the functions  $h_{ij}$ . The definition of  $v(\tau)$  in  $[0, 3T]$  states that  $u(\tau, \xi, \bar{\xi}) = u(\tau + T, e^{-i2\pi/3}\xi, e^{i2\pi/3}\bar{\xi})$ . Therefore,

$$\sum_{k,l} \frac{1}{k!l!} h_{kl}(\tau) \xi^k \bar{\xi}^l = \sum_{k,l} \frac{1}{k!l!} h_{kl}(\tau + T) (e^{-i2\pi/3})^k \xi^k (e^{i2\pi/3})^l \bar{\xi}^l,$$

and thus

$$h_{kl}(\tau) = h_{kl}(\tau + T) (e^{-i2\pi/3})^k (e^{i2\pi/3})^l,$$

for  $\tau \in [0, T]$ . This for example implies that  $h_{kk}$  is  $T$ -periodic. These periodicity properties allow us to just concentrate on the interval  $[0, T]$

The adjoint eigenfunction  $\varphi^*$  corresponding to the trivial multiplier is the unique  $T$ -periodic solution of BVP (29). The adjoint eigenfunction  $v^*$  satisfies

$$(88) \quad \begin{cases} \dot{v}^*(\tau) + A^T(\tau)v^* &= 0, \quad \tau \in [0, T], \\ v^*(T) - e^{i\frac{2\pi}{3}}v^*(0) &= 0, \\ \int_0^T \langle v^*, v \rangle d\tau - 1 &= 0. \end{cases}$$

Similarly, we obtain  $\bar{v}^*$ .

We now write down the homological equation and compare term by term. The constant and linear terms give us as usual

$$\dot{u}_0 = F(u_0), \quad \dot{v} - A(\tau)v = 0, \quad \dot{\bar{v}} - A(\tau)\bar{v} = 0.$$

From the  $\xi^2$ - or  $\bar{\xi}^2$ -terms we obtain the following equation (or its complex conjugate)

$$\dot{h}_{20} - A(\tau)h_{20} = B(\tau; v, v) - 2\bar{b}\bar{v},$$

to be solved in the space of functions satisfying  $h_{20}(T) = e^{i\frac{4\pi}{3}}h_{20}(0)$ . In this space the operator  $\frac{d}{d\tau} - A(\tau)$  has a range space with codimension one which is orthogonal to  $\bar{v}^*$ . So one Fredholm solvability condition is involved, namely

$$\int_0^T \langle \bar{v}^*, B(\tau; v, v) - 2\bar{b}\bar{v} \rangle d\tau = 0,$$

which makes it possible to obtain the value of the coefficient  $b$ . In fact,

$$(89) \quad b = \frac{1}{2} \int_0^T \langle v^*, B(\tau; \bar{v}, \bar{v}) \rangle d\tau.$$

Using this value for  $b$  we can find  $h_{20}$  as the unique solution of the BVP

$$(90) \quad \begin{cases} \dot{h}_{20} - A(\tau)h_{20} - B(\tau; v, v) + 2\bar{b}\bar{v} &= 0, \quad \tau \in [0, T], \\ h_{20}(T) - e^{i\frac{4\pi}{3}}h_{20}(0) &= 0, \\ \int_0^T \langle \bar{v}^*, h_{20} \rangle d\tau &= 0. \end{cases}$$

By collecting the  $\xi\bar{\xi}$ -terms we obtain an equation for  $h_{11}$

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - \alpha_1 \dot{u}_0,$$

to be solved in the space of functions satisfying  $h_{11}(T) = h_{11}(0)$ . The Fredholm solvability condition with  $\varphi^*$  gives us the value of  $\alpha_1$

$$(91) \quad \alpha_1 = \int_0^T \langle \varphi^*, B(\tau; v, \bar{v}) \rangle d\tau.$$

With  $\alpha_1$  defined in this way, let  $h_{11}$  be the unique solution of the BVP

$$(92) \quad \begin{cases} \dot{h}_{11} - A(\tau)h_{11} - B(\tau; v, \bar{v}) + \alpha_1 \dot{u}_0 &= 0, \quad \tau \in [0, T], \\ h_{11}(T) - h_{11}(0) &= 0, \\ \int_0^T \langle \varphi^*, h_{11} \rangle d\tau &= 0. \end{cases}$$

Finally, collecting the  $\xi^2\bar{\xi}$ -terms gives an equation for  $h_{21}$

$$\dot{h}_{21} - A(\tau)h_{21} = C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2cv - 2\bar{b}h_{02} - 2\alpha_1 \dot{v},$$



to be solved in the space of the functions satisfying  $h_{21}(T) = e^{i\frac{2\pi}{3}} h_{21}(0)$ . Therefore, there must hold that

$$\int_0^T \langle v^*, C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2cv - 2\bar{b}h_{02} - 2\alpha_1 \dot{v} \rangle d\tau = 0,$$

such that parameter  $c$  of (10) is determined by

$$c = \frac{1}{2} \int_0^T \langle v^*, C(\tau; v, v, \bar{v}) + 2B(\tau; v, h_{11}) + B(\tau; \bar{v}, h_{20}) - 2\alpha_1 A v \rangle d\tau,$$

where  $\alpha_1$  and  $b$  are defined by (91) and (89), respectively, and  $v$ ,  $h_{11}$  and  $h_{20}$  are the unique solutions of the BVPs (87), (92) and (90).

Since we have to check the solvability of all the equations up to the maximal order of the normal form, we also collect the  $\xi^3$ -terms and obtain

$$\dot{h}_{30} - A(\tau)h_{30} = C(\tau; v, v, v) + 3B(\tau; v, h_{20}) - 6\bar{b}h_{11} - 6\alpha_2 \dot{u}_0,$$

to be solved in the space of the functions satisfying  $h_{30}(T) = h_{30}(0)$ . Therefore, there must hold that

$$\int_0^T \langle \varphi^*, C(\tau; v, v, v) + 3B(\tau; v, h_{20}) - 6\bar{b}h_{11} - 6\alpha_2 \dot{u}_0 \rangle d\tau = 0,$$

which determines the value of  $\alpha_2$ , namely

$$\alpha_2 = \int_0^T \langle \varphi^*, C(\tau; v, v, v) + 3B(\tau; v, h_{20}) \rangle d\tau.$$

Remark that as in the Chenciner case  $v$  is not uniquely determined. Indeed, when  $v$  is a solution of (87) and  $\gamma \in \mathbb{C}$  with  $\gamma^H \gamma = 1$ , then  $\gamma v$  is also a solution. Then the adjoint function is given by  $\gamma v^*$ , and  $b$  and  $h_{20}$  are replaced by  $\bar{\gamma}^3 b$  and  $\gamma^2 h_{20}$ , respectively. The normal form coefficient  $c$  stays the same. However, the normal form coefficient  $b$  is multiplied with  $\bar{\gamma}^3$ . This doesn't affect the bifurcation analysis since this normal form coefficient only has to be different from zero, and obviously  $\gamma \neq 0$ . Moreover, the analysis around the bifurcation point is independent from the sign of  $b$ .

### 3.7 Strong resonance 1:4 bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the R4 bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(93) \quad u = u_0 + \xi v + \bar{\xi} \bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad \tau \in [0, 4T], \quad \xi \in \mathbb{C},$$

where the real function  $H$  satisfies  $H(4T, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi})$  and has the Taylor expansion

$$(94) \quad \begin{aligned} H(\tau, \xi, \bar{\xi}) &= \frac{1}{2} h_{20} \xi^2 + h_{11} \xi \bar{\xi} + \frac{1}{2} h_{02} \bar{\xi}^2 + \frac{1}{6} h_{30} \xi^3 + \frac{1}{2} h_{21} \xi^2 \bar{\xi} \\ &+ \frac{1}{2} h_{12} \xi \bar{\xi}^2 + \frac{1}{6} h_{03} \bar{\xi}^3 + O(|\xi|^4), \end{aligned}$$

with  $h_{ij}(4T) = h_{ij}(0)$  and  $h_{ij} = \bar{h}_{ji}$  so that  $h_{ii}$  is real, while  $v$  is defined by

$$(95) \quad \begin{cases} \dot{v} - A(\tau)v &= 0, \quad \tau \in [0, T], \\ v(T) - e^{i\frac{\pi}{2}} v(0) &= 0, \\ \int_0^T \langle v, v \rangle d\tau - 1 &= 0, \end{cases}$$

extended on  $[0, 4T]$  using the equivariance property of the normal form, i.e.

$$\begin{aligned} v(\tau + T) &:= e^{i\frac{\pi}{2}} v(\tau) = iv(\tau), \\ v(\tau + 2T) &:= e^{i\pi} v(\tau) = -v(\tau), \\ v(\tau + 3T) &:= e^{i\frac{3\pi}{2}} v(\tau) = -iv(\tau), \end{aligned}$$

for  $\tau \in [0, T]$ .

The definition of the conjugate  $\bar{v}$  follows from this. These functions exist due to Lemma 2 of [23]. As usual the functions  $h_{ij}$  can be found by solving appropriate BVPs, assuming that (2) restricted to  $W^c(\Gamma)$  has the periodic normal form (11). Similar to the R1:3 case, there holds that

$$h_{kl}(\tau) = h_{kl}(\tau + T)(e^{-i\pi/2})^k (e^{i\pi/2})^l,$$

for  $\tau \in [0, T]$ .

The adjoint eigenfunction  $\varphi^*$  is defined by the  $T$ -periodic solution of (29) and  $v^*$  satisfies

$$(96) \quad \begin{cases} \dot{v}^*(\tau) + A^T(\tau)v^* &= 0, \quad \tau \in [0, T], \\ v^*(T) - e^{i\frac{\pi}{2}}v^*(0) &= 0, \\ \int_0^T \langle v^*, v \rangle d\tau - 1 &= 0. \end{cases}$$

Similarly, we obtain  $\bar{v}^*$ .

The constant and the linear terms give the identities

$$\dot{u}_0 = F(u_0), \quad \dot{v} - A(\tau)v = 0, \quad \dot{\bar{v}} - A(\tau)\bar{v} = 0.$$

From the  $\xi^2$ - or  $\bar{\xi}^2$ -terms the following equation (or its complex conjugate) is obtained

$$(97) \quad \dot{h}_{20} - A(\tau)h_{20} = B(\tau; v, v).$$

Notice that this equation is non-singular in the space of functions satisfying  $h_{20}(T) = -h_{20}(0)$ . So  $h_{20}$  is given as the unique solution of the BVP

$$(98) \quad \begin{cases} \dot{h}_{20} - A(\tau)h_{20} - B(\tau; v, v) &= 0, \quad \tau \in [0, T], \\ h_{20}(T) + h_{20}(0) &= 0. \end{cases}$$

By collecting the  $\xi\bar{\xi}$ -terms we obtain an equation for  $h_{11}$

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - \alpha_1 \dot{u}_0,$$

to be solved in the space of functions satisfying  $h_{11}(T) = h_{11}(0)$ . The Fredholm solvability condition

$$\int_0^T \langle \varphi^*, B(\tau; v, \bar{v}) - \alpha_1 \dot{u}_0 \rangle d\tau = 0$$

gives us the possibility to obtain the value of  $\alpha_1$ , namely given by (91). With this value of  $\alpha_1$ ,  $h_{11}$  is the unique solution of BVP (92).

The  $\xi\bar{\xi}^2$ -terms give an equation for  $h_{12}$

$$\dot{h}_{12} - A(\tau)h_{12} = C(\tau; v, \bar{v}, \bar{v}) + B(\tau; v, h_{02}) + 2B(\tau; \bar{v}, h_{11}) - 2\bar{c}\bar{v} - 2\alpha_1 \dot{\bar{v}},$$

to be solved in the space of functions satisfying  $h_{12}(T) = -ih_{12}(0)$ . The Fredholm solvability condition leads us to the value of  $\bar{c}$ , namely

$$(99) \quad \bar{c} = \frac{1}{2} \int_0^T \langle \bar{v}^*, C(\tau; v, \bar{v}, \bar{v}) + B(\tau; v, h_{02}) + 2B(\tau; \bar{v}, h_{11}) - 2\alpha_1 A(\tau)\bar{v} \rangle d\tau,$$

where  $\alpha_1$  is defined in (91), and  $v$ ,  $h_{11}$  and  $h_{02}$  are the unique solutions of the BVPs (95), (92) and the complex conjugate of (98). Taking the complex conjugate gives us the critical coefficient  $c$ .

Finally, by collecting the  $\xi^3$ -terms we obtain an equation for  $h_{03}$

$$\dot{h}_{03} - A(\tau)h_{03} = C(\tau; \bar{v}, \bar{v}, \bar{v}) + 3B(\tau; \bar{v}, h_{02}) - 6dv,$$

to be solved in the space of the functions satisfying  $h_{03}(T) = ih_{03}(0)$ . The non-trivial Fredholm solvability condition

$$\int_0^T \langle v^*, C(\tau; \bar{v}, \bar{v}, \bar{v}) + 3B(\tau; \bar{v}, h_{02}) - 6dv \rangle d\tau = 0$$

gives us the value of the critical coefficient  $d$  of (11), namely

$$(100) \quad d = \frac{1}{6} \int_0^T \langle v^*, C(\tau; \bar{v}, \bar{v}, \bar{v}) + 3B(\tau; \bar{v}, h_{02}) \rangle d\tau.$$

So we finally obtain the value of

$$a = \frac{c}{|d|}$$

which makes it possible to understand which bifurcation scenario of the R4 resonance we have.

Also in this case  $v$  is not uniquely determined, since for every  $\gamma \in \mathbb{C}$  with  $\gamma^H \gamma = 1$ ,  $\gamma v$  is a solution. Then the adjoint eigenfunction is given by  $\gamma v^*$ , and  $h_{20}$  is replaced by  $\gamma^2 h_{20}$ . The normal form coefficient  $c$  stays the same, but instead of  $d$  we get  $\bar{\gamma}^4 d$ . However, this again doesn't influence the bifurcation analysis since the study is determined by the above defined  $a$  for which we need only  $|d|$ .

### 3.8 Fold-Flip bifurcation

The three-dimensional critical center manifold  $W^c(\Gamma)$  at the LPPD bifurcation can be parametrized locally by  $(\tau, \xi)$  as

$$(101) \quad u = u_0 + \xi_1 v_1 + \xi_2 v_2 + H(\tau, \xi), \quad \tau \in [0, 2T], \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where  $H$  satisfies  $H(2T, \xi) = H(0, \xi)$  and has the Taylor expansion

$$(102) \quad H(\tau, \xi) = \frac{1}{2} h_{20} \xi_1^2 + h_{11} \xi_1 \xi_2 + \frac{1}{2} h_{02} \xi_2^2 + \frac{1}{6} h_{30} \xi_1^3 + \frac{1}{2} h_{21} \xi_1^2 \xi_2 + \frac{1}{2} h_{12} \xi_1 \xi_2^2 + \frac{1}{6} h_{03} \xi_2^3 + O(\|\xi\|^4),$$

while the eigenfunctions  $v_1$  and  $v_2$  are given by

$$(103) \quad \begin{cases} \dot{v}_1 - A(\tau)v_1 - F(u_0) &= 0, \quad \tau \in [0, T], \\ v_1(T) - v_1(0) &= 0, \\ \int_0^T \langle v_1, F(u_0) \rangle d\tau &= 0, \end{cases}$$

and

$$(104) \quad \begin{cases} \dot{v}_2 - A(\tau)v_2 &= 0, \quad \tau \in [0, T], \\ v_2(T) + v_2(0) &= 0, \\ \int_0^T \langle v_2, v_2 \rangle d\tau - 1 &= 0 \end{cases}$$

with

$$(105) \quad v_1(\tau + T) := v_1(\tau) \text{ and } v_2(\tau + T) := -v_2(\tau) \text{ for } \tau \in [0, T].$$

The functions  $v_1$  and  $v_2$  exist because of Lemma 2 and Lemma 5 of [23]. The functions  $h_{ij}$  can be found by solving appropriate BVPs, assuming that (2) restricted to  $W^c(\Gamma)$  has the periodic normal form (12). Moreover, similar as before,  $u(\tau, \xi_1, \xi_2) = u(\tau + T, \xi_1, -\xi_2)$  such that

$$h_{ij}(\tau) = (-1)^j h_{ij}(\tau + T),$$

for  $\tau \in [0, T]$ . As before, we will reduce all computations to the interval  $[0, T]$ .

The coefficients of the normal form arise from the solvability conditions for the BVPs as integrals of scalar products over the interval  $[0, T]$ . Specifically, those scalar products involve among other things the quadratic and cubic terms of (3) near the periodic solution  $u_0$ , the generalized eigenfunction  $v_1$  and eigenfunction  $v_2$ , and the adjoint eigenfunctions  $\varphi^*$ ,  $v_1^*$  and  $v_2^*$  as solution of the problems

$$(106) \quad \begin{cases} \dot{\varphi}^* + A^T(\tau)\varphi^* &= 0, \tau \in [0, T], \\ \varphi^*(T) - \varphi^*(0) &= 0, \\ \int_0^T \langle \varphi^*, v_1 \rangle d\tau - 1 &= 0, \end{cases}$$

$$(107) \quad \begin{cases} \dot{v}_1^* + A^T(\tau)v_1^* + \varphi^* &= 0, \tau \in [0, T], \\ v_1^*(T) - v_1^*(0) &= 0, \\ \int_0^T \langle v_1^*, v_1 \rangle d\tau &= 0, \end{cases}$$

and

$$(108) \quad \begin{cases} \dot{v}_2^* + A^T(\tau)v_2^* &= 0, \tau \in [0, T], \\ v_2^*(T) - v_2^*(0) &= 0, \\ \int_0^T \langle v_2^*, v_2 \rangle d\tau - 1 &= 0. \end{cases}$$

Note that the integral conditions are possible due to the spectral assumptions at the LPPD point. The following orthogonality conditions hold automatically

$$(109) \quad \begin{aligned} \int_0^T \langle \varphi^*, F(u_0) \rangle d\tau &= \int_0^T \langle \varphi^*, v_2 \rangle d\tau = \int_0^T \langle v_1^*, v_2 \rangle d\tau = \\ &= \int_0^T \langle v_2^*, v_1 \rangle d\tau = \int_0^T \langle v_2^*, F(u_0) \rangle d\tau = 0, \end{aligned}$$

and since we have normalized the adjoint eigenfunction associated to multiplier 1 with the last generalized eigenfunction, we have for free that

$$(110) \quad \begin{aligned} 1 &= \int_0^T \langle \varphi^*, v_1 \rangle d\tau = - \int_0^T \left\langle \left( \frac{d}{d\tau} + A^T(\tau) \right) v_1^*, v_1 \right\rangle d\tau \\ &= \int_0^T \langle v_1^*, \left( \frac{d}{d\tau} - A(\tau) \right) v_1 \rangle d\tau = \int_0^T \langle v_1^*, F(u_0) \rangle d\tau. \end{aligned}$$

As usual, to derive the normal form coefficients we write down the homological equation and compare term by term.

By collecting the constant and linear terms we get the identities

$$\dot{u}_0 = F(u_0), \quad \dot{v}_1 = A(\tau)v_1 + F(u_0), \quad \dot{v}_2 = A(\tau)v_2,$$

and the complex conjugate of the last equation.

By collecting the  $\xi_1^2$ -terms we find an equation for  $h_{20}$

$$(111) \quad \dot{h}_{20} - A(\tau)h_{20} = B(\tau; v_1, v_1) - 2a_{20}v_1 - 2\alpha_{20}\dot{u}_0 + 2\dot{v}_1,$$

to be solved in the space of functions satisfying  $h_{20}(T) = h_{20}(0)$ . In this space, the differential operator  $\frac{d}{d\tau} - A(\tau)$  is singular and its null-space is spanned by  $\dot{u}_0$ . The Fredholm solvability condition

$$\int_0^T \langle \varphi^*, B(\tau; v_1, v_1) - 2a_{20}v_1 - 2\alpha_{20}\dot{u}_0 + 2\dot{v}_1 \rangle d\tau = 0$$

gives us the possibility to calculate parameter  $a_{20}$  of our normal form, i.e.

$$(112) \quad a_{20} = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v_1, v_1) + 2A(\tau)v_1 \rangle d\tau.$$

With  $a_{20}$  tuned in this way equation (111) is solvable, for any value of parameter  $\alpha_{20}$ . As in the Cusp of cycles case, we are free to choose parameter  $\alpha_{20}$  as we want, and we take  $\alpha_{20} = 0$ . This choice will not influence our final conclusion about the kind of situation we are in.

In order to make the solution of (111) unique, we have to fix the projection on the null-space of the operator, more specifically in the direction of  $F(u_0)$ . Therefore we impose the orthogonality with the adjoint generalized eigenfunction  $v_1^*$ , and obtain  $h_{20}$  as the unique solution of the BVP

$$(113) \quad \begin{cases} \dot{h}_{20} - A(\tau)h_{20} - B(\tau; v_1, v_1) + 2a_{20}v_1 + 2\alpha_{20}F(u_0) - 2A(\tau)v_1 - 2F(u_0) &= 0, \quad \tau \in [0, T], \\ h_{20}(T) - h_{20}(0) &= 0, \\ \int_0^T \langle v_1^*, h_{20} \rangle d\tau &= 0. \end{cases}$$

By collecting the  $\xi_1\xi_2$ -terms we obtain a singular equation for  $h_{11}$

$$\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v_1, v_2) - b_{11}v_2 + \dot{v}_2,$$

to be solved in the space of the functions that satisfy  $h_{11}(T) = -h_{11}(0)$ . The Fredholm solvability condition

$$\int_0^T \langle v_2^*, B(\tau; v_1, v_2) - b_{11}v_2 + \dot{v}_2 \rangle d\tau = 0$$

gives us the possibility, using (104) and (108), to calculate coefficient  $b_{11}$  of our normal form

$$(114) \quad b_{11} = \int_0^T \langle v_2^*, B(\tau; v_1, v_2) + A(\tau)v_2 \rangle d\tau.$$

With  $b_{11}$  defined in this way we can compute  $h_{11}$  as the unique solution of the BVP

$$(115) \quad \begin{cases} \dot{h}_{11} - A(\tau)h_{11} - B(\tau; v_1, v_2) + b_{11}v_2 - A(\tau)v_2 &= 0, \quad \tau \in [0, T], \\ h_{11}(T) + h_{11}(0) &= 0, \\ \int_0^T \langle v_2^*, h_{11} \rangle d\tau &= 0. \end{cases}$$

Collecting the  $\xi_2^2$ -terms gives a singular equation for  $h_{02}$

$$(116) \quad \dot{h}_{02} - A(\tau)h_{02} = B(\tau; v_2, v_2) - 2a_{02}v_1 - 2\alpha_{02}\dot{u}_0$$

and, since this equation has to be solvable, the following Fredholm solvability condition is involved

$$\int_0^T \langle \varphi^*, B(\tau; v_2, v_2) - 2a_{02}v_1 - 2\alpha_{02}\dot{u}_0 \rangle d\tau = 0,$$

from which we obtain an equation for  $a_{02}$

$$(117) \quad a_{02} = \frac{1}{2} \int_0^T \langle \varphi^*, B(\tau; v_2, v_2) \rangle d\tau.$$

So (116) is solvable, for any value of the parameter  $\alpha_{02}$ . For simplicity, we take  $\alpha_{02} = 0$ . Notice that also here, the solution of (116) is orthogonal to the adjoint eigenfunction  $\varphi^*$ . Since we have to fix the projection in the direction of eigenfunction  $\dot{u}_0$ , we define  $h_{02}$  as the unique solution of

$$(118) \quad \begin{cases} \dot{h}_{02} - A(\tau)h_{02} - B(\tau; v_2, v_2) + 2a_{02}v_1 + 2\alpha_{02}F(u_0) &= 0, \tau \in [0, T], \\ h_{02}(T) - h_{02}(0) &= 0, \\ \int_0^T \langle v_1^*, h_{02} \rangle d\tau &= 0. \end{cases}$$

By collecting the  $\xi_1^3$ -terms we get a singular equation for  $h_{30}$

$$\dot{h}_{30} - A(\tau)h_{30} = C(\tau; v_1, v_1, v_1) + 3B(\tau; h_{20}, v_1) - 6a_{20}h_{20} - 6a_{30}v_1 + 3\dot{h}_{20} - 6\alpha_{30}\dot{u}_0 - 6\alpha_{20}\dot{v}_1,$$

where the Fredholm solvability condition

$$\int_0^T \langle \varphi^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; h_{20}, v_1) - 6a_{20}h_{20} - 6a_{30}v_1 + 3\dot{h}_{20} - 6\alpha_{30}\dot{u}_0 - 6\alpha_{20}\dot{v}_1 \rangle d\tau = 0$$

gives us the value of  $a_{30}$

$$(119) \quad a_{30} = \frac{1}{6} \int_0^T \langle \varphi^*, C(\tau; v_1, v_1, v_1) + 3B(\tau; h_{20}, v_1) - 6a_{20}h_{20} + 3(A(\tau)h_{20} + B(\tau; v_1, v_1)) + 6(1 - \alpha_{20})A(\tau)v_1 \rangle d\tau - a_{20}.$$

Similarly, by collecting the  $\xi_1^2\xi_2$ -terms we get a singular equation for  $h_{21}$

$$\begin{aligned} \dot{h}_{21} - A(\tau)h_{21} &= C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) - 2a_{20}h_{11} \\ &\quad - 2b_{11}h_{11} - 2b_{21}v_2 + 2\dot{h}_{11} - 2\alpha_{20}\dot{v}_2, \end{aligned}$$

where the Fredholm solvability condition

$$\begin{aligned} \int_0^T \langle v_2^*, C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) \\ - 2a_{20}h_{11} - 2b_{11}h_{11} - 2b_{21}v_2 + 2\dot{h}_{11} - 2\alpha_{20}\dot{v}_2 \rangle d\tau = 0 \end{aligned}$$

produces the value of  $b_{21}$

$$(120) \quad b_{21} = \frac{1}{2} \int_0^T \langle v_2^*, C(\tau; v_1, v_1, v_2) + B(\tau; h_{20}, v_2) + 2B(\tau; h_{11}, v_1) - 2a_{20}h_{11} - 2b_{11}h_{11} + 2(A(\tau)h_{11} + B(\tau; v_1, v_2)) + 2(1 - \alpha_{20})A(\tau)v_2 \rangle d\tau - b_{11}.$$

By collecting the  $\xi_1\xi_2^2$ -terms we obtain a singular equation for  $h_{12}$

$$\begin{aligned} \dot{h}_{12} - A(\tau)h_{12} &= C(\tau; v_1, v_2, v_2) + B(\tau; h_{02}, v_1) + 2B(\tau; h_{11}, v_2) \\ &\quad - 2b_{11}h_{02} - 2a_{02}h_{20} - 2a_{12}v_1 + \dot{h}_{02} - 2\alpha_{12}\dot{u}_0 - 2\alpha_{02}\dot{v}_1, \end{aligned}$$

where its solvability requires the following equality

$$\begin{aligned} \int_0^T \langle \varphi^*, C(\tau; v_1, v_2, v_2) + B(\tau; h_{02}, v_1) + 2B(\tau; h_{11}, v_2) \\ - 2b_{11}h_{02} - 2a_{02}h_{20} - 2a_{12}v_1 + \dot{h}_{02} - 2\alpha_{12}\dot{u}_0 - 2\alpha_{02}\dot{v}_1 \rangle d\tau = 0, \end{aligned}$$

such that

(121)

$$a_{12} = \frac{1}{2} \int_0^T \langle \varphi^*, C(\tau; v_1, v_2, v_2) + B(\tau; h_{02}, v_1) + 2B(\tau; h_{11}, v_2) - 2b_{11}h_{02} - 2a_{02}h_{20} + A(\tau)h_{02} + B(\tau; v_2, v_2) - 2\alpha_{02}A(\tau)v_1 \rangle d\tau - a_{02}.$$

Finally, the  $\xi_2^3$ -terms give us the value of the last needed critical coefficient. The equation for  $h_{03}$  is

$$\dot{h}_{03} - A(\tau)h_{03} = C(\tau; v_2, v_2, v_2) + 3B(\tau; h_{02}, v_2) - 6a_{02}h_{11} - 6b_{03}v_2 - 6\alpha_{02}\dot{v}_2,$$

with Fredholm solvability condition

$$\int_0^T \langle v_2^*, C(\tau; v_2, v_2, v_2) + 3B(\tau; h_{02}, v_2) - 6a_{02}h_{11} - 6b_{03}v_2 - 6\alpha_{02}\dot{v}_2 \rangle d\tau = 0,$$

and thus

$$(122) \quad b_{03} = \frac{1}{6} \int_0^T \langle v_2^*, C(\tau; v_2, v_2, v_2) + 3B(\tau; h_{02}, v_2) - 6a_{02}h_{11} - 6\alpha_{02}A(\tau)v_2 \rangle d\tau.$$

## 4 Implementation issues

Numerical implementation of the formulas derived in the preceding sections requires the evaluation of integrals of scalar functions over  $[0, T]$  and the solution of nonsingular linear BVPs with integral constraints. Such tasks can be carried out with continuation software such as AUTO [14], CONTENT [31], and MATCONT [11, 10]. In these software packages, periodic solutions to (1) are computed with the method of *orthogonal collocation* with piecewise polynomials applied to properly formulated BVPs.

The standard BVP for the periodic solutions is formulated on the unit interval  $[0, 1]$ , so that the period  $T$  becomes a parameter, and it involves an integral phase condition:

$$(123) \quad \begin{cases} \dot{x}(\tau) - Tf(x(\tau), \alpha) &= 0, \quad \tau \in [0, 1], \\ x(0) - x(1) &= 0, \\ \int_0^1 \langle x(\tau), \dot{\xi}(\tau) \rangle d\tau &= 0, \end{cases}$$

where  $\xi$  is a previously calculated periodic solution to a nearby problem, rescaled to  $[0, 1]$ .

In the orthogonal collocation method [4], problem (123) is replaced by the following discretization:

$$(124) \quad \begin{cases} \sum_{j=0}^m x_{i,j} \dot{\ell}_{i,j}(\zeta_{i,k}) - Tf\left(\sum_{j=0}^m x_{i,j} \ell_{i,j}(\zeta_{i,k}), \alpha\right) &= 0, \\ x_{0,0} - x_{N-1,m} &= 0, \\ \sum_{i=0}^{N-1} \sum_{j=0}^{m-1} \sigma_{i,j} \langle x_{i,j}, \dot{\xi}_{i,j} \rangle + \sigma_{N,0} \langle x_{N,0}, \dot{\xi}_{N,0} \rangle &= 0. \end{cases}$$

The points  $x_{i,j}$  form the approximation of  $x(\tau)$  with  $m+1$  equidistant mesh points

$$\tau_{i,j} = \tau_i + \frac{j}{m}(\tau_{i+1} - \tau_i), \quad j = 0, 1, \dots, m,$$

in each of the  $N$  intervals  $[\tau_i, \tau_{i+1}]$ , where

$$0 = \tau_0 < \tau_1 < \dots < \tau_N = 1.$$

The  $\ell_{i,j}(\tau)$ 's are the Lagrange basis polynomials, while the points  $\zeta_{i,j}$  ( $j = 1, \dots, m$ ) are Gauss points [9], i.e. the roots of the Legendre polynomial of degree  $m$ , all relative to the interval  $[\tau_i, \tau_{i+1}]$ .

With this choice of collocation points  $\zeta_{i,j}$ , the approximation error at the mesh points has order of accuracy  $m$ ,

$$\|x(\tau_{i,j}) - x_{i,j}\| = \mathcal{O}(h^m),$$

where  $h = \max_{i=1,2,\dots,N} \{t_i\}$ ,  $t_i = \tau_i - \tau_{i-1}$  ( $i = 1, \dots, N$ ), while for the coarse mesh points  $\tau_i$  the error has order of accuracy  $2m$ ,

$$\|x(\tau_i) - x_{i,0}\| = \mathcal{O}(h^{2m})$$

(“superconvergence”).

The integration weight  $\sigma_{i,j}$  of  $\tau_{i,j}$  is given by  $w_{j+1}t_{i+1}$  for  $0 \leq i \leq N-1$  and  $0 < j < m$ . For  $i = 0, \dots, N-2$ , the integration weight of  $\tau_{i,m}$  ( $\tau_{i,m} = \tau_{i+1,0}$ ) is given by  $\sigma_{i,m} = w_{m+1}t_{i+1} + w_1t_{i+2}$ , and the integration weights of  $\tau_0$  and  $\tau_N$  are given by  $w_1t_1$  and  $w_{m+1}t_N$ , respectively. In the above expressions,  $w_{j+1}$  is the Lagrange quadrature coefficient.

The numerical continuation of solutions of (124) leads to structured, sparse linear systems, which in AUTO [14] and CONTENT [31] are solved by an efficient, specially adapted elimination algorithm that computes the multipliers as a by-product, without explicitly using the Poincaré map. To detect codim 1 bifurcations, one can specify test functions that are based on computing multipliers [13, 14] or on solving appropriate bordered linear BVPs [15].

## 4.1 Discretization symbols

It is convenient to discretize all computed functions using the same mesh as in (124). For a given vector function  $\eta \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$  we consider three different discretizations:

- $\eta_M \in \mathbb{R}^{(Nm+1)n}$ , the vector of the function values at the mesh points;
- $\eta_C \in \mathbb{R}^{Nmn}$ , the vector of the function values at the collocation points;
- $\eta_W = \begin{bmatrix} \eta_{W_1} \\ \eta_{W_2} \end{bmatrix} \in \mathbb{R}^{Nmn} \times \mathbb{R}^n$ , where  $\eta_{W_1}$  is the vector of the function values at the collocation points multiplied by the Gauss-Legendre weights and the lengths of the corresponding mesh intervals, and  $\eta_{W_2} = \eta(0)$ .

Formally we also introduce the structured sparse matrix  $L_{C \times M}$  that converts a vector  $\eta_M$  of function values at the mesh points into a vector  $\eta_C$  of its values at the collocation points, namely,  $\eta_C = L_{C \times M} \eta_M$ . This matrix is never formed explicitly; its entries are approximated by the  $\ell_{i,j}(\zeta_{i,k})$ -coefficients in (124). We also need a matrix  $A_{C \times M}$  such that  $A_{C \times M} \eta_M = (A(t)\eta(t))_C$ . Again this matrix need not be formed explicitly. On the other hand, we do need the matrix  $(D - TA(t))_{C \times M}$  explicitly; it is defined by  $(D - TA(t))_{C \times M} \eta_M = (\dot{\eta}(t) - TA(t)\eta(t))_C$ . Finally, let the tensors  $B_{C \times M \times M}$  and  $C_{C \times M \times M \times M}$  be defined by  $B_{C \times M \times M} \eta_{1M} \eta_{2M} = (B(t; \eta_1(t), \eta_2(t)))_C$  and

$$C_{C \times M \times M \times M} \eta_{1M} \eta_{2M} \eta_{3M} = (C(t; \eta_1(t), \eta_2(t), \eta_3(t)))_C$$

for all  $\eta_i \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ . (These tensors are not formed explicitly.)

Let  $f(t), g(t) \in \mathcal{C}^0([0, 1], \mathbb{R})$  be two scalar functions. Then the integral  $\int_0^1 f(t)dt$  is represented by  $\sum_{i=0}^{N-1} \sum_{j=1}^m \omega_j (f_C)_{i,j} t_{i+1} = \sum_{i=0}^{N-1} \sum_{j=1}^m (f_{W_1})_{i,j}$ , where  $(f_C)_{i,j} = f(\zeta_{i,j})$  and  $\omega_j$  is the Gauss-Legendre quadrature coefficient. The integral  $\int_0^1 f(t)g(t)dt$  is approximated with Gauss-Legendre by  $f_{W_1}^T g_C \approx f_{W_1}^T L_{C \times M} g_M$ , where equality holds if  $g(t)$  is a piecewise polynomial of degree  $m$  or less on the given mesh. For vector functions  $f(t), g(t) \in \mathcal{C}^0([0, 1], \mathbb{R}^n)$ , the integral  $\int_0^1 \langle f(t), g(t) \rangle dt$  is formally approximated by the same expression:  $f_{W_1}^T g_C \approx f_{W_1}^T L_{C \times M} g_M$ , where again we have equality if  $g(t)$  is a piecewise polynomial of degree  $m$  or less on the given mesh. Concerning the accuracy of the quadrature formulas, we first note that accuracy is not an important issue for the phase integral in (123), as this equation only selects a specific solution from the continuum of solutions obtained by phase shifts. Similarly, the discretization of the normalization



integrals does not affect the inherent accuracy, including superconvergence at the main mesh points  $\tau_i$  of the solution of the discretized BVP. Discretization of integrals, as specified above, follows the standard Gauss quadrature error, which has order of accuracy  $2m$  if, as mentioned, the function  $g(t)$  is a piecewise polynomial of degree  $m$  or less on the given mesh and if  $f(t)$  is sufficiently smooth (in a piecewise sense). Otherwise, still assuming sufficient piecewise smoothness, the order of accuracy of the numerical integrals is  $m+1$  if  $m$  is odd, and  $m+2$  if  $m$  is even. In particular, for the often used choice  $m=4$ , the integrals would then have order of accuracy 6.

## 4.2 Cusp of cycles bifurcation

The first task is to rescale the computed functions to the interval  $[0, 1]$ . We start by defining  $u_1(t) = u_0(Tt) = u_0(\tau)$  for  $t \in [0, 1]$ . The linear BVP's (15) and (21) are replaced by

$$(125) \quad \begin{cases} \dot{v}_1(t) - TA(t)v_1(t) - TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ v_1(1) - v_1(0) &= 0, \\ \int_0^1 \langle v_1(t), F(u_1(t)) \rangle dt &= 0, \end{cases}$$

with  $v(\tau) = v_1(\tau/T)$  and

$$\begin{cases} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ , respectively. We also need to rescale the adjoint generalized eigenfunction defined by (22):

$$\begin{cases} \dot{v}_1^*(t) + TA^T(t)v_1^*(t) + T\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ v_1^*(1) - v_1^*(0) &= 0, \\ \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt &= 0, \end{cases}$$

with  $v^*(\tau) = v_1^*(\tau/T)/T$ . Now,  $\alpha_1 = 0$  and  $h_{2,1}$  is the unique solution of the BVP

$$(126) \quad \begin{cases} \dot{h}_{2,1}(t) - TA(t)h_{2,1}(t) - TB(t; v_1(t), v_1(t)) - 2TA(t)v_1(t) - 2TF(u_1(t)) + 2\alpha_1 TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{2,1}(1) - h_{2,1}(0) &= 0, \\ \int_0^1 \langle v_1^*(t), h_{2,1}(t) \rangle dt &= 0, \end{cases}$$

where  $h_2(\tau) = h_{2,1}(\tau/T)$ . Therefore, we obtain

$$(127) \quad c = \frac{1}{6} \int_0^1 \langle \varphi_1^*(t), -6\alpha_1 A(t)v_1(t) + 3A(t)h_{2,1}(t) + 3B(t; v_1(t), v_1(t)) + 6A(t)v_1(t) + 3B(t; h_{2,1}(t), v_1(t)) + C(t; v_1(t), v_1(t)) \rangle dt.$$

We now determine the matrix solutions for the several functions. We compute  $v_{1M}$  by solving the discretization of (125)

$$(128) \quad \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ g_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} v_{1M} \\ a \end{bmatrix} = \begin{bmatrix} Tf_C \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

where  $a$  equals zero since the  $M \times M$  upper left part of the big matrix is singular,  $g(t) = F(u_1(t))$ , and  $p$  is obtained by solving the following system

$$\begin{bmatrix} p^T & b \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & r_1 \\ \delta_0 - \delta_1 & 0 \\ r_2 & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

with  $r_1$  and  $r_2$  random vectors. In the solution of this system  $b = 0$ ; in (128) we then use the normalized  $p$ . This technique guarantees that we are working with well-defined systems.

We will compute  $\varphi_{1W}^*$  instead of  $\varphi_{1M}^*$  since  $\varphi_{1W}^*$  can be computed with a matrix which is very similar to the matrix from (128). Formally, the computation of  $\varphi_{1W}^*$  is based on Proposition C.1 from the appendix since

$$\begin{bmatrix} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) \\ \varphi_1^*(1) - \varphi_1^*(0) \end{bmatrix} = 0,$$

we have that  $\begin{bmatrix} \varphi_1^* \\ \varphi_1^*(0) \end{bmatrix}$  is orthogonal to the range of  $\begin{bmatrix} D - TA(t) \\ \delta_0 - \delta_1 \end{bmatrix}$ . By discretization we obtain

$$(\varphi_1^*)^T_W \begin{bmatrix} (D - TA(t))_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix} = 0.$$

Therefore,  $\varphi_{1W}^*$  can be obtained by solving

$$\begin{bmatrix} (\varphi_1^*)^T_W & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & q^T \\ & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

where  $a$  equals zero and  $q$  is the normalized right null-vector of  $\begin{bmatrix} (D - TA(t))_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix}$ . We then approximate

$I = \int_0^1 \langle \varphi_1^*(t), v_1(t) \rangle dt$  by  $I_1 = (\varphi_1^*)^T_{W_1} L_{C \times M} v_{1M}$ .  $\varphi_{1W}^*$  is then rescaled to ensure that  $I_1 = 1$ .

It is more efficient to compute  $v_{1W}^*$  than  $v_{1M}^*$ , since  $v_1^*$  will be used only to compute integrals of the form  $\int_0^1 \langle v_1^*(t), \zeta(t) \rangle dt$ . Proposition C.5 learns us how to determine  $v_1^*$ . Indeed,

$$\left\langle \begin{bmatrix} v_1^* \\ v_1^*(0) \end{bmatrix}, \begin{bmatrix} \dot{h} - TA h \\ h(0) & -h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} -T\varphi_1^* \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle,$$

for all appropriate functions  $h$ , thus  $v_1^*$  can be obtained by solving

$$\begin{bmatrix} (v_1^*)^T_W & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & v_{1C} \\ \delta_0 - \delta_1 & 0_{n \times 1} \\ q^T & 0 \end{bmatrix} = \begin{bmatrix} (T\varphi_1^*)^T_{W_1} L_{C \times M} & 0 \end{bmatrix},$$

where  $a$  equals zero and  $p$  is defined above.

Next,  $(h_{2,1})_M$  is found by solving the discretization of (126), namely,

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (v_1^*)^T_{W_1} L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{2,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} v_{1M} + 2TA_{C \times M} v_{1M} + 2Tg_C \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with  $a = 0$  and  $p$  defined above.

Finally, (127) is approximated by

$$\begin{aligned} c &= \frac{1}{6} (\varphi_1^*)^T_{W_1} (3A_{C \times M} h_{2,1M} + 3B_{C \times M \times M} v_{1M} v_{1M} \\ &\quad + 6A_{C \times M} v_{1M} + 3B_{C \times M \times M} h_{2,1M} v_{1M} + C_{C \times M \times M \times M} v_{1M} v_{1M} v_{1M}). \end{aligned}$$

### 4.3 Generalized period-doubling bifurcation

As done in Section 4.2, we first rescale the computed quantities to the interval  $[0, 1]$ . The linear BVPs (28) and (29) are replaced by

$$\begin{cases} \dot{v}_1(t) - TA(t)v_1(t) &= 0, \quad t \in [0, 1], \\ v_1(1) + v_1(0) &= 0, \\ \int_0^1 \langle v_1(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $v(\tau) = v_1(\tau/T)/\sqrt{T}$ , and

$$\begin{cases} \varphi_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt - 1 &= 0, \end{cases}$$

with  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ . This leads to the expression

$$\alpha_{1,1} = \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), B(t; v_1(t), v_1(t)) \rangle dt,$$

with  $\alpha_{1,1} = T\alpha_1$ .

The adjoint eigenfunction, defined by (30), determines the following rescaled  $v_1^*$ :

$$(129) \quad \begin{cases} \dot{v}_1^*(t) + TA^T(t)v_1^*(t) &= 0, \quad t \in [0, 1], \\ v_1^*(1) - v_1^*(0) &= 0, \\ \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$ .

Let  $h_{2,1}$  be the unique solution of the BVP

$$\begin{cases} \dot{h}_{2,1}(t) - TA(t)h_{2,1}(t) - TB(t; v_1(t), v_1(t)) + 2\alpha_{1,1}TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{2,1}(1) - h_{2,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{2,1}(t) \rangle dt &= 0, \end{cases}$$

where  $h_2(\tau) = h_{2,1}(\tau/T)/T$ , and  $h_{3,1}$  be the unique solution of the BVP

$$\begin{cases} \dot{h}_{3,1}(t) - TA(t)h_{3,1}(t) - TC(t; v_1(t), v_1(t), v_1(t)) - 3TB(t; v_1(t), h_{2,1}(t)) + 6\alpha_{1,1}TA(t)v_1(t) &= 0, \quad t \in [0, 1], \\ h_{3,1}(1) - h_{3,1}(0) &= 0, \\ \int_0^1 \langle v_1^*(t), h_{3,1}(t) \rangle dt &= 0, \end{cases}$$

where  $h_3(\tau) = h_{3,1}(\tau/T)/(\sqrt{T}T)$ .

Therefore, we obtain

$$\alpha_{2,1} = \frac{1}{24} \int_0^1 \langle \varphi_1^*(t), D(t; v_1(t), v_1(t), v_1(t), v_1(t)) + 6C(t; v_1(t), v_1(t), h_{2,1}(t)) + 3B(t; h_{2,1}(t), h_{2,1}(t)) + 4B(t; v_1(t), h_{3,1}(t)) - 12\alpha_{1,1}(A(t)h_{2,1}(t) + B(t; v_1(t), v_1(t))) \rangle dt + \alpha_{1,1}^2,$$

with  $\alpha_{2,1} = T^2\alpha_2$ .

Now,  $h_{4,1}$  is obtained as unique solution of the following BVP

$$\begin{cases} \dot{h}_{4,1}(t) - TA(t)h_{4,1}(t) - TD(t; v_1(t), v_1(t), v_1(t), v_1(t)) - 6TC(t; v_1(t), v_1(t), h_{2,1}(t)) - 3TB(t; h_{2,1}(t), h_{2,1}(t)) - 4TB(t; v_1(t), h_{3,1}(t)) + 12\alpha_{1,1}T(A(t)h_{2,1}(t) + B(t; v_1(t), v_1(t)) - 2\alpha_{1,1}F(u_1(t))) + 24\alpha_{2,1}TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{4,1}(1) - h_{4,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{4,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_4(\tau) = h_{4,1}(\tau/T)/T^2$ .

Finally, we can write down the critical normal form coefficient

$$\begin{aligned} e = \frac{1}{120T^2} \int_0^1 & \langle v_1^*(t), E(t; v_1(t), v_1(t), v_1(t), v_1(t)) + 10D(t; v_1(t), v_1(t), v_1(t), h_{2,1}(t)) \\ & + 15C(t; v_1(t), h_{2,1}(t), h_{2,1}(t)) + 10C(t; v_1(t), v_1(t), h_{3,1}(t)) + 10B(t; h_{2,1}(t), h_{3,1}(t)) \\ & + 5B(t; v_1(t), h_{4,1}(t)) - 120\alpha_{2,1}A(t)v_1(t) - 20\alpha_{1,1}A(t)h_{3,1}(t) \rangle dt. \end{aligned}$$

We now come to the implementation details in MatCont. We compute  $v_{1M}$  by solving

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & \\ q_1^T & 0 \end{bmatrix} \begin{bmatrix} v_{1M} \\ a_1 \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix},$$

with  $p_1$  and  $q_1$  the rescaled unit vectors of

$$\begin{bmatrix} (D - TA(t))_{C \times M} & r_1 \\ \delta_0 + \delta_1 & \\ r_2^T & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} p_1^T & a_3 \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & r_1 \\ \delta_0 + \delta_1 & \\ r_2^T & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

where  $r_1$  and  $r_2$  are random vectors. Every  $a_i$  is equal to zero. We normalize  $v_{1M}$  by requiring  $\sum_{i=0}^{N-1} \sum_{j=0}^m \sigma_j \langle (v_{1M})_{i,j}, (v_{1M})_{i,j} \rangle = 1$ , where  $\sigma_j$  is the Lagrange quadrature coefficient.

The discretization of (129) can be computed with the same matrix, see Proposition C.2 of the appendix,

$$\begin{bmatrix} (v_1^*)_{W_1}^T & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & \\ q_1^T & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

where  $a = 0$ . We then approximate  $I = \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt$  by  $I_1 = (v_1^*)_{W_1}^T L_{C \times M} v_{1M}$ .  $v_{1W}^*$  is then rescaled to ensure that  $I_1 = 1$ .

An analogous matrix is used to compute  $\varphi_{1W}^*$ :

$$\begin{bmatrix} (\varphi_1^*)_{W_1}^T & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ q^T & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

where  $q$  is the normalized right null-vector and  $p$  the normalized left null-vector of  $\begin{bmatrix} (D - TA(t))_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix}$ , and  $a$  is equal to zero. In what follows we will use these definitions for  $p, q, p_1$  and  $q_1$ . We then approximate  $I = \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt$  by  $I_1 = (\varphi_1^*)_{W_1}^T g_C$  and normalize  $\varphi_{1W}^*$  to ensure that  $I_1 = 1$ .

This then leads to the discretization of the expression for  $\alpha_{1,1}$ :

$$(130) \quad \alpha_{1,1} = \frac{1}{2} (\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{1M} v_{1M}.$$

Now,  $h_{2,1}$  and  $h_{3,1}$  are found by solving the following systems:

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{2,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} v_{1M} - 2\alpha_{1,1} T g_C \\ 0_{n \times 1} \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & \\ (v_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{3,1M} \\ b \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

respectively, with

$$\text{rhs} = TC_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} + 3TB_{C \times M \times M} v_{1M} h_{2,1M} - 6\alpha_{1,1} TA_{C \times M} v_{1M}.$$

$a$  and  $b$  will be zero.

Thus,

$$(131) \quad \alpha_{2,1} = \frac{1}{24} (\varphi_1^*)_{W_1}^T \left( D_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} v_{1M} + 6C_{C \times M \times M \times M} v_{1M} v_{1M} h_{2,1M} \right. \\ \left. + 3B_{C \times M \times M} h_{2,1M} h_{2,1M} + 4B_{C \times M \times M} v_{1M} h_{3,1M} \right. \\ \left. - 12\alpha_{1,1} (A_{C \times M} h_{2,1M} + B_{C \times M \times M} v_{1M} v_{1M}) \right) + \alpha_{1,1}^2.$$

Then,  $h_{4,1}$  is found by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{4,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with

$$\begin{aligned} \text{rhs} &= TD_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} v_{1M} + 6TC_{C \times M \times M \times M} v_{1M} v_{1M} h_{2,1M} \\ &+ 3TB_{C \times M \times M} h_{2,1M} h_{2,1M} + 4TB_{C \times M \times M} v_{1M} h_{3,1M} \\ &- 12\alpha_{1,1} T(A_{C \times M} h_{2,1M} + B_{C \times M \times M} v_{1M} v_{1M} - 2\alpha_{1,1} g_C) - 24\alpha_{2,1} Tg_C \end{aligned}$$

and  $a = 0$ .

Now, we have all ingredients for the computation of the normal form coefficient

$$(132) \quad e = \frac{1}{120T^2} (v_1^*)_{W_1}^T \left( E_{C \times M \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} v_{1M} v_{1M} + 10D_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} h_{2,1M} \right. \\ \left. + 15C_{C \times M \times M \times M} v_{1M} h_{2,1M} h_{2,1M} + 10C_{C \times M \times M \times M} v_{1M} v_{1M} h_{3,1M} + 10B_{C \times M \times M} h_{2,1M} h_{3,1M} \right. \\ \left. + 5B_{C \times M \times M} v_{1M} h_{4,1M} - 120\alpha_{2,1} A_{C \times M} v_{1M} - 20\alpha_{1,1} A_{C \times M} h_{3,1M} \right).$$

#### 4.4 Chenciner bifurcation

As before, we rescale the computed quantities to the interval  $[0, 1]$ . The linear BVPs (41), (29) and (42) are replaced by

$$(133) \quad \begin{cases} \dot{v}_1(t) - TA(t)v_1(t) + i\omega T v_1(t) &= 0, \quad t \in [0, 1], \\ v_1(1) - v_1(0) &= 0, \\ \int_0^1 \langle v_1(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v(\tau) = v_1(\tau/T)/\sqrt{T}$ ,

$$\begin{cases} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt - 1 &= 0, \end{cases}$$

with  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ , and

$$\begin{cases} \dot{v}_1^*(t) + TA^T(t)v_1^*(t) + i\omega T v_1^*(t) &= 0, \quad t \in [0, 1], \\ v_1^*(1) - v_1^*(0) &= 0, \\ \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$ , respectively.

Then  $h_{20}$  is approximated by

$$(134) \quad \begin{cases} \dot{h}_{20,1}(t) - TA(t)h_{20,1}(t) + 2i\omega Th_{20,1}(t) - TB(t; v_1(t), v_1(t)) &= 0, \quad t \in [0, 1], \\ h_{20,1}(1) - h_{20,1}(0) &= 0, \end{cases}$$

where  $h_{20}(\tau) = h_{20,1}(\tau/T)/T$ .

Before being able to compute the approximation to  $h_{11}$  we need  $\alpha_1$ , defined by (44):

$$\alpha_{1,1} = \int_0^1 \langle \varphi_1^*(t), B(t; v_1(t), \bar{v}_1(t)) \rangle dt,$$

with  $\alpha_{1,1} = T\alpha_1$ . Therefore, we get

$$\begin{cases} \dot{h}_{11,1}(t) - TA(t)h_{11,1}(t) - TB(t; v_1(t), \bar{v}_1(t)) + \alpha_{1,1}TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{11,1}(1) - h_{11,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{11,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{11}(\tau) = h_{11,1}(\tau/T)/T$ .

Now, we can compute normal form coefficient  $c$ :

$$c_1 = -\frac{i}{2} \int_0^1 \langle v_1^*(t), C(t; v_1(t), v_1(t), \bar{v}_1(t)) + 2B(t; v_1(t), h_{11,1}(t)) + B(t; \bar{v}_1(t), h_{20,1}(t)) - 2\alpha_{1,1}A(t)v_1(t) \rangle dt + \alpha_{1,1}\omega,$$

with  $c_1 = Tc$ . With  $c_1$  defined in this way,  $h_{21M}$  can be computed as follows

$$\begin{cases} \dot{h}_{21,1}(t) - TA(t)h_{21,1}(t) + i\omega Th_{21,1}(t) - TC(t; v_1(t), v_1(t), \bar{v}_1(t)) - 2TB(t; v_1(t), h_{11,1}(t)) \\ \quad - TB(t; h_{20,1}(t), \bar{v}_1(t)) + 2ic_1Tv_1(t) + 2\alpha_{1,1}T(A(t)v_1(t) - i\omega v_1(t)) &= 0, \quad t \in [0, 1], \\ h_{21,1}(1) - h_{21,1}(0) &= 0, \\ \int_0^1 \langle v_1^*(t), h_{21,1}(t) \rangle dt &= 0, \end{cases}$$

where  $h_{21}(\tau) = h_{21,1}(\tau/T)/(\sqrt{T}T)$ .

Next, the rescaling of  $h_{30}$  gives us

$$\begin{cases} \dot{h}_{30,1}(t) - TA(t)h_{30,1}(t) + 3i\omega Th_{30,1}(t) - TC(t; v_1(t), v_1(t), v_1(t)) - 3TB(t; v_1(t), h_{20,1}(t)) &= 0, \quad t \in [0, 1], \\ h_{30,1}(1) - h_{30,1}(0) &= 0, \end{cases}$$

with  $h_{30}(\tau) = h_{30,1}(\tau/T)/(\sqrt{T}T)$ .

Now, we need the rescaled  $h_{31,1}$  before being able to compute coefficient  $\alpha_{2,1}$

$$(135) \quad \begin{cases} \dot{h}_{31,1}(t) - TA(t)h_{31,1}(t) + 2i\omega Th_{31,1}(t) - TD(t; v_1(t), v_1(t), v_1(t), \bar{v}_1(t)) \\ \quad - 3TC(t; v_1(t), v_1(t), h_{11,1}(t)) - 3TC(t; v_1(t), \bar{v}_1(t), h_{20,1}(t)) - 3TB(t; h_{11,1}(t), h_{20,1}(t)) \\ \quad - 3TB(t; v_1(t), h_{21,1}(t)) - TB(t; \bar{v}_1(t), h_{30,1}(t)) \\ \quad + 6ic_1Th_{20,1}(t) + 3\alpha_{1,1}T(A(t)h_{20,1}(t) - 2i\omega h_{20,1}(t) + B(t; v_1(t), v_1(t))) &= 0, \quad t \in [0, 1], \\ h_{31,1}(1) - h_{31,1}(0) &= 0, \end{cases}$$

where  $h_{31}(\tau) = h_{31,1}(\tau/T)/T^2$ , so

$$\begin{aligned} \alpha_{2,1} = & \frac{1}{4} \int_0^1 \langle \varphi_1^*(t), D(t; v_1(t), v_1(t), \bar{v}_1(t), \bar{v}_1(t)) + C(t; v_1(t), v_1(t), h_{02,1}(t)) + 4C(t; v_1(t), \bar{v}_1(t), h_{11,1}(t)) \\ & + C(t; \bar{v}_1(t), \bar{v}_1(t), h_{20,1}(t)) + 2B(t; h_{11,1}(t), h_{11,1}(t)) + 2B(t; v_1(t), h_{12,1}(t)) + B(t; h_{02,1}(t), h_{20,1}(t)) \\ & + 2B(t; \bar{v}_1(t), h_{21,1}(t)) - 4\alpha_{1,1}(A(t)h_{11,1}(t) + B(t; v_1(t), \bar{v}_1(t))) \rangle dt + \alpha_{1,1}^2, \end{aligned}$$

with  $\alpha_{2,1} = T^2\alpha_2$ .

Now, we still need  $h_{22,1}(t)$ :

$$\left\{ \begin{aligned} & \dot{h}_{22,1}(t) - TA(t)h_{22,1}(t) - TD(t; v_1(t), v_1(t), \bar{v}_1(t), \bar{v}_1(t)) - TC(t; v_1(t), v_1(t), h_{02,1}(t)) \\ & - 4TC(t; v_1(t), \bar{v}_1(t), h_{11,1}(t)) - TC(t; \bar{v}_1(t), \bar{v}_1(t), h_{20,1}(t)) - 2TB(t; h_{11,1}(t), h_{11,1}(t)) \\ & - 2TB(t; v_1(t), h_{12,1}(t)) - TB(t; h_{02,1}(t), h_{20,1}(t)) - 2TB(t; \bar{v}_1(t), h_{21,1}(t)) \\ & + 4\alpha_{1,1}T(A(t)h_{11,1}(t) + B(t; v_1(t), \bar{v}_1(t)) - \alpha_{1,1}F(u_1(t))) + 4\alpha_{2,1}TF(u_1(t)) = 0, \quad t \in [0, 1], \\ & h_{22,1}(1) - h_{22,1}(0) = 0, \\ & \int_0^1 \langle \varphi_1^*(t), h_{22,1}(t) \rangle dt = 0, \end{aligned} \right.$$

where  $h_{22}(\tau) = h_{22,1}(\tau/T)/T^2$ .

Therefore, we can compute the critical coefficient  $e$ :

$$\begin{aligned} e = & \frac{1}{12T^2} \int_0^1 \langle \varphi_1^*(t), E(t; v_1(t), v_1(t), v_1(t), \bar{v}_1(t), \bar{v}_1(t)) + D(t; v_1(t), v_1(t), v_1(t), h_{02,1}(t)) \\ & + 6D(t; v_1(t), v_1(t), \bar{v}_1(t), h_{11,1}(t)) + 3D(t; v_1(t), \bar{v}_1(t), \bar{v}_1(t), h_{20,1}(t)) \\ & + 6C(t; v_1(t), h_{11,1}(t), h_{11,1}(t)) + 3C(t; v_1(t), v_1(t), h_{12,1}(t)) \\ & + 3C(t; v_1(t), h_{02,1}(t), h_{20,1}(t)) + 6C(t; \bar{v}_1(t), h_{11,1}(t), h_{20,1}(t)) + 6C(t; v_1(t), \bar{v}_1(t), h_{21,1}(t)) \\ & + C(t; \bar{v}_1(t), \bar{v}_1(t), h_{30,1}(t)) \\ & + 3B(t; h_{12,1}(t), h_{20,1}(t)) + 6B(t; h_{11,1}(t), h_{21,1}(t)) + 3B(t; v_1(t), h_{22,1}(t)) \\ & + B(t; h_{02,1}(t), h_{30,1}(t)) + 2B(t; \bar{v}_1(t), h_{31,1}(t)) - 12\alpha_{2,1}A(t)v_1(t) \\ & - 6\alpha_{1,1}(A(t)h_{21,1}(t) + 2B(t; v_1(t), h_{11,1}(t)) + C(t; v_1(t), v_1(t), \bar{v}_1(t)) \\ & + B(t; h_{20,1}(t), \bar{v}_1(t)) - 2\alpha_{1,1}A(t)v_1(t)) \rangle dt + \alpha_{2,1}i\frac{\omega}{T^2} + \alpha_{1,1}i\frac{c_1}{T^2} - \alpha_{1,1}^2i\frac{\omega}{T^2}. \end{aligned}$$

We now impose the matrix solutions for the functions. We compute  $v_{1M}$  by solving the discretization of (133)

$$\begin{bmatrix} (D - TA(t) + i\omega TL)_{C \times M} & p_2 \\ \delta_0 - \delta_1 & \\ q_2^H & 0 \end{bmatrix} \begin{bmatrix} v_{1,1M} \\ a \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix},$$

with  $a = 0$ , where  $q_2$  is the normalized right null-vector of the complex matrix  $K = \begin{bmatrix} (D - TA(t) + i\omega TL)_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix}$

and  $p_2$  the normalized right null-vector of  $K^H$ . This vector is then rescaled so that  $\int_0^1 \langle v_1(t), v_1(t) \rangle dt = 1$ . For the computation of  $\varphi_{1M}$ , we use Proposition C.1 to obtain

$$[(\varphi_1^*)_W^T \quad a] \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ q^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

where  $a$  equals zero. We then approximate  $I = \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt$  by  $I_1 = (\varphi_1^*)_W^T L_{C \times M} g_M$  and we rescale  $\varphi_{1W}^*$  so that  $I_1 = 1$ .

For the computation of  $v_1^*$  we apply Proposition C.3 from the Appendix. Since  $v_1^*$  lies in the kernel of the over there defined  $\phi_2$ , we have that  $\begin{bmatrix} v_1^* \\ v_1^*(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0, 1], \mathbb{C}^n))$ . The eigenfunction  $v_1^*$  is thus computed by solving

$$\begin{bmatrix} (v_1^*)_W^H & a \end{bmatrix} \begin{bmatrix} (D - TA(t) + i\omega TL)_{C \times M} & p_2 \\ \delta_0 - \delta_1 & 0 \\ q_2^H & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix}.$$

We then approximate  $I = \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt$  by  $I_1 = (v_1^*)_W^H L_{C \times M} v_{1M}$  and we rescale  $v_{1W}^*$  so that  $I_1 = 1$ . (134) is approximated by

$$\begin{bmatrix} (D - TA(t) + 2i\omega TL)_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix} h_{20,1M} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} v_{1M} \\ 0_{n \times 1} \end{bmatrix}.$$

The coefficient  $\alpha_{1,1}$  can be approximated as

$$\alpha_{1,1} = (\varphi_1^*)_W^T B_{C \times M \times M} v_{1M} \bar{v}_{1M}$$

which gives then all the information to determine the real function  $h_{11,1}$ :

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ (\varphi_1^*)_W^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{11,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} \bar{v}_{1M} - \alpha_{1,1} Tg_C \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

where  $a$  equals zero.

Then an approximation for the rescaled normal form coefficient  $c_1$  is given by

$$c_1 = -\frac{i}{2} (v_1^*)_W^H (C_{C \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} + 2B_{C \times M \times M} v_{1M} h_{11,1M} + B_{C \times M \times M} \bar{v}_{1M} h_{20,1M} - 2\alpha_{1,1} A_{C \times M} v_{1M}) + \alpha_{1,1} \omega.$$

Next, we determine the third order coefficients of the center manifold expansion, namely

$$\begin{bmatrix} (D - TA(t) + i\omega TL)_{C \times M} & p_2 \\ \delta_0 - \delta_1 & 0 \\ (v_1^*)_W^H L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{21,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} \text{rhs} = & TC_{C \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} + 2TB_{C \times M \times M} v_{1M} h_{11,1M} + TB_{C \times M \times M} h_{20,1M} \bar{v}_{1M} - 2ic_1 TL_{C \times M} v_{1M} \\ & - 2\alpha_{1,1} T(A_{C \times M} v_{1M} - i\omega L_{C \times M} v_{1M}) \end{aligned}$$

and  $a = 0$ , and

$$\begin{bmatrix} (D - TA(t) + 3i\omega TL)_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix} h_{30,1M} = \begin{bmatrix} TC_{C \times M \times M \times M} v_{1M} v_{1M} v_{1M} + 3TB_{C \times M \times M} v_{1M} h_{20,1M} \\ 0_{n \times 1} \end{bmatrix}.$$

The approximation to (135) is given by

$$\begin{bmatrix} (D - TA(t) + 2i\omega TL)_{C \times M} \\ \delta_0 - \delta_1 \end{bmatrix} h_{31,1M} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \end{bmatrix},$$

with

$$\begin{aligned} \text{rhs} = & TD_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} \bar{v}_{1M} + 3TC_{C \times M \times M \times M} v_{1M} v_{1M} h_{11,1M} + 3TC_{C \times M \times M \times M} v_{1M} \bar{v}_{1M} h_{20,1M} \\ & + 3TB_{C \times M \times M} h_{11,1M} h_{20,1M} + 3TB_{C \times M \times M} v_{1M} h_{21,1M} + TB_{C \times M \times M} \bar{v}_{1M} h_{30,1M} \\ & - 6ic_1 TL_{C \times M} h_{20,1M} - 3\alpha_{1,1} T(A_{C \times M} h_{20,1M} - 2i\omega L_{C \times M} h_{20,1M} + B_{C \times M \times M} v_{1M} v_{1M}) \end{aligned}$$



while

$$\begin{aligned} \alpha_{2,1} = & \frac{1}{4}(\varphi_1^*)_{W_1}^T (D_{C \times M \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} \bar{v}_{1M} + C_{C \times M \times M \times M} v_{1M} v_{1M} h_{02,1M} + 4C_{C \times M \times M \times M} v_{1M} \bar{v}_{1M} h_{11,1M} \\ & + C_{C \times M \times M \times M} \bar{v}_{1M} \bar{v}_{1M} h_{20,1M} + 2B_{C \times M \times M} h_{11,1M} h_{11,1M} + 2B_{C \times M \times M} v_{1M} h_{12,1M} + B_{C \times M \times M} h_{02,1M} h_{20,1M} \\ & + 2B_{C \times M \times M} \bar{v}_{1M} h_{21,1M} - 4\alpha_{1,1}(A_{C \times M} h_{11,1M} + B_{C \times M \times M} v_{1M} \bar{v}_{1M})) + \alpha_{1,1}^2. \end{aligned}$$

Now, we are able to compute  $h_{22,1}$  by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{22,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with

$$\begin{aligned} \text{rhs} = & TD_{C \times M \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} \bar{v}_{1M} + TC_{C \times M \times M \times M} v_{1M} v_{1M} h_{02,1M} - 4\alpha_{2,1} Tg_C \\ & + 4TC_{C \times M \times M \times M} v_{1M} \bar{v}_{1M} h_{11,1M} + 2TB_{C \times M \times M} h_{11,1M} h_{11,1M} + 2TB_{C \times M \times M} v_{1M} h_{12,1M} \\ & + TC_{C \times M \times M \times M} \bar{v}_{1M} \bar{v}_{1M} h_{20,1M} + TB_{C \times M \times M} h_{02,1M} h_{20,1M} + 2TB_{C \times M \times M} \bar{v}_{1M} h_{21,1M} \\ & - 4\alpha_{1,1} T(A_{C \times M} h_{11,1M} + B_{C \times M \times M} v_{1M} \bar{v}_{1M} - \alpha_1 g_C) \end{aligned}$$

and  $a = 0$ . For the fifth order coefficient of the normal form, we then obtain

$$\begin{aligned} e = & \frac{1}{12T^2} (v_1^*)_{W_1}^H (E_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} \bar{v}_{1M} \bar{v}_{1M} + D_{C \times M \times M \times M \times M} v_{1M} v_{1M} v_{1M} h_{02,1M} \\ & + 6D_{C \times M \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} h_{11,1M} + 3D_{C \times M \times M \times M \times M} v_{1M} \bar{v}_{1M} \bar{v}_{1M} h_{20,1M} \\ & + 6C_{C \times M \times M \times M} v_{1M} h_{11,1M} h_{11,1M} + 3C_{C \times M \times M \times M} v_{1M} v_{1M} h_{12,1M} \\ & + 3C_{C \times M \times M \times M} v_{1M} h_{02,1M} h_{20,1M} + 6C_{C \times M \times M \times M} \bar{v}_{1M} h_{11,1M} h_{20,1M} + 6C_{C \times M \times M \times M} v_{1M} \bar{v}_{1M} h_{21,1M} \\ & + C_{C \times M \times M \times M} \bar{v}_{1M} \bar{v}_{1M} h_{30,1M} + 3B_{C \times M \times M} h_{12,1M} h_{20,1M} \\ & + 6B_{C \times M \times M} h_{11,1M} h_{21,1M} + 3B_{C \times M \times M} v_{1M} h_{22,1M} \\ & + B_{C \times M \times M} h_{02,1M} h_{30,1M} + 2B_{C \times M \times M} \bar{v}_{1M} h_{31,1M} - 12\alpha_{2,1} A_{C \times M} v_{1M} \\ & - 6\alpha_{1,1} (A_{C \times M} h_{21,1M} + 2B_{C \times M \times M} v_{1M} h_{11,1M} + C_{C \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} \\ & + B_{C \times M \times M} h_{20,1M} \bar{v}_{1M} - 2\alpha_{1,1} A_{C \times M} v_{1M})) + \alpha_{2,1} i \frac{\omega}{T^2} + \alpha_{1,1} i \frac{c_1}{T^2} - \alpha_{1,1}^2 i \frac{\omega}{T^2}. \end{aligned}$$

## 4.5 Strong resonance 1:1 bifurcation

Again, we rescale the computed quantities to the interval  $[0, 1]$ . The linear BVPs (56) and (57) are replaced by

$$\begin{cases} \dot{v}_{1,1}(t) - TA(t)v_{1,1}(t) - TF(u_1(t)) & = 0, \quad t \in [0, 1], \\ v_{1,1}(1) - v_{1,1}(0) & = 0, \\ \int_0^1 \langle v_{1,1}(t), F(u_1(t)) \rangle dt & = 0, \end{cases}$$

where  $v_1(\tau) = v_{1,1}(\tau/T)$ , and

$$\begin{cases} \dot{v}_{2,1}(t) - TA(t)v_{2,1}(t) + Tv_{1,1}(t) & = 0, \quad t \in [0, 1], \\ v_{2,1}(1) - v_{2,1}(0) & = 0, \\ \int_0^1 \langle v_{2,1}(t), F(u_1(t)) \rangle dt & = 0, \end{cases}$$

with  $v_2(\tau) = v_{2,1}(\tau/T)$ , respectively.

The rescaling of the adjoint eigenfunction, defined by (60), gives us

$$\begin{cases} \varphi_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), v_{2,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ . This leads to the expression

$$a = \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), 2A(t)v_{1,1}(t) + B(t; v_{1,1}(t), v_{1,1}(t)) \rangle dt.$$

Definition (61) of the first generalized adjoint eigenfunction is rescaled as

$$\begin{cases} \dot{v}_{1,1}^*(t) + TA^T(t)v_{1,1}^*(t) - T\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ v_{1,1}^*(1) - v_{1,1}^*(0) &= 0, \\ \int_0^1 \langle v_{1,1}^*(t), v_{2,1}(t) \rangle dt &= 0, \end{cases}$$

with  $v_1^*(\tau) = v_{1,1}^*(\tau/T)/T$ , which then gives all the information we need to compute the critical coefficient  $b$

$$(136) \quad b = \int_0^1 \langle \varphi_1^*(t), B(t; v_{1,1}(t), v_{2,1}(t)) + A(t)v_{2,1}(t) \rangle dt + \int_0^1 \langle v_{1,1}^*(t), 2A(t)v_{1,1}(t) + B(t; v_{1,1}(t), v_{1,1}(t)) \rangle dt.$$

The generalized eigenfunctions are computed as

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ g_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} v_{1,1M} \\ a \end{bmatrix} = \begin{bmatrix} Tg_C \\ 0_{n \times 1} \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ g_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} v_{2,1M} \\ a \end{bmatrix} = \begin{bmatrix} -Tv_{1,1C} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with  $p$  the left null-vector, as before.

As always, we compute the adjoint eigenfunctions by means of the transposed of the usual matrix, and obtain here  $\varphi_{1W}^*$  instead of  $\varphi_{1M}^*$ . Formally, the computation of  $\varphi_{1W}^*$  is based on Proposition C.1 from the appendix and as in the cusp of cycles case we obtain then

$$[(\varphi_1^*)_W^T \quad a] \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ q^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

where  $a$  equals zero. We approximate  $I = \int_0^1 \langle \varphi_1^*(t), v_{2,1}(t) \rangle dt$  by  $I_1 = (\varphi_1^*)_W^T L_{C \times M} v_{2,1M}$  and rescale  $\varphi_{1W}^*$  to ensure that  $I_1 = 1$ .

Having found  $v_{1,1M}$  and  $\varphi_{1W}^*$ , the first normal form coefficient of interest can be computed as

$$a = \frac{1}{2} (\varphi_1^*)_W^T (2A_{C \times M} v_{1,1M} + B_{C \times M \times M} v_{1,1M} v_{1,1M}).$$

Now we still need  $v_{1,1}^*$ . The technique of Proposition C.5 from the appendix is used to obtain  $(v_{1,1}^*)_W$ , namely

$$[(v_{1,1}^*)_W^T \quad a] \begin{bmatrix} (D - TA(t))_{C \times M} & v_{2,1C} \\ \delta_0 - \delta_1 & 0_{n \times 1} \\ q^T & 0 \end{bmatrix} = [-T(\varphi_1^*)_W^T L_{C \times M} \quad 0],$$

where  $a = 0$ .

Finally (136) is approximated by

$$b = (\varphi_1^*)_W^T (B_{C \times M \times M} v_{1,1M} v_{2,1M} + A_{C \times M} v_{2,1M}) + (v_{1,1}^*)_W^T (2A_{C \times M} v_{1,1M} + B_{C \times M \times M} v_{1,1M} v_{1,1M}).$$

#### 4.6 Strong resonance 1:2 bifurcation

As before, we rescale the computed quantities to the interval  $[0, 1]$ . The linear BVPs (70) and (71) are replaced by

$$(137) \quad \begin{cases} \dot{v}_{1,1}(t) - TA(t)v_{1,1}(t) &= 0, \quad t \in [0, 1], \\ v_{1,1}(1) + v_{1,1}(0) &= 0, \\ \int_0^1 \langle v_{1,1}(t), v_{1,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v_1(\tau) = v_{1,1}(\tau/T)/\sqrt{T}$ , and

$$\begin{cases} \dot{v}_{2,1}(t) - TA(t)v_{2,1}(t) + Tv_{1,1}(t) &= 0, \quad t \in [0, 1], \\ v_{2,1}(1) + v_{2,1}(0) &= 0, \\ \int_0^1 \langle v_{2,1}(t), v_{1,1}(t) \rangle dt &= 0, \end{cases}$$

where  $v_2(\tau) = v_{2,1}(\tau/T)/\sqrt{T}$ .

The rescaling of the adjoint eigenfunction gives us

$$\begin{cases} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt - 1 &= 0, \end{cases}$$

where  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ , so we can compute

$$\alpha_1 = \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), B(t; v_{1,1}(t), v_{1,1}(t)) \rangle dt,$$

with  $\alpha_1 = T\alpha$ . With  $\alpha_1$  defined in this way, let  $h_{20,1}$  be the unique solution of the BVP

$$(138) \quad \begin{cases} \dot{h}_{20,1}(t) - TA(t)h_{20,1}(t) - TB(t; v_{1,1}(t), v_{1,1}(t)) + 2\alpha_1 TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{20,1}(1) - h_{20,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{20,1}(t) \rangle dt &= \int_0^1 \langle \varphi_1^*(t), B(v_{1,1}(t), v_{2,1}(t)) \rangle dt, \end{cases}$$

where  $h_{20}(\tau) = h_{20,1}(\tau/T)/T$ .

The rescaling of the adjoint eigenfunction corresponding with multiplier  $-1$  and the adjoint generalized eigenfunction gives us

$$\begin{cases} \dot{v}_{1,1}^*(t) + TA^T(t)v_{1,1}^*(t) &= 0, \quad t \in [0, 1], \\ v_{1,1}^*(1) + v_{1,1}^*(0) &= 0, \\ \int_0^1 \langle v_{1,1}^*(t), v_{2,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v_1^*(\tau) = v_{1,1}^*(\tau/T)/\sqrt{T}$ , and

$$\begin{cases} \dot{v}_{2,1}^*(t) + TA^T(t)v_{2,1}^*(t) - Tv_{1,1}^*(t) &= 0, \quad t \in [0, 1], \\ v_{2,1}^*(1) + v_{2,1}^*(0) &= 0, \\ \int_0^1 \langle v_{2,1}^*(t), v_{2,1}^*(t) \rangle dt &= 0, \end{cases}$$

where  $v_2^*(\tau) = v_{2,1}^*(\tau/T)/\sqrt{T}$ , respectively.

The rescaled critical coefficient is then

$$(139) \quad a_1 = \frac{1}{6} \int_0^1 \langle v_{1,1}^*(t), C(t; v_{1,1}(t), v_{1,1}(t), v_{1,1}(t)) + 3B(t; v_{1,1}(t), h_{20,1}(t)) - 6\alpha_1 A(t)v_{1,1}(t) \rangle dt,$$

with  $a_1 = Ta$ .

We replace (80) by

$$\begin{cases} \dot{h}_{11,1}(t) - TA(t)h_{11,1}(t) - TB(t; v_{1,1}(t), v_{2,1}(t)) + Th_{20,1}(t) &= 0, \quad t \in [0, 1], \\ h_{11,1}(1) - h_{11,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{11,1}(t) \rangle dt &= \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), B(v_{2,1}(t), v_{2,1}(t)) \rangle dt, \end{cases}$$

with  $h_{11}(\tau) = h_{11,1}(\tau/T)/T$ , to finally obtain that

$$\begin{aligned} b = \frac{1}{2T} \int_0^1 & \langle v_{1,1}^*(t), -2\alpha_1 A(t)v_{2,1}(t) + C(t; v_{1,1}(t), v_{1,1}(t), v_{2,1}(t)) \\ & + B(t; h_{20,1}(t), v_{2,1}(t)) + 2B(t; h_{11,1}(t), v_{1,1}(t)) \rangle dt \\ & + \frac{1}{2T} \int_0^1 \langle v_{2,1}^*(t), C(t; v_{1,1}(t), v_{1,1}(t), v_{1,1}(t)) + 3B(t; v_{1,1}(t), h_{20,1}(t)) - 6\alpha_1 A(t)v_{1,1}(t) \rangle dt. \end{aligned}$$

We now come to the second part of the implementation details and define the matrix solutions for the several functions. We compute  $v_{1,1M}$  by solving the discretization of (137)

$$(140) \quad \begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & 0 \\ q_1^T & 0 \end{bmatrix} \begin{bmatrix} v_{1,1M} \\ a \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix},$$

where  $a = 0$ . We then normalize  $v_{1,1M}$  by requiring  $\sum_{i=0}^{N-1} \sum_{j=0}^m \sigma_j \langle (v_{1,1M})_{i,j}, (v_{1,1M})_{i,j} \rangle = 1$ , where  $\sigma_j$  is the Lagrange quadrature coefficient.

We have now obtained the value of  $v_{1,1}$  in the mesh points. However, since  $v_{1,1}$  is used in the integral condition for  $v_{2,1}$ , we have to transfer this vector to the collocation points and multiply it with the Gauss-Legendre weights and the lengths of the corresponding intervals, giving us vector  $(v_{1,1})_{W_1}$ . Then,  $v_{2,1M}$  can be found by solving the following system

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & 0 \\ (v_{1,1})_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} v_{2,1M} \\ a \end{bmatrix} = \begin{bmatrix} -Tv_{1,1C} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

where  $a$  equals zero.

The adjoint eigenfunction corresponding to the trivial eigenvalue is computed with an analogous matrix as in (140):

$$[(\varphi_1^*)_{W_1}^T \quad a_1] \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ q^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

while the adjoint eigenfunction corresponding to eigenvalue  $-1$  can be found by

$$[(v_{1,1}^*)_{W_1}^T \quad a_2] \begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & 0 \\ q_1^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

where  $a_1$  and  $a_2$  are equal to zero.  $\varphi_{1W}^*$  is then rescaled to make sure that  $(\varphi_1^*)_{W_1}^T L_{C \times M} g_M = 1$  and  $v_{1,1W}^*$  so that  $(v_{1,1}^*)_{W_1}^T L_{C \times M} v_{2,1M} = 1$ .

Making use of Proposition C.6, we obtain the adjoint generalized eigenfunction by solving

$$[(v_{2,1}^*)_{W_1}^T \quad a] \begin{bmatrix} (D - TA(t))_{C \times M} & v_{2,1C} \\ \delta_0 + \delta_1 & 0_{n \times 1} \\ q_1^T & 0 \end{bmatrix} = [-T(v_{1,1}^*)_{W_1}^T L_{C \times M} \quad 0].$$

Having found  $\varphi_{1W_1}^*$  and  $v_{1,1M}$ ,  $\alpha_1$  can be computed as

$$\alpha_1 = \frac{1}{2}(\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{1,1M} v_{1,1M}.$$

Now,  $h_{20,1M}$  is found by solving the discretization of (138), namely,

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{20,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1,1M} v_{1,1M} - 2\alpha_1 T g_C \\ 0_{n \times 1} \\ (\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{1,1M} v_{2,1M} \end{bmatrix},$$

and (139) is approximated by

$$a_1 = \frac{1}{6}(v_{1,1}^*)_{W_1}^T (C_{C \times M \times M \times M} v_{1,1M} v_{1,1M} v_{1,1M} + 3B_{C \times M \times M} v_{1,1M} h_{20,1M} - 6\alpha_1 A_{C \times M} v_{1,1M}).$$

Next,  $h_{11,1M}$  is found by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{11,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1,1M} v_{2,1M} - Th_{20,1C} \\ 0_{n \times 1} \\ \frac{1}{2}(\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{2,1M} v_{2,1M} \end{bmatrix}.$$

Finally, we obtain

$$\begin{aligned} b = \frac{1}{2T} & (v_{1,1}^*)_{W_1}^T (-2\alpha_1 A_{C \times M} v_{2,1M} \\ & + C_{C \times M \times M \times M} v_{1,1M} v_{1,1M} v_{2,1M} + B_{C \times M \times M} h_{20,1M} v_{2,1M} + 2B_{C \times M \times M} h_{11,1M} v_{1,1M}) \\ & + \frac{1}{2T} (v_{2,1}^*)_{W_1}^T (C_{C \times M \times M \times M} v_{1,1M} v_{1,1M} v_{1,1M} + 3B_{C \times M \times M} v_{1,1M} h_{20,1M} - 6\alpha_1 A_{C \times M} v_{1,1M}). \end{aligned}$$

#### 4.7 Strong resonance 1:3 bifurcation

As before, we rescale the computed quantities to the interval  $[0, 1]$ . The BVPs for the eigenfunction and its adjoint belonging to eigenvalue  $e^{i\frac{2\pi}{3}}$  are replaced by

$$(141) \quad \begin{cases} \dot{v}_1(t) - TA(t)v_1(t) &= 0, \quad t \in [0, 1], \\ v_1(1) - e^{i\frac{2\pi}{3}}v_1(0) &= 0, \\ \int_0^1 \langle v_1(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v(\tau) = v_1(\tau/T)/\sqrt{T}$ , and

$$\begin{cases} \dot{v}_1^*(t) + TA^T(t)v_1^*(t) &= 0, \quad t \in [0, 1], \\ v_1^*(1) - e^{i\frac{2\pi}{3}}v_1^*(0) &= 0, \\ \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$ . The rescaling of the adjoint eigenfunction corresponding to the trivial multiplier gives

$$(142) \quad \begin{cases} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), F(u_1(t)) \rangle dt - 1 &= 0, \end{cases}$$

with  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ .

These eigenfunctions make it already possible to compute the following 2 rescaled normal form coefficients

$$(143) \quad \alpha_{1,1} = \int_0^1 \langle \varphi_1^*(t), B(v_1(t), \bar{v}_1(t)) \rangle dt,$$

where  $\alpha_{1,1} = T\alpha_1$ , and

$$b_1 = \frac{1}{2} \int_0^1 \langle v_1^*(t), B(\bar{v}_1(t), \bar{v}_1(t)) \rangle dt,$$

with  $b_1 = \sqrt{T}b$ .

The rescaled second order functions in the center manifold expansion are solutions of

$$\begin{cases} \dot{h}_{20,1}(t) - TA(t)h_{20,1}(t) - TB(v_1(t), v_1(t)) + 2\bar{b}_1 T \bar{v}_1(t) &= 0, \quad t \in [0, T], \\ h_{20,1}(1) - e^{i\frac{4\pi}{3}} h_{20,1}(0) &= 0, \\ \int_0^1 \langle \bar{v}_1^*(t), h_{20,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{20}(\tau) = h_{20,1}(\tau/T)/T$ , and

$$(144) \quad \begin{cases} \dot{h}_{11,1}(t) - TA(t)h_{11,1} - TB(v_1(t), \bar{v}_1(t)) + \alpha_{1,1} TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{11,1}(1) - h_{11,1}(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), h_{11,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{11}(\tau) = h_{11,1}(\tau/T)/T$ . This all results then in

$$c = \frac{1}{2T} \int_0^1 \langle v_1^*(t), C(v_1(t), v_1(t), \bar{v}_1(t)) + 2B(v_1(t), h_{11,1}(t)) + B(\bar{v}_1(t), h_{20,1}(t)) - 2\alpha_{1,1} A v_1(t) \rangle dt.$$

We now come to the implementation details in MatCont. Eigenfunction  $v_1$ , determined by (141), is computed by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_3 \\ \delta_0 - e^{-i\frac{2\pi}{3}} \delta_1 & 0 \\ q_3^H & 0 \end{bmatrix} \begin{bmatrix} v_{1M} \\ a \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix},$$

with  $a = 0$ . We then normalize  $v_{1M}$  by requiring  $\sum_{i=0}^{N-1} \sum_{j=0}^m \sigma_j \langle (v_{1M})_{i,j}, (v_{1M})_{i,j} \rangle = 1$ , where  $\sigma_j$  is the Lagrange quadrature coefficient.  $q_3$  is the normalized right null-vector of  $K = \begin{bmatrix} (D - TA(t))_{C \times M} \\ \delta_0 - e^{-i\theta} \delta_1 \end{bmatrix}$  and  $p_3$  the normalized right null-vector of  $K^H$ , with  $\theta = \frac{2\pi}{3}$ .

To compute the adjoint eigenfunction  $v_1^*$ , we apply Proposition C.4 from the appendix with  $\theta = \frac{2\pi}{3}$ . Since  $v_1^* \in \text{Ker}(\phi_2)$ , this function can be obtained by solving

$$(145) \quad \begin{bmatrix} (v_1^*)_{W_1}^H & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p_3 \\ \delta_0 - e^{-i\frac{2\pi}{3}} \delta_1 & 0 \\ q_3^H & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix}.$$

$v_{1W}^*$  is rescaled such that  $(v_1^*)_{W_1}^H L_{C \times M} v_{1M} = 1$ . The adjoint eigenfunction corresponding to eigenvalue 1 is discretized by

$$(146) \quad \begin{bmatrix} (\varphi_1^*)_{W_1}^T & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ q^T & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

where  $a$  equals zero.  $\varphi_{1W}^*$  is rescaled so that  $(\varphi_1^*)_{W_1}^T L_{C \times M} g_M = 1$ .

The normal form coefficients  $\alpha_{1,1}$  and  $b_1$  become

$$(147) \quad \alpha_{1,1} = (\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{1M} \bar{v}_{1M}$$

and

$$b_1 = \frac{1}{2} (v_1^*)_{W_1}^H B_{C \times M \times M} \bar{v}_{1M} \bar{v}_{1M}.$$

By computing first the complex conjugate of  $h_{20,1}$  we can use the same matrix as in (145), except for the last line which represents the integral condition, to get

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_3 \\ \delta_0 - e^{-i\frac{2\pi}{3}} \delta_1 & \\ (v_1^*)_{W_1}^H L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} \bar{h}_{20,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} \bar{v}_{1M} \bar{v}_{1M} - 2b_1 T v_{1C} \\ 0_{n \times 1} \\ 0 \end{bmatrix}.$$

To obtain the discretization of  $h_{11,1}$ , the following system is solved

$$(148) \quad \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (\varphi_1^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{11,1M} \\ a \end{bmatrix} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} \bar{v}_{1M} - \alpha_{1,1} T g_C \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

which gives us

$$c = \frac{1}{2T} (v_1^*)_{W_1}^H (C_{C \times M \times M \times M} v_{1M} v_{1M} \bar{v}_{1M} + 2B_{C \times M \times M} v_{1M} h_{11,1M} + B_{C \times M \times M} \bar{v}_{1M} h_{20,1M} - 2\alpha_{1,1} A_{C \times M} v_{1M}).$$

#### 4.8 Strong resonance 1:4 bifurcation

The eigenfunction and the adjoint eigenfunction corresponding to multiplier  $e^{i\frac{\pi}{2}}$  are given by the solution of

$$\begin{cases} \dot{v}_1(t) - TA(t)v_1(t) &= 0, \quad t \in [0, 1], \\ v_1(1) - e^{i\frac{\pi}{2}} v_1(0) &= 0, \\ \int_0^1 \langle v_1(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $v(\tau) = v_1(\tau/T)/\sqrt{T}$  and

$$\begin{cases} \dot{v}_1^*(t) + TA^T(t)v_1^*(t) &= 0, \quad t \in [0, 1], \\ v_1^*(1) - e^{i\frac{\pi}{2}} v_1^*(0) &= 0, \\ \int_0^1 \langle v_1^*(t), v_1(t) \rangle dt - 1 &= 0, \end{cases}$$

where  $v^*(\tau) = v_1^*(\tau/T)/\sqrt{T}$ , respectively.  $\varphi^*$ ,  $h_{11}$  and  $\alpha_1$  are replaced by  $\varphi_1^*$ ,  $h_{11,1}$  and  $\alpha_{1,1}$ , defined by (142), (144) and (143), respectively.

The rescaling of (98) gives

$$\begin{cases} \dot{h}_{20,1}(t) - TA(t)h_{20,1}(t) - TB(v_1(t), v_1(t)) &= 0, \quad t \in [0, 1], \\ h_{20,1}(1) + h_{20,1}(0) &= 0, \end{cases}$$

with  $h_{20}(\tau) = h_{20,1}(\tau/T)/T$ .

The critical coefficients are then given by

$$\bar{c} = \frac{1}{2T} \int_0^1 \langle \bar{v}_1^*(t), C(v_1(t), \bar{v}_1(t), \bar{v}_1(t)) + B(v_1(t), h_{02,1}(t)) + 2B(\bar{v}_1(t), h_{11,1}(t)) - 2\alpha_{11} A(t) \bar{v}_1(t) \rangle dt$$

and

$$d = \frac{1}{6T} \int_0^1 \langle v_1^*(t), C(\bar{v}_1(t), \bar{v}_1(t), \bar{v}_1(t)) + 3B(\bar{v}_1(t), h_{02,1}(t)) \rangle dt.$$

The eigenfunction and its adjoint, corresponding to the complex eigenvalue, are discretized by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_3 \\ \delta_0 - e^{-i\frac{\pi}{2}}\delta_1 & 0 \\ q_3^H & 0 \end{bmatrix} \begin{bmatrix} v_{1M} \\ a \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} (v_1^*)_{W_1}^H & a \end{bmatrix} \begin{bmatrix} (D - TA(t))_{C \times M} & p_3 \\ \delta_0 - e^{-i\frac{\pi}{2}}\delta_1 & 0 \\ q_3^H & 0 \end{bmatrix} = \begin{bmatrix} 0_{M \times 1} & 1 \end{bmatrix},$$

respectively. For the computation of the adjoint function we have applied Proposition C.4 with  $\theta = \frac{\pi}{2}$ . We then normalize  $v_{1M}$  by requiring  $\sum_{i=0}^{N-1} \sum_{j=0}^m \sigma_j \langle (v_{1M})_{i,j}, (v_{1M})_{i,j} \rangle = 1$ , where  $\sigma_j$  is the Lagrange quadrature coefficient.  $v_{1M}^*$  is rescaled so that  $(v_1^*)_{W_1}^H L_{C \times M} v_{1M} = 1$ .

(146) and the corresponding normalization, (148) and (147) determine  $\varphi_1^*, h_{11,1}$  and  $\alpha_{1,1}$ , respectively. An approximation to  $h_{20}$  is obtained by

$$\begin{bmatrix} (D - TA(t))_{C \times M} \\ \delta_0 + \delta_1 \end{bmatrix} h_{20,1M} = \begin{bmatrix} TB_{C \times M \times M} v_{1M} v_{1M} \\ 0_{n \times 1} \end{bmatrix}.$$

We are now able to compute the two needed normal form coefficients:

$$\bar{c} = \frac{1}{2T} (v_1^*)_{W_1}^T (C_{C \times M \times M \times M} v_{1M} \bar{v}_{1M} \bar{v}_{1M} + B_{C \times M \times M} v_{1M} h_{02,1M} + 2B_{C \times M \times M} \bar{v}_{1M} h_{11,1M} - 2\alpha_{1,1} A_{C \times M} \bar{v}_{1M})$$

and

$$d = \frac{1}{6T} (v_1^*)_{W_1}^H (C_{C \times M \times M \times M} \bar{v}_{1M} \bar{v}_{1M} \bar{v}_{1M} + 3B_{C \times M \times M} \bar{v}_{1M} h_{02,1M}).$$

## 4.9 Fold-Flip bifurcation

The rescaling of the eigenfunctions (103) and (104) gives us

$$(149) \quad \begin{cases} \dot{v}_{1,1}(t) - TA(t)v_{1,1}(t) - TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ v_{1,1}(1) - v_{1,1}(0) &= 0, \\ \int_0^1 \langle v_{1,1}(t), F(u_1(t)) \rangle dt &= 0, \end{cases}$$

and

$$(150) \quad \begin{cases} \dot{v}_{2,1}(t) - TA(t)v_{2,1}(t) &= 0, \quad t \in [0, 1], \\ v_{2,1}(1) + v_{2,1}(0) &= 0, \\ \int_0^1 \langle v_{2,1}(t), v_{2,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

respectively, with  $v_1(\tau) = v_{1,1}(\tau/T)$  and  $v_2(\tau) = v_{2,1}(\tau/T)/\sqrt{T}$ .

The rescaled adjoint eigenfunctions can be obtained by solving

$$\begin{cases} \dot{\varphi}_1^*(t) + TA^T(t)\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ \varphi_1^*(1) - \varphi_1^*(0) &= 0, \\ \int_0^1 \langle \varphi_1^*(t), v_{1,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

$$\begin{cases} \dot{v}_{1,1}^*(t) + TA^T(t)v_{1,1}^*(t) + T\varphi_1^*(t) &= 0, \quad t \in [0, 1], \\ v_{1,1}^*(1) - v_{1,1}^*(0) &= 0, \\ \int_0^1 \langle v_{1,1}^*(t), v_{1,1}(t) \rangle dt &= 0, \end{cases}$$



$$\begin{cases} \dot{v}_{2,1}^*(t) + TA^T(t)v_{2,1}^*(t) &= 0, \quad t \in [0, 1], \\ v_{2,1}^*(1) + v_{2,1}^*(0) &= 0, \\ \int_0^1 \langle v_{2,1}^*(t), v_{2,1}(t) \rangle dt - 1 &= 0, \end{cases}$$

with  $\varphi^*(\tau) = \varphi_1^*(\tau/T)/T$ ,  $v_1^*(\tau) = v_{1,1}^*(\tau/T)/T$  and  $v_2^*(\tau) = v_{2,1}^*(\tau/T)/\sqrt{T}$ .

The two coefficients in front of the  $\xi_1^2$ -terms are given by

$$a_{20} = \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), B(v_{1,1}(t), v_{1,1}(t)) + 2A(t)v_{1,1}(t) \rangle dt$$

and  $\alpha_{20} = 0$ .

The second order functions of the center manifold expansion are defined by the BVPs

$$\begin{cases} \dot{h}_{20,1}(t) - TA(t)h_{20,1}(t) - TB(v_{1,1}(t), v_{1,1}(t)) + 2a_{20}Tv_{1,1}(t) + 2\alpha_{20}TF(u_1(t)) \\ \quad - 2TA(t)v_{1,1}(t) - 2TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{20,1}(1) - h_{20,1}(0) &= 0, \\ \int_0^1 \langle v_{1,1}^*(t), h_{20,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{20}(\tau) = h_{20,1}(\tau/T)$ ,

$$\begin{cases} \dot{h}_{11,1}(t) - TA(t)h_{11,1}(t) - TB(v_{1,1}(t), v_{2,1}(t)) + Tb_{11}v_{2,1}(t) - TA(t)v_{2,1}(t) &= 0, \quad t \in [0, 1], \\ h_{11,1}(1) + h_{11,1}(0) &= 0, \\ \int_0^1 \langle v_{2,1}^*(t), h_{11,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{11}(\tau) = h_{11,1}(\tau/T)/\sqrt{T}$  and

$$\begin{cases} \dot{h}_{02,1}(t) - TA(t)h_{02,1}(t) - TB(v_{2,1}(t), v_{2,1}(t)) + 2a_{02,1}Tv_{1,1}(t) + 2\alpha_{02,1}TF(u_1(t)) &= 0, \quad t \in [0, 1], \\ h_{02,1}(1) - h_{02,1}(0) &= 0, \\ \int_0^1 \langle v_{1,1}^*(t), h_{02,1}(t) \rangle dt &= 0, \end{cases}$$

with  $h_{02}(\tau) = h_{02,1}(\tau/T)/T$ , where

$$b_{11} = \int_0^1 \langle v_{2,1}^*(t), B(v_{1,1}(t), v_{2,1}(t)) + A(t)v_{2,1}(t) \rangle dt,$$

$$a_{02,1} = \frac{1}{2} \int_0^1 \langle \varphi_1^*(t), B(v_{2,1}(t), v_{2,1}(t)) \rangle dt$$

with  $a_{02,1} = Ta_{02}$  and  $\alpha_{02} = 0$ .

The rescaling of the last four normal form coefficients of interest gives

$$\begin{aligned} a_{30} = \frac{1}{6} \int_0^1 \langle \varphi_1^*(t), & C(v_{1,1}(t), v_{1,1}(t), v_{1,1}(t)) + 3B(h_{20,1}, v_{1,1}(t)) - 6a_{20}h_{20,1}(t) \\ & + 3(A(t)h_{20,1}(t) + B(v_{1,1}(t), v_{1,1}(t))) + 6(1 - \alpha_{20})A(t)v_{1,1}(t) \rangle dt - a_{20}, \end{aligned}$$

$$\begin{aligned} b_{21} = \frac{1}{2} \int_0^1 \langle v_{2,1}^*(t), & C(v_{1,1}(t), v_{1,1}(t), v_{2,1}(t)) + B(h_{20,1}(t), v_{2,1}(t)) + 2B(h_{11,1}(t), v_{1,1}(t)) - 2a_{20}h_{11,1}(t) \\ & - 2b_{11}h_{11,1}(t) + 2(A(t)h_{11,1}(t) + B(v_{1,1}(t), v_{2,1}(t))) + 2(1 - \alpha_{20})A(t)v_{2,1}(t) \rangle dt - b_{11}, \end{aligned}$$

$$a_{12} = \frac{1}{2T} \int_0^1 \langle \varphi_1^*(t), C(v_{1,1}(t), v_{2,1}(t), v_{2,1}(t)) + B(h_{02,1}(t), v_{1,1}(t)) + 2B(h_{11,1}(t), v_{2,1}(t)) - 2b_{11}h_{02,1}(t) \\ - 2a_{02,1}h_{20,1}(t) + A(t)h_{02,1}(t) + B(v_{2,1}(t), v_{2,1}(t)) - 2\alpha_{02,1}A(t)v_{1,1}(t) \rangle dt - \frac{a_{02,1}}{T}$$

and

$$b_{03} = \frac{1}{6T} \int_0^1 \langle v_{2,1}^*(t), C(v_{2,1}(t), v_{2,1}(t), v_{2,1}(t)) + 3B(h_{02,1}(t), v_{2,1}(t)) - 6a_{02,1}h_{11,1}(t) - 6\alpha_{02,1}A(t)v_{2,1}(t) \rangle dt.$$

The discretization of the functions determined in (149) and (150) is given by

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ g_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} v_{1,1M} \\ a_1 \end{bmatrix} = \begin{bmatrix} Tg_C \\ 0_{n \times 1} \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & 0 \\ q_1^T & 0 \end{bmatrix} \begin{bmatrix} v_{2,1M} \\ a_2 \end{bmatrix} = \begin{bmatrix} 0_{C \times 1} \\ 0_{n \times 1} \\ 1 \end{bmatrix},$$

with  $a_1 = a_2 = 0$ . We normalize  $v_{21M}$  by requiring  $\sum_{i=0}^{N-1} \sum_{j=0}^m \sigma_j \langle (v_{2,1M})_{i,j}, (v_{2,1M})_{i,j} \rangle = 1$ , where  $\sigma_j$  is the Lagrange quadrature coefficient. The implementation of the adjoint eigenfunctions is done by

$$[(\varphi_1^*)_{W_1}^T \quad a_1] \begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ q^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

$$[(v_{1,1}^*)_{W_1}^T \quad a_2] \begin{bmatrix} (D - TA(t))_{C \times M} & (v_{1,1})_C \\ \delta_0 - \delta_1 & 0_{n \times 1} \\ q^T & 0 \end{bmatrix} = [T(\varphi_1^*)_{W_1}^T L_{C \times M} \quad 0]$$

and

$$[(v_{2,1}^*)_{W_1}^T \quad a_3] \begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & 0 \\ q_1^T & 0 \end{bmatrix} = [0_{M \times 1} \quad 1],$$

with  $a_1 = a_2 = a_3 = 0$ .  $\varphi_{1W}^*$  and  $v_{2,1W}^*$  are then rescaled to ensure that  $(\varphi_1^*)_{W_1}^T L_{C \times M} v_{1,1M} = 1$  and  $(v_{2,1}^*)_{W_1}^T L_{C \times M} v_{2,1M} = 1$ .

The first needed normal form coefficient is given by

$$a_{20} = \frac{1}{2} (\varphi_1^*)_{W_1}^T (B_{C \times M \times M} v_{1,1M} v_{1,1M} + 2A_{C \times M} v_{1,1M}).$$

The second order functions can be obtained by solving

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & 0 \\ (v_{1,1}^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{20,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with

$$\text{rhs} = TB_{C \times M \times M} v_{1,1M} v_{1,1M} - 2a_{20} T v_{1,1C} + 2T A_{C \times M} v_{1,1M} + 2T g_C,$$

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p_1 \\ \delta_0 + \delta_1 & \\ (v_{2,1}^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{11,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with

$$\text{rhs} = TB_{C \times M \times M} v_{1,1M} v_{2,1M} - T b_{11} v_{2,1C} + T A_{C \times M} v_{2,1M},$$

and

$$\begin{bmatrix} (D - TA(t))_{C \times M} & p \\ \delta_0 - \delta_1 & \\ (v_{1,1}^*)_{W_1}^T L_{C \times M} & 0 \end{bmatrix} \begin{bmatrix} h_{02,1M} \\ a \end{bmatrix} = \begin{bmatrix} \text{rhs} \\ 0_{n \times 1} \\ 0 \end{bmatrix},$$

with

$$\text{rhs} = TB_{C \times M \times M} v_{2,1M} v_{2,1M} - 2a_{02,1} T v_{1,1C}.$$

The in these functions needed coefficients are given by

$$b_{11} = (v_{2,1}^*)_{W_1}^T (B_{C \times M \times M} v_{1,1M} v_{2,1M} + A_{C \times M} v_{2,1M})$$

and

$$a_{02,1} = \frac{1}{2} (\varphi_1^*)_{W_1}^T B_{C \times M \times M} v_{2,1M} v_{2,1M}.$$

At last, we obtain

$$\begin{aligned} a_{30} = \frac{1}{6} (\varphi_1^*)_{W_1}^T & (C_{C \times M \times M \times M} v_{1,1M} v_{1,1M} v_{1,1M} + 3B_{C \times M \times M} h_{20,1M} v_{1,1M} - 6a_{20} h_{20,1C} \\ & + 3(A_{C \times M} h_{20,1M} + B_{C \times M \times M} v_{1,1M} v_{1,1M}) + 6A_{C \times M} v_{1,1M}) - a_{20}, \end{aligned}$$

$$\begin{aligned} b_{21} = \frac{1}{2} (v_{2,1}^*)_{W_1}^T & (C_{C \times M \times M \times M} v_{1,1M} v_{1,1M} v_{2,1M} + B_{C \times M \times M} h_{20,1M} v_{2,1M} + 2B_{C \times M \times M} h_{11,1M} v_{1,1M} - 2a_{20} h_{11,1C} \\ & - 2b_{11} h_{11,1C} + 2(A_{C \times M} h_{11,1M} + B_{C \times M \times M} v_{1,1M} v_{2,1M}) + 2A_{C \times M} v_{2,1M}) - b_{11}, \end{aligned}$$

$$\begin{aligned} a_{12} = \frac{1}{2T} (\varphi_1^*)_{W_1}^T & (C_{C \times M \times M \times M} v_{1,1M} v_{2,1M} v_{2,1M} + B_{C \times M \times M} h_{02,1M} v_{1,1M} + 2B_{C \times M \times M} h_{11,1M} v_{2,1M} - 2b_{11} h_{02,1C} \\ & - 2a_{02,1} h_{20,1C} + A_{C \times M} h_{02,1M} + B_{C \times M \times M} v_{2,1M} v_{2,1M}) - \frac{a_{02,1}}{T}, \end{aligned}$$

and

$$b_{03} = \frac{1}{6T} (v_{2,1}^*)_{W_1}^T (C_{C \times M \times M \times M} v_{2,1M} v_{2,1M} v_{2,1M} + 3B_{C \times M \times M} h_{02,1M} v_{2,1M} - 6a_{02,1} h_{11,1C}).$$

## 5 Examples

The computations in this section are done with MATCONT [10]. In particular, the bordering methods from [15] are used to continue the codim 1 bifurcations of limit cycles in two parameters. The algorithms described above for computing the normal form coefficients are also implemented in the current version of MATCONT.

## 5.1 Periodic predator-prey model

Our first model is a periodically forced predator-prey model, studied in [34], and described by the following differential equations

$$(151) \quad \begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - p(x, t)y, \\ \dot{y} = ep(x, t)y - dy, \end{cases}$$

where  $x$  and  $y$  are the numbers of individuals respectively of prey and predator populations or suitable (but equivalent) measures of density or biomass. The parameters present in system (151) are the intrinsic growth rate  $r$ , the carrying capacity  $K$ , the efficiency  $e$  and the death rate  $d$  of the predator. The function  $p(x, t)$  is a functional response, for which the Holling type II is chosen, with constant attack rate  $a$  and half saturation  $b(t)$  that varies periodically with as period one year, i.e.

$$p(x, t) = \frac{ax}{b(t) + x}, \quad b(t) = b_0(1 + \varepsilon \cos 2\pi t).$$

Notice that this system can be made autonomous by adding the following two differential equations

$$\dot{\alpha} = \mu\alpha - \omega\beta - (\alpha^2 + \beta^2)\alpha, \quad \dot{\beta} = \omega\alpha + \mu\beta - (\alpha^2 + \beta^2)\beta.$$

In fact, if  $\mu$  is positive then this system has as asymptotic behavior a stable circular limit cycle of radius  $\mu$  with angular velocity  $\omega$ . Therefore, we can set  $\mu = 1$  and  $\omega = 2\pi$  in order to obtain the forcing function  $\sin 2\pi t$  as the flow of one of the variables of this subsystem with a particular phase shift that depends on the initial conditions. The system becomes

$$(152) \quad \begin{cases} \dot{x} = r \left(1 - \frac{x}{K}\right) x - \frac{axy}{b_0(1+\varepsilon\alpha)+x}, \\ \dot{y} = e \frac{axy}{b_0(1+\varepsilon\alpha)+x} - dy, \\ \dot{\alpha} = \alpha - 2\pi\beta - (\alpha^2 + \beta^2)\alpha, \\ \dot{\beta} = 2\pi\alpha + \beta - (\alpha^2 + \beta^2)\beta. \end{cases}$$

We have chosen this system as first example since it allows us to check if the computation of the  $\alpha_i$  normal form coefficients is correct. In fact, in a periodically forced system the time of the normal form should not depend on the coordinate, i.e.  $d\tau/dt = 1$ , and so all the  $\alpha_i$  coefficients must vanish. With fixed  $r = 2\pi$ ,  $K = e = 1$ ,  $a = 4\pi$  and  $d = 2\pi$  we perform a bifurcation analysis in the remaining parameters  $(\varepsilon, b_0)$  obtaining the bifurcation diagram reported in Figure 1. Since the system is periodically forced, no equilibria are present. The blue curve, with label LPC2, is a limit point of cycles bifurcation curve of the second iterate, the magenta curves are supercritical Neimark-Sacker bifurcations (of the first or of the second iterate, respectively labeled with NS1 and NS2) while the brown and green curves are period-doubling bifurcations, brown when subcritical and green when supercritical (with notation PD1, PD2, PD4 and PD8).

We now analyze in detail all the detected codimension two points, reporting the scalar computed coefficients explained in Section 4.

### 5.1.1 The two GPD points

In Figure 1 the LPC2 curve is tangent to the PD1 curve in two different GPD points. In the first one, in  $(\varepsilon, b_0) = (0.319, 0.412)$ , the limit point of cycles curve is tangent to the subcritical period-doubling curve (type presented in Figure 12-(b)), while in the second one, in  $(\varepsilon, b_0) = (1.09, 0.218)$ , the LPC2 curve is tangent to the supercritical part of the PD bifurcation curve (i.e. the type presented in Figure 12-(a)).

Performing the computation of the GPD normal form coefficients at the first point we obtain:

- for the first equation of normal form (6) the two coefficients  $\alpha_1$  and  $\alpha_2$ , up to a scaling term  $T$  and  $T^2$  computed through the formula (130) and (131) are zero, up to the accuracy of the computation.

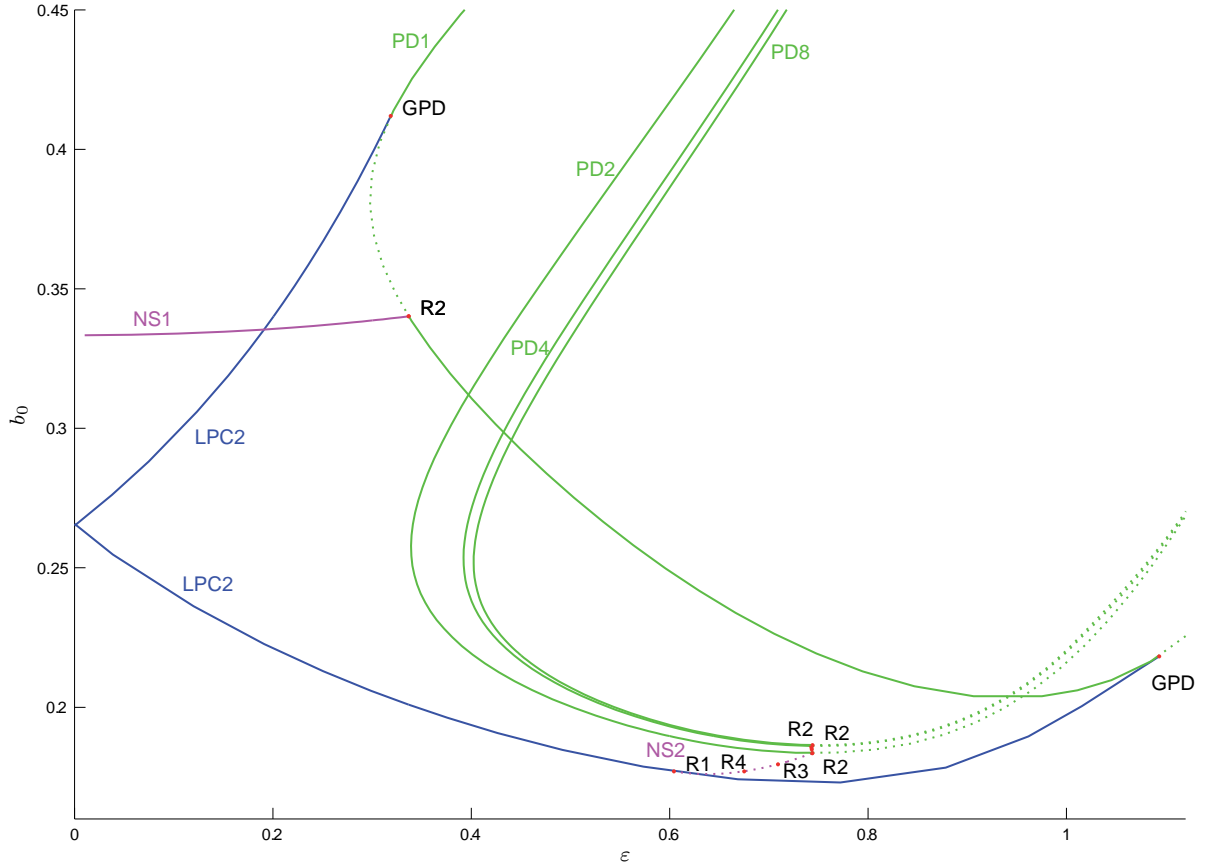


Figure 1: Bifurcation diagram of limit cycles in model (152). Blue are limit point of cycles bifurcations, green period doubling bifurcations and magenta Neimark-Sacker bifurcations. Continue/dotted curves correspond to supercritical/subcritical bifurcations.

- the normal form coefficient of the second equation, computed through formula (132) equals  $e = -58.2867$ .

Notice that these results are in agreement with what we expected, i.e. that since the system is periodically forced the time doesn't depend on the distance from the critical limit cycle, and since we are in the case presented in Figure 12-(b) the normal form coefficient  $e$  is negative.

From the computation of the GPD normal form coefficients at the second critical point we obtain:

- for the first equation of normal form (6) the two coefficients equal zero.
- the normal form coefficient of the second equation has value  $e = 41.5442$ .

Also in this case the obtained results are in agreement with the theory.

### 5.1.2 The 1:1 and 1:2 resonance points

We divide the 1:1 and 1:2 resonance points present in this model into two groups, namely the R2 point at  $(\epsilon, b_0) = (0.337, 0.34)$  and the cascade of resonance points in the lower part of the graph.

The isolated R2 point forms the intersection of the NS1 curve, the supercritical Neimark-Sacker curve of a limit cycle with period approximately equal to 1, and PD1. The situation is thus the one depicted in Figure 15-(a). Performing the normal form coefficient computation we obtain:

- for the first equation of normal form (9) holds that  $\alpha = 0$ .
- for the last equation of the normal form (9) we have  $(a, b) = (3.401426, -12.90745)$ .

Notice that the obtained results are in accordance with the theory (no secondary Neimark-Sacker curve implies that  $a > 0$  and supercritical Neimark-Sacker curve implies that  $b < 0$ ).

In the lower part of the bifurcation diagram a resonance cascade is present, which accumulates on the sequence of period-doubling curves. A zoom of this part is shown in Figure 2. Each resonance point of this

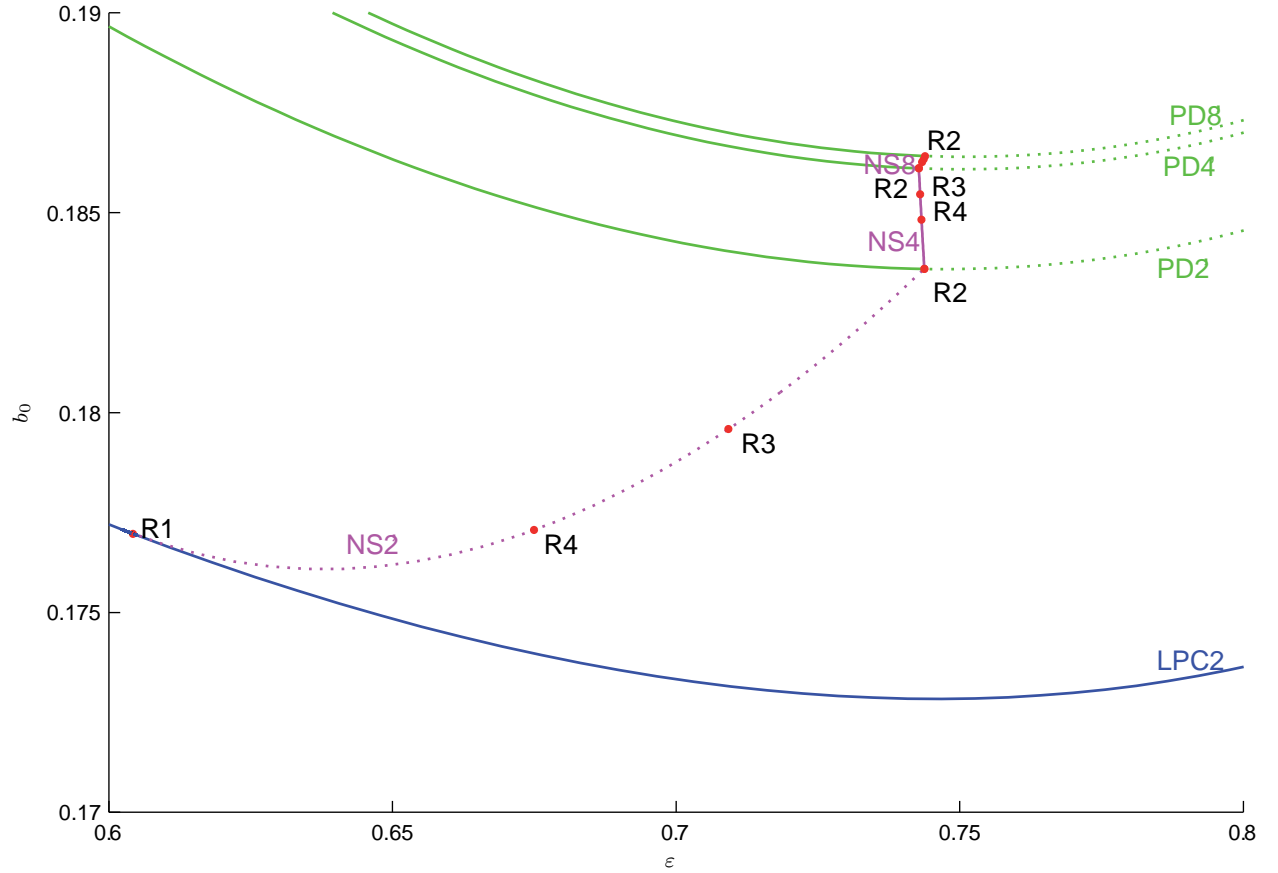


Figure 2: The resonance cascade in model (152). In blue are limit point of cycles bifurcations, green period doubling bifurcations and magenta Neimark-Sacker bifurcations. Continue/dotted curves correspond to supercritical/subcritical bifurcations.

cascade (except for the first R1 point on the LPC2 curve) can be seen in two ways: the first (and more natural way) is to see them as R2 points of the type represented in Figure 15-(b) (so with  $a < 0$  and the sign of  $b$  dependent on the criticality of the incoming Neimark-Sacker curve), the second way is to see them as R1 points represented in Figure 14-(b). Notice that the criticality of each NS curve of the cascade changes at the R2 point (as depicted in Figure 15-(b)).

As first general result we see that for the first equation of normal form (9) there holds that  $\alpha = 0$  for all points (as expected since the system is periodically forced). We remark that for the computation of the normal form coefficients of the second equation, the tolerances have to be strong enough. The results are

- (on LPC2) The R1 point is in  $(\epsilon, b_0) = (0.6044021, 0.1769608)$ . The period 2 limit cycle Neimark-Sacker curve NS2 starts tangentially to the LPC2 curve and is subcritical. In this situation we therefore expect the product of the two normal form coefficients of the last equation of (8) to be positive; the computed coefficients are  $(a, b) = (2.005489e - 6, 6.4806e + 9)$ , such that  $ab = 1.299679e + 4$ .
- (on PD2) The R2 point is in  $(\epsilon, b_0) = (0.7437713, 0.1835935)$ . On the left side of the R2 point the PD2 curve is supercritical, on the right side it is subcritical. The NS2 curve incoming in the R2 point is subcritical, while the NS4 curve outgoing at the R2 point is supercritical. We are thus in the case depicted in Figure 15-(b) with time reversed. So we expect  $b > 0$  (subcritical incoming Neimark-Sacker curve) and  $a < 0$  (there is an outgoing secondary Neimark-Sacker curve). The computed coefficients at the R2 point are  $(a, b) = (-65.76676, 16.26708)$ . Notice that this point is also a degenerate R1 point for the NS4 curve. In fact, when we compute the normal form coefficient at this 1:1 resonance bifurcation point, we obtain  $(a, b) = (-1.113774823354237e - 4, 6.116846870167980e + 12)$ . In this case the product  $ab = -6.812790e + 8$  is negative, in accordance with the fact that the NS4 curve (for which we have an R1 bifurcation) is supercritical.
- (on PD4) The R2 point is in  $(\epsilon, b_0) = (0.7427991, 0.1861098)$ . On the left side of the R2 point the PD4 curve is supercritical, on the right side it is subcritical. The NS4 curve incoming in the R2 point is supercritical, while the NS8 curve outgoing at the R2 point is subcritical. We are therefore in the case depicted in Figure 15-(b). We expect  $b < 0$  (supercritical incoming Neimark-Sacker curve) and  $a < 0$  (there is an outgoing secondary Neimark-Sacker curve). The computed coefficients at the R2 point are  $(a, b) = (-269.3681, -18.15061)$ .
- (on PD8) The R2 point is in  $(\epsilon, b_0) = (0.7439079, 0.1864190)$ . On the left side of the R2 point the PD8 curve is supercritical, on the right side it is subcritical. The NS8 curve incoming in the R2 point is subcritical: we are thus in the case depicted in Figure 15-(b) with time reversed. Thus, we expect  $b > 0$  (subcritical incoming Neimark-Sacker curve) and  $a < 0$  (there is an outgoing secondary Neimark-Sacker curve, since the cascade continues). The computed coefficients of the R2 point are  $(a, b) = (-921.7011, 16.58059)$ .

All the obtained results are in agreement with the theory.

### 5.1.3 The 1:3 resonance points

There are two 1:3 resonance points, one on NS2, the other one on NS4, as can be seen in Figure 2. These two points behave in a different way. The Neimark-Sacker curve corresponding with the first point at  $(\epsilon, b_0) = (0.709, 0.179)$  is subcritical, so we expect  $\Re(c)$  to be positive. The Neimark-Sacker curve of the second point at  $(\epsilon, b_0) = (0.743, 0.185)$  is supercritical, so  $\Re(c)$  should be negative. To check whether we are in a non degenerate case, we also have to look at  $b$ , however, as mentioned before, the sign of  $b$  is not relevant. We obtain

- for the first R3 point we have that  $(b, \Re(c)) = (4.5567 - 4.4567i, 9.155003)$ .
- for the second R3 point we have that  $(b, \Re(c)) = (0.4049 + 12.1425i, -8.819864)$ .

These results are in accordance with the theory.

### 5.1.4 The 1:4 resonance points

There are two 1:4 resonance points, one on NS2, the other one on NS4, as can be seen in Figure 2. Also these two points behave in the same way as the 1:3 resonance bifurcation points. The Neimark-Sacker curve corresponding with the first point at  $(\epsilon, b_0) = (0.675, 0.177)$  is subcritical, so here we expect  $\Re(A)$  to be positive. The Neimark-Sacker curve of the second point at  $(\epsilon, b_0) = (0.743, 0.185)$  is supercritical, so  $\Re(A)$  should be negative. Moreover, since those points are part of a resonance cascade, we should not have limit point bifurcations of non trivial equilibria, so we are in region I of Figure 17. In order to assure that we are not in a degenerate case, we also need to check that  $d \neq 0$ . We obtain

- for the first R4 point we have that  $(c, d) = (11.624 - 84.897i, 65 + 92.254i)$ , so  $A = 0.102999 - 0.752278i$ .
- for the second R4 point we have that  $(c, d) = (-8.5796 - 414.71i, -416.64 - 489.17i)$ , so  $A = -0.013352 - 0.645406i$ .

So for both bifurcation points the value of  $A$  belongs to region I, and thus the results are in accordance with the theory.

## 5.2 The Steinmetz-Larter model

The following model of the peroxidase-oxidase reaction was studied by Steinmetz and Larter [41]:

$$(153) \quad \begin{cases} \dot{A} &= -k_1 ABX - k_3 ABY + k_7 - k_{-7} A, \\ \dot{B} &= -k_1 ABX - k_3 ABY + k_8, \\ \dot{X} &= k_1 ABX - 2k_2 X^2 + 2k_3 ABY - k_4 X + k_6, \\ \dot{Y} &= -k_3 ABY + 2k_2 X^2 - k_5 Y, \end{cases}$$

where  $A, B, X, Y$  are state variables and  $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ , and  $k_{-7}$  are parameters. We fix all parameters as reported in the following table

Par.	Value	Par.	Value	Par.	Value	Par.	Value
$k_1$	0.1631021	$k_2$	1250	$k_3$	0.046875	$k_4$	20
$k_5$	1.104	$k_6$	0.001	$k_{-7}$	0.1175		

and we perform a bifurcation analysis in the parameter space  $(k_7, k_8)$ . A few curves are reported in Figure 3.

### 5.2.1 The 1:1 resonance points

The two 1:1 resonance points have different nature, since in one the Neimark-Sacker curve rooted at the bifurcation point is supercritical, while in the other one it is subcritical.

We obtain:

- for the R1 point in  $(k_7, k_8) = (1.179554, 0.7239571)$ , the two coefficients of the last equation of (8) are equal to  $(a, b) = (-0.003654362200739, 0.735048055230916)$ . Their product  $ab = -2.686132e - 3$  is negative, which corresponds with the fact that the NS curve rooted at the R1 point is supercritical.
- for the R1 point in  $(k_7, k_8) = (1.857676, 0.9304220)$ , the two coefficients of the last equation of (8) are equal to  $(a, b) = (-0.066429738171756, -2.156596806473489)$ . Their product  $ab = 0.1432622$  is positive, and indeed the NS curve rooted at the R1 point is subcritical.

So we can conclude that the results are in accordance with the theory.

### 5.2.2 The Chenciner points

As can be seen in Figure 3 we have detected a CH point at  $(k_7, k_8) = (1.757356, 0.9125773)$ . The normal form coefficient at that bifurcation point equals  $\Re(e) = 1.391931$ , hence positive so the unfolding is the one depicted in Figure 13-(b). In order to verify if the normal form computation is correct, one should use tori continuation techniques [37], either recurring to Poincaré maps [26, 8] or to the so-called *invariance equation* [12, 39, 35, 38]. However, these techniques are not stable, especially in critical cases like the one we have. In order to validate our result we thus have to do simulations.

The obtained result is shown in Figure 4. The indicated regions correspond with the regions from Figure 13. The green curve between regions 2 and 3 is the supercritical Neimark-Sacker curve, the red one between region 1 and 2 is the subcritical Neimark-Sacker curve. For each point of the grid, we have started time integration from a point close to the original limit cycle (a 1 % perturbation) until an attractor was found.



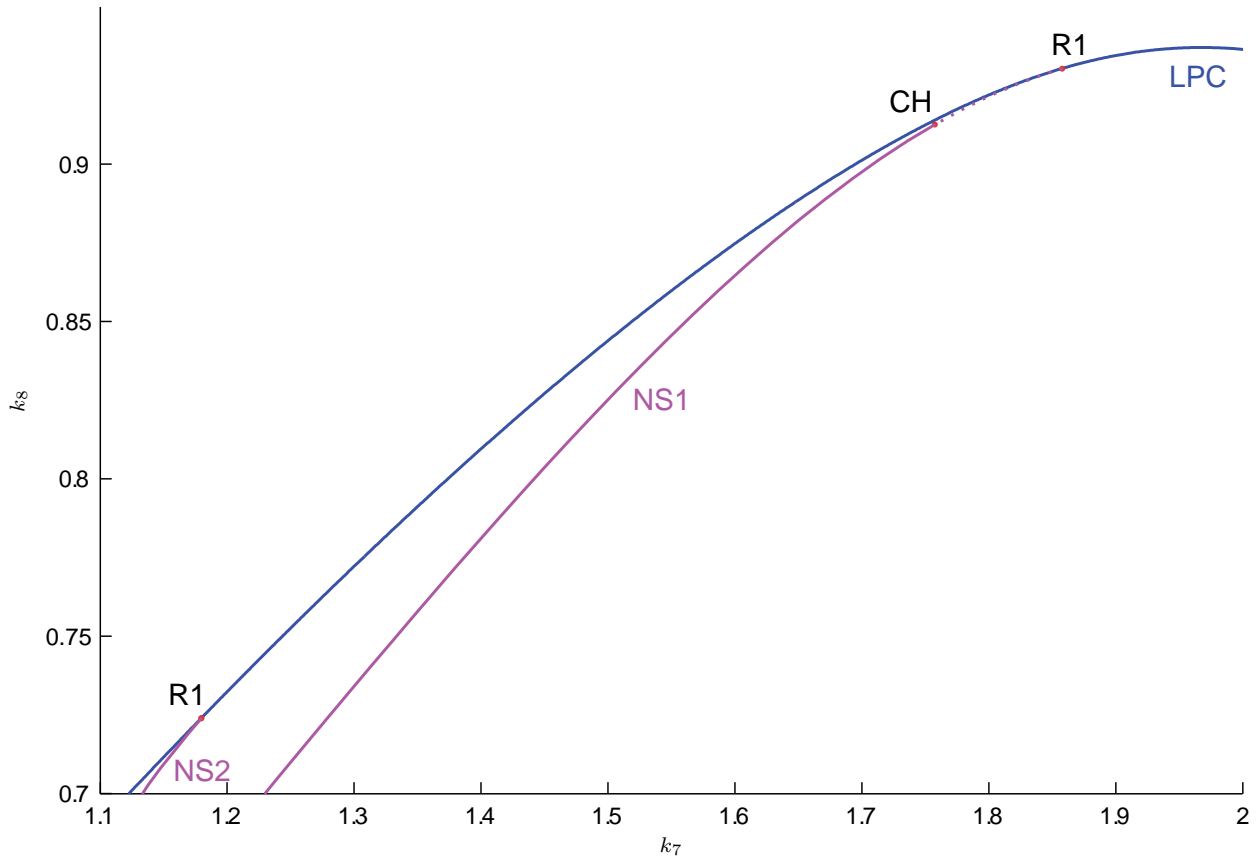


Figure 3: Bifurcation diagram of a limit cycle in model (153). In blue are limit point of cycles bifurcations, green period doubling bifurcations and magenta Neimark-Sacker bifurcations. Continue/dotted curves correspond to supercritical/subcritical bifurcations.

The 1-norm of the  $X$ -coordinate of an orbit with time length 1000 along the attractor is shown in the colormap. In region 2 this attractor is the original limit cycle, in region 3 it is the inner torus arisen through the supercritical Neimark-Sacker curve. In region 1 the original limit cycle is unstable, and so the trajectory which starts nearby converges to another attractor. Between region 1 and 3 and region 1 and 2 happens a catastrophic bifurcation, i.e. a drastic change of the attractor, identified from the change of color which varies from blue to red. Right above the Chenciner point, the catastrophic bifurcation is the subcritical NS curve, while left below it is the limit point of tori ( $T_c$ ) curve. Figure 4 shows that we obtain the scenario which corresponds with a positive second Lyapunov coefficient.

### 5.3 The Lorenz1984 system

This model, taken from [36], is a meteorological model written by Lorenz in 1984 in order to describe the atmosphere. The equations of the model are

$$(154) \quad \begin{cases} \dot{x} = -y^2 - z^2 - ax + aF, \\ \dot{y} = xy - bxz - y + G, \\ \dot{z} = bxy + xz - z, \end{cases}$$

where  $(a, b, F, G)$  are parameters, with  $a = 0.25, b = 4$ . This model, as depicted in [40], has most of the analyzed codimension two bifurcations of limit cycles. We report in Figure 5 a bifurcation diagram obtained

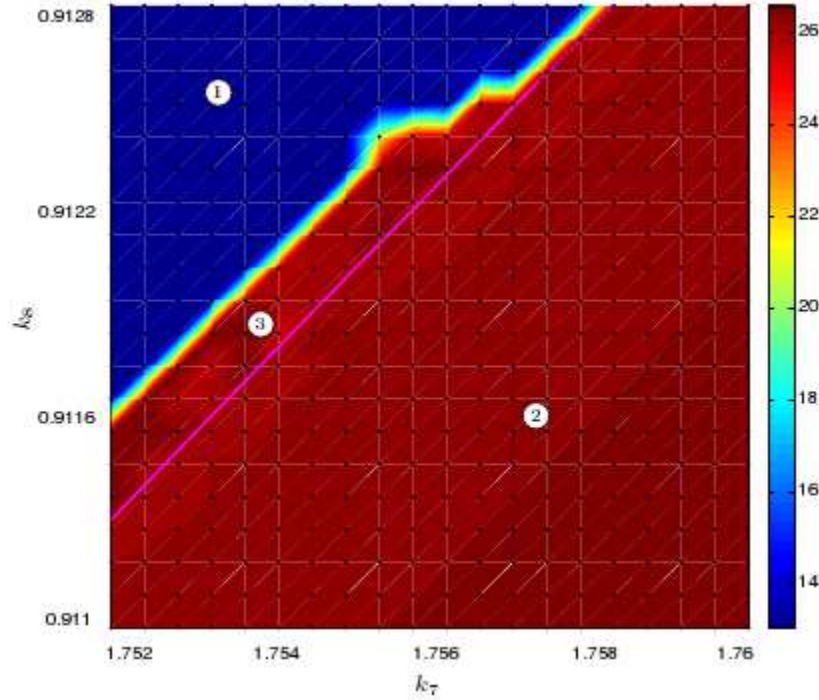


Figure 4: Simulations on a parameter grid (black points) of system (153). The magenta continue/dotted line is the supercritical/subcritical Neimark-Sacker curve. The color represents the value of the maximum of the first coordinate of the attractor reached through simulation from a point close to the limit cycle. A sketch of the state portrait is reported in Figure 13-(b).

with MATCONT in which bifurcations of equilibria (LP stands for limit point, H for Andronov-Hopf) are thicker and limit cycle bifurcations are thin. In particular, the blue curve is a limit point of cycles (LPC) bifurcation curve, the green ones are period-doubling (PD) bifurcations curves and the magenta ones are Neimark-Sacker (NS) curves. The codimension two points are marked with a red dot, and, as can be seen in the figure, almost all cases (except the Chenciner bifurcation and the fold-flip bifurcation) discussed in Section 3 are present in this model. In the sequel of this section we will investigate the normal form coefficients of each bifurcation.

### 5.3.1 The Swallowtail bifurcation

The first degeneracy we want to analyze is the vanishing of the coefficient  $c$  in the cusp of cycles normal form (5). This bifurcation, named Swallowtail bifurcation, is characterized, in our case, by the collision and disappearance of two cusp points of limit cycle. In order to get this codimension three bifurcation we analyze part of the blue dashed curve in Figure 5 for different parameter values of  $b$ . The result is shown in Figure 6: in the graph part of the limit point of cycles manifold is plotted in the  $(F, G)$ -plane for different values of parameter  $b \in [2.91, 2.95]$  (from blue to red). In the table we can see the behavior of the critical normal form coefficient  $c$ , where it exists (the colors of the lines correspond with the bifurcation diagram). Notice how the behavior of this codim 3 bifurcation is captured by a smooth vanishing of the normal form coefficient.

### 5.3.2 The degenerate generalized period doubling bifurcation

On the green curve PD4 of Figure 5 there are two generalized period-doubling (GPD) points. In the first one the flip bifurcation curve passes from subcritical to supercritical, in the second one the opposite happens.

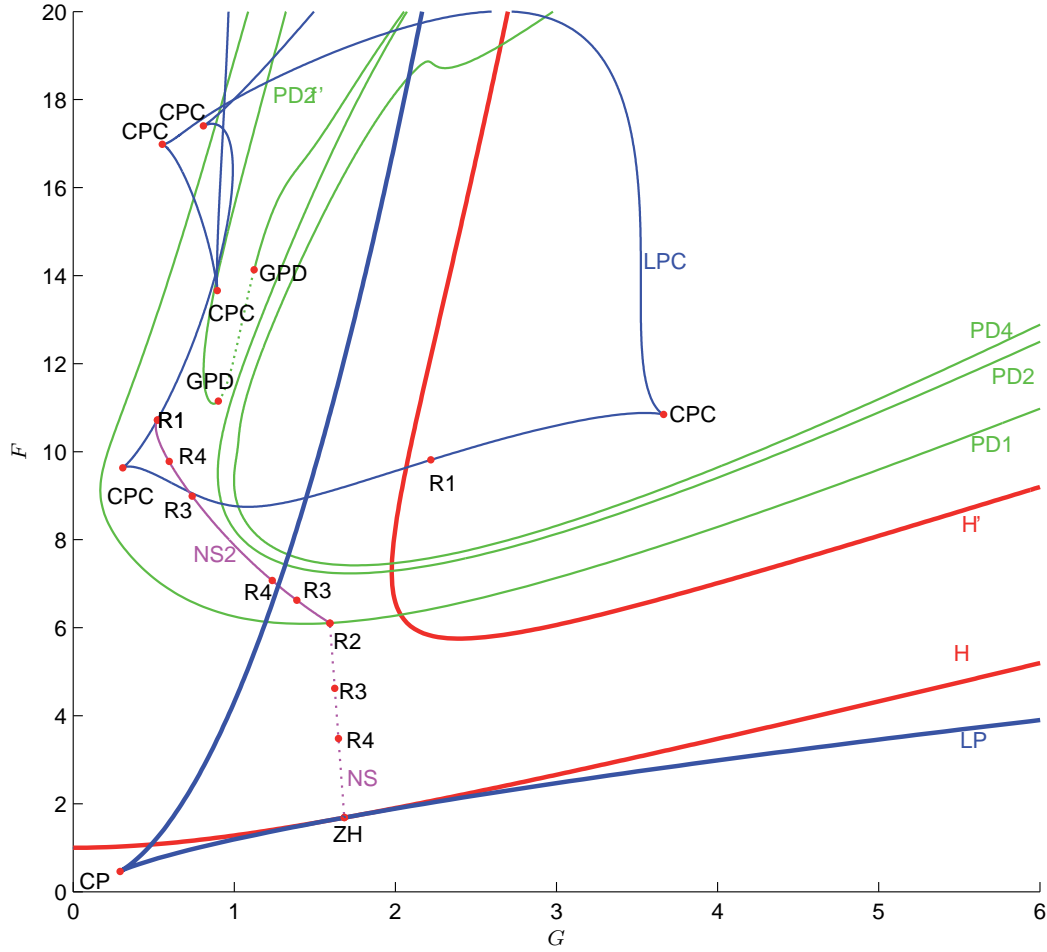


Figure 5: Bifurcation diagram of model (154). The thicker curves are bifurcation curves of equilibria, the thin curves are bifurcation curves of limit cycles and invariant tori (blue, limit point of cycles, green period doubling and magenta Neimark-Sacker). Continue/dotted curves correspond to supercritical/subcritical bifurcations.

Computing the normal form coefficient in the first case gives  $e = -1.317656e - 3 < 0$ , therefore there is a limit point of cycles bifurcation curve that starts rightward tangent to the supercritical part of the period-doubling manifold, while in the second case  $e = 2.895460e - 3 > 0$ , and so the limit point of cycles bifurcation starts leftward tangent to the subcritical part of the PD curve. These conclusions can be seen in Figure 7, where the period-doubling curve is black, dotted when supercritical; in the upper panels are sketched the Poincaré map of the limit cycle involved in the bifurcation. On the yellow curve the limit cycles sketched in green and red collide and disappear, while on the violet curve the two involved limit cycles are sketched in red and blue.

### 5.3.3 The 1:1 resonance points

Two R1 points are located on the LPC curve. Those two points should have different product of the normal form coefficients. In fact, in the first one (in  $(F, G) = (10.72, 0.522)$ ) the Neimark-Sacker curve rooted at the bifurcation point is supercritical (i.e. the system has the behavior depicted in 14-(a)), while in the second one (in  $(F, G) = (9.81, 2.22)$ ) the NS curve is subcritical (behavior similar to Figure 14-(b)). If we apply our analysis we obtain:

- for the first R1 point  $(a, b) = (2.577, -1.2659)$ , so the product  $ab = -3.26237$  is negative.
- for the second R1 point  $(a, b) = (-9.887, -2.005)$ , so the product  $ab = 19.81858e$  is positive.

These results are in accordance with the theory. The blue curve on Figure 8 is the limit point of cycles curve. The violet curves are the Neimark-Sacker curves of first iterate, the green curve is the Neimark-Sacker curve of second iterate.

### 5.3.4 The 1:2 resonance points

At the unique R2 point at  $(F, G) = (10.72, 0.522)$  shown in Figure 5, the incoming Neimark-Sacker curve, namely NS, is subcritical (and so  $b > 0$ ), while the outgoing curve (which exists and so  $a < 0$ ), namely NS2, is

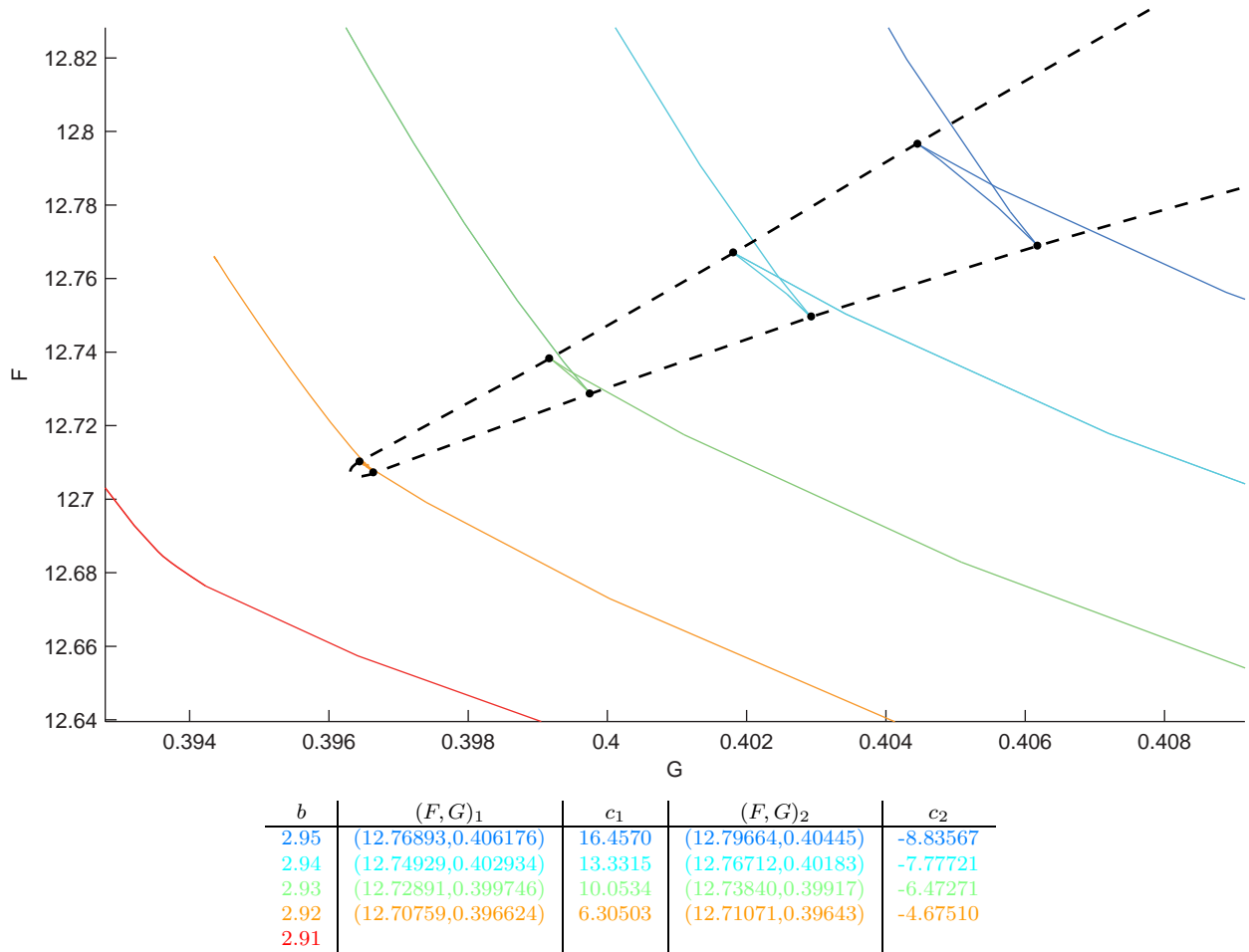


Figure 6: Different limit point of cycles bifurcation curves in the  $(F, G)$ -plane for different values of the third parameter  $b$ . The parameter values are reported in the table.

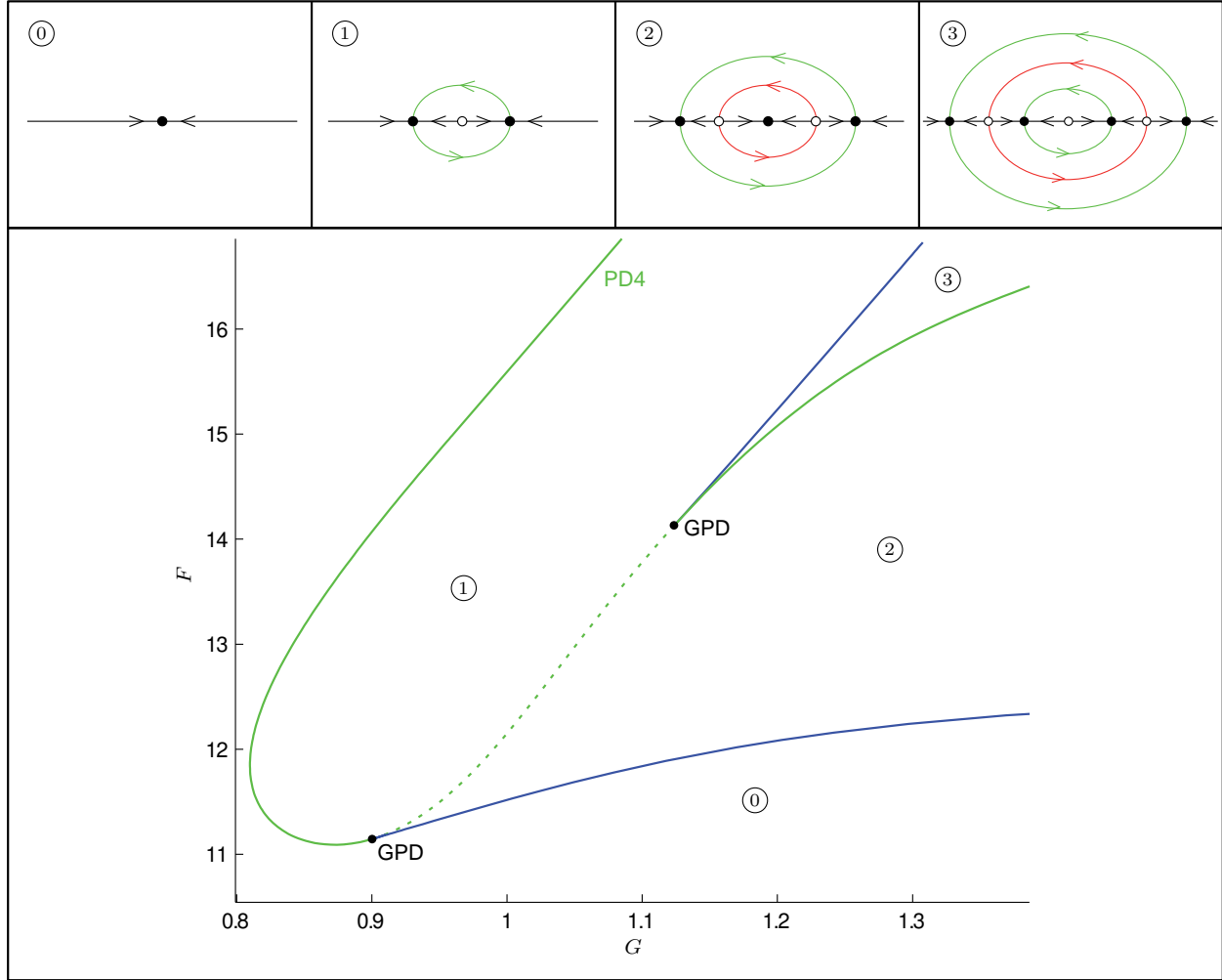


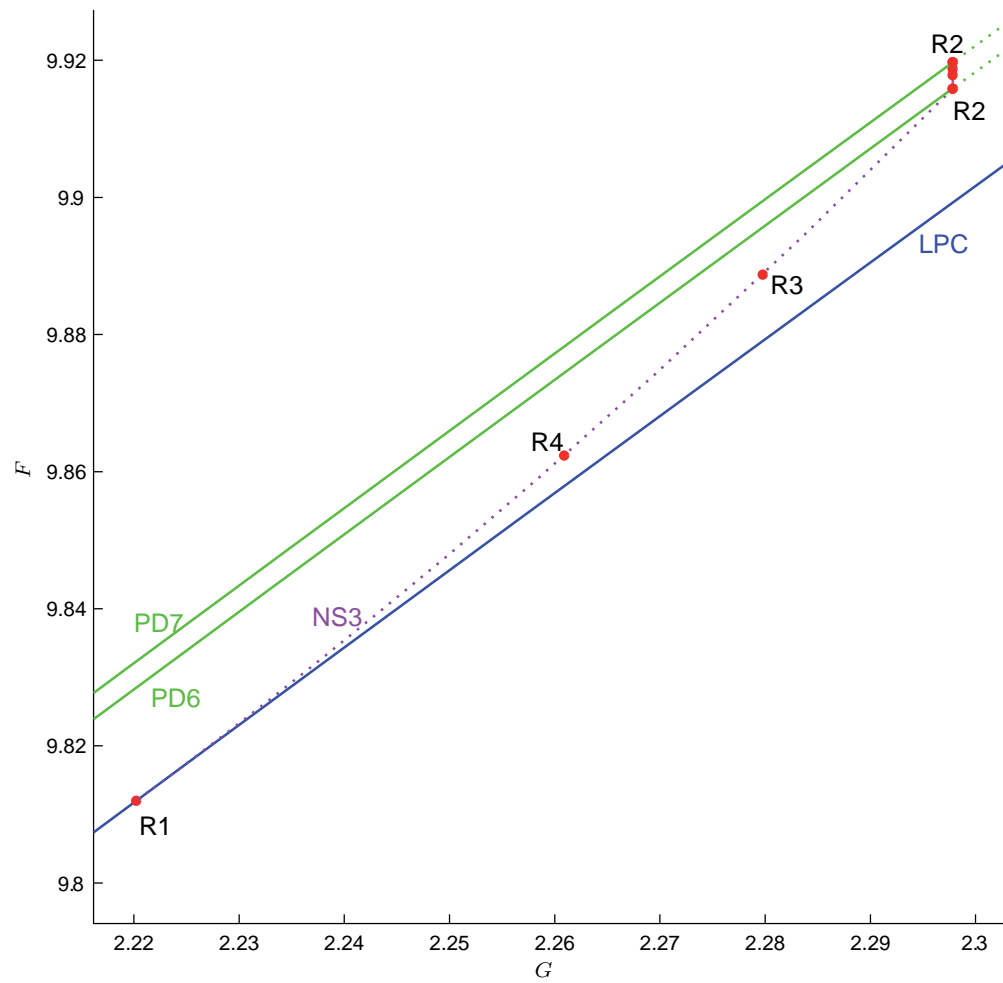
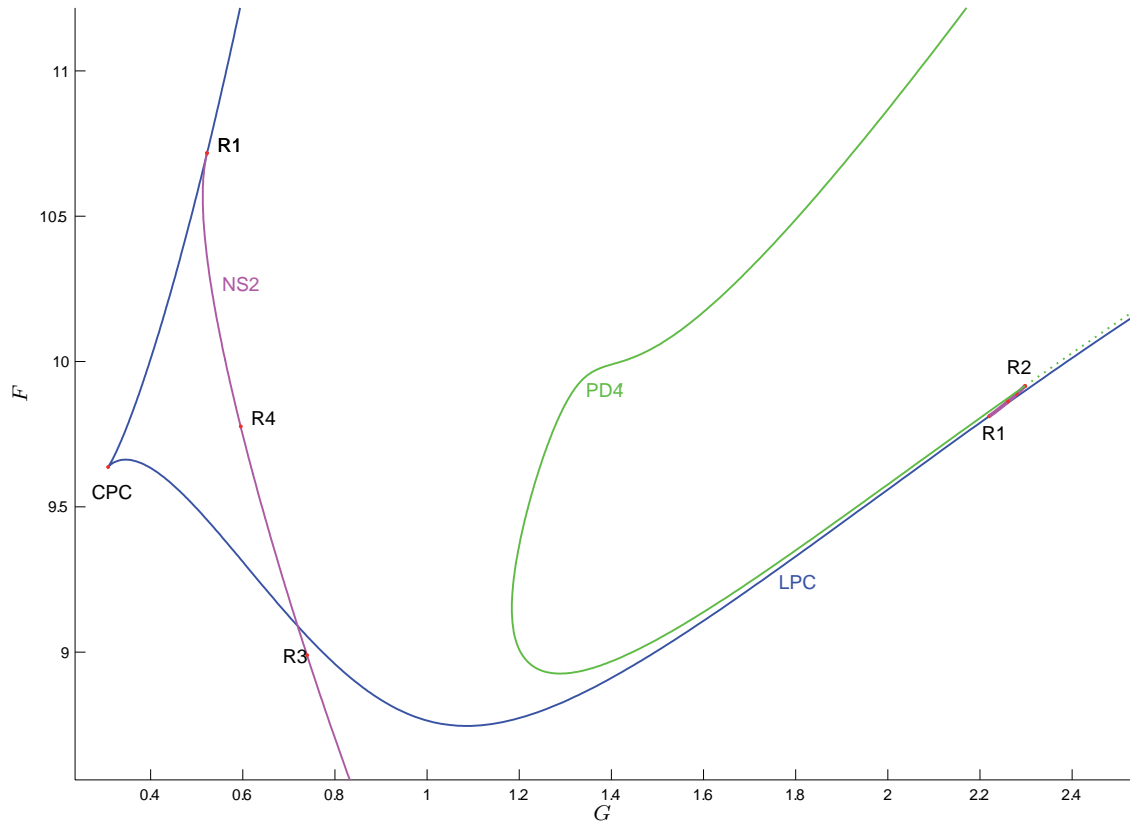
Figure 7: Two generalized period-doubling points with different normal form coefficients on the period-doubling bifurcation curve PD4 of Figure 5.

supercritical, i.e. we are in the case reported in Figure 15-(b) with time reversed. The coefficients computed at the 1:2 resonance point are  $(a, b) = (-0.6329965, 0.1785283)$ , in accordance with the theory.

At the R1 point located at  $(F, G) = (9.81, 2.22)$  starts a resonance cascade, as shown in Figure 8-(b). On that cascade we can find many resonance points which we will analyze in what follows. In particular, since the R2 points belong to a cascade they are of the same type as presented in Figure 15-(b) (so  $a < 0$ ), with at each step a change of criticality of the incoming NS curve. As mentioned before, the first NS curve, born at the R1 point, is subcritical, so for the first R2 point we expect that  $b > 0$ , while for the second one  $b < 0$ . The obtained numerical results are

- for the first R2 point at  $(F, G) = (9.9158, 2.2978)$  we have that  $(a, b) = (-1.3157, 0.11076)$ .
- for the second R2 point at  $(F, G) = (9.9197, 2.2978)$  we have that  $(a, b) = (-2.6228, -0.0564)$ .

Results are in accordance with the theory.



### 5.3.5 The 1:3 resonance points

We have several 1:3 resonance points at which we can have a closer look. There is one **R3** point located on the NS curve and two **R3** points on the NS2 curve. The **R3** point corresponding to the first iterate is at  $(F, G) = (4.628, 1.624)$ , with a positive normal form coefficient of the Neimark-Sacker curve such that we are in the case represented in Figure 16-(b). The **R3** points corresponding to the second iterate are at  $(F, G) = (7.072, 1.235)$  and  $(F, G) = (8.989, 0.7394)$ , where the Neimark-Sacker curve is both times supercritical, so we are in the situation depicted in Figure 16-(a).

- for the **R3** point at  $(F, G) = (4.628, 1.624)$  we have that  $(b, \Re(c)) = (0.1913 - 0.5464i, 6.185900e - 2)$
- for the **R3** point at  $(F, G) = (7.072, 1.235)$  we have that  $(b, \Re(c)) = (-0.4461 - 0.1901i, -3.612302e - 2)$ .
- for the **R3** point at  $(F, G) = (8.989, 0.7394)$  we have that  $(b, \Re(c)) = (-0.1285 + 0.0168i, -1.950822e - 2)$ .

These results are in accordance with the theory.

There are also **R3** points on the cascade (see Figure 8-(b)). The first one corresponds with a subcritical NS curve, while the second one corresponds with a supercritical NS curve.

- for the first **R3** point, in  $(F, G) = (9.8888, 2.2798)$  we have that  $(b, \Re(c)) = (-2.9582 - 0.3599i, 0.7383)$ .
- for the second **R3** point, in  $(F, G) = (9.9187, 2.2978)$  we have that  $(b, \Re(c)) = (2.7447 + 3.5391i, -0.3847)$ .

Also in this case all results are in accordance with the theory.

### 5.3.6 The 1:4 resonance points

There are 5 1:4 resonance points at which we will we have a look. One is located on the NS curve, two others on the NS2 curve and the last two lie on the resonance cascade (see Figure 8-(b)). We obtain

- for the **R4** point at  $(F, G) = (3.376, 1.647)$  we have that  $(c, d) = (0.0501 - 0.0746i, -0.0577 - 0.5422i)$  and so  $A = 0.0918 - 0.1368i$  (subcritical NS curve, case I).
- for the **R4** point at  $(F, G) = (9.777, 0.595)$  we have that  $(c, d) = (-0.0151 - 0.1348i, -0.0266 - 0.0411i)$  and so  $A = -0.307829 - 2.752998i$  (supercritical NS curve, case VIII).
- for the **R4** point at  $(F, G) = (6.620, 1.390)$  we have that  $(c, d) = (-0.0417 - 0.9915i, -0.4283 - 1.0826i)$  and so  $A = -0.035841 - 0.851653i$  (supercritical NS curve, case I).

For the first and the last point no further bifurcation analysis is possible to confirm the correctness of the results (present curves rooted at the point are global bifurcations of limit cycles). Instead it is possible to continue all local bifurcations of limit cycles rooted at the second **R4** point, obtaining the result shown in Figure 9. Note that we haven't made the distinction between region VII and VIII since we have not computed the fold of torus curve typical for region VIII.

For the cascade (see Figure 8-(b)) we have that the first point is on a subcritical NS curve, while the second one is on a supercritical NS curve. Moreover, since they lie on a cascade, they should be of type I.

- for the first **R4** point at  $(F, G) = (9.9159, 2.2978)$  we have that  $(c, d) = (0.0518 - 1.7633i, -2.0143 + 0.4546i)$  and so  $A = 0.025102 - 0.853919i$  (subcritical NS curve, case I).
- for the first **R4** point at  $(F, G) = (9.9197, 2.2978)$  we have that  $(c, d) = (-0.0282 - 6.8152i, -10.8446 + 2.1458i)$  and so  $A = -0.002550 - 0.616491i$  (supercritical NS curve, case I).

Also in this case the results are in accordance with the theory.

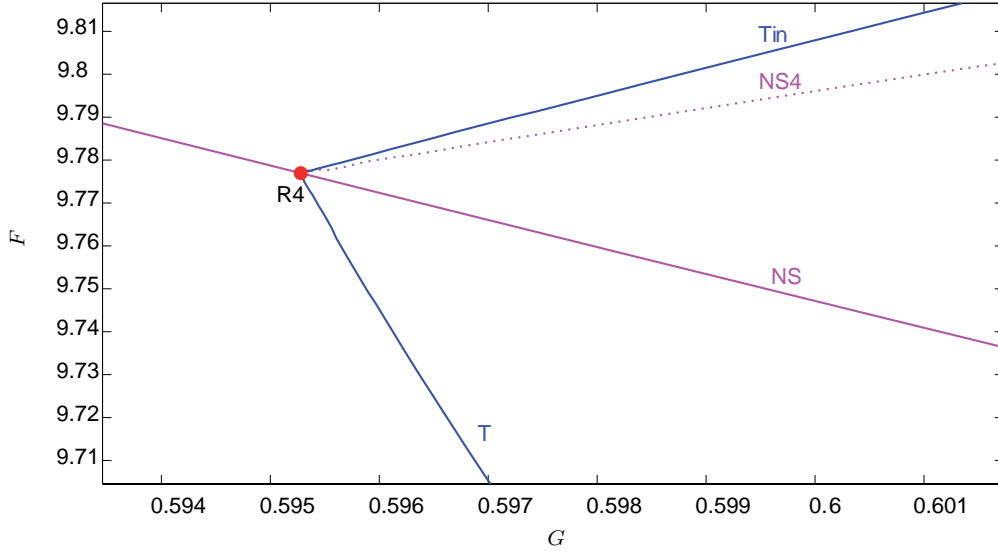


Figure 9: Bifurcation diagram at the R4 point at  $(F, G) = (9.777, 0.595)$ . In blue are the limit point of cycles bifurcation curves, in violet the Neimark-Sacker curves. Continue/dotted curves correspond to supercritical/subcritical curves.

## 5.4 The Extended Lorenz1984 system

As done in [33], it is possible to extend the Lorenz1984 system (154) by adding a fourth variable which takes the influence on the jet stream and the baroclinic waves of external parameters like the temperature of the sea surface into account. The obtained system is

$$(155) \quad \begin{cases} \dot{x} = -y^2 - z^2 - ax + aF - \gamma u^2, \\ \dot{y} = xy - bxz - y + G, \\ \dot{z} = bxy + xz - z, \\ \dot{u} = -\delta u + \gamma ux + K. \end{cases}$$

We use the parameter values mentioned in [33], i.e.

$$a = 0.25, b = 1, G = 0.2, \delta = 1.04, \gamma = 0.987, F = 1.75, K = 0.0003.$$

Simulating this system from the trivial initial condition leads to a limit cycle. In a continuation in  $K$  that limit cycle undergoes a subcritical period-doubling bifurcation. Now, we can do a two parameter continuation in  $(F, K)$  and draw the bifurcation diagram reported in Figure 10.

### 5.4.1 The fold-flip point

As can be seen in Figure 10, a fold-flip point is detected for  $(F, K) = (1.7620, 0.2806 \times 10^{-3})$ . Since there is a Neimark-Sacker curve of the period doubled limit cycle rooted at the bifurcation point and the NS curve and the LPC curve lie on different sides of the PD curve, we are in the case represented in Figure 19-(a), i.e. we have  $a_{20}b_{11} < 0$  and  $a_{02}b_{11} < 0$ . Moreover, since the NS curve is supercritical,  $L_{NS}$  should be negative. Numerically, we obtain that  $b_{11} = 562.2215$ ,  $a_{20} = -0.5756$ ,  $a_{02} = -0.1036e - 3$ ,  $L_{NS} = -178.8596$ . Hence, these results are in accordance with the theory.



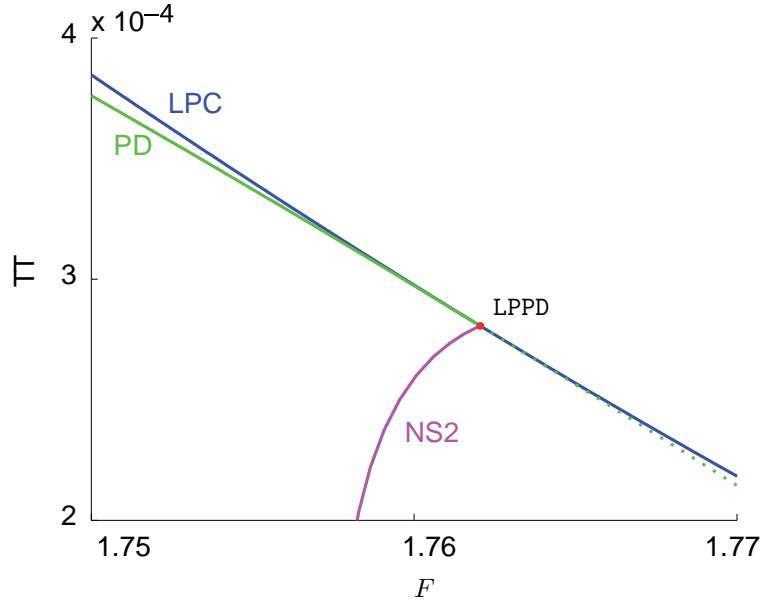


Figure 10: Bifurcation diagram of a limit cycle in model (155). The blue curve is a limit point of cycles bifurcation, the green curve is a period-doubling curve (continue/dotted curve correspond to supercritical/subcritical curves) and the magenta curve is a supercritical Neimark-Sacker bifurcation curve of the period doubled limit cycle.

## A Derivation of the normal forms

### A.1 Notation

Let  $M \in \mathbb{R}^{n \times n}$  be the monodromy matrix. In all codimension 2 cases all critical multipliers, i.e. all multipliers with modulus 1, have non-degenerate Jordan blocks. Let  $M_0$  be the critical Jordan structure, i.e. the block diagonal matrix consisting of the critical Jordan blocks, starting with the block of the trivial multiplier 1. Let  $\mu_k = e^{i\theta_k}$  ( $0 \leq \theta_k < 2\pi$ ) be a critical multiplier with multiplicity  $m_k$ . The matrix  $L_k \in \mathbb{R}^{m_k \times m_k}$  is defined as

$$L_k = \begin{pmatrix} \sigma_k & 1 & \dots & 0 \\ 0 & \sigma_k & \dots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \sigma_k \end{pmatrix},$$

where  $\sigma_k$  is the *Floquet exponent of the multiplier*  $\mu_k$ , with  $\sigma_k = i\theta_k/T$  in the case of a positive real multiplier or a complex multiplier  $\mu_k$  and  $\sigma_k = 0$  for  $\mu_k = -1$ . The matrix  $L_0$  is the block diagonal matrix formed from the blocks  $L_k$  for which  $|\mu_k| = 1$ , starting with the block that corresponds with multiplier 1. The matrix  $\tilde{L}_0$  is the matrix  $L_0$  without the first row and the first column.

### A.2 Bifurcations with 2 critical eigenvalues

#### A.2.1 CPC

At the CPC bifurcation the monodromy matrix has the critical Jordan structure

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

i.e. the multiplier  $\mu = 1$  is double non semi-simple. According to the proposed notation  $\sigma = 0$  and thus

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{L}_0 = 0.$$

We are in a situation where we can apply Theorem 2 from [23]. In particular, we have that

$$\frac{d\tau}{dt} = 1 + \xi + p(\tau, \xi), \quad \frac{d\xi}{d\tau} = \tilde{L}_0\xi + P(\tau, \xi),$$

where  $\tau$  plays the role of phase coordinate along the orbit. The polynomials  $p$  and  $P$  are  $T$ -periodic in  $\tau$  and at least quadratic in  $\xi$  such that

$$\frac{d}{d\tau}p(\tau, \xi) - \frac{d}{d\xi}p(\tau, \xi)\tilde{L}_0^*\xi = 0, \quad \frac{d}{d\tau}P(\tau, \xi) + \tilde{L}_0^*P(\tau, \xi) - \frac{d}{d\xi}P(\tau, \xi)\tilde{L}_0^*\xi = 0$$

for all  $\tau$  and  $\xi \in \mathbb{R}$ . Putting  $\tilde{L}_0 = 0$  we obtain

$$\frac{d}{d\tau}p(\tau, \xi) = 0, \quad \frac{d}{d\tau}P(\tau, \xi) = 0,$$

i.e. the two polynomials are independent from  $\tau$ . So, by a Taylor expansion of the two polynomials the normal form becomes

$$\begin{cases} \frac{d\tau}{dt} = 1 + \xi + p(\xi) = 1 + \xi + \alpha_1\xi^2 + \alpha_2'\xi^3 + \dots, \\ \frac{d\xi}{d\tau} = P(\xi) = b\xi^2 + c\xi^3 + \dots \end{cases}$$

Applying the chain rule gives

$$\begin{aligned} \frac{d\xi}{dt} &= (b\xi^2 + c\xi^3 + \dots)(1 + \xi + \alpha_1\xi^2 + \alpha_2'\xi^3 + \dots) \\ &= b\xi^2 + b\xi^3 + c\xi^3 + \dots \\ &= c\xi^3 + \dots, \end{aligned}$$

because of the cusp degeneracy condition. In order to obtain the proposed normal form (5), we do the substitution  $\xi \mapsto -\xi$  and find

$$\begin{cases} \frac{d\tau}{dt} = 1 - \xi + \alpha_1\xi^2 + \alpha_2\xi^3 + \dots, \\ \frac{d\xi}{dt} = c\xi^3 + \dots, \end{cases}$$

with  $\alpha_2 = -\alpha_2'$ .

### A.2.2 GPD

At the GPD bifurcation the matrices described in Section A.1 are

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{L}_0 = 0.$$

We are in a case in which we can apply Theorem 3 from [23]. So we have the following  $2T$ -periodic normal form (using the formula of Theorem 1 from [23])

$$\frac{d\tau}{dt} = 1 + p(\tau, \xi), \quad \frac{d\xi}{d\tau} = \tilde{L}_0\xi + P(\tau, \xi),$$

with polynomials  $p$  and  $P$   $2T$ -periodic in  $\tau$  and at least quadratic in  $\xi$  such that

$$\begin{aligned} \frac{d}{d\tau}p(\tau, \xi) - \frac{d}{d\xi}p(\tau, \xi)\tilde{L}_0^*\xi &= 0, & \frac{d}{d\tau}P(\tau, \xi) + \tilde{L}_0^*P(\tau, \xi) - \frac{d}{d\xi}P(\tau, \xi)\tilde{L}_0^*\xi &= 0, \\ p(\tau + T, \xi) &= p(\tau, -\xi), & P(\tau + T, -\xi) &= -P(\tau, \xi). \end{aligned}$$

Putting  $\tilde{L}_0 = 0$  in the first two formulas brings us in the same situation as of the previous case

$$\frac{d}{d\tau}p(\tau, \xi) = 0, \quad \frac{d}{d\tau}P(\tau, \xi) = 0,$$

i.e. the two polynomials are independent of  $\tau$ . This makes it possible to rewrite the last two formulas as

$$p(\xi) = p(-\xi), \quad P(-\xi) = -P(\xi),$$

so polynomial  $p$  is even ( $p = \phi(\xi^2)$ ) and polynomial  $P$  is odd ( $P = \xi\psi(\xi^2)$ ). Therefore, taking the GPD degenerate condition into account, we can write down the first approximation of our normal form

$$\begin{cases} \frac{d\tau}{dt} = 1 + \phi(\xi^2) = 1 + \alpha_1\xi^2 + \alpha_2\xi^4 + \dots, \\ \frac{d\xi}{d\tau} = \xi\psi(\xi^2) = c\xi^3 + e\xi^5 + \dots \end{cases}$$

Applying the chain rules gives

$$\begin{aligned} \frac{d\xi}{dt} &= (c\xi^3 + e\xi^5 + \dots)(1 + \alpha_1\xi^2 + \alpha_2\xi^4 + \dots) \\ &= c\xi^3 + \alpha_1c\xi^5 + e\xi^5 + \dots \\ &= e\xi^5 + \dots, \end{aligned}$$

because of the GPD degeneracy condition. So we obtain the normal form presented in (6), namely

$$\begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1\xi^2 + \alpha_2\xi^4 + \dots, \\ \frac{d\xi}{dt} = e\xi^5 + \dots \end{cases}$$

### A.3 Bifurcations with 3 critical eigenvalues

#### A.3.1 CH

In the CH case the Jordan block associated to the trivial multiplier is one-dimensional. We have

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega T} & 0 \\ 0 & 0 & e^{-i\omega T} \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix} \quad \tilde{L}_0 = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}.$$

This puts us in a situation in which we can apply Theorem 1 from [23]. If we assume that  $\frac{\omega T}{2\pi} \notin \mathbb{Q}$ , then it follows immediately from the results of Example III.9 from [24] that the normal form is given by

$$\begin{cases} \frac{d\tau}{dt} = 1 + \phi(|\xi|^2), \\ \frac{d\xi}{d\tau} = i\omega\xi + \xi\psi(|\xi|^2), \end{cases}$$

where the polynomials  $\phi$  and  $\psi$  are at least linear in their argument.  $\phi$  is real, while  $\psi$  takes values in  $\mathbb{C}$ . We expand the polynomials up to the fifth order, namely

$$\begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1|\xi|^2 + \alpha_2|\xi|^4 + \dots, \\ \frac{d\xi}{d\tau} = i\omega\xi + c'\xi|\xi|^2 + e'\xi|\xi|^4 + \dots, \end{cases}$$

such that the chain rule gives

$$\begin{aligned} \frac{d\xi}{dt} &= (i\omega\xi + c'\xi|\xi|^2 + e'\xi|\xi|^4 + \dots)(1 + \alpha_1|\xi|^2 + \alpha_2|\xi|^4 + \dots) \\ &= i\omega\xi + i\omega\alpha_1\xi|\xi|^2 + c'\xi|\xi|^2 + i\omega\alpha_2\xi|\xi|^4 + \alpha_1c'\xi|\xi|^4 + e'\xi|\xi|^4 + \dots \\ &= i\omega\xi + ic\xi|\xi|^2 + e\xi|\xi|^4 + \dots \end{aligned}$$

Here  $i\omega\alpha_1 + c' = ic$  since the Lyapunov coefficient of the Neimark-Sacker bifurcation is purely imaginary and  $i\omega\alpha_2 + \alpha_1c' + e' = e$ . This gives us normal form (7).

### A.3.2 R1

At the R1 bifurcation the matrices described in Section A.1 are

$$M_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We are in a case in which we can apply Theorem 2 from [23]. So we can define a  $T$ -periodic normal form

$$\frac{d\tau}{dt} = 1 + \xi_1 + p(\tau, \xi), \quad \frac{d\xi}{d\tau} = \tilde{L}_0\xi + P(\tau, \xi),$$

where  $\xi = (\xi_1, \xi_2)$ . The polynomials  $p$  and  $P$  are  $T$ -periodic in  $\tau$  and at least quadratic in  $(\xi_1, \xi_2)$  such that

$$\frac{d}{d\tau}p(\tau, \xi) - \frac{d}{d\xi}p(\tau, \xi)\tilde{L}_0^*\xi = 0, \quad \frac{d}{d\tau}P(\tau, \xi) + \tilde{L}_0^*P(\tau, \xi) - \frac{d}{d\xi}P(\tau, \xi)\tilde{L}_0^*\xi = 0.$$

If we write the polynomials in a Fourier expansion, namely

$$p(\tau, \xi) = \sum_{l=-\infty}^{\infty} p_l(\xi)e^{i\frac{2\pi l\tau}{T}}, \quad P(\tau, \xi) = \sum_{l=-\infty}^{\infty} P_l(\xi)e^{i\frac{2\pi l\tau}{T}},$$

we obtain for any  $l \in \mathbb{Z}$  the following differential equations

$$\begin{aligned} \frac{d}{d\xi}p_l(\xi)\tilde{L}_0^*\xi - i\frac{2\pi l}{T}p_l(\xi) &= 0, \\ \frac{d}{d\xi}P_l(\xi)\tilde{L}_0^*\xi - i\frac{2\pi l}{T}P_l(\xi) - \tilde{L}_0^*P_l(\xi) &= 0. \end{aligned}$$

Putting our  $\tilde{L}_0$  into the equations and writing  $P_l(\xi_1, \xi_2) = (P_l^{(1)}(\xi_1, \xi_2), P_l^{(2)}(\xi_1, \xi_2))$  we can rewrite them as a set of differential equations in variable  $\xi_2$

$$\begin{aligned} \frac{d}{d\xi_2}p_l(\xi_1, \xi_2) &= i\frac{2\pi l}{T\xi_1}p_l(\xi_1, \xi_2), \\ \frac{d}{d\xi_2}P_l^{(1)}(\xi_1, \xi_2) &= i\frac{2\pi l}{T\xi_1}P_l^{(1)}(\xi_1, \xi_2), \\ \frac{d}{d\xi_2}P_l^{(2)}(\xi_1, \xi_2) &= \frac{1}{\xi_1} \left( i\frac{2\pi l}{T}P_l^{(2)}(\xi_1, \xi_2) + P_l^{(1)}(\xi_1, \xi_2) \right). \end{aligned}$$

Since  $p_l(\xi_1, \xi_2)$ ,  $P_l^{(1)}(\xi_1, \xi_2)$  and  $P_l^{(2)}(\xi_1, \xi_2)$  are polynomials, if  $l \neq 0$  the only solution is the trivial one. So  $l$  equals zero and thus the polynomials are  $\tau$  independent. We obtain

$$\frac{d}{d\xi_2} p_0(\xi_1, \xi_2) = \frac{d}{d\xi_2} P_0^{(1)}(\xi_1, \xi_2) = 0, \quad \frac{d}{d\xi_2} P_0^{(2)}(\xi_1, \xi_2) = \frac{1}{\xi_1} P_0^{(1)}(\xi_1, \xi_2).$$

The first two equations show that  $p_0$  and  $P_0^{(1)}$  are independent from  $\xi_2$ , thus

$$p_0(\xi_1) = \phi_0(\xi_1), \quad P_0^{(1)}(\xi_1) = \xi_1 \chi(\xi_1).$$

Integrating the last differential equation gives

$$P_0^{(2)}(\xi_1, \xi_2) = \xi_2 \chi(\xi_1) + \psi(\xi_1).$$

Now we can further simplify our normal form. In fact, since the homological operator associated with  $\tilde{L}_0$  has a two-dimensional null-space, we can make a change of variables such that polynomial  $P_0^{(1)}$  vanishes (see [24]). So we can write

$$\tilde{P}_0^{(1)}(\xi_1) = 0, \quad \tilde{P}_0^{(2)}(\xi_1, \xi_2) = \xi_2 \phi_1(\xi_1) + \phi_2(\xi_1),$$

where  $\phi_1$  and  $\phi_2$  are polynomials satisfying  $\phi_1(0) = \phi_2(0) = \frac{d\phi_2}{d\xi_1} \Big|_{\xi_1=0} = 0$ .

Assembling all the information gives us the following normal form

$$\begin{cases} \frac{d\tau}{dt} = 1 + \xi_1 + \phi_0(\xi_1) = 1 + \xi_1 + \alpha\xi_1^2 + \dots, \\ \frac{d\xi_1}{d\tau} = \xi_2, \\ \frac{d\xi_2}{d\tau} = \xi_2 \phi_1(\xi_1) + \phi_2(\xi_1) = a\xi_1^2 + b\xi_1\xi_2 + \dots \end{cases}$$

Note that the polynomials  $\phi_0$  and  $\phi_2$  are at least quadratic in  $\xi_1$ , while  $\phi_1$  is at least linear in its argument. In order to obtain the normal form presented in (8) we apply the chain rule which gives

$$\begin{aligned} \frac{d\xi_1}{dt} &= \xi_2(1 + \xi_1 + \alpha\xi_1^2 + \dots) \\ &= \xi_2 + \xi_1\xi_2 + \dots \end{aligned}$$

and

$$\begin{aligned} \frac{d\xi_2}{dt} &= (a\xi_1^2 + b\xi_1\xi_2 + \dots)(1 + \xi_1 + \alpha\xi_1^2 + \dots) \\ &= a\xi_1^2 + b\xi_1\xi_2 + \dots \end{aligned}$$

### A.3.3 R2

At the R2 bifurcation the matrices described in Section A.1 are

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We are in a case in which we can apply Theorem 3 from [23]. So we have a  $2T$ -periodic normal form

$$\frac{d\tau}{dt} = 1 + p(\tau, \xi), \quad \frac{d\xi}{d\tau} = \tilde{L}_0\xi + P(\tau, \xi),$$

where  $\xi = (\xi_1, \xi_2)$ . The polynomials  $p$  and  $P$  are  $2T$ -periodic in  $\tau$  and at least quadratic in their argument with

$$(156) \quad \frac{d}{d\tau}p(\tau, \xi) - \frac{d}{d\xi}p(\tau, \xi)\tilde{L}_0^*\xi = 0, \quad \frac{d}{d\tau}P(\tau, \xi) + \tilde{L}_0^*P(\tau, \xi) - \frac{d}{d\xi}P(\tau, \xi)\tilde{L}_0^*\xi = 0,$$

$$(157) \quad p(\tau + T, \xi) = p(\tau, -\xi), \quad P(\tau + T, -\xi) = -P(\tau, \xi).$$

Similar as in the R1 case (since the  $\tilde{L}_0$  matrix is the same) we obtain that all polynomials are independent from  $\tau$ ,  $l$  has to be equal to zero and the first two polynomials  $p$  and  $P^{(1)}$  are independent from  $\xi_2$ .

Since the polynomials obtained are independent from  $\tau$ , we can rewrite (157) as:

$$p(\xi) = p(-\xi), \quad P(-\xi) = -P(\xi),$$

obtaining that polynomial  $p$  is even ( $p(\xi_1) = \phi_0(\xi_1^2)$ ) and the polynomials  $P(\xi_1, \xi_2)$  are odd ( $P^{(1)}(\xi_1) = \xi_1\tilde{\phi}_1(\xi_1^2)$  and  $P^{(2)}(\xi_1, \xi_2) = \xi_2\tilde{\phi}_1(\xi_1^2) + \xi_1\tilde{\phi}_2(\xi_1^2)$ ). Now we can simplify our normal form by changing variables as discussed in the previous section. So we can write

$$\tilde{P}^{(1)}(\xi_1) = 0, \quad \tilde{P}^{(2)}(\xi_1, \xi_2) = \xi_2\phi_1(\xi_1^2) + \xi_1\phi_2(\xi_1^2).$$

Putting all information in the normal form equations gives the system

$$\begin{cases} \frac{d\tau}{dt} = 1 + \phi_0(\xi_1^2) = 1 + \alpha\xi_1^2 + \dots, \\ \frac{d\xi_1}{d\tau} = \xi_2, \\ \frac{d\xi_2}{d\tau} = \xi_2\phi_1(\xi_1^2) + \xi_1\phi_2(\xi_1^2) = a\xi_1^3 + b\xi_1^2\xi_2 + \dots \end{cases}$$

In order to obtain normal form (9) we apply the chain rule

$$\begin{aligned} \frac{d\xi_1}{dt} &= \xi_2(1 + \alpha\xi_1^2 + \dots) \\ &= \xi_2 + \alpha\xi_1^2\xi_2 + \dots, \\ \frac{d\xi_2}{dt} &= (a\xi_1^3 + b\xi_1^2\xi_2 + \dots)(1 + \alpha\xi_1^2 + \dots) \\ &= a\xi_1^3 + b\xi_1^2\xi_2 + \dots \end{aligned}$$

#### A.3.4 R3

This is a simple case, since the Jordan block associated with the trivial multiplier is one-dimensional and -1 is not a multiplier of the critical limit cycle. So we can write

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\frac{2\pi}{3T} & 0 \\ 0 & 0 & -i\frac{2\pi}{3T} \end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix} i\frac{2\pi}{3T} & 0 \\ 0 & -i\frac{2\pi}{3T} \end{pmatrix}.$$

We are in a case in which we can apply Theorem 1 from [23]. So we can define the following  $T$ -periodic normal form

$$\frac{d\tau}{dt} = 1 + p(\tau, z), \quad \frac{dz}{d\tau} = \tilde{L}_0 z + P(\tau, z),$$

where  $z = (z_1, \bar{z}_1)$ . The polynomials  $p$  and  $P$  are  $T$ -periodic in  $\tau$  and at least quadratic in their argument such that

$$\frac{d}{d\tau}p(\tau, z) - \frac{d}{dz}p(\tau, z)\tilde{L}_0^*z = 0, \quad \frac{d}{d\tau}P(\tau, z) + \tilde{L}_0^*P(\tau, z) - \frac{d}{dz}P(\tau, z)\tilde{L}_0^*z = 0.$$

We apply the results obtained in Example III.9 from [24] with  $\omega T/2\pi = 1/3$  to obtain

$$\begin{cases} \frac{d\tau}{dt} = 1 + P_0(|z_1|^2, \bar{z}_1^3 e^{i2\pi\tau/T}, z_1^3 e^{-i2\pi\tau/T}), \\ \frac{dz_1}{d\tau} = \frac{i2\pi}{3T} z_1 + z_1 Q_0(|z_1|^2, \bar{z}_1^3 e^{-i2\pi\tau/T}) + \bar{z}_1^2 e^{i2\pi\tau/T} Q_1(|z_1|^2, \bar{z}_1^3 e^{i2\pi\tau/T}). \end{cases}$$

Defining a new variable  $\xi = e^{-i\frac{2\pi\tau}{3T}} z_1$ , this system can be rewritten as

$$\begin{cases} \frac{d\tau}{dt} = 1 + \phi_0(|\xi|^2, \bar{\xi}^3, \xi^3), \\ \frac{d\xi}{d\tau} = \xi \phi_1(|\xi|^2, \xi^3) + \bar{\xi}^2 \phi_2(|\xi|^2, \bar{\xi}^3), \end{cases}$$

with polynomials  $\phi_0$  and  $\phi_1$  at least linear in their arguments, while  $\phi_2(0) \neq 0$ . Notice that this system is autonomous and equivariant under the rotations of angle  $2\pi/3$ . Expanding the polynomials gives

$$\begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1 |\xi|^2 + \alpha_2 \xi^3 + \alpha_3 \bar{\xi}^3 + \dots, \\ \frac{d\xi}{d\tau} = b \bar{\xi}^2 + c \xi |\xi|^2 + \dots, \end{cases}$$

and thus

$$\begin{aligned} \frac{d\xi}{dt} &= (b \bar{\xi}^2 + c \xi |\xi|^2 + \dots)(1 + \alpha_1 |\xi|^2 + \alpha_2 \xi^3 + \alpha_3 \bar{\xi}^3 + \dots) \\ &= b \bar{\xi}^2 + c \xi |\xi|^2 + \dots, \end{aligned}$$

so normal form (10) is obtained.

### A.3.5 R4

As in the previous case the Jordan block associated with the trivial multiplier is one-dimensional. The matrices in Section A.1 are

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\pi}{2}} & 0 \\ 0 & 0 & e^{-i\frac{\pi}{2}} \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\frac{\pi}{2T} & 0 \\ 0 & 0 & -i\frac{\pi}{2T} \end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix} i\frac{\pi}{2T} & 0 \\ 0 & -i\frac{\pi}{2T} \end{pmatrix}.$$

We can apply Theorem 1 from [23] and define a  $T$ -periodic normal form

$$\frac{d\tau}{dt} = 1 + p(\tau, z), \quad \frac{dz}{d\tau} = \tilde{L}_0 z + P(\tau, z),$$

where  $z = (z_1, \bar{z}_1)$ . The polynomials  $p$  and  $P$  are  $T$ -periodic in  $\tau$  and at least quadratic in their argument such that

$$\frac{d}{d\tau} p(\tau, z) - \frac{d}{dz} p(\tau, z) \tilde{L}_0^* z = 0, \quad \frac{d}{d\tau} P(\tau, z) + \tilde{L}_0^* P(\tau, z) - \frac{d}{dz} P(\tau, z) \tilde{L}_0^* z = 0.$$

Again, we use Example III.9 from [24] with  $\omega T/2\pi = 1/4$  and obtain

$$\begin{cases} \frac{d\tau}{dt} = 1 + P_0(|z_1|^2, \bar{z}_1^4 e^{i2\pi\tau/T}, z_1^4 e^{-i2\pi\tau/T}), \\ \frac{dz_1}{d\tau} = \frac{i\pi}{2T} z_1 + z_1 Q_0(|z_1|^2, \bar{z}_1^4 e^{-i2\pi\tau/T}) + \bar{z}_1^3 e^{i2\pi\tau/T} Q_1(|z_1|^2, \bar{z}_1^4 e^{i2\pi\tau/T}). \end{cases}$$

Defining a new variable  $\xi = e^{-i\frac{\pi\tau}{2T}} z_1$ , the system can be rewritten as

$$\begin{cases} \frac{d\tau}{dt} = 1 + \phi_0(|\xi|^2, \bar{\xi}^4, \xi^4), \\ \frac{d\xi}{d\tau} = \xi\phi_1(|\xi|^2, \xi^4) + \bar{\xi}^3\phi_2(|\xi|^2, \bar{\xi}^4), \end{cases}$$

with polynomials  $\phi_0$  and  $\phi_1$  at least linear in their arguments, while  $\phi_2(0) \neq 0$ . Notice that this system is autonomous and equivariant under the rotations of angle  $\pi/2$ . Expanding the polynomials gives

$$\begin{cases} \frac{d\tau}{dt} = 1 + \alpha_1|\xi|^2 + \alpha_2\xi^4 + \alpha_3\bar{\xi}^4 + \dots, \\ \frac{d\xi}{d\tau} = c\xi|\xi|^2 + d\bar{\xi}^3 + \dots, \end{cases}$$

We still have to transform the second equation, namely

$$\begin{aligned} \frac{d\xi}{dt} &= (c\xi|\xi|^2 + d\bar{\xi}^3 + \dots)(1 + \alpha_1|\xi|^2 + \alpha_2\xi^4 + \alpha_3\bar{\xi}^4 + \dots) \\ &= c\xi|\xi|^2 + d\bar{\xi}^3 + \dots, \end{aligned}$$

so normal form (11) is obtained.

### A.3.6 LPPD

At the LPPD bifurcation the matrices from Section A.1 are

$$M_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{L}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We are in a case in which we can apply Theorem 3 from [23]. So we can define a  $2T$ -periodic normal form

$$\frac{d\tau}{dt} = 1 + \xi_1 + p(\tau, \xi), \quad \frac{d\xi}{d\tau} = \tilde{L}_0\xi + P(\tau, \xi),$$

where  $\xi = (\xi_1, \xi_2)$ . The polynomials  $p$  and  $P$  are  $2T$ -periodic in  $\tau$  and at least quadratic in their argument such that

$$(158) \quad \frac{d}{d\tau}p(\tau, \xi) - \frac{d}{d\xi}p(\tau, \xi)\tilde{L}_0^*\xi = 0, \quad \frac{d}{d\tau}P(\tau, \xi) + \tilde{L}_0^*P(\tau, \xi) - \frac{d}{d\xi}P(\tau, \xi)\tilde{L}_0^*\xi = 0,$$

$$(159) \quad p(\tau + T, \xi_1, \xi_2) = p(\tau, \xi_1, -\xi_2),$$

$$(160) \quad P^{(1)}(\tau + T, \xi_1, -\xi_2) = P^{(1)}(\tau, \xi_1, \xi_2), \quad P^{(2)}(\tau + T, \xi_1, -\xi_2) = -P^{(2)}(\tau, \xi_1, \xi_2).$$

By putting  $\tilde{L}_0$  into (158), we obtain

$$\frac{d}{d\tau}p(\tau, \xi_1, \xi_2) = \frac{d}{d\tau}P^{(1)}(\tau, \xi_1, \xi_2) = \frac{d}{d\tau}P^{(2)}(\tau, \xi_1, \xi_2) = 0,$$

so our polynomials are independent from  $\tau$ . Then, using (159) and (160), there holds

$$p(\xi_1, \xi_2) = p(\xi_1, -\xi_2), \quad P^{(1)}(\xi_1, -\xi_2) = P^{(1)}(\xi_1, \xi_2), \quad P^{(2)}(\xi_1, -\xi_2) = -P^{(2)}(\xi_1, \xi_2),$$

so the polynomials are of the following form

$$\begin{aligned} p &= \chi_1(\xi_1) + \chi_2(\xi_2^2)(1 + \chi_3(\xi_1)), \\ P^{(1)} &= \psi_1(\xi_1) + \psi_2(\xi_2^2)(1 + \psi_2(\xi_1)), \\ P^{(2)} &= \xi_2\varphi_1(\xi_1) + \xi_2\varphi_2(\xi_2^2)(1 + \varphi_3(\xi_1)), \end{aligned}$$



with  $\chi_1$  and  $\psi_1$  at least quadratic in their argument and all the other polynomials at least linear in their argument.

Assembling all the information gives the following system

$$\left\{ \begin{array}{l} \frac{d\tau}{dt} = 1 + \xi_1 + \chi_1(\xi_1) + \chi_2(\xi_2^2)(1 + \chi_3(\xi_1)) \\ \quad = 1 + \xi_1 + \alpha_{20}\xi_1^2 + \alpha_{02}\xi_2^2 + \alpha'_{30}\xi_1^3 + \alpha'_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_1}{d\tau} = \psi_1(\xi_1) + \psi_2(\xi_2^2)(1 + \psi_3(\xi_1)) \\ \quad = a'_{20}\xi_1^2 + a'_{02}\xi_2^2 + a'_{30}\xi_1^3 + a'_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_2}{d\tau} = \xi_2\varphi_1(\xi_1) + \xi_2\varphi_2(\xi_2^2)(1 + \varphi_3(\xi_1)) \\ \quad = b'_{11}\xi_1\xi_2 + b'_{21}\xi_1^2\xi_2 + b_{03}\xi_2^3 + \dots. \end{array} \right.$$

Applying the chain rule gives

$$\begin{aligned} \frac{d\xi_1}{dt} &= a'_{20}\xi_1^2 + a'_{02}\xi_2^2 + a'_{30}\xi_1^3 + a'_{20}\xi_1^3 + a'_{12}\xi_1\xi_2^2 + a'_{02}\xi_1\xi_2^2 + \dots \\ &= a'_{20}\xi_1^2 + a'_{02}\xi_2^2 + a_{30}\xi_1^3 + a_{12}\xi_1\xi_2^2 + \dots, \end{aligned}$$

with  $a_{30} = a'_{30} + a'_{20}$ ,  $a_{12} = a'_{12} + a'_{02}$ , and

$$\begin{aligned} \frac{d\xi_2}{dt} &= b'_{11}\xi_1\xi_2 + b'_{21}\xi_1^2\xi_2 + b'_{11}\xi_1^2\xi_2 + b_{03}\xi_2^3 + \dots \\ &= b'_{11}\xi_1\xi_2 + b_{21}\xi_1^2\xi_2 + b_{03}\xi_2^3 + \dots, \end{aligned}$$

with  $b_{21} = b'_{21} + b'_{11}$ . Now, we do the substitution  $\xi_1 \mapsto -\xi_1$  which gives

$$\left\{ \begin{array}{l} \frac{d\tau}{dt} = 1 - \xi_1 + \alpha_{20}\xi_1^2 + \alpha_{02}\xi_2^2 - \alpha'_{30}\xi_1^3 - \alpha'_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_1}{dt} = -a'_{20}\xi_1^2 - a'_{02}\xi_2^2 + a_{30}\xi_1^3 + a_{12}\xi_1\xi_2^2 + \dots, \\ \frac{d\xi_2}{dt} = -b'_{11}\xi_1\xi_2 + b_{21}\xi_1^2\xi_2 + b_{03}\xi_2^3 + \dots. \end{array} \right.$$

By putting  $\alpha_{30} = -\alpha'_{30}$ ,  $\alpha_{12} = -\alpha'_{12}$ ,  $a_{20} = -a'_{20}$ ,  $a_{02} = -a'_{02}$ ,  $b_{11} = -b'_{11}$ , we obtain (12).

## B Poincaré maps of the periodic normal forms

### B.1 Bifurcations with 2 critical eigenvalues

#### B.1.1 CPC

If we reparametrize time, we obtain the system

$$\left\{ \begin{array}{l} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi}{d\tau} = \frac{c\xi^3}{1 - \xi + \alpha_1\xi^2 + \alpha_2\xi^3} + \dots. \end{array} \right.$$

Note that we can define a Poincaré map out of system (5) evaluating the solution of this system starting from a point  $(0, \eta)$  at time  $t = T$ . Since the two equations are uncoupled we can consider only the second one. Making one Picard iteration [29], we can construct an approximation of this map as follows

$$\xi_0 = \eta, \quad \xi_1 = \xi_0 + \int_0^T \frac{c\xi_0^3}{1 - \xi_0 + \alpha_1\xi_0^2 + \alpha_2\xi_0^3} dt$$

and expanding  $\xi_1$  in a Taylor series we obtain

$$(161) \quad \eta \mapsto \eta + cT\eta^3 + O(\eta^4).$$

Further iterations do not change this expansion and so (161) is the Poincaré map of system (5). Note that this Poincaré map is similar to the one for the cusp point of fixed points [29]. Therefore, we can conclude that the behavior of the system in the neighborhood of the bifurcation is the same. In particular, referring always to [29], we can draw the bifurcation diagram of the Poincaré map and obtain Figure 11. On the two drawn curves, with label  $T_1$  and  $T_2$ , two limit cycles collide and disappear. The output given by MatCont is the normal form coefficient  $c$ .

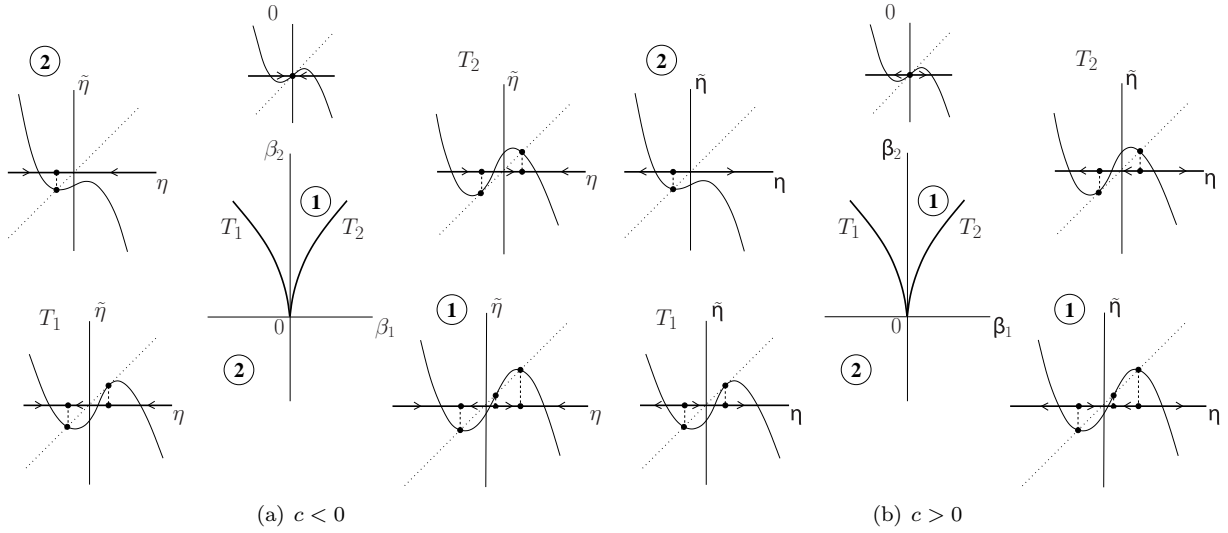


Figure 11: Bifurcation diagram of the cusp bifurcation of the fixed point normal form.

### B.1.2 GPD

First, we have to calculate the second iterate of the normal form of the Generalized flip bifurcation of fixed points:

$$(162) \quad g(v) = -v + \frac{1}{120}\tilde{e}v^5 \implies g^2(\omega) = v - \frac{1}{60}\tilde{e}v^5.$$

Reparametrizing the time of (6) and doing Taylor expansion up to the fifth order gives

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi}{d\tau} = e\xi^5 + \dots, \end{cases}$$

where the dots are  $O(\xi^6)$  terms  $2T$ -periodic in  $\tau$ . Doing one Picard iteration of the second equation up to  $2T$  we obtain the Poincaré map of our normal form

$$\xi \mapsto \xi + 2Te\xi^5 + \dots$$

and see that it is the same map as the second iterate of the generalized period-doubling normal form of fixed points. In particular, since the coefficient of the fifth order term of the second iterate of the normal form of fixed points has opposite sign than the one of limit cycles, we can conclude that the behavior of the system

at the bifurcation is the same but with opposite sign of the normal form coefficient. Moreover, since  $T$  is nonzero, the non-degeneracy condition is  $e \neq 0$ . In fact, if  $e > 0$  we obtain the bifurcation diagram reported in Figure 12-(a), in which the limit point bifurcation of the period doubled limit cycles ( $T^{(2)}$ ) is tangent to the supercritical period-doubling branch (the one in which the normal form coefficient is positive, so labeled as  $F_+^{(1)}$ ), while if  $e$  is negative we are in the opposite situation, reported in Figure 12-(b). The output given by MatCont is the normal form coefficient  $e$ .

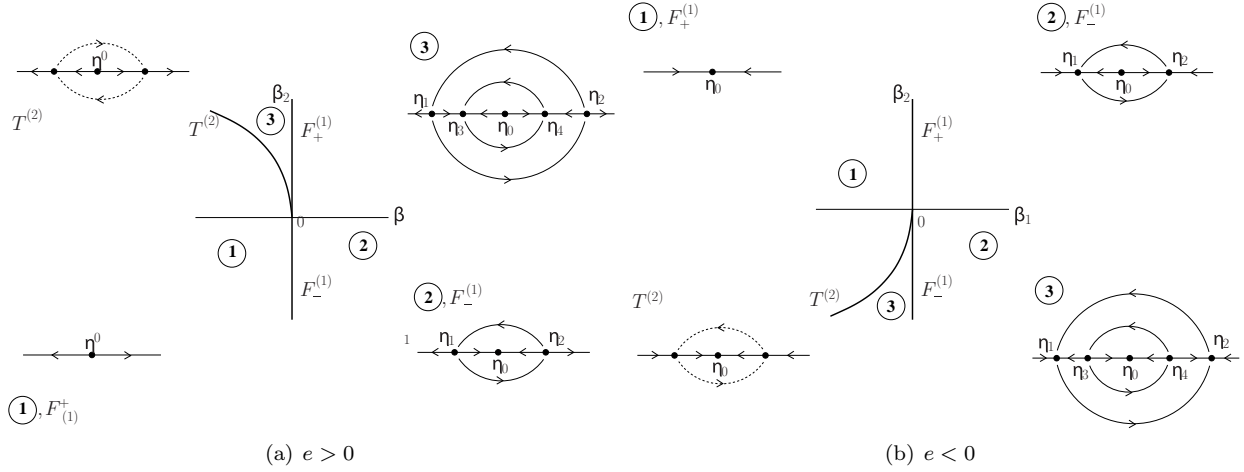


Figure 12: Bifurcation diagram of the degenerate period-doubling point bifurcation of the fixed point normal form.

Remark that the normal form coefficient of the period-doubling bifurcation of limit cycles, computed through periodic normalization as done in [30], has opposite sign of the one for fixed points. Also in this case, the normal form coefficient of the GPD bifurcation of limit cycles has opposite sign of the one for fixed points.

## B.2 Bifurcations with 3 critical eigenvalues

### B.2.1 CH

In this section we will show how the periodic normal form (7) is related to the normal form of the Chenciner bifurcation of fixed points, and how the non-degeneracy condition of the Chenciner bifurcation of limit cycles is related to the Chenciner bifurcation of maps. First note that, if we scale the time, we can rewrite system (7) as

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi}{d\tau} = i\omega\xi + i(c - \alpha_1\omega)\xi|\xi|^2 + (e - i(\alpha_1c - \alpha_1^2\omega + \alpha_2\omega))\xi|\xi|^4 + \dots \end{cases}$$

We need to make a change of variables in order to obtain a quasi-identity flow. Introducing the new complex variable  $z = e^{-i\omega\tau}\xi$ , the second equation can be rewritten as

$$\frac{dz}{d\tau} = i(c - \alpha_1\omega)z^2\bar{z} + (e - i(\alpha_1c - \alpha_1^2\omega + \alpha_2\omega))z^3\bar{z}^2 + \dots$$

Doing two Picard iterations up to time  $T$ , we obtain the rotating Poincaré map of the system

$$\xi \mapsto \xi + iT(c - \alpha_1\omega)\xi^2\bar{\xi} + T\left(e - \frac{c^2T}{2} + \alpha_1cT\omega - \frac{1}{2}\alpha_1^2T\omega^2 + i(\alpha_1^2\omega - \alpha_1c - \alpha_2\omega)\right)\xi^3\bar{\xi}^2 + \dots$$

Note that this system has the same Poincaré map as the Chenciner bifurcation in the fixed point case [29], so we expect the same bifurcation scenario on the Poincaré map of the system. Notice that the real part of the first Lyapunov coefficient is 0, since the Neimark-Sacker bifurcation is degenerate. The sign of the second Lyapunov coefficient  $L_2$  (as defined on page 420 of [29]) determines the bifurcation scenario. However, from (7) we can derive that  $\Re(e) < 0$  corresponds with a stable critical limit cycle and  $\Re(e) > 0$  with an unstable critical limit cycle. Therefore, the case  $\Re(e) < 0$  corresponds with the case  $L_2 < 0$  and  $\Re(e) > 0$  corresponds with  $L_2 > 0$ . So  $\Re(e)$  and the second Lyapunov coefficient  $L_2$  as defined in [29] have the same sign and vanish at the same time. Since both coefficients have the same effect and  $L_2$  requires more computations, we compute  $\Re(e)$  to determine the bifurcation scenario, and in this paper we will call  $\Re(e)$  the second Lyapunov coefficient. The bifurcation diagram in the neighborhood of this codim 2 point can be found by looking at the sign of this value. When  $\Re(e) < 0$  the outer invariant curve is stable and the limit point of tori curve is tangent to the subcritical Neimark-Sacker branch, as shown in Figure 13-(a). When  $\Re(e) > 0$  the outer invariant curve is unstable and the limit point of tori curve is tangent to the supercritical Neimark-Sacker branch, see Figure 13-(b). The output given by MatCont is  $\Re(e)$ .

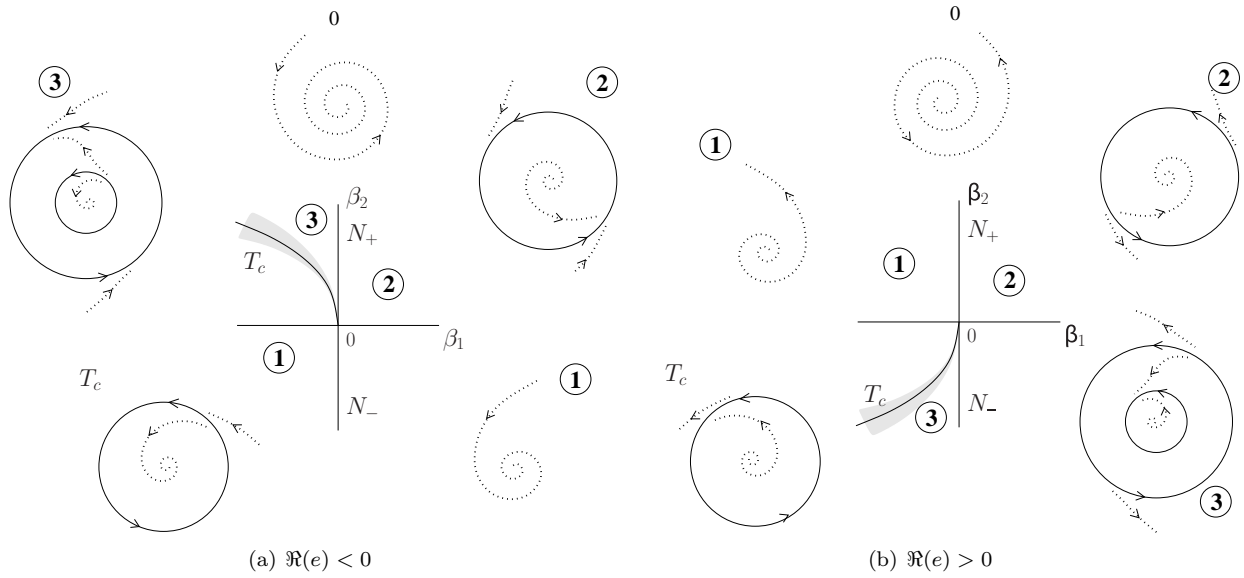


Figure 13: Bifurcation diagram of the generalized Neimark-Sacker bifurcation of the fixed point normal form.

### B.2.2 R1

In this section we will show how the periodic normal form (8) is related to the normal form for the 1:1 resonance of fixed points, which allows us to formulate the non-degeneracy condition. First note that, if we scale the time, we can rewrite system (8) as

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi_1}{d\tau} = \frac{\xi_2 + \xi_1\xi_2}{1 - \xi_1 + \alpha\xi_1^2} + \dots, \\ \frac{d\xi_2}{d\tau} = \frac{a\xi_1^2 + b\xi_1\xi_2}{1 - \xi_1 + \alpha\xi_1^2} + \dots \end{cases}$$

Doing Picard iterations up to time  $T$ , it's possible to show that the generated map is the same, up to second order terms, of the one generated by Picard iterations up to time 1 of the truncated system

$$\begin{cases} \dot{\xi}_1 = T(\xi_2 + 2\xi_1\xi_2), \\ \dot{\xi}_2 = T(a\xi_1^2 + b\xi_1\xi_2). \end{cases}$$

This last system is topological equivalent to the one-time shift ODE which represents the first iteration of the normal form of 1:1 resonance of fixed points, i.e. the system

$$\begin{cases} \dot{\zeta}_1 = \zeta_2, \\ \dot{\zeta}_2 = a_1\zeta_1^2 + b_1\zeta_1\zeta_2, \end{cases}$$

since we can transform one in the other using the following change of variables

$$\begin{cases} \zeta_1 = \xi_1 - \xi_1^2, \\ \zeta_2 = T\xi_2. \end{cases}$$

The bifurcation phenomena in the 1:1 resonance of cycles are the same of those which appear in the 1:1 resonance of fixed points. Notice that

$$a_1 = T^2a, \quad b_1 = Tb,$$

so the non-degeneracy conditions are

$$a \neq 0, \quad b \neq 0,$$

and the cases depend on the sign of the product of  $a$  and  $b$ . In particular, as shown in figure 14, if the two coefficients have different sign the Neimark-Sacker bifurcation (labeled  $H$ ) is supercritical (with negative normal form coefficient), while in the other case it is subcritical. The output given by MatCont is the product of the coefficients  $a$  and  $b$ .

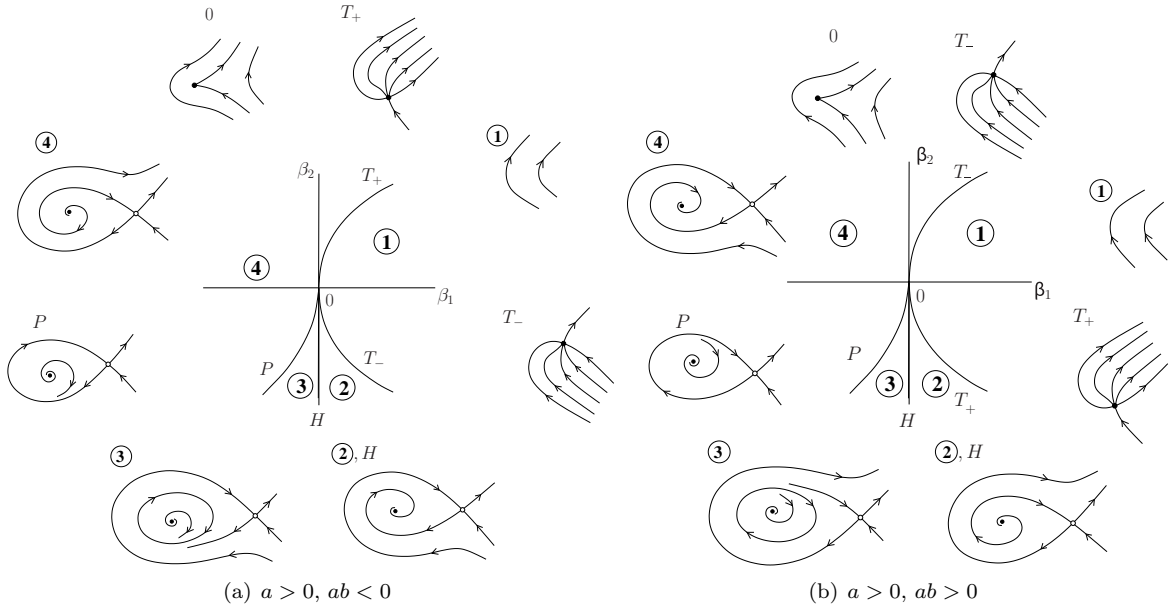


Figure 14: Bifurcation diagram of the 1:1 resonance bifurcation of the fixed point normal form. The other two cases in which  $a < 0$  can be obtained by a reflection around the origin of the state portraits and a horizontal flip of the bifurcation diagrams.

### B.2.3 R2

In this section we will show how the periodic normal form (9) is related with the normal form for the 1:2 resonance of fixed points, which allows us to formulate the non-degeneracy condition. First note that, if we scale the time, we can rewrite system (9) as

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi_1}{d\tau} = \frac{\xi_2 + \alpha\xi_1^2\xi_2}{1 + \alpha\xi_1^2} + \dots, \\ \frac{d\xi_2}{d\tau} = \frac{a\xi_1^3 + b\xi_1^2\xi_2}{1 + \alpha\xi_1^2} + \dots \end{cases}$$

Doing Picard iterations up to time  $2T$ , it's possible to show that the generated map is the same, up to third order terms, of the one generated by Picard iterations up to time 1 of the truncated system

$$\begin{cases} \dot{\xi}_1 = 2T\xi_2, \\ \dot{\xi}_2 = 2T(a\xi_1^3 + b\xi_1^2\xi_2). \end{cases}$$

This last system is topological equivalent to the one-time shift ODE which represents the second iteration of the normal form of 1:2 resonance of fixed points, i.e. the system

$$\begin{cases} \dot{\zeta}_1 = \zeta_2, \\ \dot{\zeta}_2 = a_1\zeta_1^3 + b_1\zeta_1^2\zeta_2, \end{cases}$$

since we can transform one in the other using the following change of variables

$$\begin{cases} \zeta_1 = \xi_1, \\ \zeta_2 = 2T\xi_2. \end{cases}$$

The bifurcation phenomena in the 1:2 resonance of cycles are the same that appear in the corresponding bifurcation of fixed points. Notice that in this case

$$a_1 = 4T^2a, \quad b_1 = 2Tb.$$

so the non-degeneracy conditions are

$$a \neq 0, \quad b \neq 0.$$

We have four different unfoldings as possible bifurcation diagrams, determined by the signs of the coefficients. The ones with negative  $b$  are reported in Figure 15. The other two cases can be obtained by reversing the arrows of the phase portraits and making a vertical flip both of the state portraits and of the bifurcation diagrams. The primary Neimark-Sacker bifurcation (labeled  $H^{(1)}$ ) is supercritical (with negative normal form coefficient) if our coefficient  $b$  is negative, subcritical otherwise. Moreover if  $a < 0$  a secondary Neimark-Sacker bifurcation ( $H^{(2)}$ ) is rooted at the 1:2 resonance point with opposite criticality of the primary one. The output given by MatCont is  $(a, b)$ .

### B.2.4 R3

In this section we will show how the periodic normal form (10) is related with the normal form for the 1:3 resonance of fixed points, which allows us to formulate the non-degeneracy condition. First note that, if we scale the time, we can rewrite system (10) as

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi}{d\tau} = b\bar{\xi}^2 + c\xi|\xi|^2 + \dots \end{cases}$$

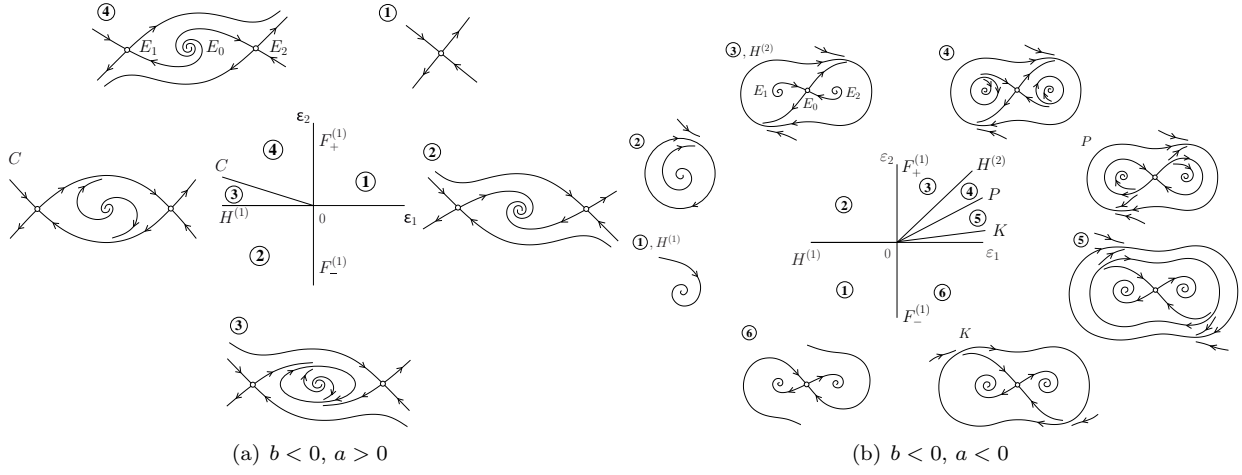


Figure 15: Bifurcation diagram of the 1:2 resonance bifurcation of the fixed point normal form. The other two possible cases in which  $b > 0$  can be obtained by reversing time and making a vertical flip both of the state portraits and of the bifurcation diagrams.

Doing Picard iterations up to time  $3T$ , it's possible to show that the generated map is the same, up to cubic terms, of the one generated by Picard iterations up to time 1 of the truncated system

$$\dot{\xi} = 3Tb\bar{\xi}^2 + 3Tc\xi|\xi|^2.$$

This system is topological equivalent to the one-time shift ODE which represents the third iteration of the normal form of 1:3 resonance of fixed points, i.e. the system

$$\dot{\zeta} = b_1\bar{\zeta}^2 + c_1\zeta|\zeta|^2.$$

The bifurcation phenomena in the 1:3 resonance of cycles are the same of the ones which appear in the 1:3 resonance of fixed points, if the non-degeneracy conditions are satisfied, i.e.

$$b \neq 0, \quad \Re(c) \neq 0.$$

As can be seen in Figure 16, if  $\Re(c) < 0$  the Neimark-Sacker bifurcation (labeled  $N$ ) is supercritical (with negative normal form coefficient), while in the other case it is subcritical. The output given by MatCont is  $(b, \Re(c))$ .

### B.2.5 R4

In this section we will show how the periodic normal form (11) is related with the normal form for the 1:4 resonance of fixed points, which allows us to formulate the non-degeneracy condition. First note that, if we scale the time, we can rewrite system (11) as

$$\begin{cases} \frac{d\tau}{d\tau} = 1, \\ \frac{d\xi}{d\tau} = c\xi|\xi|^2 + d\bar{\xi}^3 + \dots \end{cases}$$

Doing Picard iterations up to time  $4T$ , it's possible to show that the generated map is the same, up to cubic terms, of the one generated by Picard iterations up to time 1 of the truncated system

$$\dot{\xi} = 4Tc\xi|\xi|^2 + 4Td\bar{\xi}^3.$$

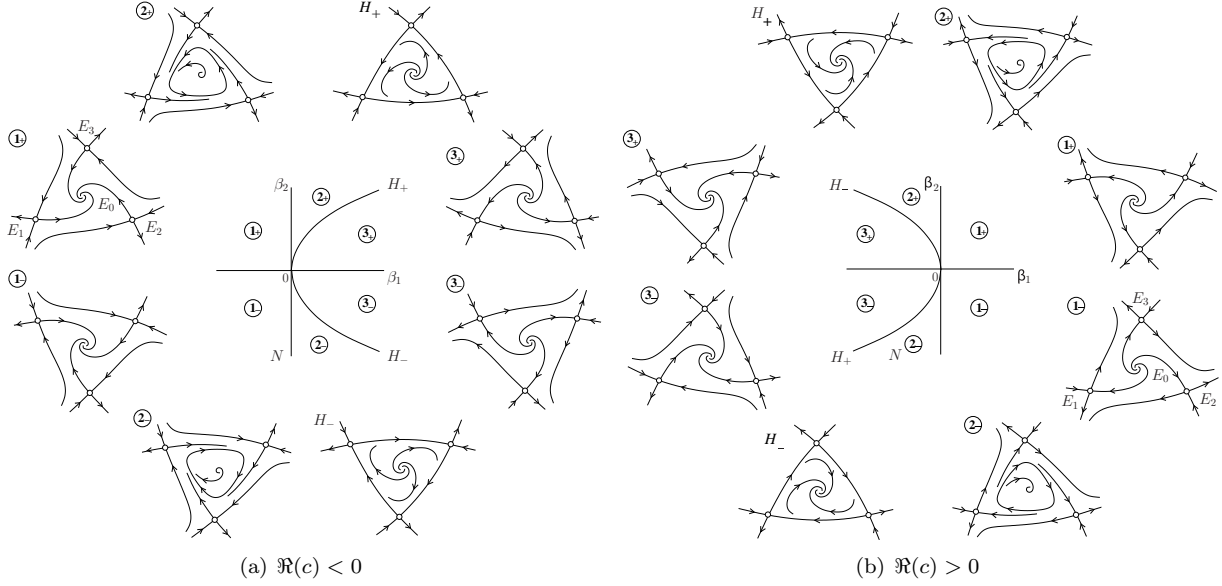


Figure 16: Bifurcation diagram of the 1:3 resonance bifurcation of the fixed point normal form.

This system is topologically equivalent to the one-time shift ODE which represents the fourth iteration of the normal form of 1:4 resonance of fixed points, i.e. the system

$$\dot{\zeta} = c_1 \zeta |\zeta|^2 + d_1 \bar{\zeta}^3.$$

The bifurcation phenomena in the 1:4 resonance of cycles are the ones which appear in the corresponding bifurcation of fixed points, if the non-degeneracy condition is satisfied, i.e.

$$d \neq 0.$$

After defining

$$A = \frac{c}{|d|},$$

we can determine which bifurcations occur by looking at the place of  $A$  in the Gauss plane (see Figure 17). We here report the different possible bifurcation diagrams. A complete investigation can be found in the literature [27, 29].

Many topologically different bifurcation diagrams can be found near a 1:4 resonance point. The analysis, if one excludes higher codimension situations, can be reduced to 22 different cases, which, as mentioned before, depend on the value of  $A$ . First of all, analyzing the normal form, one can divide the Gauss plane into two big regions: in the semiplane  $\Re(A) < 0$  the primary Neimark-Sacker bifurcation is supercritical, in the semiplane  $\Re(A) > 0$  it is subcritical. What happens in the semiplane  $\Re(A) > 0$  can therefore be obtained by inverting the direction of the vector fields. We can further reduce the analysis to the third quadrant of the Gauss plane, since the 12 possible cases are topologically equivalent paired through the transformation  $\zeta \mapsto \bar{\zeta}$ . The different regions are reported in Figure 17, in which only some curves (the continuous lines) are known analytically, the dashed curves are computed numerically.

Figure 18 shows the possible bifurcation diagrams with the sketches of the phase portraits for the Poincaré maps in the case that  $\Re(A) < 0$ . Many local and global bifurcations are involved in the different scenarios. We use the following notation, consistent with the rest of the text:

N: Neimark-Sacker bifurcation. In regions VII and VIII we also have a Neimark-Sacker bifurcation of the period-4 limit cycle.



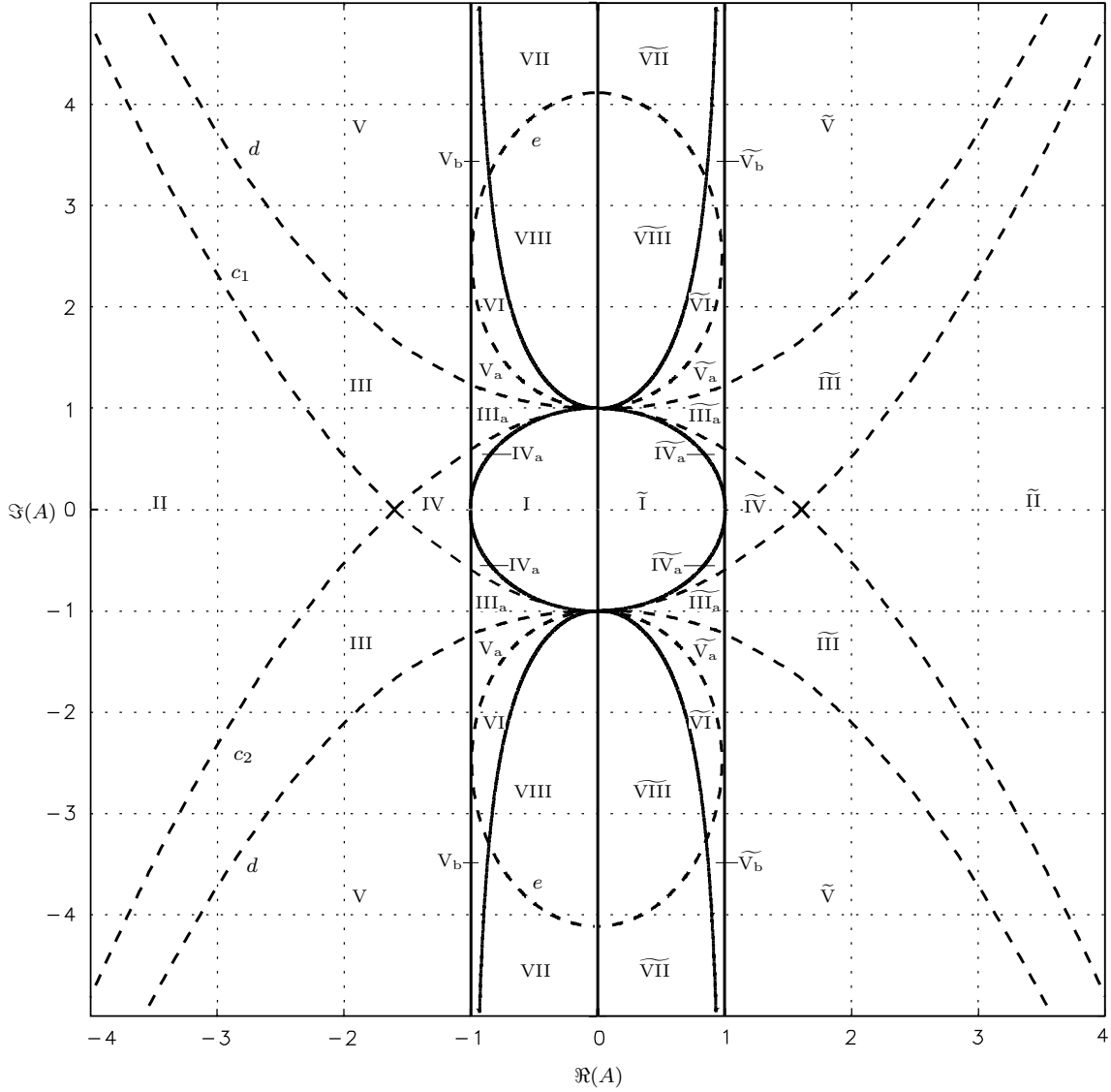


Figure 17: Partitioning of the  $A$  plane into topologically different regions.

T: Fold bifurcation of the period-4 limit cycles. There are three possibilities. Superscript *in*, *on* or *out* means that the bifurcation happens inside, on or outside the invariant curve.

H: Homoclinic connection of the period-4 saddle limit cycle. Superscript *S* means that the born invariant curve is smaller than the limit cycle (a square looking homoclinic connection), *C* that it is bigger (a clover looking homoclinic connection), and *L* means that the born invariant curve is around the period-4 limit cycle; subscript *+* (*-*) means that the saddle quantity is positive (negative), so the born invariant curve is repelling (attracting).

F: Fold bifurcation of the tori.

The output given by MatCont is  $(A, d)$ .

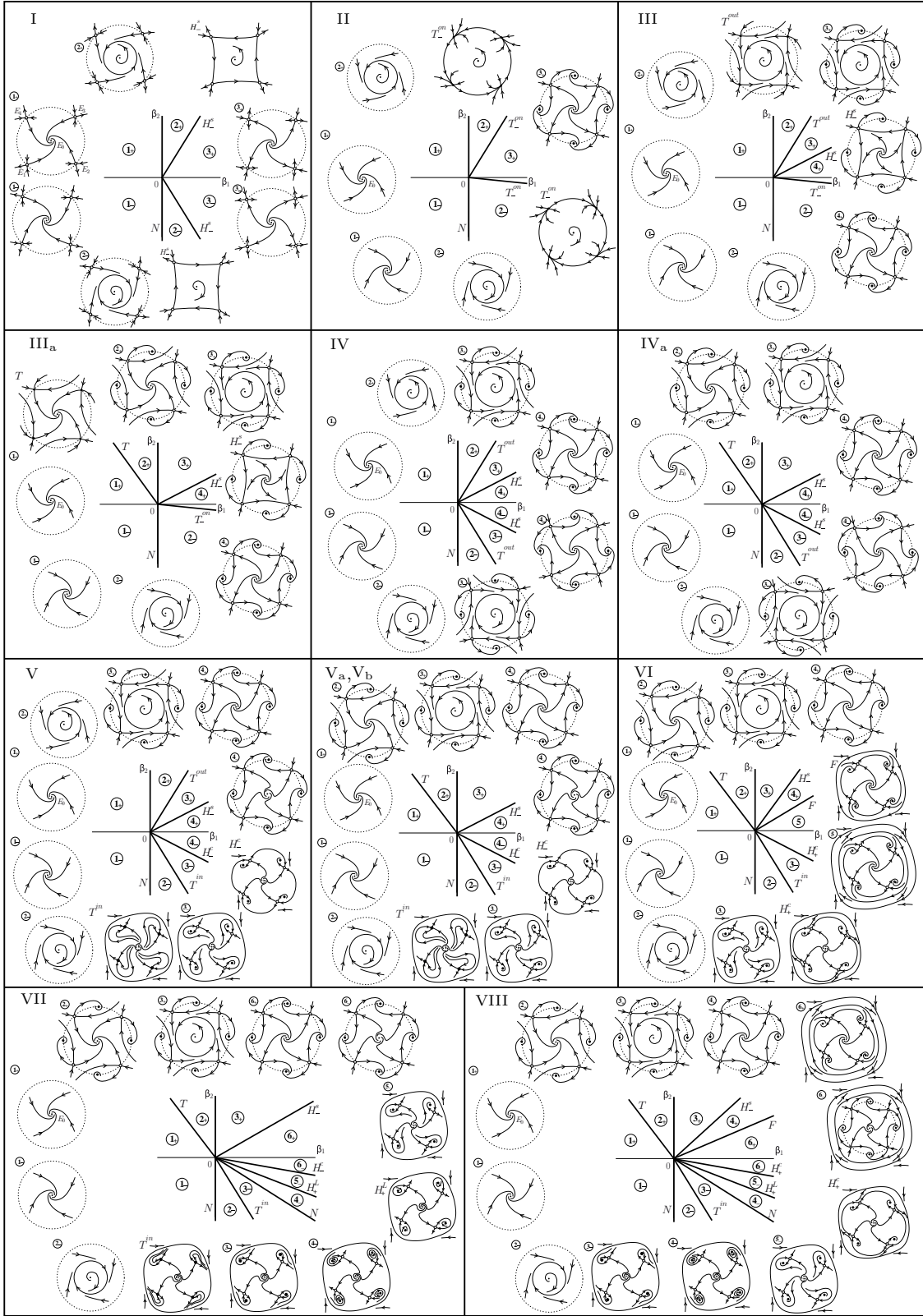


Figure 18: Bifurcation diagrams locally to the 1:4 resonance bifurcation in the different regions of figure 17. The cases in which  $Re(A) > 0$  can be obtained with the transformation  $t \rightarrow -t$ ,  $\beta \rightarrow -\beta$ .

### B.2.6 LPPD

In this section we will show how the periodic normal form (12) is related to the normal form for the fold-flip of fixed points, which allows us to formulate the non-degeneracy condition. As for the R2 case we show the topological equivalence of the  $2T$ -shift map of our system with the 1-shift map of the approximating vector field of the fold-flip normal form for fixed points. First of all we can prove, by scaling the time and using Picard iterations, that the Poincaré map of our normal form is the same, up to cubic terms, as the one-shift map of the truncated system

$$\begin{cases} \dot{\xi}_1 = 2Ta_{20}\xi_1^2 + 2Ta_{02}\xi_2^2 + 2T(a_{30} + a_{20})\xi_1^3 + 2T(a_{12} + a_{02})\xi_1\xi_2^2, \\ \dot{\xi}_2 = 2Tb_{11}\xi_1\xi_2 + 2T(b_{21} + b_{11})\xi_1^2\xi_2 + 2Tb_{03}\xi_2^3. \end{cases}$$

This system is topologically equivalent with the one-time shift ODE which represents the second iteration of the normal form of the fold-flip bifurcation of fixed points, i.e. the system

$$(163) \quad \begin{cases} \dot{\zeta}_1 = a_1\zeta_1^2 + b_1\zeta_2^2 + (c_1 - a_1^2)\zeta_1^3 + (d_1 - a_1b_1 + b_1)\zeta_1\zeta_2^2, \\ \dot{\zeta}_2 = -\zeta_1\zeta_2 + \frac{1}{2}(a_1 - 1)\zeta_1^2\zeta_2 + \frac{1}{2}b_1\zeta_2^3, \end{cases}$$

since this second system can be obtained (neglecting higher order terms) from the first one using the transformation

$$\begin{cases} \zeta_1 = -2b_{11}T\xi_1 - 2T(b_{11} + b_{21} + a_{20}b_{11}T + b_{11}^2T)\xi_1^2 - 2T(b_{03} + a_{02}b_{11}T)\xi_2^2, \\ \zeta_2 = 2b_{11}T\xi_2. \end{cases}$$

This transformation should be invertible, so one non-degeneracy condition is involved, namely

$$b_{11} \neq 0.$$

If this condition is satisfied, then the system can be put in the form (163), where the constants are defined as

$$\begin{aligned} a_1 &= -\frac{a_{20}}{b_{11}}, & b_1 &= -\frac{a_{02}}{b_{11}}, & c_1 &= \frac{a_{20} + a_{30} + 2a_{20}^2T}{2b_{11}^2T}, \\ d_1 &= \frac{-2a_{20}b_{03} + 3a_{02}b_{11} + a_{12}b_{11} + 2b_{03}b_{11} + 2a_{02}b_{21} + 2a_{02}a_{20}b_{11}T + 6a_{02}b_{11}^2T}{2b_{11}^3T} \end{aligned}$$

and from those values we can understand which types of bifurcation the system has. In particular (see [29, 33] for more details) three more non-degeneracy conditions are involved

- if  $a_{20} \neq 0$  there are two limit cycles that collide and disappear
- if  $a_{02} \neq 0$  a period doubled limit cycle born in this point.

Moreover if  $\frac{a_{02}}{b_{11}} < 0$  a torus bifurcation occurs on the period doubled orbit, with Lyapunov coefficient

$$L_{NS} = -2a_{20}^2b_{03} - 3a_{02}a_{30}b_{11} + a_{20}(a_{12}b_{11} + 2b_{03}b_{11} + 2a_{02}b_{21})$$

and so, in order to avoid degeneracy, we also assume  $L_{NS} \neq 0$ .

In Figure 19 the four possible scenarios are reported depending on the sign of the normal form coefficients. The output given by MatCont is  $(b_{11}, a_{20}, a_{02}, L_{NS})$ .

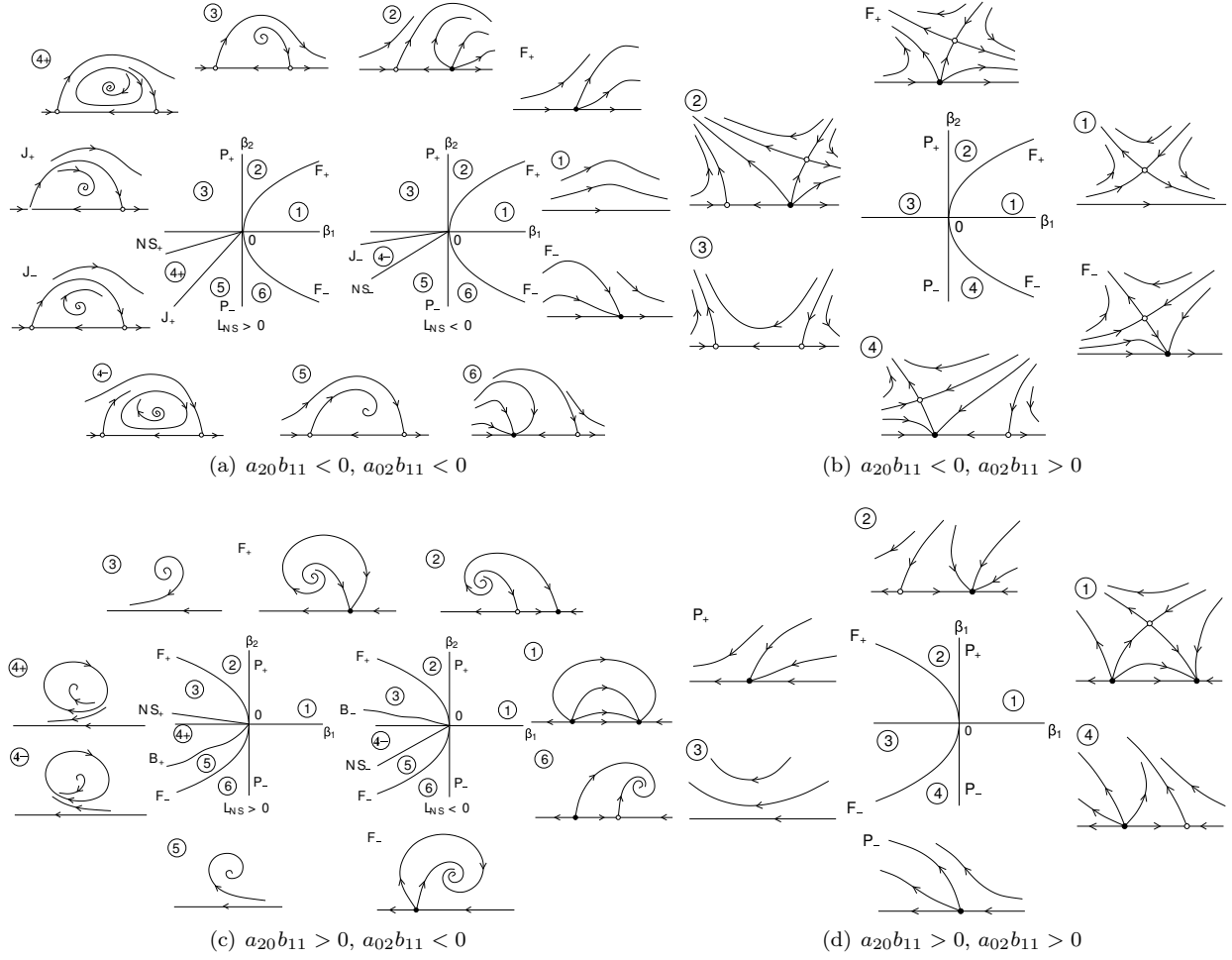


Figure 19: Bifurcation diagrams of a fold-flip bifurcation of the fixed point normal form.

## C Kernels of some differential-difference operators

In Section 4 we used the orthogonality with respect to the following inner product: if  $\zeta_1, \zeta_2 \in \mathcal{C}^0([0, 1], \mathbb{C}^n)$  and  $\eta_1, \eta_2 \in \mathbb{C}^n$ , then

$$\left\langle \begin{bmatrix} \zeta_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} \zeta_2 \\ \eta_2 \end{bmatrix} \right\rangle = \int_0^1 \langle \zeta_1(t), \zeta_2(t) \rangle dt + \langle \eta_1, \eta_2 \rangle = \int_0^1 \zeta_1^H(t) \zeta_2(t) dt + \eta_1^H \eta_2.$$

If this inner product vanishes, then we say that the corresponding vectors are orthogonal and write

$$\begin{bmatrix} \zeta_1 \\ \eta_1 \end{bmatrix} \perp \begin{bmatrix} \zeta_2 \\ \eta_2 \end{bmatrix}.$$

In Section 4 we used some propositions, from which we will give the proof in this appendix.

**Proposition C.1.** *Consider two differential-difference operators*

$$\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{R}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n,$$

with

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ , then  $\zeta \in \text{Ker}(\phi_1)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0, 1], \mathbb{R}^n)),$$

and  $\zeta \in \text{Ker}(\phi_2)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0, 1], \mathbb{R}^n)).$$

*Proof.* We will focus on the first assertion. If  $\zeta$  is in the kernel of  $\phi_1$ , then  $\dot{\zeta} - TA(t)\zeta = 0$  and  $\zeta(0) - \zeta(1) = 0$ . For all  $g \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$  we have

$$\begin{aligned} & \int_0^1 g^T(t)\dot{\zeta}(t)dt - \int_0^1 Tg^T(t)A(t)\zeta(t)dt = 0 \\ \Rightarrow & g^T(t)\zeta(t)|_0^1 - \int_0^1 \dot{g}^T(t)\zeta(t)dt - \int_0^1 Tg^T(t)A(t)\zeta(t)dt = 0 \\ \Rightarrow & g^T(1)\zeta(1) - g^T(0)\zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t))^T \zeta(t)dt = 0 \\ \Rightarrow & -(g(0) - g(1))^T \zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t))^T \zeta(t)dt = 0 \\ \Rightarrow & \left\langle \begin{bmatrix} \dot{g} + TA^T(t)g \\ g(0) - g(1) \end{bmatrix}, \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \right\rangle = 0. \end{aligned}$$

Conversely, assume that  $\left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{g} + TA^T(t)g \\ g(0) - g(1) \end{bmatrix} \right\rangle = 0$  for all  $g \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ . Then,

$$\begin{aligned} & \int_0^1 \zeta^T(t)(\dot{g}(t) + TA^T(t)g(t))dt + \zeta^T(0)(g(0) - g(1)) = 0 \\ \Rightarrow & \zeta^T(1)g(1) - \zeta^T(0)g(0) - \int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^T g(t)dt + \zeta^T(0)(g(0) - g(1)) = 0 \\ \Rightarrow & -(\zeta(0) - \zeta(1))^T g(1) - \int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^T g(t)dt = 0. \end{aligned}$$

If  $\dot{\zeta}(t) - TA(t)\zeta(t) \neq 0$ , then there exists a  $g(t)$  with  $g(1) = 0$  such that

$$\int_0^1 (\dot{\zeta}(t) - TA(t)\zeta(t))^T g(t)dt \neq 0.$$

This is impossible, so  $\dot{\zeta}(t) - TA(t)\zeta(t) = 0$ . Hence  $(\zeta(0) - \zeta(1))^T g(1) = 0$  for all  $g$ ; and thus there must hold that  $\zeta(0) - \zeta(1) = 0$ . From both observations it follows that  $\zeta \in \text{Ker}(\phi_1)$ .

The proof of the second assertion is similar. □

**Proposition C.2.** Consider  $\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{R}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n$ , where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ , then  $\zeta \in \text{Ker}(\phi_1)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0, 1], \mathbb{R}^n)),$$

and  $\zeta \in \text{Ker}(\phi_2)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0, 1], \mathbb{R}^n)).$$

*Proof.* The proof is similar to the proof of Proposition C.1. □

**Proposition C.3.** Consider  $\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{C}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{C}^n) \times \mathbb{C}^n$ , where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta + i\theta\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta + i\theta\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{C}^n)$ , then  $\zeta \in \text{Ker}(\phi_1)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0, 1], \mathbb{C}^n)),$$

and  $\zeta \in \text{Ker}(\phi_2)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0, 1], \mathbb{C}^n)).$$

*Proof.* If  $\zeta$  is in the kernel of  $\phi_1$ , then  $\dot{\zeta} - TA(t)\zeta + i\theta\zeta = 0$  and  $\zeta(0) - \zeta(1) = 0$ . For all  $g \in \mathcal{C}^1([0, 1], \mathbb{C}^n)$  we have

$$\begin{aligned} & \int_0^1 g^H(t)\dot{\zeta}(t)dt - \int_0^1 Tg^H(t)A(t)\zeta(t)dt + \int_0^1 i\theta g^H(t)\zeta(t)dt = 0 \\ \Rightarrow & g^H(t)\zeta(t)|_0^1 - \int_0^1 \dot{g}^H(t)\zeta(t)dt - \int_0^1 Tg^H(t)A(t)\zeta(t)dt + \int_0^1 i\theta g^H(t)\zeta(t)dt = 0 \\ \Rightarrow & g^H(1)\zeta(1) - g^H(0)\zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t) + i\theta g(t))^H \zeta(t)dt = 0 \\ \Rightarrow & -(g(0) - g(1))^H \zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t) + i\theta g(t))^H \zeta(t)dt = 0 \\ \Rightarrow & \left\langle \begin{bmatrix} \dot{g} + TA^T(t)g + i\theta g \\ g(0) - g(1) \end{bmatrix}, \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \right\rangle = 0. \end{aligned}$$

The proofs of the reverse implication and the second assertion are similar.  $\square$

**Proposition C.4.** Consider  $\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{C}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{C}^n) \times \mathbb{C}^n$ , where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta \\ \zeta(0) - e^{-i\theta}\zeta(1) \end{bmatrix}, \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta \\ \zeta(0) - e^{-i\theta}\zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{C}^n)$ , then  $\zeta \in \text{Ker}(\phi_1)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_2(\mathcal{C}^1([0, 1], \mathbb{C}^n)),$$

and  $\zeta \in \text{Ker}(\phi_2)$  if and only if

$$\begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \perp \phi_1(\mathcal{C}^1([0, 1], \mathbb{C}^n)).$$

*Proof.* If  $\zeta$  is in the kernel of  $\phi_1$ , then  $\dot{\zeta} - TA(t)\zeta = 0$  and  $\zeta(0) - e^{-i\theta}\zeta(1) = 0$ . For all  $g \in \mathcal{C}^1([0, 1], \mathbb{C}^n)$  we have

$$\begin{aligned} & \int_0^1 g^H(t)\dot{\zeta}(t)dt - \int_0^1 Tg^H(t)A(t)\zeta(t)dt = 0 \\ \Rightarrow & g^H(t)\zeta(t)|_0^1 - \int_0^1 \dot{g}^H(t)\zeta(t)dt - \int_0^1 Tg^H(t)A(t)\zeta(t)dt = 0 \\ \Rightarrow & g^H(1)\zeta(1) - g^H(0)\zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t))^H \zeta(t)dt = 0 \\ \Rightarrow & -(g(0) - e^{-i\theta}g(1))^H \zeta(0) - \int_0^1 (\dot{g}(t) + TA^T(t)g(t))^H \zeta(t)dt = 0 \\ \Rightarrow & \left\langle \begin{bmatrix} \dot{g} + TA^T(t)g \\ g(0) - e^{-i\theta}g(1) \end{bmatrix}, \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix} \right\rangle = 0. \end{aligned}$$

The proofs of the reverse implication and the second assertion are similar.  $\square$

**Proposition C.5.** Consider two differential-difference operators  $\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{R}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n$ , where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}, \quad \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta \\ \zeta(0) - \zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ , then

$$\phi_1(\zeta) = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

if and only if

$$\left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} + TA^T(t)h \\ h(0) - h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle,$$

for all  $h \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ . Furthermore

$$\phi_2(\zeta) = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

if and only if

$$\left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} - TA(t)h \\ h(0) - h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle,$$

for all  $h \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ .

*Proof.* We focus on the first assertion. Suppose that  $\dot{\zeta}(t) - TA(t)\zeta(t) = g(t)$  and  $\zeta(0) - \zeta(1) = 0$ . For all  $h \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$  we have

$$\begin{aligned} & \int_0^1 h^T(t)\dot{\zeta}(t)dt - \int_0^1 Th^T(t)A(t)\zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & h^T(t)\zeta(t)|_0^1 - \int_0^1 \dot{h}^T(t)\zeta(t)dt - \int_0^1 Th^T(t)A(t)\zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & h^T(1)\zeta(1) - h^T(0)\zeta(0) - \int_0^1 (\dot{h}(t) + TA^T(t)h(t))^T \zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & \int_0^1 \zeta^T(t)(\dot{h}(t) + TA^T(t)h(t))dt + \zeta^T(0)(h(0) - h(1)) = - \int_0^1 g^T(t)h(t)dt \\ \Rightarrow & \left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} + TA^T(t)h \\ h(0) - h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle. \end{aligned}$$

The proofs of the reverse implication and the second assertion are similar.  $\square$

**Proposition C.6.** Consider two differential-difference operators  $\phi_{1,2} : \mathcal{C}^1([0, 1], \mathbb{R}^n) \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n$ , where

$$\phi_1(\zeta) = \begin{bmatrix} \dot{\zeta} - TA(t)\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}, \quad \phi_2(\zeta) = \begin{bmatrix} \dot{\zeta} + TA^T(t)\zeta \\ \zeta(0) + \zeta(1) \end{bmatrix}.$$

If  $\zeta \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ , then

$$\phi_1(\zeta) = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

if and only if

$$\left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} + TA^T(t)h \\ h(0) + h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle,$$

$\forall h \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ . Furthermore

$$\phi_2(\zeta) = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

if and only if

$$\left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} - TA(t)h \\ h(0) + h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle,$$

$\forall h \in \mathcal{C}^1([0, 1], \mathbb{R}^n)$ .

*Proof.* Suppose that  $\dot{\zeta}(t) - TA(t)\zeta(t) = g(t)$  and  $\zeta(0) + \zeta(1) = 0$ . For all  $h \in C^1([0, 1], \mathbb{R}^n)$  we have

$$\begin{aligned} & \int_0^1 h^T(t)\dot{\zeta}(t)dt - \int_0^1 Th^T(t)A(t)\zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & h^T(t)\zeta(t)|_0^1 - \int_0^1 \dot{h}^T(t)\zeta(t)dt - \int_0^1 Th^T(t)A(t)\zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & h^T(1)\zeta(1) - h^T(0)\zeta(0) - \int_0^1 (\dot{h}(t) + TA^T(t)h(t))^T \zeta(t)dt = \int_0^1 h^T(t)g(t)dt \\ \Rightarrow & \int_0^1 \zeta^T(t)(\dot{h}(t) + TA^T(t)h(t))dt + \zeta^T(0)(h(0) + h(1)) = - \int_0^1 g^T(t)h(t)dt \\ \Rightarrow & \left\langle \begin{bmatrix} \zeta \\ \zeta(0) \end{bmatrix}, \begin{bmatrix} \dot{h} + TA^T(t)h \\ h(0) + h(1) \end{bmatrix} \right\rangle = - \left\langle \begin{bmatrix} g \\ 0 \end{bmatrix}, \begin{bmatrix} h \\ 0 \end{bmatrix} \right\rangle. \end{aligned}$$

The proofs of the reverse implication and the second assertion are similar.  $\square$

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