# A MAJORIZATION BOUND FOR THE EIGENVALUES OF SOME GRAPH LAPLACIANS 

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#### Abstract

It is conjectured that the Laplacian spectrum of a graph is majorized by its conjugate degree sequence. In this paper, we prove that this majorization holds for a class of graphs including trees. We also show that a generalization of this conjecture to graphs with Dirichlet boundary conditions is equivalent to the original conjecture.


## 1. Introduction

One way to extract information about the structure of a graph is to encode the graph in a matrix and study the invariants of that matrix, such as the spectrum. In this note, we study the spectrum of the "Combinatorial Laplacian" matrix of a graph.

The Combinatorial Laplacian of a simple graph $G=(V, E)$ on the set of $n$ vertices $V=$ $\left\{v_{1}, \ldots, v_{n}\right\}$ is the $n \times n$ matrix $L(G)$ defined by:

$$
L(G)_{i j}= \begin{cases}\operatorname{deg}\left(v_{i}\right) & \text { if } i=j \\ -1 & \text { if }\{i, j\} \in E \\ 0 & \text { otherwise }\end{cases}
$$

Here $\operatorname{deg}(v)$ is the degree of $v$, that is number of edges on $v$. The matrix $L(G)$ is positive semidefinite, so its eigenvalues are real and non-negative. We list them in non-increasing order and with multiplicity:

$$
\lambda_{1}(L(G)) \geq \lambda_{2}(L(G)) \geq \ldots \geq \lambda_{n-1}(L(G)) \geq \lambda_{n}(L(G))=0
$$

When the context is clear, we can write $\lambda_{i}(G)$ or simply $\lambda_{i}$. We abbreviate the sequence of $n$ eigenvalues as $\lambda(L(G))$.

We are interested in the conjecture of Grone and Merris ("GM") that the spectrum $\lambda(L(G))$ is majorized by the conjugate partition of the (non-increasing) sequence of vertex degrees of $G$ [7]. This question is currently being studied (see for example [6]), but has yet to be resolved. In this paper, we extend the class of graphs for which the conjecture is known to hold. We also show that if GM holds for graph Laplacians, it also holds for more general "Dirichlet Laplacians" (cf. [3]) as conjectured by Duval [5].

## 2. BACKGROUND AND DEFINITIONS

2.1. Graphs. Given a graph $G=(V, E)$ with $n=|V|$ vertices and $m=|E|$ edges, there are several ways to represent $G$ as a matrix. There is the edge-incidence matrix, a $n \times m$ matrix that records in each column the two vertices incident on a given edge. For directed graphs we can consider a signed edge-incidence matrix:

$$
\partial(G)_{v e}= \begin{cases}1 & \text { if } v \text { is the head of edge } e \\ -1 & \text { if } v \text { is the tail of edge } e \\ 0 & \text { otherwise }\end{cases}
$$

There is also a $n \times n$ matrix $A(G)$ called the adjacency matrix which is defined by:

$$
A(G)_{i j}= \begin{cases}1 & \text { if }(i, j) \in E \\ 0 & \text { otherwise }\end{cases}
$$

The diagonal of $A(G)$ is zero.
We can encode the (vertex) degree sequence of $G$ in non-increasing order as a vector $d(G)$ of length $n$, and in an $n \times n$ matrix $D(G)$ whose diagonal is $d(G)$ and whose off-diagonal elements are 0 . Then the Combinatorial Laplacian $L(G)$ that we study in this paper is simply $D(G)-A(G)$. It is easy to check that if we (arbitrarily) orient $G$ and consider the matrix $\partial(G)$ above, we also have $L(G)=\partial(G) \partial(G)^{t}$.

When the graph in question is clear from context, we may abbreviate the above terms: $L, A, d, D$.

Remark 2.1.1. The Laplacian is sometimes defined with entries normalized by dividing by the square roots of the degrees. However, we do not do that here.
2.2. Graph spectra. The field of spectral graph theory is the study of the structure of graphs through the spectra (eigenvalues) of matrices encoding $G$. Several surveys are available, including [1] and [4]. Besides theoretical aspects of spectral graph theory, these books describe a wide range of applications of the subject to chemistry and physics as well as to problems in other branches of mathematics such as random walks and isoperimetric problems.

In the case of $L(G)$, there has been considerable effort to study the eigenvalue $\lambda_{n-1}$, which is known as the algebraic connectivity of $G$. It can be shown that $\lambda_{n-1}(L(G))=0$ if and only if $G$ is disconnected. Bounds on $\lambda_{n-1}(L(G))$ then give information on how well connected a graph is, and are useful, for example, in showing the existence of expander graphs. This and other applications are discussed in [1].

Currently, little is known about the middle terms of the spectrum. This is partly because it varies widely depending on the graph. However, Grone and Merris [7] conjecture that the conjugate partition of the degree sequence majorizes the spectrum, and showed that the majorization inequalities are tight on the class of threshold graphs. This conjecture has been extended to simplicial complexes in recent work by Duval and Reiner [6].
2.3. Majorization. We recall that a partition $p=p(i)$ is a non-increasing sequence of natural numbers, and its conjugate is the sequence $p^{T}(j):=|\{i: p(i) \leq j\}|$. Then $p^{T}$ has exactly $p(1)$ non-zero elements. When convenient, we can add or drop trailing zeros in a partition. For non-increasing real sequences $s$ and $t$ of length $n$, we say that $s$ is majorized by $t$ (denoted $s \unlhd t$ ) if for all $k \leq n$ :

$$
\begin{equation*}
\sum_{i=1}^{k} s_{i} \leq \sum_{i=1}^{k} t_{i} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{n} s_{i}=\sum_{i=1}^{n} t_{i} \tag{2.2}
\end{equation*}
$$

The concept of majorization extends to vectors by comparing the non-increasing vectors produced by sorting the elements of the vector into non-increasing order. Given a vector $v$, call the sorted vector $v^{\prime}$ which contains the elements of $v$ sorted in non-increasing order (with multiplicity) sort $(v)$.

In the context of majorization of unsorted vectors, we will often want to refer to the concatenation of two vectors $x$ and $y$ (ie. the vector which contains the elements of $x$ followed the elements of $y$ ). This is denoted $x, y$ as for example in Lemma 2.3.2 below.

There is a rich theory of majorization inequalities which occur throughout mathematics, see for example [11]. Matrices are an important source of majorization inequalities. Notably, the relationship between the diagonal and spectrum of a Hermitian matrix is characterized by majorization (see for example [9]).

We will use the following lemmas about majorization which can be found in [11]:
Lemma 2.3.1. If $x$ and $y$ are vectors and $P$ is a doubly-stochastic matrix and $x=P y$, then $x \unlhd y$.

This yields two simple corollaries:
Lemma 2.3.2. For any vectors $x \unlhd y$ and any vector $z$ we have: $x, z \unlhd y, z$.
Lemma 2.3.3. If $x$ and $y$ non-increasing sequences, and $x=y$ except that at indices $i<j$ we have $x_{i}=y_{i}-a$ and $x_{j}=y_{j}+a$ where $a \geq 0$ then $x \unlhd y$.

Lemma.2.3.3 says that for non-increasing sequences transferring units from lower to higher indices reduces the vector in the majorization partial order. In particular, if $x, x^{\prime}, y, y^{\prime}$ are all non-increasing sequences, $x^{\prime} \unlhd x$ and $y^{\prime} \unlhd y$, then

$$
\begin{equation*}
x^{\prime}+y^{\prime} \unlhd x^{\prime}+y \unlhd x+y \tag{2.3}
\end{equation*}
$$

Let $A$ and $B$ be positive semidefinite (more generally, Hermitian) matrices. Then:

$$
\begin{equation*}
\lambda(A), \lambda(B) \unlhd \lambda(A+B) \tag{2.4}
\end{equation*}
$$

A theorem of Fan (1949) says that for positive semidefinite (more generally, Hermitian) matrices $A$ and $B$ :

$$
\begin{equation*}
\lambda(A+B) \unlhd \lambda(A)+\lambda(B) \tag{2.5}
\end{equation*}
$$

Let $A$ be an $m \times n 0-1$ (or incidence) matrix, with row sums $r_{1}, \ldots, r_{m}$ and columns sums $c_{1}, \ldots, c_{n}$ both indexed in non-increasing order. Let $r^{T}$ be the conjugate of the partition $\left(r_{1}, \ldots, r_{m}\right)$, and $c$ be the partition $\left(c_{1}, \ldots, c_{n}\right)$. Then the Gale-Ryser theorem asserts that

$$
\begin{equation*}
c \unlhd r^{T} \tag{2.6}
\end{equation*}
$$

2.4. The Grone-Merris Conjecture. The Grone-Merris conjecture (GM) is that the spectrum of the combinatorial Laplacian of a graph is majorized by its conjugate degree sequence, that is

$$
\begin{equation*}
\lambda(G) \unlhd d^{T}(G) \tag{2.7}
\end{equation*}
$$

Note that

$$
\sum_{i=1}^{n} d_{i}^{T}=\sum_{i=1}^{n} d_{i}=\operatorname{trace}(L(G))=\sum_{i=1}^{n} \lambda_{i}
$$

If we ignore isolated vertices (which contribute only zero entries to $\lambda$ and $d$ ) we will have $d_{1}^{T}=n$. Using this fact, it is possible to show that

$$
\begin{equation*}
\lambda_{1} \leq d_{1}^{T} \tag{2.8}
\end{equation*}
$$

Three short proofs of this are given in [6]. The authors then continue to prove the second majorization inequality

$$
\begin{equation*}
\lambda_{1}+\lambda_{2} \leq d_{1}^{T}+d_{2}^{T} \tag{2.9}
\end{equation*}
$$

However, their proof would be difficult to extend.
There are several other facts which fit well with the GM conjecture. One is that if the GM conjecture holds, then the instances where (2.7) holds with equality are well-understood, these would be the threshold graphs of Section 3.1. Also, since $d$ and $\lambda$ are respectively the diagonal and spectrum of $L(G)$ we have $d \unlhd \lambda$. Combining this with GM gives $d \unlhd d^{T}$, a fact that has been proved combinatorially. We refer to [6] for further discussion.

Remark 2.4.1 (Complements). Given a graph $G$, we can study its complement $\bar{G}$, the graph whose edges are exactly those not included in $G$. For a graph $G$ with $n$ vertices the $i$ th largest vertex of $G$ is the $(n-i)$ th largest vertex of $\bar{G}$, and we have $d_{i}(G)=n-1-d_{n-i}(\bar{G})$. Translating this to the conjugate partition $d^{T}$ yields: $d_{i}^{T}(G)=n-d_{n-1-i}^{T}(\bar{G})$ with $d_{n}^{T}(G)=$ $d_{n}^{T}(\bar{G})=0$.

The relationship between $\lambda(G)$ and $\lambda(\bar{G})$ is the same as between $d_{n}^{T}(G)$ and $d_{n}^{T}(\bar{G})$. This follows from the fact that $L(G)+L(\bar{G})=n I_{n}-J_{n}$ where $J_{n}$ is the $n \times n$ matrix of ones. We observe that the matrix $n I_{n}-J_{n}$ sends the special eigenvector $e_{n}$ ( $n$ ones) to 0 , and acts as the scalar $n$ on $e_{n}^{\perp}$. Both $L(G)$ and $L(\bar{G})$ also send $e_{n}$ to 0 , giving us $\lambda_{n}(G)=\lambda_{n}(\bar{G})=0$. Since $L(G)$ and $L(\bar{G})$ sum to $n I_{n}$ on $e_{n}^{\perp}$ they have the same set of eigenvectors on $e_{n}^{\perp}$, and and for each eigenvector the corresponding eigenvalues for $L(G)$ and $L(\bar{G})$ sum to $n$. Thus $\lambda_{i}(G)=n-\lambda_{n-1-i}(\bar{G})$. As a consequence, GM holds for $G$ if and only if GM holds for $\bar{G}$.

## 3. Grone-Merris on classes of graphs

In this section we give further evidence for the Grone-Merris conjecture by remarking that it holds for several classes of graphs including threshold graphs, regular graphs and trees.
3.1. Threshold graphs. The GM conjecture was originally formulated in the context of threshold graphs, which are a class of graphs with several extremal properties. An introduction to threshold graphs is [10]. Threshold graphs are the graphs that can be constructed recursively by adding isolated vertices and taking graph complements. It turns out that they are also characterized by degree sequences: the convex hull of possible (unordered) degree sequences of an $n$ vertex graph defines a polytope. The extreme points of this polytope are the degree sequences that have a unique labelled realization, and these are exactly the threshold graphs.

Threshold graphs are interesting from the point of view of spectra. Both Kelmans and Hammer [8] and Grone and Merris [7] investigated the question of which graphs have integer spectra. They found that threshold graphs are one class of graphs that have integer spectra and showed for these graphs that $\lambda(G)=d^{T}(G)$.

In the process of showing this equality for threshold graphs, Grone and Merris observed that for non-threshold graphs, the majorization inequality $\lambda(G) \unlhd d^{T}(G)$ appears to hold, and made their conjecture. We could describe the conjecture as saying that threshold graphs are extreme in terms of spectra, and that the these extreme spectra can be interpreted as conjugate degree sequences.
3.2. Regular and nearly regular graphs. For some small classes of graphs, it can be easily shown that the GM conjecture holds. Consider a $k$-regular graph $G$ on $n$ vertices (in a $k$ regular graph, all vertices have degree $k$ ). Then the degree sequence $d(G)$ is $k$ repeated $n$ times, and its conjugate $d^{T}(G)$ is $n$ repeated $k$ times followed by $n-k$ zeros. Thus $d^{T}$ majorizes every non-negative sequence of sum $k n$ whose largest terms is at most $n$, and in particular $\lambda \unlhd d^{T}$. Indeed, this proof shows that GM holds for what we might call nearly regular graphs, that is graphs whose vertices have degree either $k$ or $(k-1)$.
3.3. Graphs with low maximum degree. Using facts about the initial GM inequalities we can prove that GM must hold for graphs with low maximal degree. For example, if a graph has maximum vertex degree 2 , then $d_{3}^{T}=d_{4}^{T}=\ldots=d_{n}^{T}=0$, so for $k=2,3, \ldots, n$ :

$$
\sum_{i=1}^{k} \lambda_{i} \leq \sum_{i=1}^{n} \lambda_{i}=\sum_{i=1}^{n} d_{i}^{T}=\sum_{i=1}^{k} d_{i}^{T}
$$

More generally, the GM inequalities for $k \geq \max \_d e g(G)$ hold trivially. Thus GM holds for graphs of maximum degree 2 by (2.8). Using Duval and Reiner's result (2.9), we get that GM holds for graphs of maximum degree 3 .
3.4. Trees and more. It is tempting to try to prove GM inductively by breaking graphs into simpler components on which GM clearly holds. In this section, we show that if $G$ is "almost" the union of two smaller graphs on which GM holds then GM holds for $G$ as well. We apply this construction to show that GM holds for trees.

Take two graphs $A=\left(V_{A}, E_{A}\right)$ and $B=\left(V_{B}, E_{B}\right)$ on disjoint vertex sets $V_{A}$ and $V_{B}$. Define their disjoint sum to be $A+B=\left(V_{A} \cup V_{B}, E_{A} \cup E_{B}\right)$. Assuming $V_{A}$ and $V_{B}$ are not empty this is a disconnected graph. Now take two graphs $G=\left(V, E_{G}\right)$ and $H=\left(V, E_{H}\right)$ on the same vertex set $V$. Define their union as $G \cup H=\left(V, E_{G} \cup E_{H}\right)$.

Given the spectra and conjugate degree sequences of $A$ and $B$, the spectrum of $A+B$ is (up to ordering) $\lambda(A+B)=(\lambda(A), \lambda(B)$, while the conjugate degree sequence of $A+B$ is $d^{T}(A+B)=d^{T}(A)+d^{T}(B)$ (taking each vector to have length $n$ ). Thus by 2.4 if $\lambda(A) \unlhd d^{T}(A)$ and $\lambda(B) \unlhd d^{T}(B)$ we will have $\lambda(A+B) \unlhd d^{T}(A+B)$.

In a typical situation, where neither $A$ or $B$ is very small, we would expect the above majorization inequality to hold with considerable slack. We can use this slack to show that if we add a few more edges to $A+B$ the majorization will still hold.

Theorem 3.4.1. Take graphs $A$ or $B$ on disjoint vertex sets $V_{A}$ and $V_{B}$. Let $G=A+B$ and on $V=V_{A} \cup V_{B}$ let $C$ be a graph of "new edges" between $V_{A}$ and $V_{B}$. Assume that $G M$ holds on $A, B$ and $C$, ie. that $\lambda(A) \unlhd d^{T}(A), \lambda(B) \unlhd d^{T}(B)$ and $\lambda(C) \unlhd d^{T}(C)$. Additionally, assume that $d_{i}^{T}(C) \leq d_{i}^{T}(A), d_{i}^{T}(B)$ for all $i$, and that $d_{1}^{T}(B) \leq d_{m}^{T}(A)$ where $m$ is the largest non-zero index of $d^{T}(C)$ (equivalently, $m$ is the maximum vertex degree in $C$ ). Let $H=C \cup G$. Then:

$$
\begin{equation*}
\lambda(H) \unlhd d^{T}(H) \tag{3.1}
\end{equation*}
$$

Proof. Let $k$ be the larger of max_deg $(A)$ and max_deg $(B)$. Note that
$d^{T}(G)=d^{T}(A)+d^{T}(B)=\left(d_{1}^{T}(A)+d_{1}^{T}(B), d_{2}^{T}(A)+d_{2}^{T}(B), \ldots, d_{k}^{T}(A)+d_{k}^{T}(B), 0, \ldots, 0\right)$
Claim 3.4.2.

$$
d^{T}(H) \unrhd\left(d_{1}^{T}(G), d_{2}^{T}(G), \ldots, d_{k}^{T}(G), d_{1}^{T}(C), \ldots, d_{m}^{T}(C)\right)
$$

Proof of Claim. The term on the right is the concatenation of two partitions, $d^{T}(G)$ and $d^{T}(C)$. The columns of $d^{T}(G)$ index the vertices of $G$ and the length of a column gives the degree of the corresponding vertex. Since this claim is purely about the combinatorics of of degree sequences, we introduce a series of intermediate "partial graphs" where edges are allowed to have only one end. Degree sequences and their conjugates are still well defined for such objects.

Let $G_{0}=G$ and $C_{0}=C$. Define $G_{i}$ by moving one end of an edge from every non-isolated vertex of $C_{i-1}$ to $G_{i-1}$, and let $C_{i}$ contain whatever is left. Iterating this, for some $l \geq 0$ we
will have $G_{l}=H$ and $C_{l}$ consisting entirely of isolated vertices. Then the claim will follow if we can show that:

$$
d^{T}\left(G_{0}\right), d^{T}\left(C_{0}\right) \unlhd d^{T}\left(G_{1}\right), d^{T}\left(C_{1}\right) \unlhd \ldots \unlhd d^{T}\left(G_{l}\right), d^{T}\left(C_{l}\right)
$$

Compare the partitions at the $(i-1)$ st majorization: we remove the first row of $d^{T}\left(C_{i-1}\right)$ and put each element from that row into a separate column (representing a distinct vertex in $G$ ) of $d^{T}\left(G_{i-1}\right)$. Where there are columns of equal length in $d^{T}\left(G_{i-1}\right)$ they should be ordered so that those acquiring new elements come first. To see that this operation increases the partition in the majorization partial order, observe that after ignoring the (unchanged) contents of $d^{T}\left(C_{i}\right)$ it is equivalent to sorting the new row into the partition, using Lemma 2.3.3 to move its final (rightmost) element to the proper column and repeating as necessary.

This completes the proof of the Claim 3.4.2 and gives us:

$$
d^{T}(H) \unrhd\left(d_{1}^{T}(A)+d_{1}^{T}(B), d_{2}^{T}(A)+d_{2}^{T}(B), \ldots, d_{k}^{T}(A)+d_{k}^{T}(B), d_{1}^{T}(C), \ldots, d_{m}^{T}(C)\right)
$$

If we sort the vector on the right into non-increasing order, the first $m$ terms will remain fixed by the assumptions that $d_{m}^{T}(A) \geq d_{1}^{T}(B) \geq d_{1}^{T}(C)$. Since we have assumed that $d_{i}^{T}(C) \leq$ $d_{i}^{T}(B)$ for all $i$, we can apply Lemma 2.3.3 to the reordered sequence to get:

$$
\begin{array}{r}
d^{T}(H) \unrhd\left(d_{1}^{T}(A)+d_{1}^{T}(C), d_{2}^{T}(A)+d_{2}^{T}(C), \ldots, d_{m}^{T}(A)+d_{m}^{T}(C),\right. \\
\left.d_{m+1}^{T}(A), \ldots, d_{k}^{T}(A), d_{1}^{T}(B), \ldots, d_{k}^{T}(B)\right)
\end{array}
$$

The right hand term decomposes as:

$$
\left(d_{1}^{T}(A), \ldots, d_{k}^{T}(A), d_{1}^{T}(B), \ldots, d_{k}^{T}(B)\right)+\left(d_{1}^{T}(C), \ldots, d_{m}^{T}(C), 0, \ldots, 0\right)
$$

Since we assume $d_{m}^{T}(A) \geq d_{1}^{T}(B)$, the first $m$ entries of $\left(d^{T}(A), d^{T}(B)\right)$ will remain unchanged if the vector is sorted. Thus:

$$
\begin{equation*}
d^{T}(H) \unrhd \operatorname{sort}\left(d^{T}(A), d^{T}(B)\right)+d^{T}(C) \tag{3.2}
\end{equation*}
$$

By (2.3) we can apply the majorizations of $\lambda$ by $d^{T}$ for $A, B, C$ to the above terms to get:

$$
\begin{aligned}
d^{T}(H) & \unrhd \operatorname{sort}\left(d^{T}(A), d^{T}(B)\right)+d^{T}(C) \unrhd \operatorname{sort}\left(d^{T}(A), d^{T}(B)\right)+\lambda(C) \\
& \unrhd \operatorname{sort}\left(d^{T}(A), \lambda(B)\right)+\lambda(C) \unrhd \operatorname{sort}(\lambda(A), \lambda(B))+\lambda(C)
\end{aligned}
$$

The two terms on the right side of this equation are spectra of $L(G)$ and $L(C)$ respectively. Hence by Fan's theorem (2.5) their sum majorizes the spectrum of $\mathrm{L}(\mathrm{G})+\mathrm{L}(\mathrm{C})=\mathrm{L}(\mathrm{H})$ :

$$
d^{T}(H) \unrhd \lambda(G)+\lambda(C) \unrhd \lambda(H)
$$

More generally, we could replace the conditions in the statement of Theorem 3.4.1 with the condition (3.2), which can be checked combinatorially. The conditions in the theorem statement and equation (3.2) are most likely to be satisfied if $C$ is small relative to $A$ and $B$.

A useful case is when $C$ consists of $k$ disjoint edges. Then $m=1$ and $d_{1}^{T}(C)=2 k$. Without loss of generality we can take $d_{1}(A) \geq d_{1}(B)$ and the only condition that we will need to check is that $d_{1}(A), d_{1}(B) \geq d_{1}(C)$, ie. both $A$ and $B$ must have at least $2 k$ non-isolated vertices.

The strategy for applying Theorem 3.4.1 to show that a given graph $H$ satisfies GM is to find a "cut" $C$ for it that contains few edges and divides $H$ into relatively large components. For example we have the following result:

Corollary 3.4.3. The GM conjecture holds for trees.

Proof. Proceed by induction on the diameter of the graph. If $T$ has diameter 1 or 2 , then there is a vertex $v$ which is the neighbour of all the remaining vertices and $T$ is a threshold graph. So GM holds with equality for $T$.

Otherwise, we can find some edge $e$ that does not have a leaf vertex. Since $T$ is a tree, $e$ is a cut edge and divides $T$ into two non-trivial connected components, $A$ and $B$. We apply induction to $A$ and $B$ and apply Theorem 3.4.1 to $H=(A+B) \cup C$ where $C$ is the graph on the vertex set of $T$ containing the single edge $e$.

Remark 3.4.4 (Small Graphs). The facts in this section allow us to check that GM holds for some small graphs without directly computing eigenvalues. For example, since the GM condition is closed under complement (see 2.4.1) for graphs on up to 5 vertices it is enough to observe that either $G$ or $\bar{G}$ has maximum degree $\leq 3$. Out of 156 graphs on 6 vertices, 146 can be decomposed into smaller graphs $(A+B) \cup C$ using Theorem 3.4.1 Calculating the eigenvalues of the remaining 10 does not yield a counterexample.

## 4. Simplices and pairs

The most recent work relating to the GM conjecture has been to study the spectra of more general structures than graphs, such as simplicial complexes and simplicial family pairs. In this section we show that the generalization of GM to graphs with Dirichlet boundary conditions is equivalent to the original conjecture and may be useful in approaching GM.
4.1. Simplicial complexes. In [6], the authors look at simplicial complexes, which are higher dimensional analogues of simple graphs (see for example [12]). A set of faces of a given dimension $i$ is called an $i$-family. Given a simplicial complex $\Delta$ we can denote the $i$-family of all faces in $\Delta$ of dimension $i$ as $\Delta^{(i)}$. For example, a graph is a 1 -dimensional complex, and its edge set is the 1 -family $\Delta^{(1)}$. Define the degree sequence $d$ of an $i$-family to be the list of the numbers of $i$-faces from the family incident on each vertex, and sorted into non-increasing order. We can then define $d(\Delta, i)$ as the degree sequence of $\Delta^{(i)}$, which we can abbreviate to $d(\Delta)$ or $d$ when the context is clear.

We define the chain group $C_{i}(\Delta)$ of formal linear combinations of elements of $\Delta^{(i)}$, and generalize the signed incidence matrix $\partial$ of Section 2.1 to a signed boundary map $\partial_{i}: C_{i}(\Delta) \rightarrow$ $C_{i-1}(\Delta)$. This allows us to define a Laplacian on $C_{i}(\Delta)$, namely $L_{i}(\Delta)=\partial_{i} \partial_{i}^{T}$, and study its corresponding spectrum $s(\Delta, i)$ sometimes abbreviated $s(\Delta)$ or $s$.

Duval and Reiner [6] looked at shifted simplicial complexes, which are a generalization of threshold graphs to complexes. They showed that for a shifted complex $\Delta$ and any $i$, we have $s(\Delta, i)=d^{T}(\Delta, i)$. They then conjectured that GM also holds for complexes, ie. that for any complex and any $i$ we have:

$$
\begin{equation*}
s(\Delta, i) \unlhd d^{T}(\Delta, i) \tag{4.1}
\end{equation*}
$$

They also show that some related facts, such as equation (2.8) generalize to complexes.
4.2. Simplicial pairs. In [5], Duval continues by studying relative (family) pairs ( $K, K^{\prime}$ ) where the set $K=\Delta^{(i)}$ for some $i$ is taken modulo a family of $(i-1)$-faces $K^{\prime} \subseteq \Delta^{(i-1)}$. When $K^{\prime}=\emptyset$, this reduces to the situation of the previous section.

Remark 4.2.1. In the case $i=1$ this is the edge set of a graph ( $K$ ) with a set of deleted boundary vertices $K^{\prime}$. An edge attached to a deleted vertex will not be removed - it remains as part of the pair, but we now think of the edge as having a hole on one (or both) ends.

This type of graph with a boundary appears in conformal invariant theory. In this language, the relative Laplacian of an (edge, vertex) pair is sometimes referred to as a Dirichlet Laplacian and its eigenvalues as Dirichlet eigenvalues, see for example [3]. Recently [2] used the spectrum of the Dirichlet Laplacian in the analysis of "chip-firing games", which are processes on graphs that have an absorbing (Dirichlet) boundary at some vertices.

We can form chain groups $C_{i}(K)$ and $C_{i-1}\left(K, K^{\prime}\right)$ and use these to define a (signed) boundary operator on the pair $\partial\left(K, K^{\prime}\right): C_{i}(K) \rightarrow C_{i-1}\left(K, K^{\prime}\right)$. Hence we get a Laplacian for family pairs $L\left(K, K^{\prime}\right)=\partial\left(K, K^{\prime}\right) \partial\left(K, K^{\prime}\right)^{T}$. Considered as a matrix, $L\left(K, K^{\prime}\right)$ will be the principal submatrix of $L(K)$ whose rows are indexed by the $i$-faces in $\Delta^{(i-1)}-K^{\prime}$. Finally, we get a spectrum $s\left(K, K^{\prime}\right)$ for family pairs from the eigenvalues of $L\left(K, K^{\prime}\right)$.

Duval defines the degree $d_{v}\left(K, K^{\prime}\right)$ of vertex $v$ (in the case of a graph, $v$ is allowed to be in $K^{\prime}$ ) relative to the pair $\left(K, K^{\prime}\right)$ as the number of faces in $K$ that contain $v$ such that $K-\{v\}$ is in $\Delta^{(i-1)}-K^{\prime}$. This allows him to define the degree sequence $d\left(K, K^{\prime}\right)$ for pairs, and to conjecture that GM holds for relative pairs:

$$
\begin{equation*}
s\left(K, K^{\prime}\right) \unlhd d^{T}\left(K, K^{\prime}\right) \tag{4.2}
\end{equation*}
$$

4.3. The Grone-Merris conjecture for relative pairs. It turns out that at least in the case of (edge, vertex) pairs that (4.2) follows from the original GM conjecture for graphs.

Theorem 4.3.1. GM for graphs $\Rightarrow$ GM for (edge, vertex) pairs.
Proof. Let $G=(V, E)$ be a graph with $D \subseteq V$ a set of "deleted" vertices. Let $U=V-D$ be the remaining "undeleted" vertices. We will assume that GM holds only on the undeleted part of the graph, ie. $\left.G\right|_{U}$. So we have $s\left(\left.G\right|_{U}\right) \unlhd d^{T}\left(\left.G\right|_{U}\right)$. We can ignore the edges in $\left.G\right|_{D}$ completely, since they have no effect on either $s(G)$ or $d(G)$. The remaining edges connect vertices in $D$ to vertices in $U$. Define $G^{\prime}$ to be the graph on $V$ whose edge are exactly the edges of $G$ between $D$ and $U$. Let $a$ be the degree sequence of the deleted vertices in $G^{\prime}$ and $b$ be the degree sequence of the undeleted vertices in $G^{\prime}$.

We can compute $d^{T}(E, D)$ in terms of the degree sequences and spectra of $\left.G\right|_{U}, G^{\prime}$ and $\left.G\right|_{D}$ since $d_{i}^{T}(E, D)$ is the number of vertices (deleted or not) attached to at least $i$ non-deleted vertices. The number of such vertices in $U$ will be $d_{i}^{T}\left(\left.G\right|_{U}\right)$, and the number in $D$ will be $d_{i}^{T}\left(G^{\prime}\right)=a^{T}$. Hence $d^{T}(E, D)=d_{i}^{T}\left(\left.G\right|_{U}\right)+a^{T}$.

Now consider the Laplacian $L(E, D)$. This is the submatrix of $L(G)$ indexed by $U$. An edge $(i, j)$ in $\left.G\right|_{U}$ contributes to entries $i i, i j, j i, j j$ in both $L(E, D)$ and $L(G)$. An edge in $G^{\prime}$, say from $i \in U$ to $j \in D$ contributes only to entry $i i$, and an edge in $\left.G\right|_{D}$ does not affect $L(E, D)$. So, we have $L(E, D)=L\left(\left.G\right|_{U}\right)+\operatorname{Diag}(b)$, and by (2.5) we have:

$$
\begin{equation*}
s(E, D) \unlhd s\left(\left.G\right|_{U}\right)+b \tag{4.3}
\end{equation*}
$$

We complete our equivalence by appealing to the Gale-Ryser theorem (2.6) to claim that $b \unlhd a^{T}$. This follows from the fact that $a$ and $b$ are row and column sums (in non-increasing order) of the $|D| \times|U|$ bipartite incidence matrix for $G^{\prime}$. Combining with the assumption that $s\left(\left.G\right|_{U}\right) \unlhd d^{T}\left(\left.G\right|_{U}\right)$ and (4.3) we get:

$$
s(E, D) \unlhd s\left(\left.G\right|_{U}\right)+b \unlhd d^{T}\left(\left.G\right|_{U}\right)+a^{T}=d^{T}(E, D)
$$

This proof relies on the bipartite structure of $G^{\prime}$, so it is not immediately obvious how to extend it to higher dimensional complexes. It would be interesting to do this.

Remark 4.3.2. Because the induction used to prove Theorem4.3.1requires only that the "undeleted" part of the graph satisfy GM, it is tempting to attack the original GM conjecture by showing if GM holds for a pair $(G,\{v\})$ then GM holds for $G$.

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