# Well-posedness of a multiscale model for concentrated suspensions 

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#### Abstract

In a previous work [1], three of us have studied a nonlinear parabolic equation arising in the mesoscopic modelling of concentrated suspensions of particles that are subjected to a given time-dependent shear rate. In the present work we extend the model to allow for a more physically relevant situation when the shear rate actually depends on the macroscopic velocity of the fluid, and as a feedback the macroscopic velocity is influenced by the average stress in the fluid. The geometry considered is that of a planar Couette flow. The mathematical system under study couples the one-dimensional heat equation and a nonlinear Fokker-Planck type equation with nonhomogeneous, nonlocal and possibly degenerate, coefficients. We show the existence and the uniqueness of the global-in-time weak solution to such a system.


## 1 Mechanical context and setting of the equations

We consider here a concentrated suspension of particles in a Couette flow. Examples of such suspensions are numerous: tooth pastes, cements, the blood. As opposed to some other complex fluids such as polymeric liquids for which elaborate rheological models, based on fine mesoscopic physical descriptions, are available (see e.g. [8]), the modelling of concentrated suspensions is still in its infancy. The specific model considered here however raises interesting mathematical issues, mainly related to the various nonlinearities present and the coupling of equations at different scales. Such features are likely to be shared by a large variety of models, which motivates, and enlarges the scope of, the present mathematical study.

Let us begin with some basics on the mechanical context. Depending on the concentration, a suspension of particles may exhibit different rheological behaviors. At low concentration, the suspension behaves like a newtonian fluid at rest or under weak stresses. On the other hand, when the suspension becomes more concentrated, the motion of each particle becomes strongly perturbed by the presence of the others and one observes a so-called jamming transition
where the sample adopts a pastelike behavior. In this transition, a macroscopic yield stress appears [7].

It is well known that when simple fluids are sheared, stress and shear rate are linked by a linear relation. The linear response coefficients and their relation to the microstructure of the fluid are well understood 5]. On the contrary, complex fluids exhibit highly nonlinear properties far from being understood. These nonlinear properties occur not only at high shear rates, where one does expect that linear response theory fails, but also at very low shear rates, which is more surprising. It is for instance commonly observed that for some materials (yield stress fluids) the shear stress $\sigma$ goes to a non-zero value when the shear rate goes to zero.

In [6], Hébraud and Lequeux proposed a model of the rheological behavior of complex fluids based on elementary physical processes. The system is divided in mesoscopic blocks whose size is large enough for the stress and strain tensors to be defined for each block. The size is however small compared to the characteristic length scale of the stress field. A mesoscopic evolution equation of the stress of each block is then written:
i. at low shear, each particle keeps the same neighbors, and a block behaves as an Einstein elastic solid, in which the elasticity arises from interactions between neighboring particles ;
ii. then, deformation induces local reorganization of the particles, at a given stress threshold $\sigma_{c}$. Above this threshold, the block flows as an Eyring fluid : the configuration reached by shearing the suspension relaxes with a characteristic time $T_{0}$ towards a completely relaxed state, where no stress is stored ;
iii. lastly, coupling between the flow of neighboring blocks must be included. This is taken into account by the introduction of a diffusion term in the evolution equation, where it is assumed that the diffusion coefficient is proportional to the number of reorganizations per unit time.

The equation proposed by Hébraud and Lequeux (HL equation in short) is written for a given shear rate $\dot{\gamma}$, which only depends on time:

$$
\begin{align*}
\partial_{t} p(t, \sigma)=-G_{0} \dot{\gamma}(t) & \partial_{\sigma} p(t, \sigma)+D(p(t)) \partial_{\sigma \sigma}^{2} p(t, \sigma) \\
& -\frac{\mathbb{1}_{\mathbb{R} \backslash\left[-\sigma_{c}, \sigma_{c}\right]}(\sigma)}{T_{0}} p(t, \sigma)+\frac{D(p(t))}{\alpha} \delta_{0}(\sigma) \tag{1.1}
\end{align*}
$$

with

$$
\begin{equation*}
D(p(t))=\frac{\alpha}{T_{0}} \int_{|\sigma|>\sigma_{c}} p(t, \sigma) d \sigma \tag{1.2}
\end{equation*}
$$

In the model, each block carries a given shear stress $\sigma$ ( $\sigma$ is a real number; it is in fact an extradiagonal term of the stress tensor in convenient coordinates). The evolution of the blocks is described through a probability density $p(t, \sigma)$ which represents the distribution of stress in the assembly of blocks at time $t$. In equation (1.1), $\mathbb{1}_{\mathbb{R} \backslash\left[-\sigma_{c}, \sigma_{c}\right]}$ denotes the characteristic function of the open set $\mathbb{R} \backslash\left[-\sigma_{c}, \sigma_{c}\right]$ and $\delta_{0}$ the Dirac delta function on $\mathbb{R}$. The three terms arising in the right-hand side of equation (1.1) correspond to the three physical features described above. When a block is submitted to the shear rate $\dot{\gamma}$, the stress of this block evolves with a variation rate $G_{0} \dot{\gamma}$ where $G_{0}$ is an elasticity constant. When the modulus of the stress overcomes the critical positive value $\sigma_{c}$, the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time $T_{0}$. This property is expressed by
the last two terms in (1.1). This relaxation phenomenon induces a rearrangement of the other blocks and this is finally modelled through the (nonlinear) diffusion term $D(p(t)) \partial_{\sigma \sigma}^{2} p$. The diffusion coefficient $D(p(t))$ as given by (1.2) is assumed to be proportional to the density of blocks that relax during time $T_{0}$. The parameter $\alpha$ depends on the microscopic properties of the sample and is supposed to model the "mechanical fragility" of the material. This nonlinear diffusion term emphasizes the importance of collective effects in such materials.

As mentioned above, the shear rate $\dot{\gamma}$ inserted in the original HL equation depends only on time, and not on the space variable. It is however known from experiment that the shear rate in Couette flows of non-newtonian fluids is not homogeneous in space. In order to better describe the coupling of the macroscopic flow with the evolution of the mesostructure, we therefore introduce a space-dependent shear rate $\dot{\gamma}$ given by the velocity gradient (which immediately implies that an equation of HL type holds at each point of the sample) and propose here the following multiscale model for planar Couette flows of concentrated suspensions (see Fig below) :

$$
\left\{\begin{array}{l}
\rho \partial_{t} U(t, y)=\partial_{y} \tau(t, y)+\mu \partial_{y y}^{2} U(t, y) ;  \tag{1.3a}\\
\partial_{t} p(t, y, \sigma)=-G_{0} \partial_{y} U(t, y) \partial_{\sigma} p(t, y, \sigma)+D(p(t, y)) \partial_{\sigma \sigma}^{2} p(t, y, \sigma) \\
\quad-\frac{\mathbb{1}_{\mathbb{R} \backslash\left[-\sigma_{c}, \sigma_{c}\right]}(\sigma)}{T_{0}} p(t, y, \sigma)+\frac{D(p(t, y))}{\alpha} \delta_{0}(\sigma) ; \\
\tau(t, y)=\int_{\mathbb{R}} \sigma p(t, y, \sigma) d \sigma ; \\
D(p(t, y))=\frac{\alpha}{T_{0}} \int_{|\sigma|>\sigma_{c}} p(t, y, \sigma) d \sigma ; \\
p \geq 0 ; \\
U(0, y)=u_{0}(y), \quad p(0, y, \sigma)=p_{0}(y, \sigma) ; \\
t \in(0 ; T), y \in(0 ; L), \sigma \in \mathbb{R} .
\end{array}\right.
$$

This system is supplied with the initial condition for the probability density:

$$
\begin{equation*}
p_{0} \geq 0, \quad \int_{\mathbb{R}} p_{0}(y, \sigma) d \sigma=1, \quad \text { for almost every } y \in(0 ; L) \tag{1.4}
\end{equation*}
$$

and the no-slip boundary conditions

$$
\begin{equation*}
U(t, 0)=0, \quad U(t, L)=V(t), \quad \text { for almost all } t \tag{1.5}
\end{equation*}
$$

In the above equations, $U(t, y)$ denotes the component along $e_{x}$ of the velocity field (the flow being laminar and incompressible, the velocity field is of the form $\left.\vec{U}=U(t, y) e_{x}\right), \rho$ is the volumic mass of the fluid and $\mu$ some non-negative viscosity coefficient. The function $V$ which appears in the boundary condition (1.5) is a continuous function on $\mathbb{R}$ such that $V \in L^{\infty}(\mathbb{R}) \cap H_{\mathrm{loc}}^{1}(\mathbb{R})$ and $V(0)=0$. The initial velocity $u_{0}$ lies in $L^{2}(0 ; L)$.

The mathematical analysis of the original HL model (1.1) has been the subject of [1], where a more detailed presentation of the physical background and some additional references may be found. The main difficulties of course come from the nonlinearity in the diffusion term, from the presence of the singular Dirac mass as a source term, and foremost from the fact that the parabolic equation degenerates if the viscosity coefficient $D(p)$ vanishes. In particular,


Figure 1: Planar Couette flow
it has been shown that such a degeneracy may only occur if it is already the case at start, that is $D\left(p_{0}\right)=0$. And then, the situation becomes very intricate since in some particular cases several solutions may exist (see [1). In the coupled system (1.3), we have to deal with additional difficulties due to the multiscale coupling: there are several HL-equations (roughly stated, one for each $y$ ) and all of them are coupled through the macroscopic equation (1.3a). Proving the well-posedness of the Cauchy problem for the coupled system is the main purpose of this article.

Before we get to the heart of the matter, we would like to comment on the diffusion term $\mu \partial_{y y}^{2} u$ in the equation of motion (1.3a). Let us emphasize that this artificial viscosity has been added only for mathematical purposes: there is no physical reason why the fluid should be considered viscous. In the absence of such a regularizing term, we are unfortunately unable to conduct the mathematical analysis in the whole generality.

There is however one particular situation, namely that when $\sigma_{c}=0$, where we are indeed able to study the system even without the diffusion term (i.e. with $\mu=0$ ). This is the subject of the short Section 3.

Notation: For given positive constants $T$ and $L$, we denote $\Omega=[0 ; L]$ and $\Omega_{T}=(0 ; T) \times \Omega$. In the sequel, $C$ a generic positive constant that may depend on the data but that is independent of $t, y$ and $\sigma$. Also to simplify the notation we shall use in the proofs the shorthands $L_{T}^{p}, L_{y}^{p}$, and $L_{\sigma}^{p}$ for the functional spaces $L^{p}(0 ; T), L^{p}(\Omega), L^{p}(\mathbb{R})$, respectively.

## 2 Existence and uniqueness of weak solutions

In this section, we assume that $\sigma_{c}>0$, which is the physically relevant case. With an appropriate change of scales in the coordinates and variables (see Appendix), we may equivalently assume that $L=T_{0}=\sigma_{c}=1$. In addition, without loss of generality, we take $\mu=1$ in (1.3a), for, we recall, $\mu$ is only here for mathematical convenience and is needed to be strictly positive.

Let us define the velocity field

$$
\tilde{U}(t, y)=V(t) y
$$

as a lifting of the boundary condition (1.5), and denote for $T>0$

$$
\mathcal{U}_{T}=C^{0}\left([0, T] ; L^{2}(\Omega)\right) \cap L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)
$$

and

$$
\mathcal{P}_{T}=C^{0}\left([0, T] ; L^{2}(\mathbb{R})\right) \cap L^{2}\left([0, T] ; H^{1}(\mathbb{R})\right) .
$$

Setting $U=u+\tilde{U}$ and denoting by $\dot{V}$ the derivative of $V$ with respect to time, the problem under consideration now reads :

Find $u \in \mathcal{U}_{T}$ and $p \in L^{\infty}\left([0, T] \times \Omega ; L^{1}(\mathbb{R})\right) \cap L^{\infty}\left(\Omega ; \mathcal{P}_{T}\right)$ solutions to

$$
\begin{gather*}
\left\{\begin{array}{l}
\rho \partial_{t} u-\partial_{y y}^{2} u=\partial_{y} \tau-\rho \dot{V}(t) y \\
\tau=\tau(t, y)=\int_{\mathbb{R}} \sigma p(t, y, \sigma) d \sigma ; \\
u(0, y)=u_{0}(y) ;
\end{array}\right.  \tag{2.1a}\\
\left\{\begin{array}{l}
\partial_{t} p+G_{0}\left(\partial_{y} u+V(t)\right) \partial_{\sigma} p-D(p(t, y)) \partial_{\sigma \sigma}^{2} p+\mathbb{1}_{\mathbb{R} \backslash[-1,1]}(\sigma) p=\frac{D(p(t, y))}{\alpha} \delta_{0}(\sigma) ; \\
D(p(t, y))=\alpha \int_{|\sigma|>1} p(t, y, \sigma) d \sigma ; \\
p(0, y, \sigma)=p_{0}(y, \sigma) .
\end{array}\right.
\end{gather*}
$$

Our main result is the following :
Theorem 2.1 Let $u_{0}$ be in $L^{2}(\Omega)$ and let $p_{0}$ satisfy the conditions

$$
\left\{\begin{array}{l}
p_{0} \geq 0, \quad \int_{\mathbb{R}} p_{0}(y, \sigma) d \sigma=1, \quad \text { for almost every } y \in \Omega  \tag{2.3}\\
p_{0} \in L^{\infty}(\Omega \times \mathbb{R}), \int_{\mathbb{R}}|\sigma| p_{0} d \sigma \in L^{2}(\Omega),
\end{array}\right.
$$

together with:

$$
\left\{\begin{array}{l}
\text { There exists a positive constant } \eta \text { such that }  \tag{2.4}\\
\alpha \inf _{\substack{y \in \Omega \\
\chi \in \mathbb{R}}} \int_{|\sigma+\chi|>1} p_{0}(y, \sigma) d \sigma \geq \eta>0
\end{array}\right.
$$

Then, there exists a unique global-in-time weak solution ( $u ; p$ )

$$
\begin{gather*}
u \in C^{0}\left(\mathbb{R}^{+} ; L^{2}(\Omega)\right) \cap L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{+} ; H_{0}^{1}(\Omega)\right), \\
p \in L^{\infty}\left(\mathbb{R}^{+} \times \Omega ; L^{1}(\mathbb{R})\right) \quad p \in L^{\infty}\left(\Omega ; \mathcal{P}_{T}\right), \quad \forall T>0 \tag{2.5}
\end{gather*}
$$

to (2.1)-(2.2). In addition, for such a solution, $p \in L_{\mathrm{loc}}^{\infty}\left(\mathbb{R}^{+} ; L^{\infty}(\Omega \times \mathbb{R})\right)$ and we have

$$
\tau \in L^{2}\left(\Omega ; L_{\mathrm{loc}}^{\infty}(0 ; T)\right) \cap C^{0}\left([0, T] ; L^{2}(\Omega)\right)
$$

$$
p \in C^{0}\left([0, T] ; L^{2}(\Omega \times \mathbb{R})\right) \quad, \quad \int_{\mathbb{R}} p(t, y, \sigma) d \sigma=1, \quad \text { for all } t \geq 0 \text { and } y \in \Omega
$$

and

$$
\inf _{\substack{0 \leq t \leq T \\ y \in \Omega}} D(p(t, y)) \geq \frac{\eta}{2} e^{-T}
$$

Some comments regarding the assumption (2.4) are immediately in order.
Condition (2.4) obviously implies that $D\left(p_{0}\right)$ is bounded away from zero independently of $y$. The aim of this condition on the initial data $p_{0}$ is to ensure that, at any time, the viscosity term $D(p)$ in (2.2a) is also bounded away from zero, so that the nonlinear parabolic equation (2.2a) satisfied by $p$ is non-degenerate at any time (see (2.18) in Lemma 2.2 below). The condition is satisfied for example when $p_{0}$ is a Gaussian-like function.

Such an assumption seems very demanding, and thus restrictive from the viewpoint of applications. In fact, some numerical simulations performed by one of us in 33 show that even when that assumption is not satisfied at initial time $t=0$, it is indeed satisfied for $t>0$ arbitrarily small. We are unfortunately not able to establish this fact rigorously, but the numerical evidence mentioned above heuristically shows that condition (2.4) can be considered to be always satisfied, up to a change in the choice of the origin of times.

The rest of this section is devoted to the proof of Theorem 2.1. The existence and uniqueness result is first proven on a small time interval with an argument based on the Banach fixed point Theorem. We introduce the function $\mathcal{F}_{1}$ which associates to every function $u$ in $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$ the function $\tau=\int_{\mathbb{R}} \sigma p d \sigma$ in $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$, corresponding to the (unique) solution $p$ in $L^{\infty}\left([0, T] \times \Omega ; L^{1}(\mathbb{R})\right) \cap L^{\infty}\left(\Omega ; \mathcal{P}_{T}\right)$ to (2.2). Then, we denote by $\mathcal{F}_{2}$ the mapping from $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$ to $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$, which associates to every $\tau$ in $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$, the unique solution $v\left(\right.$ in $\left.\mathcal{U}_{T}\right)$ to the heat equation :

$$
\left\{\begin{array}{l}
\rho \partial_{t} v-\partial_{y y}^{2} v=\partial_{y} \tau-\rho \dot{V}(t) y \quad \text { on } \Omega_{T}  \tag{2.6a}\\
v(0, y)=u_{0}
\end{array}\right.
$$

We next define the mapping $\mathcal{F}$ on $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$ as $\mathcal{F}=\mathcal{F}_{2} \circ \mathcal{F}_{1}$ :

$$
\begin{array}{ccccc}
\mathcal{F}: \quad L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right) & \xrightarrow[\mathcal{F}_{1}]{\longrightarrow} & L^{\infty}\left([0, T] ; L^{2}(\Omega)\right) & \xrightarrow{\mathcal{F}_{2}} & L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)  \tag{2.7}\\
u & \longmapsto & \tau & \longmapsto & v
\end{array}
$$

and our main step consists in proving the following.
Proposition 2.1 For every $T>0$, the mapping $\mathcal{F}$ from $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$ into itself is welldefined and for $T>0$ small enough, it admits a unique fixed point denoted by $u$.

The proof of Proposition 2.1 is organized as follows. We first check in Lemma 2.1 below that $\mathcal{F}_{2}$ is well-defined and that it is a Lipschitz continuous function with a Lipschitz constant that may be chosen arbitrarily small provided the length of the time interval is reduced (Section 2.1). In Section 2.2 we next prove that $\mathcal{F}_{1}$ is well-defined. We establish in Section 2.3 that $\mathcal{F}_{1}$ is a Lipschitz continuous function with a locally bounded Lipschitz constant with respect to time interval. Therefore the composed mapping $\mathcal{F}$ is contracting on small enough time interval. The existence and uniqueness of a solution on a small time interval follows by the

Banach fixed point theorem. Finally in Section 2.4 we deduce the existence and uniqueness of the global-in-time solution.

Henceforth, and unless otherwise stated, the initial condition $p_{0}$ is fixed and it satisfies the assumptions (2.3)-(2.4) of the statement of the Theorem.

### 2.1 The map $\mathcal{F}_{2}$ is a contraction on $[0, T]$ for $T$ small enough

Lemma 2.1 For every $T>0$, the mapping $\mathcal{F}_{2}$ is Lipschitz continuous from $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$ to $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$, and the Lipschitz constant goes to 0 with $T$.

Proof of Lemma 2.1\} We first observe that the mapping $\mathcal{F}_{2}$ is well-defined. Indeed, for every function $\tau$ in $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right), \partial_{y} \tau \in L^{\infty}\left([0, T] ; H^{-1}(\Omega)\right)$, and therefore, the existence and the uniqueness of a solution $v \in \mathcal{U}_{T}$ of the heat equation (2.6) is a standard result. Let now $\tau_{1}$ and $\tau_{2}$ be two functions in $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$, and let us denote $v_{1}=\mathcal{F}_{2}\left(\tau_{1}\right)$ and $v_{2}=\mathcal{F}_{2}\left(\tau_{2}\right)$. We also set $v=v_{1}-v_{2}$ and $\tau=\tau_{1}-\tau_{2}$. Then, $v$ satisfies

$$
\left\{\begin{array}{l}
\rho \partial_{t} v-\partial_{y y}^{2} v=\partial_{y} \tau \text { on } \Omega_{T}  \tag{2.8a}\\
v(0, y)=0 \\
v(t, 0)=v(t, 1)=0
\end{array}\right.
$$

and if we apply Equation (2.8) to $v$ and integrate over $\Omega$ we get

$$
\begin{equation*}
\frac{\rho}{2} \frac{d}{d t} \int_{\Omega}|v|^{2}+\int_{\Omega}\left|\partial_{y} v\right|^{2}=-\int_{\Omega} \tau \partial_{y} v . \tag{2.9}
\end{equation*}
$$

By the Cauchy-Schwarz and the Young inequalities, we obtain for $t \in[0 ; T]$,

$$
\rho \int_{\Omega}|v|^{2}+\int_{0}^{t}\left(\int_{\Omega}\left|\partial_{y} v\right|^{2} d y\right) d s \leq \int_{0}^{T}\left(\int_{\Omega}|\tau|^{2} d y\right) d s
$$

and therefore by the Poincaré inequality

$$
\begin{equation*}
\|v\|_{L^{2}\left([0, T] ; H^{1}(\Omega)\right)} \leq 2 \sqrt{T}\|\tau\|_{L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)} . \tag{2.10}
\end{equation*}
$$

### 2.2 The map $\mathcal{F}_{1}$ is well defined

Equation (2.2) with the variable $y$ frozen has been studied in [1]. For the sake of consistency we now recall :

Proposition 2.2 [1], Theorem 1.1] (Global-in-time existence for all $y$ )
For almost every $y$ in $\Omega$, let $b(\cdot, y)$ be a given function in $L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{+}\right)$, and let $p_{0}$ such that :

$$
\begin{equation*}
p_{0}(y, \cdot) \in L^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}), \quad p_{0}(y, \cdot) \geq 0, \quad \int_{\mathbb{R}} p_{0}(y, \sigma) d \sigma=1 \quad \text { and } \int_{\mathbb{R}}|\sigma| p_{0} d \sigma<+\infty \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
D\left(p_{0}(y)\right)>0 . \tag{2.12}
\end{equation*}
$$

Then, for every $T>0$ and for almost every $y$ in $\Omega$, there exists a unique solution $p=p(t, y, \sigma)$ in $L^{\infty}\left([0, T] ; L^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})\right) \cap L^{2}\left([0, T] ; H^{1}(\mathbb{R})\right)$ to the equation

$$
\left\{\begin{array}{l}
\partial_{t} p=-b(t, y) \partial_{\sigma} p+D(p(t, y)) \partial_{\sigma \sigma}^{2} p-\mathbb{1}_{\mathbb{R} \backslash[-1,1]}(\sigma) p+\frac{D(p(t, y))}{\alpha} \delta_{0}(\sigma)  \tag{2.13a}\\
p \geq 0 ; \\
D(p(t, y))=\alpha \int_{|\sigma|>1} p(t, y, \sigma) d \sigma \\
p(0, y, \sigma)=p_{0}(y, \sigma)
\end{array}\right.
$$

In addition, for almost every $y$ in $\Omega$, we have

- $\int_{\mathbb{R}} p(t, y, \sigma) d \sigma=1 \quad$ for all $t \geq 0$,
- for all $T>0$,

$$
\begin{equation*}
\max _{0 \leq t \leq T}\|p(t, y, \cdot)\|_{L_{\sigma}^{\infty}} \leq\left\|p_{0}(y, \cdot)\right\|_{L_{\sigma}^{\infty}}+\frac{\sqrt{\alpha} \sqrt{T}}{\sqrt{\pi}} \tag{2.14}
\end{equation*}
$$

- $p(\cdot, y) \in C^{0}\left([0, T] ; L^{1}(\mathbb{R}) \cap L^{2}(\mathbb{R})\right)$,
- $D(p(\cdot, y)) \in C^{0}([0, T])$,
- for every $T>0$ there exists a positive constant $\eta(T, y)$ such that

$$
\begin{equation*}
\min _{0 \leq t \leq T} D(p(t, y)) \geq \eta(T, y) \tag{2.15}
\end{equation*}
$$

- for almost all $y,(t, \sigma) \mapsto \sigma p(t, y, \sigma) \in L^{\infty}\left([0, T] ; L^{1}(\mathbb{R})\right)$, so that the average stress $\tau(\cdot, y)$ is well defined by (1.3c) in $L_{\mathrm{loc}}^{\infty}\left(\mathbb{R}^{+}\right)$.

We now fix some initial condition $p_{0}$ satisfying the conditions (2.3) and (2.4) (thus a fortiori the conditions (2.11) and (2.12)) and set

$$
b(t, y)=G_{0}\left(\partial_{y} u(t, y)+V(t)\right)
$$

for $u \in L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$. In view of Proposition [2.2. we know the existence and uniqueness of a solution $p$ to (1.3b) for given $u$. Our next step now consists in analyzing the dependence on the variable $y$.

Lemma 2.2 (Uniform-in-y a priori estimates on $p$ ) Let $T>0$ be given. We assume that the initial data $p_{0}$ satisfies (2.3) and

$$
\inf _{y \in \Omega} D\left(p_{0}(y)\right)>0 .
$$

Notice that (2.4) is not needed, but (2.11) and (2.12)) are fulfilled. Then, if we denote by $p$ the unique solution to (2.13) given by Proposition [2.2, we have :
(i) $p \in L^{\infty}\left([0, T] \times \Omega ; L_{\sigma}^{1} \cap L_{\sigma}^{\infty}\right)$ with

$$
\begin{equation*}
\|p\|_{L^{\infty}([0, T] \times \Omega \times \mathbb{R})} \leq\left\|p_{0}\right\|_{L^{\infty}(\Omega \times \mathbb{R})}+\frac{\sqrt{\alpha} \sqrt{T}}{\sqrt{\pi}} \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\mathbb{R}} p(t, y, \sigma) d \sigma=1 \quad \text { for all } t \geq 0, \text { for almost every } y \text { in } \Omega \tag{2.17}
\end{equation*}
$$

(ii) The stress $\tau$ is in $L^{2}\left(\Omega, L^{\infty}([0, T])\right.$ ) (hence in $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$ ).
(iii) If in addition $p_{0}$ satisfies the non-degeneracy condition (2.4), we have

$$
\begin{equation*}
\inf _{\substack{0 \leq t \leq T \\ y \in \bar{\Omega}}} D(p(t, y)) \geq \frac{1}{2} e^{-T} \eta \tag{2.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{y \in \Omega} \int_{0}^{T}\left(\int_{\mathbb{R}}\left|\partial_{\sigma} p\right|^{2} d \sigma\right) d t \leq \frac{2}{\eta} e^{T}\left(\left\|p_{0}\right\|_{L^{\infty}(\Omega \times \mathbb{R})}\left(\frac{1}{2}+T\right)+\frac{\alpha}{\sqrt{\pi}} T^{3 / 2}\right) . \tag{2.19}
\end{equation*}
$$

## Proof of Lemma 2.2,

To prove Assertion (i), we use the estimates obtained in [1] with the variable $y$ kept frozen. The assumptions on $p_{0}$ ensure that $p$ is in $L_{T, \sigma}^{\infty}$ for almost every $y$, that (2.17) holds, and that (2.14) holds by virtue of [1, Proposition 1.1, Eq.(1.8)]. Estimate (2.16) follows.

Assertion (ii) follows from [1] Proposition 1.1, Eq.(1.9)] : for almost every $y$ in $\Omega$,

$$
\sup _{0 \leq t \leq T} \int_{\mathbb{R}}|\sigma| p d \sigma \leq \int_{\mathbb{R}}|\sigma| p_{0} d \sigma+\sqrt{T}\left(\frac{2 \sqrt{\alpha}}{\sqrt{\pi}}+\|b\|_{L_{T}^{2}}\right)+\frac{2}{3} T^{3 / 2}\left(1+\frac{2 \sqrt{\alpha}}{\sqrt{\pi}}\right)
$$

with $b=b(t, y)=G_{0}\left(\partial_{y} u(t, y)+V(t)\right)$. Then

$$
\left\|\sup _{0 \leq t \leq T} \int_{\mathbb{R}}|\sigma| p d \sigma\right\|_{L_{y}^{2}} \leq\left\|\sigma p_{0}\right\|_{L_{y}^{2}\left(L_{\sigma}^{1}\right)}+C\left(T, \alpha,\|V\|_{L^{\infty}([0, T])}\right)+\sqrt{T} G_{0}\left\|\partial_{y} u\right\|_{L^{2}\left(\Omega_{T}\right)},
$$

with

$$
C\left(T, \alpha,\|V\|_{L^{\infty}([0, T])}\right)=G_{0} T\|V\|_{L^{\infty}([0, T])}+\frac{\sqrt{T}}{\sqrt{\pi}}\left(2 \sqrt{\alpha}+\frac{2}{3} T(\sqrt{\pi}+2 \sqrt{\alpha})\right) .
$$

For Assertion (iii), following [1, Proof of Lemma 3.1], we define

$$
t^{*}(y)=\inf \left\{t>0 ; \int_{\left|\sigma+\int_{0}^{t} b(s, y) d s\right|>1} p_{0}(y, \sigma) d \sigma=0\right\}
$$

Because $p_{0}$ satisfies the non-degeneracy condition (2.4), we have for all $y, t^{*}(y)>0$ and the support of $p_{0}$ is contained in the interval $\left[-1-\int_{0}^{t^{*}(y)} b(s, y) d s, 1-\int_{0}^{t^{*}(y)} b(s, y) d s[\right.$.

Moreover, for any $T=T(y)<\frac{t^{*}(y)}{2}$,

$$
\begin{equation*}
\min _{0 \leq t \leq T} D(p(t, y)) \geq \frac{\alpha}{2} e^{-T} \min _{0 \leq t \leq T} \int_{\left|\sigma+\int_{0}^{t} b(s, y) d s\right|>1} p_{0}(y, \sigma) d \sigma . \tag{2.20}
\end{equation*}
$$

The assumption (2.4) on $p_{0}$ ensures that $t^{*}(y)=+\infty$ for almost every $y$ in $\Omega$. Therefore (2.20) holds true on any time interval $T>0$ independently of $y$ and (2.18) is an immediate consequence of (2.20) by using (2.4).

Finally (2.19) follows in a very standard way from (2.18) and [1. Equation (3.7)], multiplying (1.3b) by $p$, integrating over $\mathbb{R}$ with respect to $\sigma$, and using (2.18) and the previous bounds.

### 2.3 The map $\mathcal{F}_{1}$ is Lipschitz continuous

Lemma 2.3 For every $T>0$, the mapping $\mathcal{F}_{1}$ is Lipschitz continuous from $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$ to $L^{\infty}\left([0, T] ; L^{2}(\Omega)\right)$, and the Lipschitz constant is a locally bounded function of $T$.

Proof of Lemma 2.3: Let $u_{1}$ and $u_{2}$ be two functions in $L^{2}\left([0, T] ; H_{0}^{1}(\Omega)\right)$, and let $\tau_{1}=$ $\mathcal{F}_{1}\left(u_{1}\right)$ and $\tau_{2}=\mathcal{F}_{1}\left(u_{2}\right)$. We denote by $p_{i}, i=1,2$, the unique solution to (2.2) corresponding to $u_{i}$ whose existence is guaranteed by Proposition 2.2 and Lemma 2.2. We also set $v=u_{1}-u_{2}$, $q=p_{1}-p_{2}$ and $\tau=\tau_{1}-\tau_{2}$. Recall that, for $i=1,2, \tau_{i}=\int_{\mathbb{R}} \sigma p_{i} d \sigma$. We formally multiply equation (2.2) by $\sigma$ and integrate it over $\mathbb{R}$ with respect to $\sigma$ to find

$$
\left\{\begin{array}{l}
\partial_{t} \tau_{i}+\tau_{i}=G_{0}\left(\partial_{y} u_{i}+V(t)\right)+\int_{|\sigma| \leq 1} \sigma p_{i} d \sigma \\
\tau_{i}(0, y)=\int_{\mathbb{R}} \sigma p_{0} d \sigma
\end{array}\right.
$$

The argument may be made rigorous with the help of a standard cut-off argument as in [1]. Subtracting the equations satisfied by $\tau_{1}$ and $\tau_{2}$ yields

$$
\left\{\begin{array}{l}
\partial_{t} \tau+\tau=G_{0} \partial_{y} v+\int_{|\sigma| \leq 1} \sigma q d \sigma  \tag{2.21}\\
\tau(0, y)=0
\end{array}\right.
$$

We then apply $\tau$ to (2.21) and integrate over $\Omega$ to obtain

$$
\begin{equation*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}|\tau|^{2}+\int_{\Omega}|\tau|^{2}=G_{0} \int_{\Omega} \partial_{y} v \tau+\int_{\Omega} \tau\left(\int_{|\sigma| \leq 1} \sigma q d \sigma\right) d y \tag{2.22}
\end{equation*}
$$

Using the Young and the Cauchy-Schwarz inequalities, we have

$$
\int_{\Omega}\left|\partial_{y} v \tau\right| \leq \frac{1}{2 G_{0}}\|\tau\|_{L^{2}(\Omega)}^{2}+\frac{G_{0}}{2}\left\|\partial_{y} v\right\|_{L^{2}(\Omega)}^{2}
$$

and

$$
\begin{align*}
\left|\int_{\Omega} \tau\left(\int_{|\sigma| \leq 1} \sigma q d \sigma\right) d y\right| & \leq \int_{\Omega}\left(\int_{\mathbb{R}}|q|^{2} d \sigma\right)^{1 / 2}|\tau| d y \\
& \leq \frac{1}{2} \int_{\Omega}\left(\int_{\mathbb{R}}|q|^{2} d \sigma\right) d y+\frac{1}{2}\|\tau\|_{L^{2}(\Omega)}^{2} \tag{2.23}
\end{align*}
$$

thus

$$
\frac{d}{d t} \int_{\Omega}|\tau|^{2} \leq G_{0}^{2}\left\|\partial_{y} v\right\|_{L^{2}(\Omega)}^{2}+\int_{\Omega}\left(\int_{\mathbb{R}}|q|^{2} d \sigma\right) d y
$$

and

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\|\tau\|_{L^{2}(\Omega)}^{2} \leq G_{0}^{2}\left\|\partial_{y} v\right\|_{L^{2}\left(\Omega_{T}\right)}^{2}+\|q\|_{L^{2}\left(\Omega_{T} \times \mathbb{R}\right)}^{2} \tag{2.24}
\end{equation*}
$$

Let us now admit for a while that

$$
\begin{equation*}
\|q\|_{L^{2}\left(\Omega_{T} \times \mathbb{R}\right)}^{2} \leq C(T)\left\|\partial_{y} v\right\|_{L^{2}\left(\Omega_{T}\right)}^{2}, \tag{2.25}
\end{equation*}
$$

with $C(T)$ being a locally bounded function of $T$. Inserting (2.25) into (2.24), we obtain

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\|\tau\|_{L^{2}(\Omega)}^{2} \leq\left(G_{0}^{2}+C(T)\right)\left\|\partial_{y} v\right\|_{L^{2}\left(\Omega_{T}\right)}^{2} \tag{2.26}
\end{equation*}
$$

and therefore, the mapping $\mathcal{F}_{1}$ is indeed Lipschitz continuous.
In order to establish (2.25), we subtract the equations (2.2) satisfied by $p_{1}$ and $p_{2}$ respectively to deduce that $q$ solves

$$
\left\{\begin{align*}
\partial_{t} q=- & G_{0} \partial_{y} u_{1} \partial_{\sigma} q-G_{0} \partial_{y} v \partial_{\sigma} p_{2}-G_{0} V(t) \partial_{\sigma} q+D(q) \partial_{\sigma \sigma}^{2} p_{1}  \tag{2.27a}\\
& +D\left(p_{2}\right) \partial_{\sigma \sigma}^{2} q-\mathbb{1}_{\mathbb{R} \backslash[-1,1]}(\sigma) q+\frac{D(q)}{\alpha} \delta_{0}(\sigma) ; \\
q(0, y, \sigma) & =0
\end{align*}\right.
$$

for almost every $y$ in $\Omega$. Then, we apply (2.27) to $q$, and integrate with respect to $\sigma$, to obtain

$$
\begin{align*}
\frac{1}{2} & \frac{\partial}{\partial t} \int_{\mathbb{R}} q^{2}+\int_{|\sigma|>1} q^{2}+D\left(p_{2}\right) \int_{\mathbb{R}}\left|\partial_{\sigma} q\right|^{2} \\
& =G_{0} \partial_{y} v \int_{\mathbb{R}} p_{2} \partial_{\sigma} q-D(q) \int_{\mathbb{R}} \partial_{\sigma} p_{1} \partial_{\sigma} q+\frac{D(q)}{\alpha} q(t, y, 0) \tag{2.28}
\end{align*}
$$

By the Cauchy-Schwarz and the Young inequalities and using the bound from below (2.18) on $D\left(p_{2}\right)$, we have

$$
\begin{align*}
\left|\partial_{y} v \int_{\mathbb{R}} p_{2} \partial_{\sigma} q\right| & \leq\left|\partial_{y} v\right|\left\|p_{2}\right\|_{L_{\sigma}^{2}}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}} \\
& \leq \frac{3 G_{0}}{4 D\left(p_{2}\right)}\left\|p_{2}\right\|_{L_{\sigma}^{2}}^{2}\left|\partial_{y} v\right|^{2}+\frac{D\left(p_{2}\right)}{3 G_{0}}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} \\
& \leq \frac{3 G_{0}}{2 \eta} e^{T}\left(\left\|p_{0}\right\|_{L_{y, \sigma}^{\infty}}+\frac{\sqrt{\alpha T}}{\sqrt{\pi}}\right)\left|\partial_{y} v\right|^{2}+\frac{D\left(p_{2}\right)}{3 G_{0}}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} \tag{2.29}
\end{align*}
$$

thanks to the $L^{\infty}$ bound (2.16) on $p_{2}$ and the fact that $\int_{\mathbb{R}} p_{2}(t, y, \sigma) d \sigma=1$. In a similar way, we obtain

$$
\begin{align*}
\left|D(q) \int_{\mathbb{R}} \partial_{\sigma} p_{1} \partial_{\sigma} q\right| & \leq|D(q)|\left\|\partial_{\sigma} p_{1}\right\|_{L_{\sigma}^{2}}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}} \\
& \leq \frac{3}{2 \eta} e^{T}|D(q)|^{2}\left\|\partial_{\sigma} p_{1}\right\|_{L_{\sigma}^{2}}^{2}+\frac{D\left(p_{2}\right)}{3}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} \tag{2.30}
\end{align*}
$$

As $\int_{\mathbb{R}} q=0$ (recall $\int_{\mathbb{R}} p_{1}=\int_{\mathbb{R}} p_{2}=1$ ), we may write

$$
\begin{equation*}
|D(q)|=\alpha\left|\int_{|\sigma| \geq 1} q\right|=\alpha\left|\int_{|\sigma| \leq 1} q\right| \leq \alpha \sqrt{2}\|q\|_{L_{\sigma}^{2}} . \tag{2.31}
\end{equation*}
$$

Thus, inserting (2.31) into (2.30), we obtain

$$
\begin{equation*}
\left|D(q) \int_{\mathbb{R}} \partial_{\sigma} p_{1} \partial_{\sigma} q\right| \leq \frac{3 \alpha^{2}}{\eta} e^{T}\|q\|_{L_{\sigma}^{2}}^{2}\left\|\partial_{\sigma} p_{1}\right\|_{L_{\sigma}^{2}}^{2}+\frac{D\left(p_{2}\right)}{3}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} . \tag{2.32}
\end{equation*}
$$

On the other hand, from the Sobolev embedding $H^{1}(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$, we know

$$
\begin{equation*}
\|q\|_{L_{\sigma}^{\infty}} \leq \frac{1}{\sqrt{2}}\left(\|q\|_{L_{\sigma}^{2}}^{2}+\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2}\right)^{\frac{1}{2}}, \tag{2.33}
\end{equation*}
$$

and next using successively Young's inequality, (2.31), (2.18) again and the fact that $D\left(p_{2}\right) \leq$ $\alpha$, we find

$$
\begin{align*}
\frac{1}{\alpha}|D(q) \| q(t, y, 0)| & \leq\|q\|_{L_{\sigma}^{2}}\left(\|q\|_{L_{\sigma}^{2}}^{2}+\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2}\right)^{\frac{1}{2}} \\
& \leq\left(\frac{D\left(p_{2}\right)}{3}+\frac{3}{4 D\left(p_{2}\right)}\right)\|q\|_{L_{\sigma}^{2}}^{2}+\frac{D\left(p_{2}\right)}{3}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} \\
& \leq\left(\frac{\alpha}{3}+\frac{3}{2 \eta} e^{T}\right)\|q\|_{L_{\sigma}^{2}}^{2}+\frac{D\left(p_{2}\right)}{3}\left\|\partial_{\sigma} q\right\|_{L_{\sigma}^{2}}^{2} . \tag{2.34}
\end{align*}
$$

Inserting (2.29), (2.32) and (2.34) in (2.28), we have

$$
\begin{aligned}
\frac{1}{2} \partial_{t} \int_{\mathbb{R}} q^{2} \leq & \frac{3 G_{0}^{2}}{2 \eta} e^{T}\left(\left\|p_{0}\right\|_{L_{y, \sigma}^{\infty}}+\frac{\sqrt{\alpha T}}{\sqrt{\pi}}\right)\left|\partial_{y} v\right|^{2} \\
& \quad+\left(\frac{\alpha}{3}+\frac{3}{2 \eta} e^{T}+\frac{3 \alpha^{2}}{\eta} e^{T}\left\|\partial_{\sigma} p_{1}\right\|_{L_{\sigma}^{2}}^{2}\right)\|q\|_{L_{\sigma}^{2}}^{2}
\end{aligned}
$$

for all $t$ in $[0 ; T]$ and almost every $y$ in $\Omega$. Applying the Gronwall lemma, we obtain

$$
\begin{align*}
& \int_{\mathbb{R}} q^{2}(t, y, \sigma) d \sigma \leq \frac{3 G_{0}^{2}}{\eta} e^{T}\left(\left\|p_{0}\right\|_{L_{y, \sigma}^{\infty}}+\frac{\sqrt{\alpha T}}{\sqrt{\pi}}\right) \int_{0}^{T}\left|\partial_{y} v\right|^{2} d t \\
& \quad \times \exp \left(\frac{2 \alpha T}{3}+\frac{3}{\eta} T e^{T}+\frac{6 \alpha^{2}}{\eta} e^{T} \int_{0}^{T}\left\|\partial_{\sigma} p_{1}\right\|_{L_{\sigma}^{2}}^{2} d t\right) \tag{2.35}
\end{align*}
$$

for almost every $(t, y)$ in $\Omega_{T}$. We now integrate the above inequality over $\Omega_{T}$ and we use the bound (2.19) on $\partial_{\sigma} p_{1}$ in $L_{y}^{\infty}\left(L_{T, \sigma}^{2}\right)$ to deduce (2.25) with

$$
\begin{align*}
C(T)= & \frac{3 G_{0}^{2}}{\eta} T e^{T}\left(\left\|p_{0}\right\|_{L_{y, \sigma}^{\infty}}+\frac{\sqrt{\alpha T}}{\sqrt{\pi}}\right) \times \\
& \times \exp \left(\frac{2 \alpha T}{3}+\frac{3}{\eta} T e^{T}+\frac{6 \alpha^{2}}{\eta} e^{T} C^{\prime}(T)\right), \tag{2.36}
\end{align*}
$$

with an explicit expression for $C^{\prime}(T)$ coming from the right-hand side of (2.19), namely :

$$
C^{\prime}(T)=\frac{2}{\eta} e^{T}\left(\left\|p_{0}\right\|_{L_{y, \sigma}^{\infty}}\left(\frac{1}{2}+T\right)+\frac{\alpha}{\sqrt{\pi}} T^{3 / 2}\right) .
$$

### 2.4 Global-in-time existence

Let us assume that there exists some finite $t^{*}$ such that the system admits a solution $\left(u^{*} ; p^{*}\right)$ on $\left[0, t^{*}\left[\right.\right.$ that ceases to exist after the time $t^{*}$ (at least in the appropriate functional spaces prescribed by our notion of solution).

Let $0<t_{0}<t^{*}$. We consider the following Cauchy problem starting at time $t_{0}$ :

$$
\left\{\begin{array}{l}
\rho \partial_{t} u-\partial_{y y}^{2} u=\partial_{y} \tau-\rho \dot{V}(t) y  \tag{2.37a}\\
\tau(t, y)=\int_{\mathbb{R}} \sigma p d \sigma \\
u(0, y)=\widetilde{u}_{0}(y):=u^{*}\left(t_{0}, y\right) \quad \text { on } \Omega \\
u(t, 0)=0, \quad u(t, 1)=0
\end{array}\right.
$$

which is coupled to

$$
\left\{\begin{array}{l}
\partial_{t} p+G_{0}\left(\partial_{y} u+V(t)\right) \partial_{\sigma} p-D(p(t, y)) \partial_{\sigma \sigma}^{2} p+\mathbb{1}_{\mathbb{R} \backslash[-1,1]}(\sigma) p=\frac{D(p(t, y))}{\alpha} \delta_{0}(\sigma) ;  \tag{2.38a}\\
p \geq 0 ; \\
p(0, y, \sigma)=\tilde{p}_{0}(y, \sigma):=p^{*}\left(t_{0}, y, \sigma\right)
\end{array}\right.
$$

As a first step we prove that the new initial condition ( $\tilde{u}_{0} ; \tilde{p}_{0}$ ) satisfies the assumptions in Theorem (2.1) The only point to be checked is that $\tilde{p}_{0}$ fulfills (2.4). Indeed, $\tilde{u}_{0}$ is in $L^{2}(\Omega)$ and $\tilde{p}_{0}$ satisfies (2.3), thanks to the bounds in Lemma 2.2 which hold true for $p^{*}$ on $\left[0, t^{*}\right)$. Actually we prove the more general :

Lemma 2.4 Let $\left(u^{*} ; p^{*}\right)$ be the solution to the coupled system on $\left[0 ; t^{*}\right)$ under the assumptions of Theorem 2.1 for the initial data. Then, for every $0<T<t^{*}$, we have

$$
\begin{equation*}
\inf _{\substack{(t, y) \in \Omega_{T} \\ \chi \in \mathbb{R}_{T}}} \int_{|\sigma+\chi|>1} p^{*}(t, y, \sigma) d \sigma \geq \frac{\eta}{2 \alpha} e^{-T} . \tag{2.39}
\end{equation*}
$$

Proof of Lemma 2.4: The proof follows from a comparison principle and is inspired from [1. It is reproduced here for the reader's convenience. We denote by $p_{-}$the solution to the linear equation :

$$
\left\{\begin{align*}
\partial_{t} p_{-} & =-G_{0}\left(\partial_{y} u+V(t)\right) \partial_{\sigma} p_{-}+D\left(p^{*}(t, y)\right) \partial_{\sigma \sigma}^{2} p_{-}-p_{-}  \tag{2.40}\\
p_{-}(0, y, \sigma) & =p_{0}(y, \sigma)
\end{align*}\right.
$$

It is well-known that $p_{-}$is given by

$$
\begin{equation*}
p_{-}(t, y, \sigma)=e^{-t} \int_{-\infty}^{+\infty} p_{0}\left(y, \sigma^{\prime}\right) \varphi \sqrt{2 \int_{0}^{t} D\left(p^{*}(s, y)\right) d s}\left(\sigma-\sigma^{\prime}-\xi(t, y)\right) d \sigma^{\prime}, \tag{2.41}
\end{equation*}
$$

with

$$
\left\{\begin{aligned}
\varphi_{\nu}(x) & =\frac{1}{\sqrt{2 \pi} \nu} \exp \left(-\frac{x^{2}}{2 \nu^{2}}\right) \quad \text { if } \nu>0 ; \\
\varphi_{0} & =\delta_{0}
\end{aligned}\right.
$$

and

$$
\xi(t, y)=G_{0} \int_{0}^{t}\left(\partial_{y} u(s, y)+V(s)\right) d s
$$

Since $p_{-} \leq p^{*}$, by the maximum principle, we get the bound from below

$$
\begin{align*}
& \int_{|\sigma-\chi|>1} p^{*}(t, y, \sigma) d \sigma \\
& \quad \geq \int_{|\sigma-\chi|>1} p_{-}(t, y, \sigma) d \sigma \\
& \geq e^{-t} \int_{\mathbb{R}} p_{0}\left(y, \sigma^{\prime}\right)\left(\int_{|\sigma-\chi|>1} \varphi \sqrt{2 \int_{0}^{t}\left(D\left(p^{*}(s, y)\right) d s\right.}\left(\sigma-\sigma^{\prime}-\xi(t, y)\right) d \sigma\right) d \sigma^{\prime}, \tag{2.42}
\end{align*}
$$

for every $\chi$ in $\mathbb{R}$. As in 【1, we introduce the interval $K_{\xi, \chi}=[-1-\xi(t, y)+\chi, 1-\xi(t, y)+\chi]$. The function $\sigma \mapsto \varphi \sqrt{2 \int_{0}^{t} D\left(p^{*}(s, y)\right) d s}\left(\sigma-\sigma^{\prime}-\xi(t, y)\right)$ is a Gaussian probability density with mean $\sigma^{\prime}+\xi(t, y)$ and squared width $2 \int_{0}^{t} D\left(p^{*}(s, y)\right) d s$. Therefore, for every $\sigma^{\prime} \in \mathbb{R} \backslash K_{\xi, \chi}$, we have

$$
\int_{|\sigma-\chi|>1} \varphi \sqrt{2 \int_{0}^{t} D\left(p^{*}(s, y)\right) d s}\left(\sigma-\sigma^{\prime}-\xi(t, y)\right) d \sigma \geq \frac{1}{2}
$$

which implies

$$
\text { (2.42) } \geq \frac{1}{2} e^{-T} \int_{\mathbb{R} \backslash K_{\xi, \chi}} p_{0}\left(y, \sigma^{\prime}\right) d \sigma^{\prime}=\frac{1}{2} e^{-T} \int_{\left|\sigma^{\prime}-\chi+\xi(t, y)\right|>1} p_{0}\left(y, \sigma^{\prime}\right) d \sigma^{\prime} .
$$

And we conclude using (2.4).

## Completion of the Proof of Theorem 2.1

In view of Lemma [2.4 we may apply the Banach fixed point theorem as in the proof of Proposition 2.1 and deduce the existence of a unique solution to the Cauchy problem (2.37)(2.38) on the time interval $\left[t_{0} ; t_{0}+\kappa\right]$ for some small enough $\kappa>0$. Of course, this solution coincides with $\left(u^{*} ; p^{*}\right)$ by uniqueness. We now show that $\kappa$ may be chosen independently of $t_{0}$ in $\left(0 ; t^{*}\right)$. Therefore the solution exists beyond the time $t^{*}$, which contradicts the finiteness of $t^{*}$. The constant $\kappa$ depends on the Lipschitz constant for the mapping $\mathcal{F}$. Because of (2.10) the Lipschitz constant of the mapping $\mathcal{F}_{2}$ is clearly independent of the initial time $t_{0}$ and can be made arbitrarily small using (2.10) provided the length of the time interval is taken small. We thus focus on the Lipschitz constant of $\mathcal{F}_{1}$, and now show it is bounded uniformly in $t^{*}$. Thus, the condition on $\kappa$ such that $\mathcal{F}=\mathcal{F}_{2} \circ \mathcal{F}_{1}$ is a contraction on $\left[t_{0} ; t_{0}+\kappa\right]$ is independent of $t_{0}$ in $\left(0 ; t^{*}\right)$, which concludes the proof.

We revisit carefully the proof of Lemma [2.3] which is the crucial step for checking the assumptions of the Banach fixed point theorem on small time interval. We go back to the proof of (2.25). The only modifications are in the proofs of estimates (2.29), (2.32) and (2.34) as follows. In view of the uniform estimate given by Lemma 2.4 the quantity $D\left(p^{*}\right)$ is bounded from below by $\eta \exp \left(-t^{*}\right) / 2$ and the $L^{\infty}$ and $H^{1}$ norms of $p^{*}$ in the sense of (2.16) and (2.19) are bounded uniformly in terms of $t^{*}$. (In all these bounds $T$ may obviously be bounded by $t^{*}$.) Therefore the Lipschitz constant of $\mathcal{F}_{1}$ given by (2.26) is bounded uniformly in $t^{*}$.

## 3 The case $\sigma_{c}=0$

In the situation examined so far, that is when $\sigma_{c}>0$, we have only been able to show the well-posedness of the coupled system when the macroscopic equation has a positive diffusion coefficient $\mu$. This is a mathematical artefact, apparently related to our technique of proof. Our aim in the present section, as announced in the introduction, is to mention that the coupled system is also well-posed in the particular case when $\mu \geq 0$ and $\sigma_{c}=0$.

We again scale out the variables $y$ and $t$, together with the function $u$ as explained in Appendix, which amounts to taking $T_{0}=L=1$. In the present case when $\sigma_{c}=0$, we note that, for a given $b(t, y) \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{+}, L^{2}(\Omega)\right.$ ), the unique solution of (2.13) provided by Proposition 2.2 reads

$$
\begin{align*}
p(t, y, \sigma)= & e^{-t} \int_{\mathbb{R}} p_{0}\left(\sigma^{\prime}\right) \mathcal{G}_{\alpha t}\left(\sigma-\sigma^{\prime}-\chi(t, y)\right) d \sigma^{\prime}  \tag{3.1}\\
& +\int_{0}^{t} e^{-(t-s)} \mathcal{G}_{\alpha(t-s)}(\sigma-\chi(t, y)+\chi(s, y)) d s
\end{align*}
$$

where $\chi(t, y)=\int_{0}^{t} b(s, y) d s$ and where $\left(\mathcal{G}_{t}\right)_{t \geq 0}$ denotes the heat kernel

$$
\begin{aligned}
& \mathcal{G}_{t}(\sigma)=\frac{1}{\sqrt{4 \pi t}} \exp \left(-\frac{\sigma^{2}}{4 t}\right) \quad \text { if } t>0 ; \\
& \mathcal{G}_{0}(\sigma)=\delta_{0}(\sigma)
\end{aligned}
$$

If we multiply (3.1) by $\sigma$ and integrate over the real line, we obtain

$$
\begin{equation*}
\partial_{t} \tau(t, y)+\tau(t, y)=b(t, y) . \tag{3.2}
\end{equation*}
$$

For $\sigma_{c}=0$, the multiscale Hébraud-Lequeux model is therefore equivalent to the so-called Maxwell model [9, and the coupled system under consideration reads

$$
\left\{\begin{array}{l}
\rho \partial_{t} u=\mu \partial_{y y}^{2} u+\partial_{y} \tau-\rho \dot{V}(t) y  \tag{3.3}\\
\partial_{t} \tau+\tau=G_{0} \partial_{y} u+G_{0} V(t) \\
u(\cdot, 0)=u(\cdot, 1)=0
\end{array}\right.
$$

The latter is a linear system for which it is easy to prove global existence and uniqueness in convenient functional spaces such that $\partial_{y} u \in L_{\mathrm{loc}}^{2}\left(\mathbb{R}^{+} ; L^{2}(\Omega)\right)$, whatever $\mu \geq 0$. Global existence and uniqueness for the multiscale Hébraud-Lequeux model immediately follows.

## Appendix: Non-dimensionalized equations

We first scale the space and time variables in order to work with dimensionless constants and with a reduced number of parameters. We introduce the new dimensionless variables

$$
t^{\prime}=\frac{t}{T_{0}}, \quad y^{\prime}=\frac{y}{L}, \quad \sigma^{\prime}=\frac{\sigma}{\sigma_{c}},
$$

and the dimensionless rescaled functions

$$
U^{\prime}=\frac{T_{0}}{L} U, \quad p^{\prime}=\sigma_{c} p, \quad \tau^{\prime}=\frac{\tau}{\sigma_{c}}=\int_{\mathbb{R}} \sigma^{\prime} p^{\prime} d \sigma^{\prime},
$$

together with the corresponding dimensionless parameters

$$
\rho^{\prime}=\frac{\rho L^{2}}{\sigma_{c} T_{0}^{2}}, \quad \alpha^{\prime}=\frac{\alpha}{\sigma_{c}^{2}}, \quad G_{0}^{\prime}=\frac{G_{0}}{\sigma_{c}}, \quad \mu^{\prime}=\frac{\mu}{T_{0} \sigma_{c}} .
$$

Note that $\rho^{\prime}$ is actually the so-called Reynolds number. We also define

$$
D^{\prime}\left(p^{\prime}\right)=\alpha^{\prime} \int_{\left|\sigma^{\prime}\right|>1} p^{\prime} d \sigma^{\prime}
$$

Then, equations (1.3a) and (1.3b) respectively read:

$$
\begin{equation*}
\rho \partial_{t} U^{\prime}-\mu \partial_{y y}^{2} U^{\prime}=\partial_{y} \tau^{\prime} \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{t} p^{\prime}=-G_{0} \partial_{y} U^{\prime} \partial_{\sigma} p^{\prime}+D\left(p^{\prime}\right) \partial_{\sigma \sigma}^{2} p^{\prime}-\mathbb{1}_{\mathbb{R} \backslash[-1,1]}(\sigma) p^{\prime}+\frac{D\left(p^{\prime}\right)}{\alpha} \delta_{0}(\sigma) \tag{3.5}
\end{equation*}
$$

Of course the corresponding change of scales and variables are also applied to the initial conditions $u_{0}$ and $p_{0}$. The new function $V^{\prime}$ entering in the boundary conditions of $U^{\prime}$ has to be changed according to

$$
V^{\prime}=\frac{T_{0}}{L} V,
$$

All the primes are omitted in the body of the article in order to lighten the notation.

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