

# GENERALIZED $\mathcal{S}$ -PROCEDURE FOR INEQUALITY CONDITIONS ON ONE-VECTOR-LOSSLESS SETS AND LINEAR SYSTEM ANALYSIS\*

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**Abstract.** The generalized version of the  $\mathcal{S}$ -procedure, recently introduced by Iwasaki and co-authors and Scherer independently, has proved to be very useful for robustness analysis and synthesis of control systems. In particular, this procedure provides a nonconservative way to convert specific inequality conditions on lossless sets into numerically verifiable conditions represented by linear matrix inequalities (LMIs). In this paper, we introduce a new notion, one-vector-lossless sets, and propose a generalized  $\mathcal{S}$ -procedure to reduce inequality conditions on one-vector-lossless sets into LMIs without any conservatism. By means of the proposed generalized  $\mathcal{S}$ -procedure, we can examine various properties of matrix-valued functions over some regions on the complex plane. To illustrate the usefulness, we show that full rank property analysis problems of polynomial matrices over some specific regions on the complex plane can be reduced into LMI feasibility problems. It turns out that many existing results such as Lyapunov's inequalities and LMIs for state-feedback controller synthesis readily follow from the suggested generalized  $\mathcal{S}$ -procedure.

**Key words.**  $\mathcal{S}$ -procedure, one-vector-lossless set, linear matrix inequalities

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**1. Introduction.** Recently, the generalized version of the  $\mathcal{S}$ -procedure has been introduced independently by Iwasaki and Hara [7, 8], Iwasaki, Meinsma, and Fu [9], Iwasaki and Shibata [10], and Scherer [14, 15, 16, 17]. Basically speaking, this procedure enables us to convert intractable semi-infinite parametrized linear matrix inequalities into numerically verifiable finite-dimensional linear matrix inequalities (LMIs). The scope of its application is wide and includes a variety of robustness analysis and synthesis problems in linear control system theory.

Among these recent papers, in [9, 8], the following inequality condition with respect to a Hermitian matrix  $\Theta$  and a subset  $\mathcal{S}$  of Hermitian matrices is discussed:

$$(1.1) \quad \zeta^* \Theta \zeta > 0 \quad \forall \zeta \in \mathcal{G}, \quad \mathcal{G} := \{\zeta \in \mathbf{C}^n : \zeta \neq 0, \quad \zeta^* S \zeta \geq 0 \quad \forall S \in \mathcal{S}\}.$$

It can be easily seen that a sufficient condition for (1.1) is given by

$$(1.2) \quad \exists S \in \mathcal{S} \text{ such that } \Theta \succ S.$$

The procedure to replace (1.1) by (1.2) is called the generalized  $\mathcal{S}$ -procedure in [9, 8]. Generally, this replacement introduces conservatism; the condition (1.2) is only sufficient for (1.1) and may not be necessary. The significance of the studies in [9, 8] lies in the fact that the generalized  $\mathcal{S}$ -procedure has been proved to be nonconservative if the set  $\mathcal{S}$  is *lossless*<sup>1</sup> [9, 8]. If the set  $\mathcal{S}$  is lossless, then the set  $\mathcal{S}$  is convex and hence

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<sup>1</sup>In the paper [8], the terminology “rank-one separable” is used in place of “lossless.”

the LMI condition (1.2) can be verified numerically via sophisticated interior-point methods [1, 4].

When we deal with linear system analysis and synthesis problems by working with the generalized  $\mathcal{S}$ -procedure in [9, 8], the underlying idea is that inequality conditions on matrix-valued functions  $G(\lambda)$  over *curves*  $\lambda \in \Lambda$  ( $\Lambda \subset \mathbf{C}$ ) can be reformulated into a conformable form to the condition (1.1) by considering an appropriate Hermitian matrix  $\Theta$  and a lossless set  $\mathcal{S}$  [9, 8]. When dealing with a linear system, its various properties can be characterized by inequality conditions on their transfer functions in the frequency domain [1, 20, 21]. In [9, 8], those frequency domain inequalities are reformulated in the form of (1.1) so that the generalized  $\mathcal{S}$ -procedure can be applied. It follows that we can verify various properties of linear systems without introducing any conservatism by solving LMIs resulting from the generalized  $\mathcal{S}$ -procedure.

For linear system analysis and synthesis, however, we also need to verify inequality conditions on matrix-valued functions  $G(\lambda)$  over *region*  $\lambda \in \mathcal{D}$  ( $\mathcal{D} \subset \mathbf{C}$ ). For example, full rank property analysis of polynomial matrices over some specific regions on the complex plane forms an important basis for the stability analysis of linear systems [1, 6]. In view of these facts, it should be quite natural to pose the following question: Can we verify various properties of matrix-valued functions  $G(\lambda)$  over *region*  $\lambda \in \mathcal{D}$  ( $\mathcal{D} \subset \mathbf{C}$ ) by following similar lines to the generalized  $\mathcal{S}$ -procedure?

To answer this question, in this paper, we first introduce a new notion, *one-vector-lossless sets*, and provide a nonconservative generalized  $\mathcal{S}$ -procedure for inequality conditions on this set. More precisely, by taking account of the fact that the properties of lossless sets are fully used to represent *curves* on the complex plane [9, 8], we first consider to relax the requirements for the lossless sets and define one-vector-lossless sets, which enables us to represent *regions* on the complex plane. Then, we clarify under what condition the generalized  $\mathcal{S}$ -procedure for inequality conditions on this set is nonconservative. It follows that we can provide a counterpart result of [9, 8] in the case of the one-vector-lossless sets.

To illustrate the usefulness of the proposed generalized  $\mathcal{S}$ -procedure, we show that full rank property analysis problems of polynomial matrices over some regions  $\mathcal{D} \subset \mathbf{C}$  can be reduced into LMI feasibility problems. It turns out that the well-known results such as Lyapunov's inequalities for stability analysis of linear systems [1, 5] and LMIs for state-feedback controller synthesis [1, 18] follow immediately from the full rank property analysis by means of the proposed generalized  $\mathcal{S}$ -procedure. Thus, in conjunction with the results in [8, 16], the present paper reveals that most LMI results in linear system theory can be captured in a unified fashion within the framework of the generalized  $\mathcal{S}$ -procedure.

We use the following notation in this paper. For a matrix  $A$ , its transpose and complex conjugate transpose are denoted by  $A^T$  and  $A^*$ , respectively. For a matrix  $A \in \mathbf{C}^{n \times m}$  with  $\text{rank}(A) = r < n$ ,  $A^\perp \in \mathbf{C}^{(n-r) \times n}$  is a matrix such that  $A^\perp A = 0$  and  $A^\perp (A^\perp)^* \succ 0$ . The symbols  $\mathbf{H}_n$  and  $\mathbf{P}_n$  denote the sets of  $n \times n$  Hermitian matrices and positive-definite Hermitian matrices, respectively. For matrices  $\Psi$  and  $P$ , we denote by  $\Psi \otimes P$  their Kronecker product. For  $\lambda \in \mathbf{C}$  and  $\Psi \in \mathbf{H}_2$ , we define a function  $\sigma : \mathbf{C} \times \mathbf{H}_2 \rightarrow \mathbf{R}$  by

$$\sigma(\lambda, \Psi) := \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix}.$$

**2. Generalized  $\mathcal{S}$ -procedure for inequality conditions on one-vector-lossless sets.** The notion of *one-vector-lossless sets* plays an important role in this

paper. In this section, we first describe its precise definition and provide a nonconservative generalized  $\mathcal{S}$ -procedure for inequality conditions on this set.

DEFINITION 2.1 (one-vector-lossless sets). *A subset  $\mathcal{S} \subset \mathbf{H}_n$  is said to be one-vector-lossless if it has the following properties:*

- (a)  $\mathcal{S}$  is convex.
- (b)  $S \in \mathcal{S} \Rightarrow \tau S \in \mathcal{S} \quad \forall \tau > 0$ .
- (c) For each nonzero matrix  $H \in \mathbf{C}^{n \times n}$  with rank  $r$  that satisfies

$$(2.1) \quad H = H^* \succeq 0, \quad \text{trace}(SH) \geq 0 \quad \forall S \in \mathcal{S},$$

there exist vectors  $\zeta_i \in \mathbf{C}^n$  ( $i = 1, \dots, r$ ) such that  $H = \sum_{i=1}^r \zeta_i \zeta_i^*$  and the condition  $\zeta_j^* S \zeta_j \geq 0$  ( $\forall S \in \mathcal{S}$ ) holds for at least one index  $j$ .

This definition has been introduced by relaxing the requirements for the *lossless sets* given in [9]. Indeed, Definition 2.1 becomes the requirements for the lossless sets by replacing (c) by (c') given in the following:

- (c') For each nonzero matrix  $H \in \mathbf{C}^{n \times n}$  with rank  $r$  that satisfies (2.1), there exist vectors  $\zeta_i \in \mathbf{C}^n$  ( $i = 1, \dots, r$ ) such that

$$H = \sum_{i=1}^r \zeta_i \zeta_i^*, \quad \zeta_i^* S \zeta_i \geq 0 \quad \forall i, \quad \forall S \in \mathcal{S}.$$

This property is referred to as *rank-one separable* in [8]. Detailed analysis on this property can also be found in [13].

By comparing the conditions (c) and (c'), we see that the condition  $\zeta_j^* S \zeta_j \geq 0$  ( $\forall S \in \mathcal{S}$ ) is required only for one index  $j$  in the definition of the one-vector-lossless sets. Hence, it is obvious that a lossless set is one-vector-lossless.

In the case where the set  $\mathcal{S}$  is lossless, the condition (1.1) can be converted into (1.2) without introducing any conservatism. The following theorem gives a counterpart of this result in the case where the set  $\mathcal{S}$  is one-vector-lossless.

THEOREM 2.2 (the generalized  $\mathcal{S}$ -procedure for inequality conditions on one-vector-lossless sets). *Let  $\Theta \in \mathbf{H}_n$  and a one-vector-lossless set  $\mathcal{S} \subset \mathbf{H}_n$  be given. If  $\Theta = \Theta^* \succeq 0$ , then the following statements are equivalent:*

- (i)  $\zeta^* \Theta \zeta > 0 \quad \forall \zeta \in \mathcal{G}$ ,  $\mathcal{G} := \{\zeta \in \mathbf{C}^n : \zeta \neq 0, \zeta^* S \zeta \geq 0 \quad \forall S \in \mathcal{S}\}$ .
- (ii) There exists  $S \in \mathcal{S}$  such that  $\Theta \succ S$ .

*Proof.* (ii)  $\Rightarrow$  (i). Suppose (ii) holds. Then, there exists  $S_0 \in \mathcal{S}$  such that  $\zeta^*(\Theta - S_0)\zeta > 0$  ( $\forall \zeta \neq 0$ ). This inequality implies that

$$\zeta^* \Theta \zeta > 0 \quad \forall \zeta \in \mathcal{G}_0, \quad \mathcal{G}_0 := \{\zeta \in \mathbf{C}^n : \zeta \neq 0, \quad \zeta^* S_0 \zeta \geq 0\}.$$

Since  $\mathcal{G} \subset \mathcal{G}_0$ , we can conclude that the condition (ii) implies (i).

(i)  $\Rightarrow$  (ii). Suppose (ii) does not hold, i.e., there is no  $S \in \mathcal{S}$  such that  $\Theta \succ S$ . Then, since  $\mathcal{S}$  is convex, it follows from the separating hyperplane theorem [11] that there exists a nonzero matrix  $H \in \mathbf{C}^{n \times n}$  such that

$$(2.2) \quad H = H^* \succeq 0, \quad \text{trace}((\Theta - S)H) \leq 0 \quad \forall S \in \mathcal{S}.$$

In view of the property (b) of the one-vector-lossless set, we see that the following conditions are necessary for the second condition in (2.2) to hold:

$$(2.3) \quad \text{trace}(\Theta H) \leq 0, \quad \text{trace}(SH) \geq 0 \quad \forall S \in \mathcal{S}.$$

Since  $\mathcal{S}$  is one-vector-lossless, it follows from the property (c) of Definition 2.1 that the second condition from (2.3) implies the existence of the vectors  $\zeta_i$  ( $i = 1, \dots, r$ ) such that  $H = \sum_{i=1}^r \zeta_i \zeta_i^*$  and  $\zeta_j^* S \zeta_j \geq 0$  ( $\forall S \in \mathcal{S}$ ) for some  $j$ , where  $r$  is the rank of  $H$ . For those vectors  $\zeta_i$ , the first condition in (2.3) implies  $\zeta_i^* \Theta \zeta_i = 0$  ( $i = 1, \dots, r$ ) due to the assumption  $\Theta = \Theta^* \succeq 0$ . These facts in particular imply that  $\zeta_j^* \Theta \zeta_j = 0$  and  $\zeta_j \in \mathcal{G}$  for at least one index  $j$ . This clearly contradicts the condition (i).  $\square$

We note that, in comparison with the case where the set  $\mathcal{S}$  is lossless [9], an additional condition  $\Theta = \Theta^* \succeq 0$  has been imposed in Theorem 2.2. This could be regarded as a price to pay for relaxing the requirements on the set  $\mathcal{S}$  from a lossless one to a one-vector-lossless one.

By means of the generalized  $\mathcal{S}$ -procedure in Theorem 2.2, we can convert the semi-infinite inequality condition (i) into the numerically verifiable LMI condition in (ii). Hence, when we deal with control system analysis and synthesis problems at hand, a crucial step is to reduce those problems into a form conformable to the condition (i). This step is not obvious in general. When exploring such reduction, it is indispensable to see concretely what sets are indeed one-vector-lossless. In the next theorem, we will show a class of one-vector-lossless sets that is relevant to control system analysis and synthesis.

**THEOREM 2.3.** *Let  $\Psi \in \mathbf{H}_2$  with  $\det(\Psi) < 0$  and  $\Gamma \in \mathbf{C}^{2n \times l}$  be given. Define a subset of Hermitian matrices by*

$$(2.4) \quad \mathcal{S} := \{\Gamma^*(\Psi \otimes P)\Gamma : P \in \mathbf{P}_n\}.$$

*Then the set  $\mathcal{S}$  is one-vector-lossless.*

*Proof.* The proof is rather technical and thus given in the appendix.  $\square$

It is meaningful to examine the property of one-vector-lossless set  $\mathcal{S}$  given by (2.4) in comparison with the lossless set  $\mathcal{S}_1$  discussed in [8], where

$$\mathcal{S}_1 := \{\Gamma^*(\Psi \otimes P)\Gamma : P \in \mathbf{H}_n\}.$$

To see a significant difference between these two sets, let us take  $\Psi = \text{diag}(-1, 1)$  and  $\Gamma = I_{2n}$  for simplicity and consider the following set that concerns the condition (i) in Theorem 2.2:

$$(2.5) \quad \mathcal{G} := \left\{ \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \in \mathbf{C}^{2n} : f_0, f_1 \in \mathbf{C}^n, \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \neq 0, \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}^* S \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \geq 0 \quad \forall S \in \mathcal{S} \right\}.$$

Then, we can show that the above set defined from the one-vector-lossless set  $\mathcal{S}$  coincides with

$$(2.6) \quad \mathcal{L} := \left\{ \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} \in \mathbf{C}^{2n} : f_1 = s f_0 \text{ for some } s \in \overline{\mathbf{D}} \right\},$$

where  $\overline{\mathbf{D}}$  denotes the closure of the open unit disc  $\mathbf{D}$  on the complex plane. On the other hand, if we replace the one-vector-lossless set  $\mathcal{S}$  in (2.5) by the lossless set  $\mathcal{S}_1$ , then the resulting set  $\mathcal{G}_1$  coincides with the set  $\mathcal{L}_1$  obtained by replacing  $\overline{\mathbf{D}}$  in (2.6) by  $\partial\mathbf{D}$ . These observations clearly indicate that the lossless sets are related to *curves* on the complex plane, while the one-vector-lossless sets are related to *regions* on the complex plane. This is the key observation to develop the generalized  $\mathcal{S}$ -procedure for inequality conditions on the one-vector-lossless sets. We show in the next section that full rank property analysis problems of polynomial matrices over some regions on the complex plane can be dealt with by the proposed generalized  $\mathcal{S}$ -procedure.

**3. Linear system analysis using generalized  $\mathcal{S}$ -procedure.** For given complex matrices  $M_k \in \mathbf{C}^{n \times m}$  ( $k = 0, \dots, N$ ) with  $n \geq m$ , let us consider the  $n \times m$  complex polynomial matrix represented by  $M(s) = \sum_{k=0}^N s^k M_k$ . We assume that the normal rank of  $M(s)$  is  $m$ . Following the discussions in [6, 19], we define a (finite) zero of  $M(s)$  as a complex value  $z \in \mathbf{C}$  for which the rank of  $M(s)$  drops from its normal value, i.e.,  $\text{rank}(M(z)) < m$ . In linear system analysis and synthesis, it is of great importance to determine whether the zeros of given polynomial matrix  $M(s)$  belong to a specific region  $\mathcal{D} \subset \mathbf{C}$ . This can be restated equivalently in the way that the polynomial matrix  $M(s)$  is of full-column rank for all  $s \in \mathcal{D}^c$ , where  $\mathcal{D}^c$  denotes the complement of the region  $\mathcal{D}$  in  $\mathbf{C}$ . In the subsequent discussions, we restrict our attention to the regions defined below.

DEFINITION 3.1. For given  $\Psi \in \mathbf{H}_2$  with  $\det(\Psi) < 0$ , we define a set  $\mathcal{D}_\Psi$  and its complement  $\mathcal{D}_\Psi^c$  by

$$(3.1) \quad \mathcal{D}_\Psi := \{\lambda \in \mathbf{C} : \sigma(\lambda, \Psi) < 0\}, \quad \mathcal{D}_\Psi^c := \{\lambda \in \mathbf{C} : \sigma(\lambda, \Psi) \geq 0\}.$$

By selecting the Hermitian matrix  $\Psi$  in (3.1) appropriately, we can obtain several important regions in linear system analysis and synthesis. In particular, by letting

$$(3.2) \quad \Psi_c := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi_d := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we see that  $\mathcal{D}_{\Psi_c}$  and  $\mathcal{D}_{\Psi_d}$  coincide with the open left half plane  $\mathbf{C}_-$  and the open unit disc  $\mathbf{D}$ , respectively. These regions are particularly important for stability analysis of continuous- and discrete-time linear systems.

We are now in the right position to show that the full rank property analysis problems of polynomial matrices can be reduced into LMI feasibility problems by means of the proposed generalized  $\mathcal{S}$ -procedure. We note that such reduction into LMIs is also investigated in the preceding studies, and similar results to the next theorem can also be found in the literature; see, for example, [6].

THEOREM 3.2. Let complex matrices  $M_k \in \mathbf{C}^{n \times m}$  ( $k = 0, \dots, N$ ) with  $n \geq m$  and  $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12}^* & \psi_{22} \end{bmatrix} \in \mathbf{H}_2$  with  $\det(\Psi) < 0$  be given, and define  $M(s) := \sum_{k=0}^N s^k M_k$ ,  $\mathcal{M} := [M_N \cdots M_0]$ . Suppose either of the following assumptions holds:

1.  $M_N$  is of full-column rank.
2.  $\psi_{11} < 0$ .

Then, the following conditions are equivalent:

- (i) The polynomial matrix  $M(s)$  is of full-column rank for all  $s \in \mathcal{D}_\Psi^c$ .
- (ii)

$$\begin{aligned} & f^* \mathcal{M}^* \mathcal{M} f > 0 \quad \forall f \in \mathcal{L}, \\ & \mathcal{L} := \{f \in \mathbf{C}^{(N+1)m} : f \neq 0, \quad f^* S f \geq 0 \quad \forall S \in \mathcal{S}_W\}, \\ & \mathcal{S}_W := \{W^*(\Psi \otimes P)W : P \in \mathbf{P}_{Nm}\}, \\ & W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad W_1 := \begin{bmatrix} I_{Nm} \\ 0_{m, Nm} \end{bmatrix}^*, \quad W_2 := \begin{bmatrix} 0_{m, Nm} \\ I_{Nm} \end{bmatrix}^*. \end{aligned}$$

- (iii) There exists  $P \in \mathbf{P}_{Nm}$  such that

$$(3.3) \quad \mathcal{M}^* \mathcal{M} - W^*(\Psi \otimes P)W \succ 0.$$

If the matrices  $M_k$  ( $k = 0, \dots, N$ ) and  $\Psi$  are all real, then the equivalence still holds when we restrict  $P$  to be real.

*Proof.* The proof for the equivalence of (i) and (ii) is given in the appendix. The main step of the proof, the equivalence of (ii) and (iii), follows immediately from Theorems 2.2 and 2.3. Indeed, the set  $\mathcal{S}_W$  is one-vector-lossless by Theorem 2.3 while it is clear that  $\mathcal{M}^* \mathcal{M} \succeq 0$ . Hence the generalized  $\mathcal{S}$ -procedure in Theorem 2.2 establishes the equivalence of (ii) and (iii). Noting that the real case results can be shown by following similar arguments to [9], we complete the proof.  $\square$

From this theorem, we see that full rank property of polynomial matrices can be assessed by simply solving the LMI (3.3), provided that either assumption 1 or 2 is satisfied. From the definition of  $\mathcal{D}_\Psi^c$  in (3.1), we see that the assumption  $\psi_{11} < 0$  enforces the region  $\mathcal{D}_\Psi^c$  to be bounded. To put it another way, our assumption requires that the matrix  $M_N$  is of full-column rank if the region  $\mathcal{D}_\Psi^c$  is unbounded. When studying the full rank property of polynomial matrices over unbounded regions, it is well-known that we have to take a special care on zeros at infinity [2, 3, 6, 19], and the assumption 1 is surely a sufficient condition for the absence of the zeros at infinity. Hence, under the assumption 1, delicate problems stemming from zeros at infinity have been excluded from our discussions.

It is obvious that the result in Theorem 3.2 forms an important basis for dealing with stability related issues in linear system analysis. In particular, the (generalized) Lyapunov's inequality [1, 5] is surely a special case of (3.3). In addition, existing LMI results for  $\mathcal{D}$ -stabilizability also follow from Theorem 3.2, where a matrix pair  $A \in \mathbf{C}^{n \times n}$ ,  $B \in \mathbf{C}^{n \times m}$  is said to be  $\mathcal{D}$ -stabilizable iff there exists  $K \in \mathbf{C}^{m \times n}$  such that  $sI - (A + BK)$  is nonsingular for all  $s \in \mathcal{D}^c$ . From the Popov–Belevitch–Hautus (PBH) tests [20, 21], this condition can be restated equivalently as  $[sI - A \quad B]^*$  is of full-column rank for all  $s \in \mathcal{D}^c$ . Hence, for the region  $\mathcal{D}_\Psi$ , we can conclude from Theorem 3.2 that the pair  $(A, B)$  is  $\mathcal{D}_\Psi$ -stabilizable iff there exists  $P \in \mathbf{P}_n$  such that

$$\begin{bmatrix} I_n & -A^* \\ 0_{m,n} & B^* \end{bmatrix}^* \begin{bmatrix} I_n & -A^* \\ 0_{m,n} & B^* \end{bmatrix} - \Psi^T \otimes P \succ 0.$$

From Finsler's lemma [1], this LMI can be rewritten as

$$(3.4) \quad B^\perp \begin{bmatrix} A & I \end{bmatrix} (\Psi^T \otimes P) \begin{bmatrix} A^* \\ I \end{bmatrix} (B^\perp)^* \prec 0.$$

The condition (3.4) is known as the *elimination-of-variables* type LMI condition for state-feedback stabilizing controller synthesis with respect to the stability region  $\mathcal{D}_\Psi$  [18]. In this way, we can derive existing stability-related LMI conditions straightforwardly by means of the generalized  $\mathcal{S}$ -procedure for inequality conditions on one-vector-lossless set.

**4. Conclusion.** In this paper, we first introduced a new notion, one-vector-lossless sets, and provided a nonconservative generalized  $\mathcal{S}$ -procedure for inequality conditions on the one-vector-lossless sets. We next showed that full rank property analysis problems of polynomial matrices over some regions on the complex plane can be reduced into LMI feasibility problems by means of the proposed generalized  $\mathcal{S}$ -procedure. It turned out that many existing results such as Lyapunov's inequalities for stability analysis of linear systems and LMIs for state-feedback controller synthesis can be viewed as particular cases of this result. To summarize, in conjunction with [8, 16], this paper clarified that most LMI results in linear control system theory can be grasped within the unified framework of the generalized  $\mathcal{S}$ -procedure.

## 5. Appendix.

**5.1. Proof of Theorem 2.3.** We need the following lemma for the proof.

LEMMA 5.1. For given  $F, G \in \mathbf{C}^{n \times m}$ , and  $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12}^* & \psi_{22} \end{bmatrix} \in \mathbf{H}_2$  with  $\det(\Psi) < 0$ , suppose

$$(5.1) \quad \psi_{11}FF^* + \psi_{12}^*FG^* + \psi_{12}GF^* + \psi_{22}GG^* \succeq 0.$$

Then, there exists a unitary matrix  $U \in \mathbf{C}^{m \times m}$  such that

$$(5.2) \quad \psi_{11}\tilde{f}_1\tilde{f}_1^* + \psi_{12}^*\tilde{f}_1\tilde{g}_1^* + \psi_{12}\tilde{g}_1\tilde{f}_1^* + \psi_{22}\tilde{g}_1\tilde{g}_1^* \succeq 0,$$

where

$$(5.3) \quad [\tilde{f}_1 \ \cdots \ \tilde{f}_m] := FU, \quad [\tilde{g}_1 \ \cdots \ \tilde{g}_m] := GU, \quad \tilde{f}_i, \tilde{g}_i \in \mathbf{C}^n \ (i = 1, \dots, m).$$

*Proof.* We give the proof only for the case  $\psi_{11} > 0$ . Other cases can be proved similarly. If  $\psi_{11} > 0$ , then the condition (5.1) can be rewritten equivalently as

$$\left(F + \frac{\psi_{12}}{\psi_{11}}G\right) \left(F + \frac{\psi_{12}}{\psi_{11}}G\right)^* \succeq -\psi_{11}^{-2}\det(\Psi)GG^*.$$

From [12], this condition holds iff there exists a matrix  $W \in \mathbf{C}^{m \times m}$  such that

$$(5.4) \quad \sqrt{-\psi_{11}^{-2}\det(\Psi)}G = \left(F + \frac{\psi_{12}}{\psi_{11}}G\right)W, \quad \|W\| \leq 1.$$

Since  $\|W\| \leq 1$ , for each eigenvalue  $\lambda$  of  $W$  and its associated eigenvector  $\xi$ , we have  $W\xi = \lambda\xi$ ,  $|\lambda| \leq 1$ ,  $\xi^*\xi = 1$ . Taking one such  $\xi$ , we can construct a unitary matrix  $U$  of the form  $U = \begin{bmatrix} \xi & \bar{U} \end{bmatrix}$ ,  $\bar{U} \in \mathbf{C}^{m \times (m-1)}$ . Then, we see from (5.4) that the vectors  $\tilde{f}_1$  and  $\tilde{g}_1$  defined by (5.3) with this unitary matrix  $U$  satisfy

$$\sqrt{-\psi_{11}^{-2}\det(\Psi)}\tilde{g}_1 = \left(\tilde{f}_1 + \frac{\psi_{12}}{\psi_{11}}\tilde{g}_1\right)\lambda \quad (|\lambda| \leq 1).$$

This implies (5.2).  $\square$

Now we are ready to prove Theorem 2.3.

*Proof.* It is obvious that the set  $\mathcal{S}$  given in (2.4) has the properties (a) and (b) in Definition 2.1. To prove the property (c), let  $H \in \mathbf{C}^{l \times l}$  be a nonzero matrix that satisfies (2.1). In addition, we denote the full rank factorization of  $H$  by

$$(5.5) \quad H = LL^*, \quad L \in \mathbf{C}^{l \times r}, \quad r := \text{rank}(H).$$

With this  $L$ , define

$$(5.6) \quad \begin{bmatrix} F \\ G \end{bmatrix} := \Gamma L, \quad F, G \in \mathbf{C}^{n \times r}.$$

Then, the second condition in (2.1) can be rewritten as

$$\text{trace}((\psi_{11}FF^* + \psi_{12}^*FG^* + \psi_{12}GF^* + \psi_{22}GG^*)P) \geq 0 \quad \forall P \in \mathbf{P}_n.$$

It can be seen that this condition holds iff (5.1) holds. Hence, from Lemma 5.1, there exists a unitary matrix  $U \in \mathbf{C}^{r \times r}$  that satisfies (5.2) with  $\tilde{f}_i, \tilde{g}_i \in \mathbf{C}^n$  ( $i = 1, \dots, m$ ) given by (5.3). Here, note that (5.2) is equivalent to

$$(5.7) \quad \text{trace}((\psi_{11}\tilde{f}_1\tilde{f}_1^* + \psi_{12}^*\tilde{f}_1\tilde{g}_1^* + \psi_{12}\tilde{g}_1\tilde{f}_1^* + \psi_{22}\tilde{g}_1\tilde{g}_1^*)P) \geq 0 \quad \forall P \in \mathbf{P}_n.$$

By defining  $[\zeta_1 \ \cdots \ \zeta_r] := LU$ , we have  $H = \sum_{i=1}^r \zeta_i \zeta_i^*$ . On the other hand, from (5.6) and (5.3), it is apparent that  $[\tilde{f}_1^* \ \tilde{g}_1^*]^* = \Gamma \zeta_1$ . Hence, we see from (5.7) that the

condition  $\zeta_1^* \Gamma^*(\Psi \otimes P) \Gamma \zeta_1 \geq 0$  ( $\forall P \in \mathbf{P}_n$ ) holds or, equivalently,  $\zeta_1^* S \zeta_1 \geq 0$  ( $\forall S \in \mathcal{S}$ ). This clearly shows that the set  $\mathcal{S}$  satisfies the property (c) of Definition 2.1.  $\square$

**5.2. Proof of the equivalence of (i) and (ii) in Theorem 3.2.** The condition (i) can be restated equivalently as

$$\begin{aligned} & f^* \mathcal{M}^* \mathcal{M} f > 0 \quad \forall f \in \mathcal{K}, \\ \mathcal{K} := \{ & f = [f_N^* \cdots f_0^*]^* \in \mathbf{C}^{(N+1)m} : f_0 \neq 0, \\ & \exists s \in \mathcal{D}_{\Psi}^c \text{ such that } f_{k+1} = s f_k \ (k = 0, \dots, N-1) \}. \end{aligned}$$

Hence, to prove the equivalence of (i) and (ii), it suffices to show that  $\mathcal{K} = \mathcal{L}$ . To this end, we first note that  $\mathcal{K} = \mathcal{L}'$ , where

$$\begin{aligned} \mathcal{L}' := \{ & f = [f_N^* \cdots f_0^*]^* \in \mathbf{C}^{(N+1)m} : \\ & f_k \in \mathbf{C}^m \ (k = 0, \dots, N), \\ & f_0 \neq 0, \quad f^* S f \geq 0 \quad \forall S \in \mathcal{S}_W \}. \end{aligned}$$

To see this, suppose  $f = [f_N^* \cdots f_0^*]^* \in \mathcal{K}$ , and define  $f_u, f_l \in \mathbf{C}^{Nm}$  by

$$(5.8) \quad f_u := [f_N^* \cdots f_1^*]^* = W_1 f, \quad f_l := [f_{N-1}^* \cdots f_0^*]^* = W_2 f.$$

Then, from the definition of  $\mathcal{K}$ , the following inequality holds for all  $P \in \mathbf{P}_{Nm}$ :

$$\begin{aligned} f^* W^* (\Psi \otimes P) W f &= \begin{bmatrix} f_u \\ f_l \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} f_u \\ f_l \end{bmatrix} \\ &= \begin{bmatrix} s f_l \\ f_l \end{bmatrix}^* (\Psi \otimes P) \begin{bmatrix} s f_l \\ f_l \end{bmatrix} = \sigma(s, \Psi) f_l^* P f_l \geq 0. \end{aligned}$$

This shows that  $f \in \mathcal{L}'$ , and hence  $\mathcal{K} \subset \mathcal{L}'$ . On the other hand, suppose  $f = [f_N^* \cdots f_0^*]^* \in \mathcal{L}'$ , and define  $f_u$  and  $f_l$  by (5.8). Then, from the definition of  $\mathcal{L}'$ , we have

$$\text{trace}((\psi_{11} f_u f_u^* + \psi_{12}^* f_u f_l^* + \psi_{12} f_l f_u^* + \psi_{22} f_l f_l^*) P) \geq 0 \quad \forall P \in \mathbf{P}_{Nm}.$$

It can be easily seen that the above condition implies

$$(5.9) \quad \psi_{11} f_u f_u^* + \psi_{12}^* f_u f_l^* + \psi_{12} f_l f_u^* + \psi_{22} f_l f_l^* \succeq 0.$$

Moreover, under the assumption  $f_0 \neq 0$ , we have  $f_l \neq 0$ , and hence (5.9) holds iff  $f_u = s f_l$  for some  $s \in \mathcal{D}_{\Psi}^c$  [12]. This clearly shows that  $f \in \mathcal{K}$  and hence  $\mathcal{L}' \subset \mathcal{K}$ . Thus we can conclude  $\mathcal{K} = \mathcal{L}'$ .

To complete the proof, note that  $\mathcal{L} = \mathcal{L}' \cup \mathcal{J}'$ , where

$$\begin{aligned} \mathcal{J}' := \{ & f = [f_N^* \cdots f_0^*]^* \in \mathbf{C}^{(N+1)m} : \\ & f_k \in \mathbf{C}^m \ (k = 0, \dots, N), \quad f \neq 0, \quad f_0 = 0, \quad f^* S f \geq 0 \quad \forall S \in \mathcal{S}_W \}. \end{aligned}$$

Moreover, we can show that the set  $\mathcal{J}'$  is equivalent to

$$\begin{aligned} \mathcal{J} := \{ & f = [f_N^* \cdots f_0^*]^* \in \mathbf{C}^{(N+1)m} : \\ & f_k \in \mathbf{C}^m \ (k = 0, \dots, N), \\ & f_N \neq 0, \quad f_k = 0 \ (k = 0, \dots, N-1), \quad f^* S f \geq 0 \quad \forall S \in \mathcal{S}_W \}. \end{aligned}$$

To see the equivalence of  $\mathcal{J}'$  and  $\mathcal{J}$ , let us take a vector  $f \in \mathcal{J}'$ . Furthermore, define  $f_u$  and  $f_l$  by (5.8) and suppose  $f_l \neq 0$ . Then, the vectors  $f_u$  and  $f_l$  should satisfy



(5.9), and thus  $f_u = sf_l$  holds for some  $s \in \mathcal{D}_\Psi^c$ . Since  $f_0 = 0$ , however,  $f_u = sf_l$  implies  $f_l = 0$ . This clearly contradicts the assumption  $f_l \neq 0$ . Hence, we have  $f_l = 0$  and thus  $f \in \mathcal{J}$ . This shows that  $\mathcal{J}' \subset \mathcal{J}$ . On the other hand, it is apparent that  $\mathcal{J} \subset \mathcal{J}'$ , and hence we have  $\mathcal{J}' = \mathcal{J}$ .

Summarizing the above arguments,  $\mathcal{L} = \mathcal{K} \cup \mathcal{J}$  holds. Hence, the equivalence of (i) and (ii) can be verified by showing that the condition  $f^* \mathcal{M}^* \mathcal{M} f > 0$  holds for all  $f \in \mathcal{J}$  under either assumption 1 or 2. If  $\psi_{11} < 0$ , however, it can be easily seen that the set  $\mathcal{J}$  is empty. On the other hand, if  $\psi_{11} \geq 0$ , then we have  $M_N^* M_N \succ 0$  from the assumption. This indicates that  $f^* \mathcal{M}^* \mathcal{M} f = f_N^* M_N^* M_N f_N > 0$  ( $\forall f \in \mathcal{J}$ ). Hence, we can conclude that the conditions (i) and (ii) are equivalent under either assumption 1 or 2. This completes the proof.  $\square$

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