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Discrete Ingham inequalities and applications

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Abstract

In this Note we prove a discrete version of the classical Ingham inequality for nonharmonic Fourier series whose exponents satisfy a gap condition. Time integrals are replaced by discrete sums on a discrete mesh. We prove that, as the mesh becomes finer and finer the limit of the discrete Ingham inequality is the classical continuous one. This analysis is partially motivated by control-theoretical applications. As an application we analyze the control/observation properties of numerical approximation schemes of the 1-d wave equation. The discrete Ingham inequality provides observability and controllability results which are uniform with respect to the mesh size in suitable classes of numerical solutions in which the high frequency components have been filtered. ***To cite this article: M. Negreanu, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Inégalités d’Ingham discrètes et applications. Dans cette Note nous prouvons une version discrète de l’inégalité classique d’Ingham pour les séries de Fourier non-harmoniques dont les exposants satisfont une condition de séparation ou « gap ». Les intégrales en temps sont remplacées par des sommes discrètes sur une maille. Nous prouvons que, lorsque la maille devient de plus en plus fine, la limite de l’inégalité discrète d’Ingham est l’inégalité classique continue. Cette analyse est partiellement motivée par des applications au contrôle et à l’observation des ondes. À l’aide de ce résultat, nous analysons les propriétés des schémas d’approximation numérique pour l’equation des ondes 1-d. L’inégalité discrète d’Ingham fournit des résultats d’observabilité et de contrôlabilité qui sont uniformes en ce qui concerne la maille dans les classes appropriées de solutions numériques dans lesquelles les composantes à haute fréquence ont été filtrées. ***Pour citer cet article : M. Negreanu, E. Zuazua, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Dans cette Note on démontre une version discrète de l’inégalité d’Ingham [5]. Ces dernières années, l’inégalité de Ingham et ses variantes ont été utilisées pour démontrer des nombreux résultats d’observabilité et de contrôlabilité pour des équations d’ondes, de poutres, etc. (voir, par exemple, [1–4,6]). Pour ailleurs, dans le contexte de

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l'analyse numérique de ces propriétés d'observabilité et de contrôlabilité, il a été mis en évidence que le comportement pathologique des hautes fréquences numériques peut rendre ces propriétés instables lorsque le pas de la discrétisation tend vers zéro. Ceci met en évidence la nécessité de développer des versions discrètes de l'inégalité de Ingham où les intégrales en temps sont remplacées par des sommes discrètes sur les points des maillage numérique.
Nous démontrons une version discrète de l'inégalité d'Ingham qui s'avère optimale au sens que :

- Lorsque le pas de la discrétisation tend vers zéro, nous retrouvons l'inégalité continue de Ingham.
- Nous fournissons des résultats d'observabilité et de contrôlabilité uniformes pour des familles de solutions numériques filtrées qui sont optimaux et, en particulier, le temps de contrôle obtenu est celui que l'analyse du diagramme de dispersion [11].

La démonstration de l'inégalité discrète que nous développons s'inspire de celle de [10], p. 162–163, pour l'inégalité continue. On utilise aussi de manière fondamentale un résultat de [9], p. 96, qui permet d'estimer la distance entre la transformée discrète et la transformée continue de Fourier pour le poids intervenant dans la preuve. Avec ces outils la preuve est similaire à celle de l'inégalité de Ingham classique. Il faut cependant observer que la condition de séparation ou « gap spectral » ne suffit pas et qu'il faut aussi une borne convenable (dépendant du pas du maillage) sur la distance maximale de fréquences impliquées.

1. Introduction

Families of 'nonharmonic' exponentials $\{e^{i\lambda_k t}\}$ appear in various fields of mathematics and signal processing. One of the central problems arising in all of these applications is the question of the Riesz basis property.

The following inequality for nonharmonic Fourier series due to Ingham is well known (see [5] and [10], p. 162): *Assume that the strictly increasing sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ of real numbers satisfies the 'gap' condition $\lambda_{k+1}-\lambda_k \geqslant \gamma$, for all $k\in\mathbb{Z}$, for some $\gamma>0$. Then, for all $T>2\pi/\gamma$ there exist two positive constants C_1, C_2 depending only on γ and T such that*

$$C_1(T,\gamma)\sum_{k=-\infty}^{\infty}|a_k|^2 \leqslant \int_0^T\left|\sum_{k=-\infty}^{\infty}a_k\,e^{it\lambda_k}\right|^2 dt \leqslant C_2(T,\gamma)\sum_{k=-\infty}^{\infty}|a_k|^2, \tag{1}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}}\in\ell^2$, where

$$C_1(T,\gamma)=\frac{2T}{\pi}\left(1-\frac{4\pi^2}{T^2\gamma^2}\right)>0 \quad \textit{and} \quad C_2(T,\gamma)=\frac{8T}{\pi}\left(1+\frac{4\pi^2}{T^2\gamma^2}\right)>0 \tag{2}$$

and ℓ^2 is the Hilbert space of square summable sequences, $\ell^2=\{\{a_k\}:\ \|a_k\|_{\ell^2}^2=\sum_{k\in\mathbb{N}}|a_k|^2<\infty\}$.

This result shows that the sequence of exponentials $\{e^{i\lambda_k t}\}$ forms a Riesz basis of its span for $T>2\pi/\gamma$ (see [10], Chapter 3, p. 112). In the context of partial differential equations, this Ingham theorem has been used to prove observability inequalities for the solutions of 1-d evolution equations for which the sequence of eigenfrequencies has an uniform gap [7]. Attempts to extend this technique to the case when there is not uniform gap have led to far reaching generalizations of the Ingham theorem for some sequences satisfying weakened gap conditions (see [2–4,6]). On the other hand, in the numerical analysis of those observability inequalities the need of a discrete version of this inequality arises naturally.

In this Note we prove a discrete version of (1). More precisely, given $\Delta t=T/(M+1)$, with $M\in\mathbb{N}$ we replace in (1) the integral by a discrete sum, $\Delta t\sum_{n=0}^{M}$, and we analyze the existence of two positive constants C_1, C_2 such that the resulting discrete inequality holds. Obviously, we are interested on results that remain uniform as the mesh-size Δt tends to zero.

2. The discrete Ingham inequality

The main result of this paper is:

Theorem 2.1 (Discrete Ingham inequality). *Let $\Delta t > 0$ and $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers satisfying for some $\gamma > 0$ and $0 \leqslant p < 1/2$ the conditions*

$$\lambda_{k+1} - \lambda_k \geqslant \gamma > 0, \quad \text{for all } k \in \mathbb{Z}, \tag{3}$$

$$|\lambda_k - \lambda_l| \leqslant \frac{2\pi - (\Delta t)^p}{\Delta t}, \quad \text{for all } k, l \in \mathbb{Z}. \tag{4}$$

Then, there exists a positive number $\varepsilon(\Delta t)$ such that, for every $T > T_0(\Delta t) := 2\pi/\gamma + \varepsilon(\Delta t)$ there exist two positive constants $C_j(\Delta t, T, \gamma) > 0$, $j = 1, 2$, such that

$$C_1(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2 \leqslant \Delta t \sum_{n=0}^{M} \left| \sum_{k=-N}^{N} a_k \, e^{in\Delta t\lambda_k} \right|^2 \leqslant C_2(\Delta t, T, \gamma) \sum_{k=-N}^{N} |a_k|^2, \tag{5}$$

for every complex sequence $(a_k)_{k\in\mathbb{Z}} \in \ell^2$, where $M = [T/\Delta t - 1]$ and $2N < M$. Moreover, if γ and p in (3) *and* (4) *are kept fixed, then $\varepsilon(\Delta t) = o(\Delta t)^{1-2p}$ and the constants satisfy*

$$C_j(\Delta t, T, \gamma) = C_j(T, \gamma) + \delta_j(\Delta t), \quad \text{with } \delta_j(\Delta t) \geqslant 0, \ j = 1, 2, \tag{6}$$

where $C_j(T, \gamma)$, $j = 1, 2$, are the Ingham constants (2) *and* $\lim_{\Delta t\to 0} \delta_j(\Delta t) = 0$, $j = 1, 2$.

The proof of the discrete inequality (5) follows the scheme used in [10] (p. 162–163) to prove the classical Ingham inequality (1). It is easy to see that for every $N \in \mathbb{N}$ fixed, if we pass to limit with $\Delta t \to 0$ in (5) we get the classical Ingham inequality (1).

Sketch of the proof. Let us consider the function $g : \mathbb{R} \to \mathbb{R}$, given by $g(t) = \sin(t\pi/T)\chi_{(0,T)}$ where $\chi_{(0,T)}$ is the characteristic function of the interval $(0, T)$. Its Fourier transform $G : \mathbb{R} \to \mathbb{R}$ is

$$G(\alpha) = \int_{-\infty}^{\infty} g(t) \, e^{it\alpha} \, dt = -2\cos\frac{T\alpha}{2} e^{iT\alpha/2} \frac{\pi T}{T^2\alpha^2 - \pi^2}. \tag{7}$$

We define the restriction of g to the grid $\cdots < t_{-1} < t_0 = 0 < t_1 < \cdots < t_{M+1} = T < \cdots$, with $t_n = n\Delta t$, i.e., $h(n\Delta t) = g(t_n) = \sin(n\Delta t\pi/T)\chi_{M+1}(n)$, χ_{M+1} being the characteristic function of the set $\{0, \ldots, M + 1\}$. For any $\alpha \in \mathbb{R}$ we define the function

$$H(\alpha) := \Delta t \sum_{n=-\infty}^{\infty} h(n\Delta t) e^{in\Delta t\alpha}. \tag{8}$$

For all $\alpha \in [-\pi/\Delta t, \pi/\Delta t]$, H is exactly the discrete Fourier transform of the discrete function h.

Proof of the first (so-called inverse) inequality in (5). We have

$$\begin{aligned}
\Delta t \sum_{n=0}^{M} \left| \sum_k a_k \, e^{in\Delta t\lambda_k} \right|^2 &\geqslant \Delta t \sum_{n=0}^{M} h(n\Delta t) \left| \sum_k a_k \, e^{in\Delta t\lambda_k} \right|^2 = \sum_k \sum_l a_k \bar{a}_l H(\lambda_k - \lambda_l) \\
&= H(0) \sum_k |a_k|^2 + \sum_k \sum_{l,\, l\neq k} a_k \bar{a}_l H(\lambda_k - \lambda_l) \geqslant H(0) \sum_k |a_k|^2 - \sum_k |a_k|^2 \sum_{l,\, l\neq k} \left| H(\lambda_k - \lambda_l) \right|
\end{aligned}$$

$$= H(0)\sum_k |a_k|^2 - \sum_k |a_k|^2\Bigg(\sum_{\substack{l,\ k\neq l\\ |\lambda_k-\lambda_l|\leqslant\pi/\Delta t}} \big|H(\lambda_k-\lambda_l)\big| + \sum_{\substack{l,\ k\neq l\\ |\lambda_k-\lambda_l|>\pi/\Delta t}} \big|H(\lambda_k-\lambda_l)\big|\Bigg). \tag{9}$$

To estimate the last term in (9) we use a fundamental result which describes the effect of the discretization in the Fourier transform (see [9], p. 96): *There exists $C>0$ such that*

$$\big|H(\xi)\big| \leqslant \big|G(\xi)\big| + C(\Delta t)^2, \quad \text{uniformly in all } \xi\in[-\pi/\Delta t,\pi/\Delta t] \text{ and for all } \Delta t>0. \tag{10}$$

Inequality (10) holds because $g\in H^1(\mathbb{R})$ and $g'\in BV(\mathbb{R})$. Moreover, the function H is periodic with period $2\pi/\Delta t$. Consequently, for every $k,l\in\mathbb{Z}$ with $\pi/\Delta t<|\lambda_k-\lambda_l|<2\pi/\Delta t$, there exist $m_{k,l}\in[-\pi/\Delta t,\pi/\Delta t]$ such that $|m_{k,l}| = 2\pi/\Delta t-|\lambda_k-\lambda_l|$ with the property $H(\lambda_k-\lambda_l)=H(m_{k,l})$. Therefore, using this periodicity property and applying (10) in (9) we obtain

$$\begin{aligned}
\Delta t\sum_{n=0}^{M}\Big|\sum_k a_k \mathrm{e}^{\mathrm{i}n\Delta t\lambda_k}\Big|^2 &\geqslant H(0)\sum_k |a_k|^2 - \sum_k |a_k|^2\Bigg(\sum_{\substack{l,\ k\neq l\\ |\lambda_k-\lambda_l|\leqslant\pi/\Delta t}} \big|H(\lambda_k-\lambda_l)\big| + \sum_{\substack{l,\ k\neq l\\ |m_{k,l}|\leqslant\pi/\Delta t}} \big|H(m_{k,l})\big|\Bigg)\\
&\geqslant H(0)\sum_k |a_k|^2 - \sum_k |a_k|^2\Bigg(\sum_{\substack{l,\ k\neq l\\ |\lambda_k-\lambda_l|\leqslant\pi/\Delta t}} \big|G(\lambda_k-\lambda_l)\big| + \sum_{\substack{l,\ k\neq l\\ |m_{k,l}|\leqslant\pi/\Delta t}} \big|G(m_{k,l})\big| + CN(\Delta t)^2\Bigg).
\end{aligned} \tag{11}$$

On the other hand, as pointed out in [10], p. 162, for every sequence $\{\lambda_k\}$ satisfying the gap condition (3), the function G satisfies

$$\sum_{l\neq k,\ l=-N}^{N}\big|G(\lambda_k-\lambda_l)\big| \leqslant 2\pi T\sum_{l=-\infty,\ l\neq k}^{\infty}\frac{1}{T^2(\lambda_k-\lambda_l)^2-\pi^2} \leqslant \frac{8\pi}{T\gamma^2}. \tag{12}$$

Further, for every sequence $\{\lambda_k\}$ satisfying $\pi/\Delta t<|\lambda_k-\lambda_l|<(2\pi-(\Delta t)^p)/\Delta t$ (and then, $(\Delta t)^{p-1}\leqslant|m_{k,l}|\leqslant\pi/\Delta t$), for all $k\neq l$, we have

$$\sum_{l\neq k,\ l=-N}^{N}\big|G(m_{k,l})\big| \leqslant \sum_{l\neq k,\ l=-N}^{N} 2\pi T\frac{1}{T^2(m_{k,l})^2-\pi^2} \leqslant CN\Delta t(\Delta t)^{1-2p}, \quad C>0. \tag{13}$$

Therefore, replacing (12) and (13) in (11) we obtain

$$\Delta t\sum_{n=0}^{M}\Big|\sum_{k=-N}^{N} a_k\,\mathrm{e}^{\mathrm{i}n\Delta t\lambda_k}\Big|^2 \geqslant C_1(\Delta t,T,\gamma)\sum_{k=-N}^{N}|a_k|^2,$$

with $C_1(\Delta t,T,\gamma) := H(0)-8\pi/(T\gamma^2)-(NC\Delta t^2+CN\Delta t(\Delta t)^{1-2p})$. If $0\leqslant p<1/2$, we have that $C_1(\Delta t,T,\gamma)$ tends to $C_1(T,\gamma)$ in (2), as $\Delta t\to 0$.

Proof of the second (so-called direct) inequality in (5). Following the same scheme we have

$$\begin{aligned}
\Delta t\sum_{n=0}^{M}\Big|\sum_k a_k\,\mathrm{e}^{\mathrm{i}n\Delta t\lambda_k}\Big|^2 &\leqslant 4H(0)\sum_k|a_k|^2 + 4\sum_k\sum_{l,\ l\neq k}|a_k|^2\big|H(\lambda_k-\lambda_l)\big| + \Delta t\sum_k|a_k|^2\\
&\leqslant C_2(\Delta t,T,\gamma)\sum_k|a_k|^2.
\end{aligned} \tag{14}$$

Therefore we obtain that the direct inequality in (5) holds true with $C_2(\Delta t,T,\gamma) := 4H(0)+32\pi/(T\gamma^2)+\Delta t+4CN\Delta t(\Delta t)^{1-2p}$. □

Remark 1. In both the continuous and discrete cases, the sequence $\{\lambda_k\}_k$ is required to satisfy (3), the so-called gap condition. The restriction (4) imposed on $\{\lambda_k\}_k$ in Theorem 2.1 is not needed in the classical continuous Ingham inequality. However, in the discrete case, if $\lambda_k - \lambda_l \in 2\pi\mathbb{Z}/\Delta t$, for certain values of k and l with $k \neq l$ the sequence $a_k = -a_l = 1,\ a_n = 0,\ k \neq n$ satisfies $\sum_{k=-N}^{N} a_k\, \mathrm{e}^{\mathrm{i}n\Delta t\lambda_k} = 0,\ 0 < n < M$. Then, an inequality of type (5) is impossible. So, it is natural to impose the condition $\lambda_k - \lambda_l \notin 2\pi\mathbb{Z}/\Delta t$. In our theorem this latter condition is implied by the stronger one (4).

Condition $T > 2\pi/\gamma$ is optimal for the classical Ingham inequality (see [10], p. 163). In this sense, the condition $T > 2\pi/\gamma + \varepsilon(\Delta t)$ in Theorem 2.1 is asymptotically optimal since $\varepsilon(\Delta t) \to 0$ as $\Delta t \to 0$.

3. Application to the uniform observability of full discretizations of the wave equation

In this section we study an application of the discrete Ingham inequality (5) to a finite-difference full discretization of a homogeneous 1-d wave equation.

Given $M, N \in \mathbb{N}$ we set $\Delta x = 1/(N+1)$ and $\Delta t = T/(M+1)$ and introduce the nets $0 = x_0 < \cdots < x_{N+1} = 1$, $0 = t_0 < \cdots < t_{M+1} = T$ with $x_j = j\Delta x$ and $t_n = n\Delta t,\ j = 0, 1, \ldots, N+1,\ n = 0, 1, \ldots, M+1$.

We consider the following discrete finite difference 1-d homogeneous wave equation

$$\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1} = \mu^2\big(\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n\big),\ j = 1, 2, \ldots, N; \quad n = 1, 2, \ldots, M, \tag{15}$$

with $\phi_0^n = \phi_{N+1}^n = 0,\ n = 1, 2, \ldots, M$ and $\mu = \Delta t/\Delta x \leqslant 1$.

Solutions of (15) admit the Fourier development $\bar{\phi}^n = \sum_{k=-N}^{N} a_k\, \mathrm{e}^{\mathrm{i}\lambda_k n\Delta t} \bar{\varphi}_{|k|}$, with $a_k \in \mathbb{C},\ \bar{\varphi}_k = (\varphi_{k,1}, \ldots, \varphi_{k,N})$ $= (\sin(k\pi\Delta x), \ldots, \sin(Nk\pi\Delta x))$ and $\lambda_k = 2\,\mathrm{sgn}(k)/\Delta t \arcsin(\mu \sin(|k|\pi\Delta x/2))$ being the eigenvalues of the system (15) (see [8]). Our goal is to analyze the discrete observability inequality:

$$E_0 \leqslant C\left[\Delta t \sum_{n=0}^{M} \left|\frac{\phi_N^n}{\Delta x}\right|^2\right], \tag{16}$$

where E_0 is the conserved energy of the solutions of the discrete system (15)

$$E_0 = \frac{\Delta x}{2} \sum_{j=0}^{N} \left[\left(\frac{\phi_j^1 - \phi_j^0}{\Delta t}\right)^2 + \frac{(\phi_{j+1}^1 - \phi_j^1)(\phi_{j+1}^0 - \phi_j^0)}{(\Delta x)^2}\right] \geqslant 0. \tag{17}$$

In the case when $\Delta t < \Delta x$, the gap decreases at high frequencies and it is of the order of Δx when $\Delta x \to 0$. So the uniform gap condition (3) is not satisfied and we cannot apply directly Theorem 2.1 to prove inequality (16). We need to introduce a subclass of solutions of system (15) where the high frequency components have been filtered. To do that, given $\alpha \in (0, 1)$ (filtering parameter), we consider the class of solutions $\bar{\phi}^n = \sum_{k=-\alpha N}^{\alpha N} a_k\, \mathrm{e}^{\mathrm{i}\lambda_k n\Delta t} \bar{\varphi}_{|k|}$. It is easy to check that $\lambda_{k+1} - \lambda_k \geqslant \gamma_\alpha := \pi(1-\alpha)$ and $|\lambda_k - \lambda_l| < (2\pi\alpha(1-\Delta t))/(\Delta t)$ for $|k| \leqslant \alpha N$. By choosing the filtering parameter α such that $\alpha < \alpha^*(\Delta t) := (2\pi - (\Delta t)^p)/(2\pi(1-\Delta t))$, hypothesis (4) of Theorem 2.1 is verified. Therefore, applying Theorem 2.1 and the Fourier representation of the solutions we obtain that, for all $T > 2\pi/\gamma_\alpha + \varepsilon(\Delta t)$,

$$2\cos^2 \frac{N\alpha\pi\Delta x}{2} C_1(\Delta t, T, \gamma_\alpha) E_0 \leqslant \Delta t \sum_{n=0}^{M} \left|\frac{\phi_N}{\Delta x}\right|^2 \leqslant 2C_2(\Delta t, T, \gamma_\alpha) E_0, \tag{18}$$

holds for every truncated solution of (15), with $C_j(\Delta t, T, \gamma_\alpha),\ j = 1, 2$ defined by (6).

Note that $\alpha^*(\Delta t) \nearrow 1$ as $\Delta t \to 0$. Thus, the filtering parameter α may be chosen arbitrarily in the interval $\alpha \in (0, 1)$. Observe that the gap γ_α (respectively the minimal time $2\pi/\gamma_\alpha$) tends to π (respectively to 2) when

$\alpha \searrow 0^+$ while it converges to zero (respectively to infinity) when $\gamma_\alpha \nearrow 1^-$. This coincides with the predictions one may deduce from the analysis of the dispersion diagram of the numerical scheme [11].

The uniform observability inequality (18) implies uniform controllability results for the projection of solutions of the dual controlled system. In the limit as $\Delta t \to 0$ and by choosing the filtering parameter α tending to zero one may recover the sharp controllability results of the wave equation. This problem was studied in the particular case $\Delta t = \Delta x$ in [8] in which, due to the orthogonality of the family of exponentials involved in the Fourier representation of solutions, the discrete Ingham inequality above is not needed.

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