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ON THE STABILITY OF CONVEX-VALUED MAPPINGS AND THEIR RELATIVE BOUNDARY AND EXTREME POINTS SET MAPPINGS*

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Abstract. This paper deals with the transmission of the main stability properties (lower and upper semicontinuity in Berge sense, and closedness) from a given closed–convex-valued mapping to its corresponding relative boundary and extreme point set mappings, and vice versa. The domain of the mappings considered in this paper are locally metrizable spaces and the images range on Euclidean spaces. Important examples of the class of mappings considered in this paper are the feasible set mapping and the optimal set mapping of convex optimization problems, for which the space of parameters is the result of perturbing a given nominal problem.

Key words. stability theory, set-valued mappings, convex hull mappings, relative boundary mappings, extreme points set mappings

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1. Introduction.

The main objective of the paper is to analyze the relationships between important pairs of mappings, one of them being the convex hull of the other, which frequently arise in convex optimization (convex systems), where, as a consequence of measurement or roundoff errors, the nominal problem y_0 (system y_0) is usually replaced in practice by perturbed problems (systems, respectively) having the same structure. Let us denote by Y the set of all possible perturbed problems (systems) equipped with a certain pseudometric measuring the size of the perturbations and let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be the set-valued mapping associating with each $y \in Y$ its feasible set or its optimal set (its solution set, respectively). Under mild conditions, $\mathcal{F}(y)$ is the convex hull of its boundary set $\operatorname{bd}\mathcal{F}(y)$, its relative boundary set $\operatorname{rbd}\mathcal{F}(y)$, and/or its extreme points set $\operatorname{extr}\mathcal{F}(y)$ for all $y \in Y$. We denote these mappings from Y to $\mathbb{R}^n$ as $\operatorname{bd}\mathcal{F}$, $\operatorname{rbd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$, which are called *boundary mapping*, *relative boundary mapping*, and *extreme points set mapping* of $\mathcal{F}$, respectively. The connections between the stability properties of $\mathcal{F}$, $\operatorname{bd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$ have been already analyzed in the particular context of linear semi-infinite systems ([3] and [4], respectively), where Y is equipped with the pseudometric of the uniform convergence.

Throughout this paper we consider given an arbitrary convex-valued mapping $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$, where the domain Y is a locally metrizable space (i.e., Y is equipped with the topology induced by an extended distance on Y, δ, taking values on $\mathbb{R}_+ \cup \{+\infty\}$), and its boundary mapping, relative boundary mapping, and extreme points set mapping,

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$\operatorname{bd}\mathcal{F}$, $\operatorname{rbd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$. The relationships between $\mathcal{F}$ and $\operatorname{bd}\mathcal{F}$, assuming that $\mathcal{F} = \operatorname{conv}\operatorname{bd}\mathcal{F}$, have been studied in [5]. In the same vein, this paper considers the relationships between the stability properties of $\mathcal{F}$, $\operatorname{rbd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$, assuming that $\mathcal{F} = \operatorname{conv}\operatorname{rbd}\mathcal{F}$ and $\mathcal{F} = \operatorname{conv}\operatorname{extr}\mathcal{F}$, respectively. The finite dimension of the image space plays a crucial role in those arguments based on the compactness of the unit sphere or on Carathéodory's theorem.

Some of these relationships are direct consequences of basic results about arbitrary mappings $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and their corresponding *convex hull mappings*, $\operatorname{conv}\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$, which associates to each $y \in Y$ the convex hull of $\mathcal{A}(y)$, i.e., $(\operatorname{conv}\mathcal{A})(y) = \operatorname{conv}\mathcal{A}(y)$ for all $y \in Y$. Although some results on the transmission of stability properties between $\mathcal{A}$ and $\operatorname{conv}\mathcal{A}$ are already known (see, e.g., [6] and [1]), we provide proofs of other results which will be used in what follows. Thus, for each stability property, we start analyzing the relationships between $\mathcal{A}$ and $\operatorname{conv}\mathcal{A}$, and then we exploit the properties of the images of $\mathcal{F}$, $\operatorname{rbd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$ in order to obtain the relationships between these mappings; section 3 deals with the lower semicontinuous (lsc) property and section 4 with the upper semicontinuous (usc) property and closedness.

Let us introduce some additional notation. Given $X \subset \mathbb{R}^n$, $\operatorname{aff} X$ denotes the affine hull of X. From the topological side, $\operatorname{bd} X$, $\operatorname{rbd} X$, $\operatorname{int} X$, $\operatorname{rint} X$, and $\operatorname{cl} X$ represent the boundary, the relative boundary, the interior, the relative interior, and the closure of X, respectively. If X is convex, its set of extreme points is denoted by $\operatorname{extr} X$. The Euclidean norm in $\mathbb{R}^n$ will be denoted by $\|.\|$ and the open ball centered at x and radius $\varepsilon > 0$ by $B(x;\varepsilon)$. If X is a convex set and $x \in X$, then

$$B(x;\varepsilon) \cap \operatorname{rbd} X = \emptyset \Longrightarrow B(x;\varepsilon) \cap \operatorname{aff} X \subset \operatorname{rint} X \tag{1.1}$$

for all $\varepsilon > 0$.

The standard simplex in $\mathbb{R}^{n+1}$ is

$$S := \left\{ (\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

For the sake of completeness, we recall the stability concepts and some basic results for set-valued mappings that we shall consider in this paper. Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with its domain $\operatorname{dom}\mathcal{M} := \{y \in Y \mid \mathcal{M}(y) \neq \emptyset\}$. The following semicontinuity concepts are due to Bouligand and Kuratowski (see [1, section 1.4]).

We say that $\mathcal{M}$ is *lower semicontinuous* at $y_0 \in Y$ *in the Berge sense* if, for each open set $W \subset \mathbb{R}^n$ such that $W \cap \mathcal{M}(y_0) \neq \emptyset$, there exists an open set $V \subset Y$, containing y_0, such that $W \cap \mathcal{M}(y) \neq \emptyset$ for each $y \in V$. Obviously, $\mathcal{M}$ is lsc at $y_0 \notin \operatorname{dom}\mathcal{M}$ and $y_0 \in \operatorname{int}\operatorname{dom}\mathcal{M}$ if $\mathcal{M}$ is lsc at $y_0 \in \operatorname{dom}\mathcal{M}$.

$\mathcal{M}$ is *upper semicontinuous* at $y_0 \in Y$ *in the Berge sense* if, for each open set $W \subset \mathbb{R}^n$ such that $\mathcal{M}(y_0) \subset W$, there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y) \subset W$ for each $y \in V$. If $\mathcal{M}$ is usc at $y_0 \notin \operatorname{dom}\mathcal{M}$, then $y_0 \in \operatorname{int}(Y \diagdown \operatorname{dom}\mathcal{M})$.

If $\mathcal{M}$ is simultaneously lsc and usc at y_0 we say that $\mathcal{M}$ is *continuous* at this point.

$\mathcal{M}$ is *closed* at $y_0 \in \operatorname{dom}\mathcal{M}$ if for all sequences $\{y_r\}_{r=1}^{\infty} \subset Y$ and $\{x_r\}_{r=1}^{\infty} \subset \mathbb{R}^n$ satisfying $x_r \in \mathcal{M}(y_r)$ for all $r \in \mathbb{N}$, $\lim_{r\to\infty} y_r = y_0$ and $\lim_{r\to\infty} x_r = x_0$ (in brief, $y_r \to y_0$ and $x_r \to x_0$) one has $x_0 \in \mathcal{M}(y_0)$. If $\mathcal{M}$ is usc at $y_0 \in \operatorname{dom}\mathcal{M}$ and $\mathcal{M}(y_0)$ is closed, then $\mathcal{M}$ is closed at y_0. Conversely, if $\mathcal{M}$ is closed and *locally bounded* at

$y_0 \in \operatorname{dom} \mathcal{M}$ (i.e., if there is a neighborhood of y_0, say V, and a bounded set $A \subset \mathbb{R}^n$ containing $\mathcal{M}(y)$ for every $y \in V$), then $\mathcal{M}$ is usc at y_0.

Finally, $\mathcal{M}$ is lsc (usc, closed, locally bounded) if it is lsc (usc, closed, locally bounded) at y for all $y \in Y$.

Without entering in details we would like to mention that there are other notions of lower and upper semicontinuity as lsc and usc in the sense of Hausdorff (see, e.g., [2]) or inner and outer semicontinuity (see, e.g., [8], where it is shown that the last two concepts are equivalent to lsc in Berge sense and closedness when $\mathcal{M}(y)$ is closed for all $y \in Y$).

2. Preliminaries. We say that $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ is *locally convex* at $y_0 \in Y$ if there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y)$ is convex for all $y \in V$. We shall use the following sufficient condition for $\mathcal{M}$ to be locally bounded.

PROPOSITION 2.1. *Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \operatorname{dom} \mathcal{M}$ such that $\mathcal{M}(y_0)$ is bounded and $\mathcal{M}$ is lsc, closed, and locally convex at y_0. Then $\mathcal{M}$ is locally bounded and continuous at y_0.*

Proof. Let $r_0 \in \mathbb{N}$ such that

$$\mathcal{M}(y_0) \subset B(0_n; r_0). \tag{2.1}$$

Since $\mathcal{M}$ is lsc and locally convex at y_0 there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y)$ is convex and

$$B\,(0_n; r_0) \cap \mathcal{M}(y) \neq \emptyset \text{ for each } y \in V. \tag{2.2}$$

If $\mathcal{M}$ is not locally bounded at y_0, given $r \in \mathbb{N}$ there exists $y_r \in Y$, with $\delta\,(y_r, y) \leq \frac{1}{r}$, such that $\mathcal{M}(y_r) \nsubseteq B\,(0_n; r)$. Thus there exists a sequence $\{x_r\}$ such that

$$x_r \in \mathcal{M}(y_r), \|x_r\| \geq r, r = 1, 2, \ldots .$$

Let $r_1 \geq r_0$ such that $y_r \in V$ for all $r \geq r_1$. In this case, due to (2.2), we can take $z_r \in B\,(0_n; r_0) \cap \mathcal{M}(y_r)$. Since $x_r \in \mathcal{M}(y_r) \diagdown B\,(0_n; r_0)$ and $\mathcal{M}(y_r)$ is convex, there exists $u_r \in \left]x_r, z_r\right] := \{(1-\lambda)\,x_r + \lambda z_r \mid 0 < \lambda \leq 1\}$ such that

$$u_r \in \mathcal{M}(y_r), \|u_r\| = r_0, r \leq r_1. \tag{2.3}$$

By the compactness of the spheres in $\mathbb{R}^n$, there exists a subsequence $\{u_{r_k}\}$ such that $u_{r_k} \in \mathcal{M}(y_{r_k})$, $k = 1, 2, \ldots$, and $\lim_k u_{r_k} = u_0$, with $\|u_0\| = r_0$. Since $\mathcal{M}$ is closed at y_0 and $\lim_k y_{r_k} = y_0$, we must have $u_0 \in \mathcal{M}(y_0)$, which contradicts (2.1).

We have shown that $\mathcal{M}$ is locally bounded at y_0. Since we are assuming that $\mathcal{M}$ is closed at y_0, it is also usc at y_0. Hence it is continuous at y_0. ∎

The condition of $\mathcal{M}$ being locally convex above is not superfluous as the following example shows.

Example 2.2. If $Y = [0, 1]$ and $\mathcal{M} : Y \rightrightarrows \mathbb{R}$ is defined by $\mathcal{M}(y) = \{0, 1/y\}$ for $y \neq 0$ and $\mathcal{M}(0) = \{0\}$, then $\mathcal{M}$ is neither locally bounded nor continuous at $y_0 = 0$, in spite of $\mathcal{M}(y_0)$ being bounded and being $\mathcal{M}$ lsc and closed at y_0.

The *truncated mapping* of $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ with radius $\rho > 0$ is $\mathcal{M}_\rho : Y \rightrightarrows \mathbb{R}^n$ defined such as

$$\mathcal{M}_\rho\,(y) := \mathcal{M}\,(y) \cap \operatorname{cl} B\,(0_n; \rho) \text{ for all } y \in Y.$$

The following result (Lemma 2 in [5]), which establishes the relationships between $\mathcal{M}$ and $\mathcal{M}_\rho$, will be useful in the next sections.

PROPOSITION 2.3. *Let $\mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \operatorname{dom} \mathcal{M}$. Then the following*

statements hold:
(i) $\mathcal{M}$ *is closed at* y_0 *if and only if* $\mathcal{M}_\rho$ *is closed at* y_0 *for all* $\rho > 0$ *such that* $\mathcal{M}_\rho(y_0) \neq \emptyset$.
(ii) *If* $\mathcal{M}$ *is usc at* y_0 *and* $\mathcal{M}(y_0)$ *is closed, then* $\mathcal{M}_\rho$ *is usc at* y_0 *for all* $\rho > 0$ *such that* $\mathcal{M}_\rho(y_0) \neq \emptyset$.
(iii) *If* $\mathcal{M}$ *is usc at* y_0*, then there exist a positive scalar* $\overline{\rho}$ *and an open neighborhood of* y_0, V, *such that*

$$\mathcal{M}(y)\diagdown\mathcal{M}_{\overline{\rho}}(y) \subset \mathcal{M}(y_0)\diagdown\mathcal{M}_{\overline{\rho}}(y_0) \quad \text{for all } y \in V. \tag{2.4}$$

The converse statement holds when $\mathcal{M}$ *is closed at* y_0.
(iv) *If* $\mathcal{M}_\rho$ *is lsc at* y_0 *for every* ρ *such that* $\mathcal{M}(y_0) \cap B(0_n;\rho) \neq \emptyset$, *then* $\mathcal{M}$ *is lsc at* y_0*. The converse statement holds if* $\mathcal{M}(y_0)$ *is convex.*

As an immediate consequence of the following result we obtain characterizations of the identities $\mathcal{F} = \operatorname{conv}\operatorname{bd}\mathcal{F}$, $\mathcal{F} = \operatorname{conv}\operatorname{rbd}\mathcal{F}$, and $\mathcal{F} = \operatorname{conv}\operatorname{extr}\mathcal{F}$. Recall that an edge is a one-dimensional face whereas a half-flat is the intersection of a flat (also called affine manifold) with a closed halfspace which meets it, but does not contain it.

Proposition 2.4. *Given a convex set* $F \subset \mathbb{R}^n$*, the following statements hold:*
(i) $F = \operatorname{conv}\operatorname{bd}F$ *if and only if* F *is a closed set which does not contain halfspaces.*
(ii) $F = \operatorname{conv}\operatorname{rbd}F$ *if and only if* F *is a closed set which does not contain half-flats of the same dimension.*
(iii) *If* $F = \operatorname{conv}\operatorname{extr}F$*, then* F *contains neither lines nor unbounded edges. The converse holds if* F *is closed.*

Proof. Obviously, if $F = \emptyset$, then

$$\operatorname{conv}\operatorname{bd}F = \operatorname{conv}\operatorname{rbd}F = \operatorname{conv}\operatorname{extr}F = \emptyset.$$

So we can assume that $F \neq \emptyset$ without loss of generality.

(i) It is a straightforward consequence of Lemma 2 in [3].

(ii) If $F = \operatorname{conv}\operatorname{rbd}F$, then $\operatorname{rbd}F \subset F$ and so F is closed for each $y \in Y$. If F contains a half-flat of the same dimension, then it is either a flat or a half-flat, with $\operatorname{conv}\operatorname{rbd}F \neq F$ in both cases.

Conversely, since F is a closed and convex set which is neither a flat nor a half-flat, then $F = \operatorname{conv}\operatorname{rbd}F$ by Theorem 2.6.12 in [9].

(iii) Suppose that $F = \operatorname{conv}\operatorname{extr}F$. $F \neq \emptyset$ entails $\operatorname{extr}F \neq \emptyset$ and so F does not contain lines. We shall obtain a contradiction assuming the existence of a halfline edge of F, say A.

Let $A = \{\overline{x} + \lambda v \mid \lambda \geq 0\}$ be an edge of F. Then $v \neq 0_n$ and $\overline{x} \in \operatorname{extr}F$. We shall prove that no element of $A \setminus \{\overline{x}\}$ belongs to $\operatorname{conv}\operatorname{extr}F$. We assume the contrary, i.e., that there exists $\lambda > 0$ such that $\overline{x} + \lambda v \in \operatorname{conv}\operatorname{extr}F$.

If $\overline{x} + \lambda v = x_1 \in \operatorname{extr}F$, then $x_1 = \frac{1}{2}\overline{x} + \frac{1}{2}(\overline{x} + 2\lambda v)$, with $\overline{x}, \overline{x} + 2\lambda v \in F$, making this impossible. Thus we can write $\overline{x} + \lambda v = \sum_{i=1}^{p} \lambda_i x_i$, where $p \geq 2$, $\sum_{i=1}^{p} \lambda_i = 1$ and $\lambda_i > 0$, and $x_i \in \operatorname{extr}F$, $i = 1, \ldots, p$, with $x_i \neq x_j$ if $i \neq j$. Then we can write

$$\overline{x} + \lambda v = \lambda_1 x_1 + (1 - \lambda_1) \sum_{i=2}^{p} \left(\frac{\lambda_i}{1 - \lambda_1}\right) x_i, \tag{2.5}$$

which yields $x_1, \sum_{i=2}^{p} \left(\frac{\lambda_i}{1-\lambda_1}\right) x_i \in A$ because A is a face of F. Since $A \cap \operatorname{extr}F = \{\overline{x}\}$, $x_1 = \overline{x}$, and so from (2.5) we get

$$\overline{x}+\frac{\lambda}{1-\lambda_1}v=\sum_{i=2}^{p}\left(\frac{\lambda_i}{1-\lambda_1}\right)x_i. \tag{2.6}$$

By taking into account again that A is a face of F, we get the following contradiction: $x_2,\ldots,x_p\in A\cap\operatorname{extr}F=\{\overline{x}\}$.

We have shown that $(A\setminus\{\overline{x}\})\cap\operatorname{conv}\operatorname{extr}F=\emptyset$. Since $\emptyset\neq A\setminus\{\overline{x}\}\subset F$, we conclude that $\operatorname{conv}\operatorname{extr}F\subsetneq F$.

Conversely, if F is closed and does not contain lines, it is the convex hull of its extreme points and extreme directions (Corollary 2.6.15 in [9]). Since the assumption precludes the existence of extreme directions, we have $\operatorname{conv}\operatorname{extr}F=F$. ∎

Remark 2.5. According to Proposition 2.4, if $\mathcal{F}=\operatorname{conv}\operatorname{bd}\mathcal{F}$ ($\mathcal{F}=\operatorname{conv}\operatorname{rbd}\mathcal{F}$), then we have $\mathcal{F}_\rho=\operatorname{conv}\operatorname{bd}\mathcal{F}_\rho$ ($\mathcal{F}_\rho=\operatorname{conv}\operatorname{rbd}\mathcal{F}_\rho$, respectively) for all $\rho>0$. Nevertheless, in the case of $\mathcal{F}=\operatorname{conv}\operatorname{extr}\mathcal{F}$, we need to show that $\mathcal{F}_\rho=\operatorname{conv}\operatorname{extr}\mathcal{F}_\rho$ because $\mathcal{F}$ could be not closed-valued. In order to do this, it is enough to prove that if $\mathcal{F}(y):=F=\operatorname{conv}\operatorname{extr}F$ and $x\in F_\rho$ with $\|x\|<\rho$, then $x\in\operatorname{conv}\operatorname{extr}F_\rho$. We can write

$$x=\sum_{j\in J}\lambda_j x_j,\, |J|<\infty,\sum_{j\in J}\lambda_j=1,\lambda_j>0 \text{ and } x_j\in\operatorname{extr}F \text{ for all } j\in J.$$

Let $I=\{j\in J\mid \|x_j\|>\rho\}$. If $I=\emptyset$, then $x_j\in[\operatorname{extr}F]_\rho\subset\operatorname{extr}F_\rho$ for all $j\in J$ and so $x\in\operatorname{conv}\operatorname{extr}F_\rho$. Otherwise take an arbitrary $k\in I$. Let $x_k'\in[x,x_k]\subset F$ such that $\|x_k'\|=\rho$, so that $x_k'\in\operatorname{extr}F_\rho$. If $x_k'=(1-\mu)x+\mu x_k$, with $0<\mu<1$, and we denote $y_j=x_j$ for all $j\in J$, $j\neq k$, and $y_k=x_k'$, we get an expression $x=\sum_{j\in J}\alpha_j y_j$, where $\sum_{j\in J}\alpha_j=1,\alpha_j>0$ and $y_j\in\operatorname{extr}F$ for all $j\in J$, but now the cardinality of the set $\{j\in J\mid\|y_j\|>\rho\}$ is $|I|-1$. After $|I|$ iterations of this procedure we get x expressed as a convex combination of elements of $\operatorname{extr}F_\rho$. In fact, if Φ is any operator that transforms convex sets in $\mathbb{R}^n$ into sets in $\mathbb{R}^n$ satisfying $[\Phi(\mathcal{F})]_\rho\subset\Phi(\mathcal{F}_\rho)\subset\mathcal{F}_\rho$ and $\{x\in\mathcal{F}(y)\mid\|x\|=\rho\}\subset\Phi(\mathcal{F}_\rho(y))$ for all $y\in Y$, then

$$\mathcal{F}=\operatorname{conv}\Phi(\mathcal{F})\Longrightarrow\mathcal{F}_\rho=\operatorname{conv}\Phi(\mathcal{F}_\rho).$$

Observe that $\Phi(\mathcal{F})=\operatorname{bd}\mathcal{F}$, $\operatorname{rbd}\mathcal{F}$, and $\operatorname{extr}\mathcal{F}$ satisfy these conditions.

3. Lower semicontinuity. We shall use the following classical result ([6, Proposition 2.6]).

THEOREM 3.1. *If* $\mathcal{A}:Y\rightrightarrows\mathbb{R}^n$ *is lsc at* $y_0\in\operatorname{dom}\mathcal{A}$, *then* $\operatorname{conv}\mathcal{A}$ *is also lsc at* y_0.

In particular, taking $\mathcal{A}=\operatorname{bd}\mathcal{F}$ we get the direct statement of Proposition 1 in [5], whose corresponding converse statement establishes that, if $\mathcal{F}=\operatorname{conv}\operatorname{bd}\mathcal{F}$ is lsc and closed at $y_0\in\operatorname{dom}\mathcal{F}$, then $\operatorname{bd}\mathcal{F}$ is also lsc at y_0. The next two results are counterparts of this converse statement for $\operatorname{rbd}\mathcal{F}$ and $\operatorname{extr}\mathcal{F}$ (instead of $\operatorname{bd}\mathcal{F}$). Example 3 in [5], where $\operatorname{bd}\mathcal{F}=\operatorname{rbd}\mathcal{F}=\operatorname{extr}\mathcal{F}$, shows that the closedness of $\mathcal{F}$ is not superfluous in these results. The following example shows that, in general, if $\operatorname{conv}\mathcal{A}$ is lsc and closed at y_0, then $\mathcal{A}$ is not necessarily lsc at y_0. Accordingly, the proofs must appeal to the specific properties of the sets $\operatorname{rbd}\mathcal{F}(y)$ and $\operatorname{extr}\mathcal{F}(y)$.

Example 3.2. Let $\mathcal{A}:\mathbb{R}\rightrightarrows\mathbb{R}$ such that

$$\mathcal{A}(y)=\begin{cases}\{-1,0,1\}, & y=0,\\ \{-1,1\}, & y\neq0.\end{cases}$$

It is easy to see that $\operatorname{conv}\mathcal{A}$ is constant (so that it is continuous and closed) whereas $\mathcal{A}$ is not lsc at $y_0 = 0$.

Theorem 3.3. *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be such that $\mathcal{F} = \operatorname{conv}\operatorname{rbd}\mathcal{F}$ and $\mathcal{F}$ is lsc and closed at $y_0 \in \operatorname{dom}\mathcal{F}$. Then $\operatorname{rbd}\mathcal{F}$ is lsc at y_0.*

Proof. Let us denote $\mathcal{R} = \operatorname{rbd}\mathcal{F}$. Since $\mathcal{F}(y_0)$ cannot be singleton (otherwise $\mathcal{R}(y_0) = \emptyset$, contradicting the assumptions), we have $|\mathcal{F}(y_0)| > 1$.

We assume that $\mathcal{R}$ is not lsc at y_0 and we shall obtain a contradiction. This assumption entails the existence of an open convex set W and a sequence $\{y_r\}$ such that $y_r \to y_0$,

$$W \cap \mathcal{R}(y_0) \neq \emptyset, \tag{3.1}$$

and

$$W \cap \mathcal{R}(y_r) = \emptyset, r = 1, 2, \dots . \tag{3.2}$$

Since $y_0 \in \operatorname{int}\operatorname{dom}\mathcal{F}$, we can assume that $y_r \in \operatorname{dom}\mathcal{F}, r = 1, 2, \dots$. By (3.1), we can choose a point $\widehat{x} \in W \cap \mathcal{R}(y_0)$. Fix $\overline{x} \in \operatorname{rint}\mathcal{F}(y_0)$. Then

$$\widehat{x} - \lambda(\overline{x} - \widehat{x}) \notin \mathcal{F}(y_0) \text{ for all } \lambda > 0. \tag{3.3}$$

Because $\mathcal{F}$ is lsc at y_0 and $\overline{x}, \widehat{x} \in \mathcal{F}(y_0)$, there exist two sequences, $\{\overline{x}_r\}$ and $\{\widehat{x}_r\}$, with $\overline{x}_r, \widehat{x}_r \in \mathcal{F}(y_r)$ for all r, $\overline{x}_r \to \overline{x}$, and $\widehat{x}_r \to \widehat{x}$. Let $\delta > 0$ such that $B(\widehat{x}; \delta) \subset W$ and take $r_0 \in \mathbb{N}$ such that $\widehat{x}_r \in B(\widehat{x}; \frac{\delta}{2})$ for all $r \geq r_0$. Given $r \geq r_0$, (3.2) yields $B(\widehat{x}; \frac{\delta}{2}) \cap \mathcal{R}(y_r) = \emptyset$ and so, by (1.1), $B(\widehat{x}; \frac{\delta}{2}) \cap \operatorname{aff}\mathcal{F}(y_r) \subset \mathcal{F}(y_r)$. Hence

$$\widehat{x}_r - \frac{\delta}{4\|\overline{x}_r - \widehat{x}_r\|}(\overline{x}_r - \widehat{x}_r) \in \mathcal{F}(y_r) \quad \text{for all } r \geq r_0.$$

Taking limits as $r \to \infty$ we get, by the closedness of $\mathcal{F}$ at y_0, that

$$\widehat{x} - \frac{\delta}{4\|\overline{x} - \widehat{x}\|}(\overline{x} - \widehat{x}) \in \mathcal{F}(y_0),$$

in contradiction with (3.3). □

Theorem 3.4. *Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be such that $\mathcal{F} = \operatorname{conv}\operatorname{extr}\mathcal{F}$ and $\mathcal{F}$ is lsc and closed at $y_0 \in \operatorname{dom}\mathcal{F}$. Then $\operatorname{extr}\mathcal{F}$ is lsc at y_0.*

Proof. We denote $\mathcal{E} = \operatorname{extr}\mathcal{F}$ and consider two possible cases.

Case 1. $\mathcal{F}(y_0)$ is bounded.

$\mathcal{F}$ is locally bounded at y_0 according to Proposition 2.1. Let V be an open set in Y, $y_0 \in V$, and $\rho > 0$ such that $\mathcal{F}(y) \subset \operatorname{cl} B(0_n; \rho)$ for all $y \in V$.

We assume that $\mathcal{E}$ is not lsc at y_0 and we shall get a contradiction.

Let W be an open set and let $\{y_r\} \subset V$, with $y_r \to y_0$, be such that

$$W \cap \mathcal{E}(y_0) \neq \emptyset \tag{3.4}$$

and

$$W \cap \mathcal{E}(y_r) = \emptyset \text{ for all } r \in \mathbb{N}. \tag{3.5}$$

By (3.4) we can select a point $x_0 \in W \cap \mathcal{E}(y_0)$.

Given $k \in \mathbb{N}$, since $x_0 \in B(x_0; k^{-1}) \cap \mathcal{F}(y_0)$ and $\mathcal{F}$ is lsc at y_0, there exists $r_k \in \mathbb{N}$ such that $B(x_0; k^{-1}) \cap \mathcal{F}(y_{r_k}) \neq \emptyset$. We can assume that $\{y_{r_k}\}$ is a subsequence of $\{y_r\}$. Let

$$z_k \in B(x_0; k^{-1}) \cap \mathcal{F}\left(y_{r_k}\right), \quad k = 1, 2, \ldots . \tag{3.6}$$

For any $k \in \mathbb{N}$, we can write

$$z_k = \sum_{i=1}^{n+1} \lambda_i^k e_i^k, \quad (\lambda_1^k, \ldots, \lambda_{n+1}^k) \in S, \quad e_i^k \in \mathcal{E}\left(y_{r_k}\right), \quad i = 1, \ldots, n+1, \tag{3.7}$$

because $\mathcal{F}(y_{r_k}) = \operatorname{conv} \mathcal{E}(y_{r_k})$.

By the compactness of the simplex S, we can assume without loss of generality that $(\lambda_1^k, \ldots, \lambda_{n+1}^k) \to (\lambda_1, \ldots, \lambda_{n+1}) \in S$. Analogously, since for any $i \in \{1, \ldots, n+1\}$,

$$\left\{e_i^k\right\} \subset \mathcal{E}\left(y_{r_k}\right) \subset \mathcal{F}\left(y_{r_k}\right) \subset \operatorname{cl} B\left(0_n; \rho\right),$$

we can assume that $e_i^k \to e_i \in \operatorname{cl} B\left(0_n; \rho\right)$, $i = 1, \ldots, n+1$. Since $\mathcal{F}$ is closed at y_0 and $e_i^k \in \mathcal{F}\left(y_{r_k}\right)$ for all $k \in \mathbb{N}$, we get $e_i \in \mathcal{F}\left(y_0\right)$. Now, taking $\lim_k$ in (3.7) and recalling (3.6), we obtain

$$x_0 = \sum_{i=1}^{n+1} \lambda_i e_i, \quad (\lambda_1^k, \ldots, \lambda_{n+1}^k) \in S, \quad e_i \in \mathcal{F}(y_0), \quad i = 1, \ldots, n+1. \tag{3.8}$$

Since $x_0 \in \mathcal{E}\left(y_0\right) = \operatorname{extr} \mathcal{F}\left(y_0\right)$, we must have in (3.8) all the coefficients $\lambda_i = 0$ except one, $\lambda_j = 1$, in which case $x_0 = e_j$. Since $e_j = \lim_k e_j^k$, $\{e_j^k\} \subset \mathcal{E}\left(y_{r_k}\right) \subset \mathbb{R}^n \backslash W$ by (3.5), and $\mathbb{R}^n \backslash W$ is closed, we have $x_0 = e_j \in \mathbb{R}^n \backslash W$, i.e., $x_0 \notin W$. This contradicts the selection of x_0 in $W \cap \mathcal{E}\left(y_0\right)$.

Case 2. $\mathcal{F}\left(y_0\right)$ is unbounded.

The plan of the proof is to consider the truncated mapping $\mathcal{F}_\rho$, for a certain $\rho > 0$. Since $\mathcal{F}_\rho = \operatorname{conv} \operatorname{extr} \mathcal{F}_\rho$ by Remark 2.5 and $\mathcal{F}_\rho\left(y_0\right)$ is bounded, we are in case 1 and so $\operatorname{extr} \mathcal{F}_\rho$ will be lsc at y_0. This will allow us to conclude that $\mathcal{E} = \operatorname{extr} \mathcal{F}$ is lsc at y_0.

First we show that if $\mathcal{E}_\rho$ is the truncated mapping of $\mathcal{E}$ of radius $\rho > 0$, then

$$\operatorname{extr} \mathcal{F}_\rho\left(y\right) = \mathcal{E}_\rho\left(y\right) \cup \left\{x \in \mathcal{F}\left(y\right) \mid \|x\| = \rho\right\} \text{ for all } y \in Y. \tag{3.9}$$

In fact, the inclusion $\operatorname{extr} \mathcal{F}_\rho\left(y\right) \supset \mathcal{E}_\rho\left(y\right) \cup \left\{x \in \mathcal{F}\left(y\right) \mid \|x\| = \rho\right\}$ is obvious. For the reverse inclusion take $x \in \operatorname{extr} \mathcal{F}_\rho\left(y\right)$ such that $\|x\| < \rho$. Assume that $x = \lambda u + (1-\lambda) v$ with $0 < \lambda < 1$ and $u, v \in \mathcal{F}\left(y\right), u \neq v$. We may assume without loss of generality that $\|u\|, \|v\| < \rho$ which contradicts the fact that x is an extreme point of $\mathcal{F}_\rho(y)$. Therefore, $x \in \mathcal{E}_\rho\left(y\right)$.

Now, in order to prove that $\mathcal{E}$ is lsc at y_0, assume that $\mathcal{E}$ is not. Then there exist $x_0 \in \mathcal{E}\left(y_0\right)$, $\delta > 0$, and a sequence $\{y_r\}$ such that $y_r \to y_0$ and $\mathcal{E}\left(y_r\right) \cap B\left(x_0; \delta\right) = \emptyset$ for every $r \in \mathbb{N}$. Take $\rho = \|x_0\| + \delta$ and observe that $x_0 \in \operatorname{extr} \mathcal{F}_\rho\left(y_0\right)$ according to (3.9). $\mathcal{F}_\rho$ is lsc and closed at y_0, and so, by case 1, $\operatorname{extr} \mathcal{F}_\rho$ is lsc at y_0, which implies that there exists a sequence $\{x_r\}$ such that $x_r \to x_0$, $x_r \in \operatorname{extr} \mathcal{F}_\rho\left(y_r\right)$, and $\|x_r\| < \rho$ for r large enough. This yields the contradiction $\mathcal{E}\left(y_r\right) \cap B\left(x_0; \delta\right) \neq \emptyset$. $\square$

4. Upper semicontinuity and closedness. In contrast with the lower semicontinuity, the closedness of a set-valued mapping $\mathcal{A}$ is not inherited by $\operatorname{conv}\mathcal{A}$ (even though $\mathcal{A} = \operatorname{bd}\mathcal{F}, \operatorname{rbd}\mathcal{F}, \operatorname{extr}\mathcal{F}$, as Example 3 in [5] shows). On the other hand, Proposition 4 in [5] establishes that, if $\operatorname{bd}\mathcal{F}$ is usc at y_0, then $\mathcal{F}$ is usc at y_0. In this section we shall prove that a similar statement holds for $\operatorname{rbd}\mathcal{F}$, but not for $\operatorname{extr}\mathcal{F}$ even though $\operatorname{extr}\mathcal{F}$ is either locally bounded or closed (nevertheless, according to the next Theorem 4.3, these two properties together entail the upper semicontinuity and the closedness of $\mathcal{F}$).

Example 4.1. Let $\mathcal{E} : Y \rightrightarrows \mathbb{R}^2$, where $Y = [2, +\infty[$ and

$$\mathcal{E}(y) = \left\{x \in \mathbb{R}^2 \mid \|x\| = 1, x_1 < y^{-1}\right\} \cup \{(y, 0)\} \text{ for all } y \in Y.$$

It is easy to see that $\mathcal{E}$ is locally bounded and continuous but not closed at $y_0 = 2$, and that it is the extreme points set mapping of $\mathcal{F} = \operatorname{conv}\mathcal{E}$. We shall prove that $\mathcal{F}$ is not usc at y_0. Let

$$W := \{x \in \mathbb{R}^2 \mid \sqrt{3}\,|x_2| < 2 - x_1, x_1 < 2\} \cup B\left((2,0); \frac{1}{2}\right),$$

$\mathcal{F}(y_0) \subset W$. If $y > 2$, then $\overline{x} = (1, \frac{1}{\sqrt{3}}) \in \mathcal{F}(y) \setminus W$. Observe also that $\mathcal{F}$ cannot be closed at y_0 (because $\mathcal{F}(y_0)$ is not closed).

Example 4.2. Let $\mathcal{E} : \mathbb{R} \rightrightarrows \mathbb{R}^3$ be such that

$$\mathcal{E}(y) = \left\{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_2 = x_1^2\right\} \cup \{(0, 0, y)\} \text{ for all } y \in \mathbb{R}.$$

As in the previous example, $\mathcal{E} = \operatorname{extr}\mathcal{F}$ for $\mathcal{F} = \operatorname{conv}\mathcal{E}$ and $\mathcal{E}$ is continuous at $y_0 = 0$, but now $\mathcal{E}$ is also closed and $\mathcal{E}(y_0)$ is unbounded. In order to prove that $\mathcal{F}$ is not usc at y_0, let us consider the convex plane set $C := \left\{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\right\}$ and the open set

$$W := \mathbb{R}^3 \setminus \left\{x \in \mathbb{R}^3 \mid x_3 \geq x_2^{-1}, x_2 > 0\right\}.$$

Obviously, $\mathcal{F}(y_0) = C \times \{0\} \subset W$. Moreover, if $y > 0$ and $y > 4/r^2$ for $0 \neq r \in \mathbb{R}$, we have

$$\left(0, \frac{r^2}{2}, \frac{y}{2}\right) = \frac{1}{2}(0, 0, y) + \frac{1}{4}[(-r, r^2, 0) + (r, r^2, 0)] \in \mathcal{F}(y) \setminus W,$$

so that $\mathcal{F}(y) \nsubseteq W$. Hence $\mathcal{F}$ is not usc at y_0.

Finally, we show that $\mathcal{F}$ is closed at y_0. Let $y_r \to y_0$ and $x^r \to x^0$ be such that $x^r \in \mathcal{F}(y_r)$, $r = 1, 2, \ldots$. Since $\mathcal{F}(y_r) = \operatorname{conv}[(C \times \{0\}) \cup \{(0, 0, y_r)\}]$, for any $r \in \mathbb{N}$, we can write

$$x^r = \lambda_r(c^r, 0) + (1 - \lambda_r)(0, 0, y_r) = (\lambda_r c^r, (1 - \lambda_r) y_r), c^r \in C, 0 \leq \lambda_r \leq 1.$$

Observe that $c^r \in C$ and $(0,0) \in C$ entail $\lambda_r c^r \in C$. On the other hand, $x_3^r = (1 - \lambda_r) y_r \in \operatorname{conv}\{0, y_r\}$. Taking limits we get $x^0 = \lim_r x^r \in C \times \{0\} = \mathcal{F}(y_0)$.

The next result is a reformulation of a well-known result ([1, Lemma 1.1.9]), taking into account the mentioned equivalence between closedness and outer semicontinuity.

THEOREM 4.3. *If $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ is closed and locally bounded at $y_0 \in \operatorname{dom}\mathcal{A}$, then* $\operatorname{conv}\mathcal{A}$ *is closed and usc at y_0.*

Observe that it is not possible to replace in Theorem 4.3 above the condition "$\mathcal{A}$ is closed and locally bounded at y_0" by just "$\mathcal{A}$ is closed and usc at y_0" (recall Example 4.2).

Given two set-valued mappings $\mathcal{M}, \mathcal{N} : Y \rightrightarrows \mathbb{R}^n$, we say that $\mathcal{M}$ is *contained* in $\mathcal{N}$ (in brief, $\mathcal{M} \subset \mathcal{N}$) *locally at* y_0 if there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{M}(y) \subset \mathcal{N}(y)$ for all $y \in V$. We also define the *closure* of $\mathcal{M}$ as the mapping $\operatorname{cl} \mathcal{M} : Y \rightrightarrows \mathbb{R}^n$ such that $(\operatorname{cl} \mathcal{M})(y) = \operatorname{cl} \mathcal{M}(y)$ for all $y \in Y$.

COROLLARY 4.4. *Let $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \operatorname{dom} \mathcal{A}$ be such that $\mathcal{A}(y_0)$ is bounded and $\mathcal{A}$ is usc at y_0. Then each of the following conditions guarantees that* $\operatorname{conv} \mathcal{A}$ *is closed and usc at y_0:*
(i) $\mathcal{A}(y_0)$ *is closed.*
(ii) $\operatorname{cl} \mathcal{A} \subset \operatorname{conv} \mathcal{A}$ *locally at y_0.*

Proof. (i) Since $\mathcal{A}$ is usc at y_0 and $\mathcal{A}(y_0)$ is bounded, then $\mathcal{A}$ is locally bounded at y_0. The conclusion follows from Theorem 4.3.

(ii) First we prove that $\operatorname{cl} \mathcal{A}$ is usc at y_0. In fact, given an open set W such that $\operatorname{cl} \mathcal{A}(y_0) \subset W$, we have

$$\mathcal{A}(y_0) \subset U := \operatorname{cl} \mathcal{A}(y_0) + B(0_n; \varepsilon),$$

where

$$\varepsilon := \frac{1}{2} d\left(\operatorname{cl} \mathcal{A}(y_0), \mathbb{R}^n \setminus W\right) > 0.$$

Since U is open, there exists an open set $V \subset Y$, $y_0 \in V$, such that $\mathcal{A}(y) \subset U$ for all $y \in V$. Then $\operatorname{cl} \mathcal{A}(y) \subset \operatorname{cl} U \subset W$.

Now we show that $\operatorname{conv} \mathcal{A}$ is usc at y_0.

Since $\operatorname{cl} \mathcal{A}$ is usc at y_0 and $\operatorname{cl} \mathcal{A}(y_0)$ is compact we can assert, applying statement (i) to $\operatorname{cl} \mathcal{A}$, that $\operatorname{conv} \operatorname{cl} \mathcal{A}$ is closed and usc at y_0. Since the assumption implies that $\operatorname{conv} \operatorname{cl} \mathcal{A} = \operatorname{conv} \mathcal{A}$ locally at y_0, we conclude that $\operatorname{conv} \mathcal{A}$ is closed and usc at y_0. □

The boundedness assumption in Corollary 4.4 is not superfluous even for the extreme points set mapping (recall again Example 4.2, where (i) holds).

Now, we give a condition that assures that if $\mathcal{A}$ is usc at y_0, then $\operatorname{conv} \mathcal{A}$ is usc at y_0 as well.

PROPOSITION 4.5. *Let $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and let $y_0 \in \operatorname{dom} \mathcal{A}$ be such that*

$$\operatorname{rbd} \operatorname{conv} \mathcal{A} \subset \mathcal{A} \subset \operatorname{conv} \operatorname{rbd} \operatorname{conv} \mathcal{A}$$

locally at y_0 and $\operatorname{conv} \mathcal{A}$ *is closed at y_0. If $\mathcal{A}$ is usc at y_0, then* $\operatorname{conv} \mathcal{A}$ *is usc at y_0.*

Proof. Let $\mathcal{F} := \operatorname{conv} \mathcal{A}$ and let $\mathcal{R} = \operatorname{rbd} \mathcal{F}$. We assume that $\mathcal{A}$ is usc at y_0.

Let V_1 be a neighborhood of y_0 such that $\mathcal{R}(y) \subset \mathcal{A}(y) \subset \operatorname{conv} \mathcal{R}(y)$ for all $y \in V_1$. Then we have $\mathcal{F}(y) = \operatorname{conv} \mathcal{R}(y)$ for all $y \in V_1$.

By Proposition 2.3, there exists $\overline{\rho} > 0$ and a neighborhood of y_0, $V_2 \subset V_1$, such that

$$\mathcal{A}(y) \diagdown \mathcal{A}_{\overline{\rho}}(y) \subset \mathcal{A}(y_0) \diagdown \mathcal{A}_{\overline{\rho}}(y_0) \quad \text{for all } y \in V_2. \tag{4.1}$$

We shall prove that we can replace $\mathcal{A}$ with $\mathcal{F}$ in (4.1), so that $\mathcal{F}$ will be usc at y_0 because $\mathcal{F}$ is closed at y_0 (again by Proposition 2.3). Let $\overline{y} \in V_2$ and $\overline{x}$ be such that

$$\overline{x} \in \mathcal{F}(\overline{y}) \text{ and } \|\overline{x}\| > \overline{\rho}.$$

If $\overline{x} \in \mathcal{A}(\overline{y})$, then $\overline{x} \in \mathcal{A}(\overline{y}) \diagdown \mathcal{A}_{\overline{\rho}}(\overline{y})$ and so

$$\overline{x} \in \mathcal{A}(y_0) \diagdown \mathcal{A}_{\overline{\rho}}(y_0) \subset \mathcal{A}(y_0) \subset \mathcal{F}(y_0).$$

Suppose that $\overline{x} \notin \mathcal{A}(\overline{y})$ and $\overline{x} \notin \mathcal{F}(y_0)$. Now, $\mathcal{R}(\overline{y}) \subset \mathcal{A}(\overline{y})$ implies that

$$\overline{x} \in \mathcal{F}(\overline{y}) \diagdown \mathcal{A}(\overline{y}) \subset \mathcal{F}(\overline{y}) \diagdown \mathcal{R}(\overline{y}) = \operatorname{rint} \mathcal{F}(\overline{y}). \tag{4.2}$$

Since $\mathcal{F}(y_0)$ is closed and convex, there exist $a \neq 0_n$ and a scalar α such that

$$a'\overline{x} = \alpha \text{ and } a'x < \alpha \text{ for all } x \in \mathcal{F}(y_0). \tag{4.3}$$

Consider the flat $H := \{x \in \operatorname{aff} \mathcal{F}(\overline{y}) \mid a'x = \alpha\}$. Obviously $a'c = 0$ for all $c \in H - \overline{x}$ (the linear subspace parallel to H).

We shall get a contradiction if we are able to prove that $H \subset \mathcal{F}(\overline{y})$. In fact, in this case if $a'x = \alpha$ for all $x \in \operatorname{aff} \mathcal{F}(\overline{y})$, then $H = \operatorname{aff} \mathcal{F}(\overline{y})$ and so $\mathcal{F}(\overline{y}) = \operatorname{aff} \mathcal{F}(\overline{y})$, i.e., $\mathcal{F}(\overline{y})$ is a flat. Otherwise $\mathcal{F}(\overline{y})$ is a half-flat. In both cases $\mathcal{F}(\overline{y}) \neq \operatorname{conv} \mathcal{R}(\overline{y})$ despite of $\overline{y} \in V_1$.

In order to prove that $H \subset \mathcal{F}(\overline{y})$ we associate with each $c \in (H - \overline{x}) \setminus \{0_n\}$ the halfline $S(c) := \{\overline{x} + \lambda c \mid \lambda \geq 0\} \subset H$. Now we prove that

$$S(c) \cap \operatorname{cl} B(0_n; \overline{\rho}) = \emptyset \Rightarrow S(c) \subset \operatorname{rint} \mathcal{F}(\overline{y}). \tag{4.4}$$

Assume that $S(c) \cap \operatorname{cl} B(0_n; \overline{\rho}) = \emptyset$ and $S(c) \nsubseteq \operatorname{rint} \mathcal{F}(\overline{y})$. By (4.2) we have

$$0 < \overline{\lambda} := \sup \{\lambda \in \mathbb{R}_+ \mid \overline{x} + \lambda c \in \operatorname{rint} \mathcal{F}(\overline{y})\} < +\infty.$$

Thus $\overline{x} + \overline{\lambda} c \in \mathcal{R}(\overline{y}) \subset \mathcal{A}(\overline{y})$ and, by (4.1), we have

$$\begin{aligned} \overline{x} + \overline{\lambda} c \in \mathcal{A}(\overline{y}) \diagdown \operatorname{cl} B(0_n; \overline{\rho}) &= \mathcal{A}(\overline{y}) \diagdown \mathcal{A}_{\overline{\rho}}(\overline{y}) \\ &\subset \mathcal{A}(y_0) \diagdown \mathcal{A}_{\overline{\rho}}(y_0) \subset \mathcal{F}(y_0), \end{aligned}$$

so that by (4.3) $a'\overline{x} = \alpha$ and $a'\overline{x} = a'(\overline{x} + \overline{\lambda} c) < \alpha$. This is a contradiction.

Finally, we prove that $H \subset \mathcal{F}(\overline{y})$ by means of a discussion based on the set $C := H \cap \operatorname{cl} B(0_n; \overline{\rho})$.

If $C = \emptyset$, then H is the union of halflines emanating from $\overline{x}$ in all directions parallel to H, and these halflines are contained in $\operatorname{rint} \mathcal{F}(\overline{y})$, according to (4.4). Then $H \subset \operatorname{rint} \mathcal{F}(\overline{y}) \subset \mathcal{F}(\overline{y})$.

If $|C| = 1$, then all the halflines mentioned above are contained in $\operatorname{rint} \mathcal{F}(\overline{y})$, except one. Thus $H \subset \mathcal{F}(\overline{y})$.

If $|C| > 1$, then C is a closed ball in H and all the halflines in H emanating from $\overline{x}$ which do not meet C are contained in $\operatorname{rint} \mathcal{F}(\overline{y})$. Then $\mathcal{F}(\overline{y})$ contains the complement, relative to H, of a pointed cone with apex $\overline{x}$. Hence we have again $H \subset \mathcal{F}(\overline{y})$. □

Given $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ and $\rho > 0$, we denote by $\mathcal{A}_\rho$ and by $(\operatorname{conv} \mathcal{A})_\rho$ the truncated mappings of $\mathcal{A}$ and $\operatorname{conv} \mathcal{A}$, respectively, with radius ρ. We also define the mapping $\mathcal{A}^\rho : Y \rightrightarrows \mathbb{R}^n$ such that

$$\mathcal{A}^\rho(y) = \mathcal{A}_\rho(y) \cup \{x \in \operatorname{conv} \mathcal{A}(y) \mid \|x\| = \rho\}.$$

If $\mathcal{F} = \operatorname{conv} \operatorname{rbd} \mathcal{F}$ ($\mathcal{F} = \operatorname{conv} \operatorname{bd} \mathcal{F}$), and $\mathcal{A} = \operatorname{rbd} \mathcal{F}$ ($\mathcal{A} = \operatorname{bd} \mathcal{F}$, respectively), then $(\operatorname{conv} \mathcal{A})_\rho = \operatorname{conv} \mathcal{A}^\rho$. The inclusion $(\operatorname{conv} \mathcal{A})_\rho \subset \operatorname{conv} \mathcal{A}^\rho$ follows from the fact that any convex combination $x = (1 - \lambda) u + \lambda v$, $0 \leq \lambda \leq 1$, $x, u, v \in \operatorname{conv} \mathcal{A}(y)$,

$\|x\| \leq \rho$ and $\|v\| > \rho$, can be expressed as $x = (1-\alpha)\,u + \alpha w$, where $0 \leq \alpha \leq 1$ and $w \in [x, v] \subset \operatorname{conv} \mathcal{A}(y)$, with $\|w\| = \rho$.

LEMMA 4.6. *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *and let* $y_0 \in \operatorname{dom} \mathcal{A}$ *be such that* $\mathcal{A}(y_0)$ *and* $\operatorname{conv} \mathcal{A}(y_0)$ *are closed and* $\mathcal{A}$ *is usc at* y_0*. Then* $\{\rho > 0 \mid \mathcal{A}^\rho$ *is closed at* $y_0\}$ *is unbounded.*

Proof. We will prove that, under the assumptions, $\mathcal{A}^\rho$ is closed at y_0 for all $\rho \in I := \{\rho > 0 \mid \mathcal{A}_\rho(y_0) \neq \emptyset\}$ (I is a halfline). We denote $\mathcal{F} = \operatorname{conv} \mathcal{A}$.

Let $\rho \in I$, $y_k \to y_0$ and $x_k \to x_0$ be such that $x_k \in \mathcal{A}^\rho(y_k)$, $k = 1, 2, \dots$.

Since $\mathcal{A}^\rho(y_k) \subset \operatorname{cl} B(0_n; \rho)$ for all $k \in \mathbb{N}$, $\|x_0\| \leq \rho$.

If there exists an increasing sequence $\{k_r\} \subset \mathbb{N}$ such that $x_{k_r} \in \mathcal{A}(y_{k_r})$, $r = 1, 2, \dots$, then $x_0 \in \mathcal{A}(y_0)$ (because $\mathcal{A}$ is closed at y_0) and so $x_0 \in \mathcal{A}_\rho(y_0) \subset \mathcal{A}^\rho(y_0)$.

Thus we can assume without loss of generality that $x_k \notin \mathcal{A}(y_k)$, $k = 1, 2, \dots$.

Given $k \in \mathbb{N}$, we have $x_k \in \mathcal{A}^\rho(y_k) \diagdown \mathcal{A}_\rho(y_k) \subset \{x \in \mathcal{F}(y_k) \mid \|x\| = \rho\}$. Since $\|x_k\| = \rho$ for all k, we have $\|x_0\| = \rho$.

If $x_0 \in \mathcal{F}(y_0)$, then $x_0 \in \mathcal{A}^\rho(y_0)$ and we have finished. So we assume that $x_0 \notin \mathcal{F}(y_0)$. Since this set is closed, $\varepsilon := \frac{1}{2} d(x_0, \mathcal{F}(y_0)) > 0$. Let us consider the open convex set $W := \mathcal{F}(y_0) + B(0_n; \varepsilon)$. Since $\mathcal{A}(y_0) \subset \mathcal{F}(y_0) \subset W$ and $\mathcal{A}$ is usc at y_0, there exists a neighborhood of y_0, say V, such that $\mathcal{A}(y) \subset W$ for all $y \in V$. Then, taking convex hulls, we get $\mathcal{F}(y) \subset W$ for all $y \in V$.

Let $k_0 \in \mathbb{N}$ be such that $y_k \in V$ for all $k \geq k_0$. For such a k we have $x_k \in \mathcal{A}^\rho(y_k) \subset \mathcal{F}(y_k) \subset W$ whereas $x_0 \notin \mathcal{F}(y_0)$, so that $d(x_k, x_0) \geq \varepsilon$. This contradicts $x_k \to x_0$. □

LEMMA 4.7. *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *be such that* $(\operatorname{conv} \mathcal{A})_\rho = \operatorname{conv} \mathcal{A}^\rho$ *for all* $\rho > 0$ *sufficiently large and let* $y_0 \in \operatorname{dom} \mathcal{A}$ *such that* $\{\rho > 0 \mid \mathcal{A}^\rho$ *is closed at* $y_0\}$ *is unbounded. Then* $\operatorname{conv} \mathcal{A}$ *is closed at* y_0*.*

Proof. Let $\mathcal{F} := \operatorname{conv} \mathcal{A}$. Let $y_r \to y_0$ and $x_r \to x_0$ be such that $x_r \in \mathcal{F}(y_r)$, $r = 1, 2, \dots$.

Since the convergent sequence $\{x_r\}$ is bounded, and by the assumptions on $\{\mathcal{A}^\rho \mid \rho > 0\}$, there exists $\rho > 0$ such that $\|x_r\| \leq \rho$ for all $r \in \mathbb{N}$, $\mathcal{F}_\rho = \operatorname{conv} \mathcal{A}^\rho$ and $\mathcal{A}^\rho$ is closed at y_0. Since $\mathcal{A}^\rho$ is closed and locally bounded at y_0, by Theorem 4.3, $\mathcal{F}_\rho = \operatorname{conv} \mathcal{A}^\rho$ is closed and usc at y_0. Then, since $x_r \in \mathcal{F}_\rho(y_r)$ for all $r \in \mathbb{N}$, we have $x_0 \in \mathcal{F}_\rho(y_0) \subset \mathcal{F}(y_0)$. □

PROPOSITION 4.8. *Let* $\mathcal{A} : Y \rightrightarrows \mathbb{R}^n$ *be such that* $(\operatorname{conv} \mathcal{A})_\rho = \operatorname{conv} \mathcal{A}^\rho$ *for all* $\rho > 0$ *sufficiently large and let* $y_0 \in \operatorname{dom} \mathcal{A}$ *such that* $\mathcal{A}(y_0)$ *is closed,*

$$\operatorname{rbd} \operatorname{conv} \mathcal{A} \subset \mathcal{A} \subset \operatorname{conv} \operatorname{rbd} \operatorname{conv} \mathcal{A}$$

locally at y_0 *and* $\mathcal{A}$ *is usc at* y_0*. Then* $\operatorname{conv} \mathcal{A}$ *is usc at* y_0*.*

Proof. By assumption $\operatorname{rbd} \operatorname{conv} \mathcal{A}(y_0) \subset \mathcal{A}(y_0) \subset \operatorname{conv} \mathcal{A}(y_0)$, so that $\operatorname{conv} \mathcal{A}(y_0)$ is closed. Then, by Lemma 4.6, $\{\rho > 0 \mid \mathcal{A}^\rho$ is closed at $y_0\}$ is unbounded and, by Lemma 4.7, $\operatorname{conv} \mathcal{A}$ is closed at y_0. We conclude that $\operatorname{conv} \mathcal{A}$ is usc at y_0 by Proposition 4.5. □

THEOREM 4.9. *Let* $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ *be such that* $\mathcal{F} = \operatorname{conv} \operatorname{rbd} \mathcal{F}$ *and* $\operatorname{rbd} \mathcal{F}$ *is usc at* $y_0 \in \operatorname{dom} \mathcal{F}$*. Then* $\mathcal{F}$ *is usc at* y_0*.*

Proof. It is a straightforward consequence of Proposition 4.8, taking $\mathcal{A} = \operatorname{rbd} \mathcal{F}$. □

The last four results are also valid replacing "rbd" everywhere with "bd" (see [5]). The final example illustrates the results in sections 3 and 4 and shows that there is no usc counterpart for Theorems 3.3 and 3.4.

Example 4.10. Let us identify the complex field $\mathbb{C}$ with $\mathbb{R}^2$ and let us take as Y the set of polynomials of degree $q \in \mathbb{N}$ (fixed) with complex coefficients equipped with the Euclidean distance on $\mathbb{R}^{2q+2}$. Given $y \in Y$, we denote by $\mathcal{A}(y)$ its set of complex zeros and by $\mathcal{F}(y)$ its convex hull, i.e., the polytope $\mathcal{F}(y) = \operatorname{conv} \mathcal{A}(y)$. By the fundamental theorem of algebra, $\mathcal{A}(y) \neq \emptyset$ for all $y \in Y$, so that $\operatorname{dom} \mathcal{A} = Y$. Let us denote by $\mathcal{B}$, $\mathcal{R}$, and $\mathcal{E}$ the boundary mapping, the relative boundary mapping, and the extreme points set mapping of $\mathcal{F}$, respectively. By Proposition 2.4, we have

$$\mathcal{F} = \operatorname{conv} \mathcal{B} = \operatorname{conv} \mathcal{R} = \operatorname{conv} \mathcal{E}.$$

$\mathcal{A}$ is lsc and usc as a consequence of a well-known consequence of Rolle's theorem for complex polynomials (see, e.g., [7]) and, since it has closed images, it is also closed. By Theorem 3.1 and Corollary 4.4, $\mathcal{F}$ is also lsc, usc, and closed. Consequently, $\mathcal{B}$, $\mathcal{R}$, and $\mathcal{E}$ are lsc by Propositions 1 in [4] and Theorems 3.3 and 3.4 in this paper (the direct proofs of these statements are rather involved). Now we show that $\mathcal{R}$ and $\mathcal{E}$ are neither usc nor closed if $q = 3$.

Let $y_0 = x^3 + x$, with $\mathcal{A}(y_0) = \{0, \pm i\}$, and let $y_r = x^3 - \frac{2}{r}x^2 + (1 + \frac{1}{r^2})x$, with $\mathcal{A}(y_r) = \left\{0, \frac{1}{r} \pm i\right\}$, $r = 1, 2, \dots$. Obviously, $y_r \to y_0$. Taking the constant sequence $x_r = 0$, $r = 1, 2, \dots$, we have $x_r \in \mathcal{E}(y_r) \subset \mathcal{F}(y_r)$ for all r, whereas $0 \notin \mathcal{E}(y_0) = \mathcal{R}(y_0) = \{\pm i\}$. Thus neither $\mathcal{R}$ nor $\mathcal{E}$ is closed (usc) at y_0.

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