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REGULARITY OF THE EIKONAL EQUATION WITH NEUMANN BOUNDARY CONDITIONS IN THE PLANE: APPLICATION TO FRONTS WITH NONLOCAL TERMS*

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Abstract. The first part of the paper is devoted to length estimates of the boundary of the reachable set for the plane and state constrained controlled system $x'(t) = c(t, x(t))b(t)$ (where $|b(t)| \leq 1$ a.e.). This study is motivated in the second part by the analysis of dislocation dynamics, which can be modeled as a curve $\Gamma(t)$ moving in an open set $\Omega \subset \mathbb{R}^2$ according to some nonlocal law with Neumann boundary conditions on $\partial\Omega$. The length estimates of the first part play a crucial role in the proof of the existence and uniqueness of a *viscosity* solution for this model.

Key words. dislocation dynamics, eikonal equation, Hamilton–Jacobi equations, Neumann boundary condition, discontinuous viscosity solutions, nonlocal equations

AMS subject classifications. 35F25, 35D05

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1. Introduction.

In this paper, we study the dynamics of a curve $\Gamma(t) \subset \Omega$ which moves according to the following nonlocal law:

$$V_{t,x} = \bar{c}_1(t, x) + I\left(\mathbf{1}_{K(t)}\right)(t, x), \tag{1}$$

where Ω is an open subset of $\mathbb{R}^2$, $V_{t,x}$ is the normal velocity at the point x at time t, $K(t)$ is the set enclosed by the curve $\Gamma(t)$, the map I is defined by

$$I(\rho)(t, x) := \int_\Omega \bar{c}_2(t, x, y)\rho(y)\, dy \qquad \forall \rho \in \mathbb{L}^1(\Omega),$$

and $\bar{c}_2$ solves the *elasticity equation* (see [2, 3]). Moreover, the curve Γ can touch the boundary $\partial\Omega$ only orthogonally. Our investigation originated from the so-called *phase field* model of dislocation, introduced by Rodney, Le Bouar, and Finel [13]: a dislocation is a line of crystal defect, and it moves on its slip plane with a normal velocity proportional to the Peach–Koehler force acting on the line:

$$V_{t,x} = \bar{c}_1(t, x) + \bar{c}_2 \star_x \mathbf{1}_{K(t)}(t, x), \tag{2}$$

where $\bar{c}_1$ is the contribution given by an exterior field, while $\bar{c}_2 \star_x \mathbf{1}_{K(t)}$ (the convolution is done only w.r.t. the space variable $x \in \mathbb{R}^2$) is a force created by the dislocation itself. Equations of this type have been studied only in the whole space: one of the main issues of this note is that we reduced the slip plane to a open subset Ω and addressed the dynamics' modifications given by the presence of $\partial\Omega$.

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For dynamics in the whole space, let us recall that if we set

$$\rho(t,x) = \mathbf{1}_{K(t)}(x) \equiv \begin{cases} 1 & \text{if } \quad x \in K(t), \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

then ρ is a (discontinuous) viscosity solution to the following Cauchy problem for a nonlocal Hamilton–Jacobi equation

$$\begin{cases} \frac{\partial \rho}{\partial t} = (\bar{c}_1 + \bar{c}_2 \star_x \rho)\, |\nabla \rho| & \text{in} \quad (0,+\infty) \times \mathbb{R}^2, \\ \rho(0,x) = \mathbf{1}_{K_0}(x) & \text{on} \quad \mathbb{R}^2, \end{cases}$$

where K_0 is the starting set. For $\bar{c}_2 \geq 0$, the equation fulfills the *inclusion principle*, so the existence and uniqueness of the solution follow from a result by Cardaliaguet [9]; from a physical point of view, however, this case is not interesting because $\bar{c}_2$ needs to have zero mean. For $\bar{c}_2$ changing sign, Alvarez, Hoch, Le Bouar, and Monneau [2, 3] proved the short time existence and uniqueness of the solution, provided that the dislocation starts from a Lipschitz graph. Finally, in a forthcoming paper, Alvarez, Cardaliaguet, and Monneau [1] addressed the existence and uniqueness of the solution in an arbitrary time interval under the assumption that the starting set K_0 fulfills the *interior sphere condition* of radius $r > 0$ (i.e., for every $x \in \partial K_0$, there exists a unitary vector p such that $B(x - rp, r) \subseteq K_0$) and that $\bar{c}_i$ fulfills (beside some regularity assumptions)

$$\bar{c}_1(t,x) \geq \|\bar{c}_2(t,\cdot)\|_{\mathbb{L}^1(\mathbb{R}^2)} \qquad \forall (t,x) \in (0,+\infty) \times \mathbb{R}^2$$

(which assures that the dislocation is noncontracting: $\bar{c}_1 + \bar{c}_2 \star_x \mathbf{1}_K \geq 0$ for every Borel set K). Actually, these results also apply when the space is $\mathbb{R}^N$, even if the only interesting case for the phase field model is $N = 2$.

In our study of (1), the function ρ defined by (3) is a discontinuous solution to the following problem with Neumann boundary conditions in the viscosity sense:

$$\begin{cases} \frac{\partial \rho}{\partial t} = \big(\bar{c}_1 + I(\rho)\big) |\nabla \rho| & \text{in} \quad (0,+\infty) \times \Omega, \\ \rho(0,x) = \mathbf{1}_{K_0}(x) & \text{on} \quad \Omega, \\ \frac{\partial \rho}{\partial \nu} = 0 & \text{on} \quad (0,+\infty) \times \partial\Omega \end{cases} \tag{4}$$

(ν is the outward unity normal to $\partial\Omega$). One of the main issues of this paper is that we establish the existence and uniqueness (in a sense that will be specified in section 4) of the solution to the above initial boundary value problem under some interior sphere condition on the initial set K_0 and a condition on $\bar{c}_1$ and $\bar{c}_2$ ensuring that the dislocation is expanding.

As in [1], we shall use the Banach fixed point theorem for the mapping Φ on $C^0([0,T], \mathbb{L}^1(\Omega))$ that associates to any $\rho^0 \in C^0([0,T], \mathbb{L}^1(\Omega))$ the unique discontinuous solution to

$$\begin{cases} \frac{\partial \rho}{\partial t} = c_{\rho^0} |\nabla \rho| & \text{in} \quad (0,T) \times \Omega, \\ \rho(0,x) = \mathbf{1}_{K_0}(x) & \text{on} \quad \Omega, \\ \frac{\partial \rho}{\partial \nu} = 0 & \text{on} \quad (0,T) \times \partial\Omega \end{cases}$$

with $c_{\rho^0} := \big(\bar{c}_1 + I(\rho^0)\big)$. In order to prove that Φ is a well-defined contraction, we use the *representation formula*: for $0 \leq \rho^0 \leq 1$, the level set $\{\Phi(\rho^0)(t,\cdot) = 1\}$ coincides with the *reachable set* (for the precise definition, see formula (7) below) of the following control problem with reflecting boundary:

$$\begin{cases} y'(s) = c_{\rho^0}(s, y(s)) \big[b(s) - \langle b(s); \nu(y(s)) \rangle \mathbf{1}_{\partial\Omega}(y(s)) \nu(y(s)) \big] & \text{a.e. in } [0,t], \\ y(0) \in K_0, \quad y(s) \in \overline{\Omega} \qquad \forall s \in [0,t], \end{cases} \tag{5}$$

where the control $b : [0,t] \to B(0,1)$ is a measurable map and $\nu(y(s))$ is the outward unitary normal to $\partial\Omega$ in $y(s)$; namely, $z \in \{\Phi(\rho^0)(t,\cdot) = 1\}$ if and only if there exists a measurable map $b : [0,t] \to B(0,1)$ and a solution y to (5) with $y(t) = z$.

For proving that Φ is a contraction, we are led to estimate the Lebesgue measure of the reachable set and the Hausdorff measure of its boundary. In the case $\Omega = \mathbb{R}^N$ treated in [1], these estimates were based on the fact that the interior sphere assumption stated on the initial set K_0 propagates: the reachable set of the (unconstrained) control system satisfies the interior sphere condition for all times (see [7]). This gives a bound from above for the curvature of the reachable set at each time t, from which an estimate of the length of its boundary is derived. In our case, when $\Omega \neq \mathbb{R}^2$, the interior sphere condition does *not* propagate: as we show below, the curvature of the reachable set can blow up near the boundary of Ω. The main contribution of this paper is to explain how to overcome this difficulty, at least in dimension 2. In particular we show that the boundary of the reachable set has bounded length, although not necessarily finite curvature. From this we deduce estimates on the Lebesgue measure of the reachable set and the existence and uniqueness for problem (4).

The paper is organized as follows: In the rest of this introduction we give an example in which many features of our problem appear, and we set some notation. Section 2 is devoted to the estimates of the volume and the boundary of the reachable set of problem (5). The existence and uniqueness of the solution to (4) (and, in particular, the dynamics of Γ) are established in section 4 (using some technical estimates stated in section 3). Finally, the appendix is devoted to the proof of an extension of the Pontryagin maximum principle (stated in section 2).

1.1. Toy example. We give here an elementary example showing that the reachable set of a reflected controlled system of the form (5) can have a curvature which blows up at some point.

Consider $c \equiv 1$, $\Omega = \mathbb{R}^2 \backslash B(0,1)$, and $K_0 = B((-4,0),1)$. In this case the reachable set $\mathcal{R}(t)$ for the reflected control problem (5) can be explicitly computed. For this we note that $\mathcal{R}(t)$ is just the set of points $x \in \overline{\Omega}$ which is at a *geodesic distance* (in Ω) from K_0 not larger than t.

Hence for $t \in [0,2]$, $\mathcal{R}(t) = \bar{B}\big((-4,0),1+t\big)$ (see Figure 1). For $t \in [2,\sqrt{15}]$, $\mathcal{R}(t) = \bar{B}\big((-4,0),1+t\big) \backslash B(0,1)$ (see Figure 2).

After $\bar{t} = \sqrt{15} - 1$, some geodesics need to bend around $\partial\Omega$ for reaching the points in the "cone of shade" due to the presence of $\mathcal{C}\Omega$ (see Figure 3). In this case, all the points in this cone of shade are reached by geodesics passing through either the point $P_1 = (-1/4, -\sqrt{15}/4)$ or the point $P_2 = (-1/4, \sqrt{15}/4)$. These geodesics are made up of a straight line from $(-4,0)$ to P_1 or P_2, an arc on $\partial\Omega$, and again a straight line tangent to $\partial\Omega$. For the sake of simplicity, let us consider the coordinate system centered in $(0,0)$ with the ξ-axis and η-axis given, respectively, by the vector $(-1/4, -\sqrt{15}/4)$ and $(\sqrt{15}/4, -1/4)$ (i.e., in the new system $P_1 = (1,0)$). For $t \in (\bar{t}, 4)$, the coordinates of the point of $\partial\mathcal{R}(t)$ reached by a geodesic passing through P_1 and remaining on $\partial\Omega$ from the time $\bar{t}$ to the time s (for some $s \in [\bar{t}, t]$) are given by

$$\begin{cases} \xi(s) = \cos(s-\bar{t}) - (t-s)\sin(s-\bar{t}), \\ \eta(s) = \sin(s-\bar{t}) + (t-s)\cos(s-\bar{t}). \end{cases}$$

By standard arguments, one can verify that the curvature of $\partial\mathcal{R}(t)$ at that point is $(t-s)^{-2}$; in particular, as $s \to t$, it tends to $+\infty$.

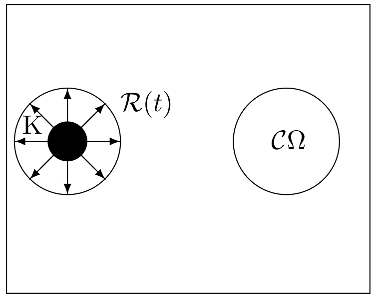

FIG. 1.

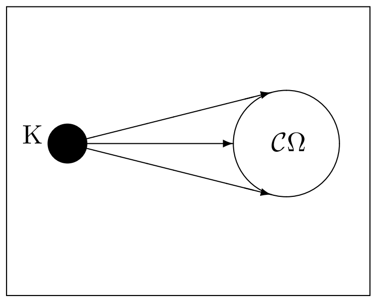

FIG. 2.

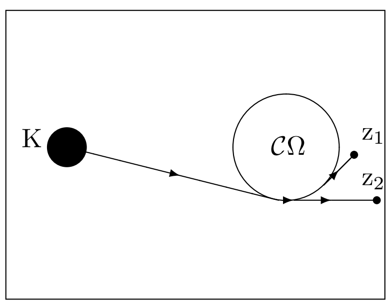

FIG. 3.

Notation. For each $K \subset \mathbb{R}^2$, $\partial_{\mathbb{R}^2} K$ and ∂K stand for its boundary in the natural topology of $\mathbb{R}^2$ and, respectively, in the induced topology of Ω; in other words, we have

$$\partial K := \partial_{\mathbb{R}^2} K \cap \Omega.$$

Moreover, $\mathcal{C}K := \mathbb{R}^2 \backslash K$, and $\mathbf{1}_K$ is the *characteristic* function of K; i.e., $\mathbf{1}_K(x) = 1$ if $x \in K$, and $\mathbf{1}_K(x) = 0$ if $x \in \mathcal{C}K$. For all sets K and H in $\mathbb{R}^2$, $d^{\mathcal{H}}(K, H)$ stands for the Hausdorff distance between the sets K and H:

$$d^{\mathcal{H}}(K, H) := \max \left\{ \max_{s \in K} \mathrm{dist}(s, H),\ \max_{u \in H} \mathrm{dist}(u, K) \right\},$$

where $\mathrm{dist}(s, H) := \min\{|s - u| : u \in H\}$. We denote by $B(p, r)$ the ball centered in $p \in \mathbb{R}^2$ with radius $r > 0$; B is the abridged notation of $B(0, 1)$. For every $x, y \in \Omega$,

$d^\Omega(x,y)$ is their *geodesic distance*; namely,

$$d^\Omega(x,y) := \inf\left\{ l \mid \begin{array}{l} \exists \text{ an AC arc, contained in } \Omega, \text{ with length } l \\ \text{and extremities in } x \text{ and } y \end{array} \right\}.$$

For a function ρ, ρ^* (resp., ρ_*) denotes the upper semicontinuous envelope (resp., the lower semicontinuous envelope) of ρ.

2. Main estimates for the reachable set. In this section we consider the reachable set for the reflected control problem

$$\text{(6)} \quad \begin{cases} y'(t) = c(t,y(t))\big[b(t) - \langle b(t), \nu(y(t))\rangle \mathbf{1}_{\partial\Omega}(y(t))\nu(y(t))\big] & \text{a.e. in } (0,T), \\ y(t) \in \overline{\Omega} \qquad \forall t \geq 0, \\ y(0) = y_0 \in K_0, \end{cases}$$

where $b : (0,T) \to B$ is a time measurable control and $\nu(y(t))$ is the outward unit normal to $\partial\Omega$ at $y(t)$. Let us recall that if Ω has a $\mathcal{C}^2$ boundary and c is sufficiently regular, then, for any measurable control $b : (0,T) \to B$ and any initial position $y_0 \in K_0$, (6) has a unique solution.

The reachable set at time $t > 0$ for this reflected control system is given by

$$\text{(7)} \qquad \mathcal{R}(t) = \{x \in \overline{\Omega} \mid \exists y \text{ solution to (6) on } [0,t] \text{ with } y(t) = x\}.$$

Our aim is to show that the boundary of this set has a finite Hausdorff measure and that its volume depends in a Lipschitz continuous way on the time.

For doing this we will need the following conditions on c, K_0, and Ω:

$$\text{(8)} \quad \begin{cases} \text{(i)} & c \text{ is continuous and derivable w.r.t. the second variable,} \\ \text{(ii)} & 0 < m \leq c(t,x) \leq M \qquad \forall (t,x) \in \mathbb{R}^+ \times \Omega, \\ \text{(iii)} & |c(t,x_1) - c(t,x_2)| \leq L_0|x_1 - x_2| \qquad \forall (t,x_1),(t,x_2) \in \mathbb{R}^+ \times \Omega, \\ \text{(iv)} & |D_x c(t,x_1) - D_x c(t,x_2)| \leq L_1|x_1 - x_2| \qquad \forall (t,x_1),(t,x_2) \in \mathbb{R}^+ \times \Omega \end{cases}$$

for some positive constants m, M, L_0, and L_1.

The initial set $K_0 \subset \Omega$ has to satisfy the following:

$$\text{(9)} \qquad K_0 \text{ is compact and fulfills the interior sphere condition with radius } r;$$

namely, for any $x \in \partial K_0$, there is some $p \in \mathbb{R}^2$, $|p| = 1$, with $B(x - rp, r) \subset K_0$. On Ω we assume that

$$\text{(10)} \qquad \begin{cases} \Omega \text{ is connected, } \partial\Omega \text{ is of class } \mathcal{C}^2, \text{ and } \mathcal{C}\Omega \text{ fulfills the} \\ \qquad \text{interior sphere condition with radius } r_1 > 0. \end{cases}$$

We also require that

$$\text{(11)} \qquad \exists\, k_0 > 0 \quad \text{such that} \quad |x - y| \leq d^\Omega(x,y) \leq k_0|x - y| \qquad \forall x, y \in \Omega$$

(the first inequality being obvious).

We finally introduce the minimal time function τ, which plays a crucial role in our study:

$$\text{(12)} \qquad \tau(x) := \inf\{t : x \in \mathcal{R}(t)\};$$

it is the time needed by the reflected controlled system to reach a point x when starting from K_0.

By standard argument (see [5]), we have the following lemma.

LEMMA 2.1. *The function τ is Lipschitz continuous with a Lipschitz constant less than k_0/m, where m and k_0 are defined in* (8) *and* (11). *Moreover, τ satisfies the Hamilton–Jacobi equation $c(\tau(x), x)|\nabla\tau(x)| = 1$ in the viscosity sense in $\Omega \setminus K_0$.*

We also note that the level sets of τ fulfill

$$\{x : \tau(x) = t\} \supseteq \partial\mathcal{R}(t); \tag{13}$$

in general, the inclusion is strict.

2.1. Extremal solutions and the Pontryagin maximum principle.

DEFINITION 2.2. *An admissible trajectory y is called extremal on $[0, t]$ if $\tau(y(t)) = t$.*

It is well known (for a detailed study of extremal trajectories, see [4, 8, 10, 15] and the references therein) that if y is an extremal trajectory on $[0, t]$, then $\tau(y(s)) = s$ for any $s \in [0, t]$. Throughout this section, $\mathcal{Y}(t)$ denotes the set of extremal trajectories on $[0, t]$:

$$\mathcal{Y}(t) = \{y \text{ extremal trajectory on } [0, t]\}.$$

The following result, proved in the appendix, is an extension of the Pontryagin maximum principle for our reflected control problem. It is inspired by similar results of Frankowska [12] on the regularity of the state and the adjoint for state constraints system. Let us note that the equation on p' is new in this context.

Here we denote by d the signed distance to the boundary of Ω:

$$d(x) = \begin{cases} -d_{\partial\Omega}(x) & \text{if } x \in \Omega, \\ d_\Omega(x) & \text{otherwise.} \end{cases}$$

From assumption (10), the function d is $\mathcal{C}^2$ in a neighborhood of $\partial\Omega$. Finally, for any $s \in \mathbb{R}$, we set $(s)_+ = \max\{s, 0\}$.

LEMMA 2.3. *Let $x(\cdot)$ be an extremal trajectory on the time interval $[0, T]$. Then there is a Lipschitz continuous function $p : [0, T] \to \mathbb{R}^N \setminus \{0\}$ and a measurable map $\lambda : [0, T] \to \mathbb{R}$ such that*

$$\begin{cases} x'(t) = c(t, x(t))\dfrac{p(t)}{|p(t)|} \qquad \forall t \in (0, T), \\ p'(t) = -\left[D_x c(t, x(t)) - \lambda(t) Dd(x(t))\right]|p(t)| \quad \textit{for a.e. } t \in (0, T), \\ \lambda(t) = \Big(- \Big\langle D^2 d(x(t)) c(t, x(t)) \frac{p(t)}{|p(t)|}, \frac{p(t)}{|p(t)|} \Big\rangle \\ \qquad\qquad + \langle Dd(x(t)), D_x c(t, x(t))\rangle \Big)_+ \mathbf{1}_{\partial\Omega}(x(t)) \quad \textit{for a.e. } t \in (0, T), \\ x(t) \in \overline{\Omega} \qquad \forall t \in (0, T). \end{cases} \tag{14}$$

The map p is called the adjoint of x.

Remark 2.4. Lemma 2.3 states, in particular, that any extremal trajectory is $\mathcal{C}^{1,1}$.

Remark 2.5. Problem (6) can be rewritten as a constraint problem; actually, its extremal trajectories coincide with those of problem

$$\begin{cases} y'(t) = c(t, y(t))b(t) \qquad \text{a.e. in } (0, T), \\ y(t) \in \overline{\Omega} \qquad \forall t \geq 0, \quad y(0) = y_0 \in K_0, \end{cases}$$

and the function λ is a multiplier. Of course, other types of Neumann boundary conditions give rise to control problems that fail this property.

We now state several consequences of Lemma 2.3.

COROLLARY 2.6. *For any $T > 0$, the set $\mathcal{Y}(T)$ is compact w.r.t. the $\mathcal{C}^1$ norm.*

Proof. From standard arguments the set $\mathcal{Y}(T)$, endowed with the $\mathcal{C}^0$ norm, is compact. Let (y_n) be a converging sequence of $\mathcal{Y}(T)$ for the $\mathcal{C}^0$ norm, and let y be its limit. We claim that y_n converges to y for the $\mathcal{C}^1$ norm.

From Lemma 2.3, for any n, there is some Lipschitz map $p_n : [0,T] \to \mathbb{R}^N\backslash\{0\}$ such that $|p_n(0)| = 1$ and (14) holds. In particular, the (p_n) are Lipschitz continuous with a Lipschitz constant depending only on the regularity of Ω and of c. Hence a subsequence, still denoted (p_n), converges uniformly to some Lipschitz continuous function p such that $|p(0)| = 1$ and p is a solution to

$$\begin{cases} p'(t) = -\left[D_x c(t,x(t)) - \lambda(t) Dd(x(t))\right]|p(t)|, \\ \lambda(t) = \Big(-\left\langle D^2 d(x(t)) c(t,x(t)) \frac{p(t)}{|p(t)|}, \frac{p(t)}{|p(t)|} \right\rangle \\ \qquad\qquad\qquad + \left\langle Dd(x(t)), D_x c(t,x(t)) \right\rangle \Big)_+ \mathbf{1}_{\partial\Omega}(x(t)) \end{cases}$$

for a.e. $t \in (0,T)$. In particular, we deduce that $p(t) \neq 0$. Therefore, since $y_n'(t) = c(t,y_n(t))p_n(t)/|p_n(t)|$, (y_n') uniformly converges to $y'(t) = c(t,y(t))p(t)/|p(t)|$.

So we have proved that for any sequence (y_n) of $\mathcal{Y}(T)$, there is a subsequence which converges in the $\mathcal{C}^1$ norm, whence we get the desired result. ◻

The next statement explains that two extremal trajectories on $[0,T]$ when crossing on $(0,T)$ necessarily have the same velocity.

LEMMA 2.7. *Let y_1 and y_2 be two extremal trajectories on $[0,T]$ for which there is some $t \in (0,T)$ with $y_1(t) = y_2(t) \in \partial\Omega$. Then $y_1'(t) = y_2'(t)$.*

Remark 2.8. Contrary to what happens in the unconstrained case ($\Omega = \mathbb{R}^2$), two different extremal trajectories of our reflected control problem can indeed cross, in particular on the boundary of Ω (as shown in Figure 3).

Proof. Since Ω has a $\mathcal{C}^2$ boundary, $y_1(t) = y_2(t) \in \partial\Omega$, and y_1 and y_2 are $\mathcal{C}^1$ and remain in $\overline{\Omega}$ on $[0,T]$, we deduce that the vectors $y_1'(t)$ and $y_2'(t)$ are necessarily tangent to $\partial\Omega$. Since, moreover, $|y_1'(t)| = c(t,y_1(t)) = |y_2'(t)|$ and we are in the plane, this leads to $y_1'(t) = \pm y_2'(t)$. Suppose for a while that $y_1'(t) = -y_2'(t)$. Then, thanks to Lemma 2.1, we have for any $h > 0$ sufficiently small

$$2h = \tau(y_2(t+h)) - \tau(y_1(t-h)) \le \frac{k_0}{m}|y_1(t-h) - y_2(t+h)| = \frac{k_0}{m} h\epsilon(h),$$

where $\epsilon(h) \to 0$ as $h \to 0^+$. This is impossible. So $y_1'(t) = y_2'(t)$. ◻

As a consequence, we have the following technical result which shall be useful in the proof of the main perimeter estimate.

COROLLARY 2.9. *Let $\delta > 0$ and $T > 0$ be fixed. There is some positive σ such that if y_1 and y_2 are two extremal trajectories on some interval $[0, t_1+\delta]$ and $[0, t_2+\delta]$, for $t_1, t_2 \in (0,T]$, and if $y_1(t_1) \in \partial\Omega$, $y_2(t_2) \in \partial\Omega$ with $|y_1(t_1) - y_2(t_2)| \le \sigma$, then*

$$\left| \frac{y_1'(t_1)}{|y_1'(t_1)|} - \frac{y_2'(t_2)}{|y_2'(t_2)|} \right| \le c_b |y_1(t_1) - y_2(t_2)|,$$

where c_b depends only on the $\mathcal{C}^2$ regularity of $\partial\Omega$.

Proof. Let us first define the constant c_b. Since we are in the plane and Ω has a $\mathcal{C}^2$ boundary, there is a constant c_b (depending on the curvature of $\partial\Omega$ in $K_0 + B_{MT}$)

such that, for any $y_1, y_2 \in \partial\Omega \cap (K_0 + B_{MT})$ and for any v_1, v_2 unit tangent vector to $\partial\Omega$ at y_1 and y_2, we have either

$$|v_1 - v_2| \le c_b |y_1 - y_2| \qquad \text{or} \qquad |v_1 + v_2| \le c_b |y_1 - y_2|.$$

We now argue by contradiction, by assuming that there is a sequence $\sigma_n \to 0^+$ and, for any n, some pair y_1^n and y_2^n of extremal trajectories on some time intervals $[0, t_1^n + \delta]$ and $[0, t_2^n + \delta]$, such that

$$|y_1^n(t_1^n) - y_2^n(t_2^n)| \le \sigma_n \tag{15}$$

and

$$\left| \frac{(y_1^n)'(t_1^n)}{|(y_1^n)'(t_1^n)|} - \frac{(y_2^n)'(t_2^n)}{|(y_2^n)'(t_2^n)|} \right| > c_b |y_1^n(t_1^n) - y_2(t_2^n)|.$$

Let us observe that by conditions (8), we have

$$y_i^n(t) \in K_0 + B_{MT} \qquad \forall n \in \mathbb{N},\, t \in [0, T] \quad (i = 1,\, 2).$$

From the definition of c_b, the previous inequality implies that

$$\left| \frac{(y_1^n)'(t_1^n)}{|(y_1^n)'(t_1^n)|} + \frac{(y_2^n)'(t_2^n)}{|(y_2^n)'(t_2^n)|} \right| \le c_b |y_1^n(t_1^n) - y_2(t_2^n)|, \tag{16}$$

because $(y_1^n)'(t_1^n)/|(y_1^n)'(t_1^n)|$ and $(y_2^n)'(t_2^n)/|(y_2^n)'(t_2^n)|$ are tangent to $\partial\Omega$ at $y_1^n(t_1^n)$ and $y_2^n(t_2^n)$, respectively.

Thanks to Corollary 2.6 we can extract subsequences of (y_1^n), (y_2^n), (t_1^n), and (t_2^n)—still denoted (y_1^n), (y_2^n), (t_1^n), and (t_2^n)—converging to some y_1, y_2, t_1, and t_2, where y_1 and y_2 are extremal, on $[0, t_1 + \delta]$ and $[0, t_2 + \delta]$. From (15), we have $y_1(t_1) = y_2(t_2)$ and therefore $t_1 = t_2$. Then (16) becomes $y_1'(t_1) = -y_2'(t_2)$ with $y'(t_1) \ne 0$ because of Lemma 2.3. This is in contradiction with Lemma 2.7. $\square$

2.2. Main estimates. We now state the main results of this section. The first one states, thanks to (13), that the length of the boundary of the reachable set—or, more precisely, its Hausdorff measure $\mathcal{H}^1$ (see [11])—remains bounded. The second one explains that the volume of the reachable set depends in a Lipschitz continuous way on the time.

PROPOSITION 2.10. *For $T > 0$, there exists a constant C such that*

$$\mathcal{H}^1\big(\{\tau = t\}\big) \le C \qquad \forall t \in [0, T].$$

The constant C depends on Ω and on the constants r, m, M, L_0, L_1, and T in assumptions (8)–(11).

As a consequence we have the following corollary.

COROLLARY 2.11. *Under the assumptions of Proposition* 2.10,

$$|\mathcal{R}(t_1) \backslash \mathcal{R}(t_0)| \le MC(t_1 - t_0) \qquad \forall 0 \le t_0 \le t_1 \le T,$$

where C is the constant in Proposition 2.10 *and M is given by* (8).

Remark 2.12. In particular, the map

$$t \to \rho(t) \equiv \mathbf{1}_{\mathcal{R}(t)}$$

is locally Lipschitz continuous in $\mathbb{L}^1(\Omega)$.

The proofs of Proposition 2.10 and Corollary 2.11 are the aim of the rest of the section. For proving Proposition 2.10, we divide the set $\{\tau = t\}$ into two sets: the first one consists of points which can be reached with extremal trajectories remaining in the interior of Ω. The second one is made up of points for which the associated extremal trajectories have to touch $\partial\Omega$. For the first set, the techniques of [1] can be adapted, although they have to be localized. For the second set, the key idea amounts to comparing its length with the length of $\partial\Omega$.

Let us introduce more precise notation: for $t \geq 0$ we set

$$E_t := \{x \in \partial\Omega \mid \exists y \in \mathcal{Y}(t),\ \exists s \in [0,t] \text{ with } y(s) = x\}\,, \tag{17}$$

$$D_t^{int} := \{x \in \Omega \mid \exists y \in \mathcal{Y}(t) \text{ with } y(t) = x \text{ and } y([0,t]) \cap \partial\Omega = \emptyset\}\,, \tag{18}$$

$$D_t^{bnd} := \{x \in \Omega \mid \exists y \in \mathcal{Y}(t),\ \exists s \in [0,t) \text{ with } y(t) = x \text{ and } y(s) \in \partial\Omega\}\,. \tag{19}$$

We note that D_t^{int} is exactly the subset of points of $\{\tau = t\}$ which can be reached with extremal trajectories remaining in the interior of Ω, while D_t^{bnd} is the set of points of $\{\tau = t\}$ reached by extremal trajectories passing by $\partial\Omega$. Furthermore,

$$\{\tau = t\} = D_t^{int} \cup D_t^{bnd}.$$

2.3. Estimates for the set reachable by passing by $\partial\Omega$. In this part we estimate the length of D_t^{bnd}. For this we heavily use the fact that we are dealing with a plane system. We do not know if similar constructions can be done in higher dimensions.

Since we cannot directly estimate the length of D_t^{bnd}, we need to introduce an approximation of this set given, for any $\delta > 0$, by

$$D_t^{bnd}(\delta) = \left\{ x \in \Omega \mid \begin{array}{l} \exists y \in \mathcal{Y}(t),\ \exists s \in [0, t-\delta) \text{ with } y(t) = x, \\ y(s) \in \partial\Omega \text{ and } y((s,t]) \cap \partial\Omega = \emptyset \end{array} \right\}.$$

We note for later use that the set D_t^{bnd} is the increasing limit of the $D_t^{bnd}(\delta)$ as δ decreases to 0^+.

LEMMA 2.13. *Let $T > 0$ be fixed. There are $C_1 > 0$ and, for any $\delta > 0$, a constant $\sigma > 0$ such that if y_1, y_2 belong to $D_t^{bnd}(\delta)$, $y_1(\cdot)$, $y_2(\cdot)$ belong to $\mathcal{Y}(t)$, and $s_1, s_2 \in [0, t-\delta)$ with (for $j = 1,2$)*

$$y_j(t) = y_j, \quad y_j(s_j) \in \partial\Omega, \quad y_j((s_j,t]) \cap \partial\Omega = \emptyset, \quad \text{and } |y_1(s_1) - y_2(s_2)| \leq \sigma,$$

then

$$|y_1 - y_2| \leq C_1 |y_1(s_1) - y_2(s_2)|.$$

The constant C_1 depends on the various constants of the problem and on T but not on δ.

Proof. From Corollary 2.9, there is some positive σ such that if y_1 and y_2 are two extremal trajectories on some interval $[0, t_1 + \delta]$ and $[0, t_2 + \delta]$ and if $y_1(t_1) \in \partial\Omega$, $y_2(t_2) \in \partial\Omega$ (for some $t_1, t_2 \leq T$) with $|y_1(t_1) - y_2(t_2)| \leq \sigma$, then

$$\left| \frac{y_1'(t_1)}{|y_1'(t_1)|} - \frac{y_2'(t_2)}{|y_2'(t_2)|} \right| \leq c_b |y_1(t_1) - y_2(t_2)|, \tag{20}$$

where c_b depends only on the $\mathcal{C}^2$ regularity of $\partial\Omega$.

Now let y_j, $y_j(\cdot)$ and s_j ($j = 1, 2$) be as in the statement. Without loss of generality, we assume that $s_1 \le s_2$. Let p_1 and p_2 be the adjoint of y_1 and y_2 defined in Lemma 2.3. On $[s_j, t]$, the pair (y_j, p_j) is the unique solution of the differential equation

$$\begin{cases} y_j'(s) = c(s, y_j(s)) \dfrac{p_j(s)}{|p_j(s)|} & \forall s \in (s_j, t), \\ p_j'(s) = -D_x c(s, y_j(s))\, |p_j(s)| & \text{for a.e. } s \in (s_j, t) \end{cases} \tag{21}$$

with initial condition $(y_j(s_j), p_j(s_j))$. Without loss of generality, we can assume that $|p_j(s_j)| = 1$. From these equations, we deduce that

$$\frac{d}{ds}\left(|p_j(s)|\right) = \frac{\langle p_j'(s), p_j(s)\rangle}{|p_j(s)|} = -\langle D_x c(s, y_j(s)), p_j(s)\rangle,$$

whence, for some constant $L > 0$,

$$-L|p_j(s)| \le \frac{d}{ds}|p_j(s)| \le L|p_j(s)|$$

and, by integration,

$$e^{-L(s-s_j)} \le |p_j(s)| \le e^{L(s-s_j)} \qquad \forall s \in [s_j, t]. \tag{22}$$

In particular, (21) can be written as a Cauchy problem with a Lipschitz right-hand side.

From the Lipschitz continuity of τ given in Lemma 2.1 we have

$$|s_2 - s_1| = \tau(y_2(s_2)) - \tau(y_1(s_1)) \le \frac{k_0}{m}|y_2(s_2) - y_1(s_1)|. \tag{23}$$

We also note that inequality (20) can be rewritten as

$$|p_1(s_1) - p_2(s_2)| \le c_b |y_1(s_1) - y_2(s_2)|. \tag{24}$$

Finally, because of (21) and the bounds on c and $D_x c$ (stated in (8)), there is some constant C_0 such that

$$|y_1(s_2) - y_1(s_1)| + |p_1(s_2) - p_1(s_1)| \le C_0|s_2 - s_1| \le \frac{C_0 k_0}{m}|y_2(s_2) - y_1(s_1)|. \tag{25}$$

Putting together (23), (24), and (25) and using the Gronwall inequality for (21) then easily gives

$$|y_1(s) - y_2(s)| + |p_1(s) - p_2(s)| \le C_1|y_1(s_1) - y_2(s_2)|$$

for any $s \in [s_2, t]$ for some constant $C_1 = C_1(T, m, c_b, k_0, M, L_0, L_1)$. Setting $s = t$ gives the desired inequality. ∎

Lemma 2.14. *Fix $T > 0$. With the same constant C_1 as in Lemma 2.13, we have*

$$\mathcal{H}^1(D_t^{bnd}) \le C_1 \mathcal{H}^1(\partial\Omega) \qquad \forall 0 \le t \le T.$$

Proof. Let C_1 and, for $\delta > 0$ fixed, σ be as in Lemma 2.13. Let $\varepsilon > 0$ and a family of sets $\{B_i\}_{i\in\mathbb{N}}$ be such that

$$\operatorname{diam} B_i \le \inf\{\varepsilon, \sigma\}, \quad \bigcup_{i=1}^{+\infty} B_i \supseteq \partial\Omega, \quad \text{and} \quad \mathcal{H}^1(\partial\Omega) \ge \sum_{i=1}^{+\infty} \operatorname{diam} B_i - \varepsilon.$$

We denote by K_i the set of points of $D_t^{bnd}(\delta)$ which can be reached by an extremal trajectory touching $\partial\Omega$ for the last time in B_i; in other words, $x \in K_i$ if and only if there exists an extremal trajectory y with $y(t) = x$ and $y(s_0) \in B_i$ for $s_0 := \max\{s \in [0, t] : y(s) \in \partial\Omega\}$.

Since we have chosen $\mathrm{diam} B_i \leq \sigma$, Lemma 2.13 states that

$$\mathrm{diam}(K_i) \leq C_1 \mathrm{diam} B_i \qquad \forall i \in \mathbb{N}. \tag{26}$$

Moreover, since

$$\bigcup_{i=1}^{+\infty} K_i = D_t^{bnd}(\delta) \qquad \text{and} \qquad \mathrm{diam}(K_i) \leq C_1 \epsilon,$$

we have

$$\mathcal{H}^1_{C_1\varepsilon}(D_t^{bnd}(\delta)) \leq \sum_{i=1}^{+\infty} \mathrm{diam}(K_i) \leq C_1 \left(\mathcal{H}^1(\partial\Omega) + \varepsilon\right)$$

(for the precise definition of $\mathcal{H}^1_\delta$, see [11, pp. 60ff.]). As $\varepsilon \to 0$, we get

$$\mathcal{H}^1(D_t^{bnd}(\delta)) \leq C_1 \mathcal{H}^1(\partial\Omega).$$

Recalling that C_1 does not depend on δ, we can let $\delta \to 0^+$ to get the result. $\square$

2.4. Estimates for the set reachable by remaining in the interior of Ω. We now aim at computing the length of D_t^{int}, the part of the boundary consisting of the points which can be reached by remaining in the interior of Ω.

Let us recall that the interior sphere condition does *not* propagate in $\Omega \neq \mathbb{R}^2$ (contrary to what happens in $\mathbb{R}^2$, eventually with a radius depending on time; see [1] and [7]). However, a local version of propagation (established in Lemma 2.15) holds also in Ω. This property and a bound on the measure of ∂K_0 (still due to the interior sphere condition) will be crucial in the proof of our estimate for $\mathcal{H}^1(D_{t_1}^{int})$ (see Lemma 2.16 and Remark 2.17).

The estimates below follow the original ideas of Cannarsa and Frankowska [7] and the computations of [1]. In particular, the two following lemmata are just a localized version of Lemmata 3.1 and 6.1 of [1].

LEMMA 2.15. *Under assumption* (9), *let $y(\cdot)$ be some extremal trajectory on the time interval $[0, t]$, with adjoint $p(\cdot)$, for which there is some $\eta > 0$ with*

$$B(y(s), \eta) \subset \Omega \qquad \forall s \in [0, t].$$

Then, if we set $\kappa = 3L_0 + L_1$, we have

$$B\left(y(t) - re^{-\kappa t}\frac{p(t)}{|p(t)|}, re^{-\kappa t}\right) \cap B\left(y(t), \eta e^{-L_0 t}\right) \subset \mathcal{R}(t).$$

Proof. We explain only the main differences with the proofs of Lemmata 3.1 and 6.1 of [1]. Without loss of generality, we assume that $|p(0)| = 1$. We define the vector field $f(s, y) = c(s, y)p(s)/|p(s)|$ and set $\bar\kappa = 2L_0 + L_1$. We consider, for any $\theta \in B(0, |p(t)|)$ such that

$$re^{-\bar\kappa t}|p(t) - \theta| \leq \eta e^{-L_0 t}, \tag{27}$$

the solution y_θ of the backward differential equation

$$\begin{cases} y_\theta'(s) = f(s, y_\theta(s)) & \forall s \in [0,t], \\ y_\theta(t) = y(t) - re^{-\bar{\kappa}t}(p(t)-\theta). \end{cases}$$

Following [1], one can show that $y_\theta(0) \in K_0$. Furthermore, a straightforward application of the Gronwall lemma yields to

$$y_\theta(s) \in B(y(s), \eta e^{-L_0 s}) \subset \Omega \qquad \forall s \in [0,t],$$

because $y_\theta(t) \in B(y(t), \eta e^{-L_0 t})$ thanks to (27). So y_θ is a solution to the control system (6) which remains in Ω on $[0,t]$, and therefore $y_\theta(t) \in \mathcal{R}(t)$. This proves that

$$B\left(y(t) - re^{-\bar{\kappa}t}p(t), re^{-\bar{\kappa}t}|p(t)|\right) \cap B\left(y(t), \eta e^{-L_0 t}\right) \subset \mathcal{R}(t).$$

Since $\kappa = \bar{\kappa} + L_0$ and $|p(s)| \geq e^{-L_0 t}$ (thanks to the differential equation satisfied by p), the proof is complete. □

LEMMA 2.16. *For $0 \leq t_0 \leq t_1 \leq T$, we have*

$$\mathcal{H}^1(D_{t_1}^{int}) \leq e^{C_2(t_1-t_0)}\mathcal{H}^1(D_{t_0}^{int}),$$

where $C_2 = L_0 + Me^{\kappa T}/r$ with $\kappa = 3L_0 + L_1$.

Remark 2.17. In particular, since K_0 satisfies the interior ball condition, $\mathcal{H}^1(\partial K_0)$ is finite (Lemma 2.2 of [1]), and we have (setting $t_0 = 0$ and $t_1 = t$)

$$\mathcal{H}^1(D_t^{int}) \leq e^{C_2 t}\mathcal{H}^1(\partial K_0) \qquad \forall t \in [0,T].$$

Proof. Without loss of generality, we can assume that $\mathcal{H}^1(D_{t_0}^{int}) < +\infty$. For $\eta > 0$ let us set

$$D_t^{int}(\eta) = \left\{ x \in \Omega \mid \begin{array}{c} \tau(x) = t \text{ and } \exists y \in \mathcal{Y}(t) \text{ with } y(t) = x \\ \text{and } d_{\partial\Omega}(y(s)) \geq \eta \text{ for any } s \in [0,t] \end{array} \right\}.$$

We note that $D_t^{int}(\eta)$ is closed and that

$$D_t^{int} \subset \bigcup_{\eta>0} D_t^{int}(\eta). \tag{28}$$

Let us fix $\epsilon > 0$ and $\delta > 0$, and let us choose an (at most) countable collection of compact sets (K_i) such that $K_i \subset D_{t_0}^{int}(\eta)$,

$$D_{t_0}^{int}(\eta) = \bigcup_i K_i, \quad \operatorname{diam}(K_i) \leq \min\left\{\eta e^{-(L_0+C_2)T}, \delta e^{-C_2 T}\right\} \qquad \forall i, \tag{29}$$

and

$$\mathcal{H}^1(D_{t_0}^{int}(\eta)) \geq \sum_i \operatorname{diam}(K_i) - \epsilon.$$

For any $s \in [t_0, t_1]$, let $K_i'(s)$ be the set of points $y \in D_s^{int}(\eta)$ for which there is an extremal trajectory $y(\cdot)$ with $y(s) = y$, $y(t_0) \in K_i$ and $d_{\partial\Omega}(y(t)) \geq \eta$ for any $t \in [0,s]$. We note for later use that

$$D_s^{int}(\eta) \subset \bigcup_i K_i'(s) \qquad \forall s \in [t_0, t_1]. \tag{30}$$

We claim that

$$\text{(31)} \qquad \operatorname{diam}(K_i'(s)) \leq \min\{\eta e^{-L_0 T}, e^{C_2 s}\operatorname{diam}(K_i)\} \qquad \forall s \in [t_0, t_1].$$

For proving the claim, let

$$\theta = \inf\{s \in [t_0, t_1] \ : \ \operatorname{diam}(K_i'(s)) \geq \eta e^{-L_0 T}\}$$

with convention $\inf \emptyset = t_1$. One readily checks from its definition that $s \to \operatorname{diam}(K_i'(s))$ is right upper semicontinuous and left continuous. Therefore $\theta > t_0$ because

$$\operatorname{diam}(K_i'(t_0)) = \operatorname{diam}(K_i) < \eta e^{-L_0 T}.$$

For $y_1, y_2 \in K_i'(\theta)$, let $y_1(\cdot)$ and $y_2(\cdot)$ be two extremal trajectories on $[0, \theta]$ with $y_j(\theta) = y_j$, $d_{\partial\Omega}(y_j(t)) \geq \eta$ for any $t \in [0, \theta]$ and $y_j(t_0) \in K_i$ for $j = 1, 2$. We denote by $p_j(\cdot)$ the adjoint of $y_j(\cdot)$. Since $y_j(s) \in K_i'(s)$ for $s \in [t_0, \theta)$, and from the definition of θ, we have

$$|y_1(s) - y_2(s)| \leq \operatorname{diam}(K_i'(s)) < \eta e^{-L_0 T} \qquad \forall s \in [t_0, \theta).$$

In particular, $y_2(s)$ belongs to $B(y_1(s), \eta e^{-L_0 s})$. Since $d_{\partial\Omega}(y_1(s)) \geq \eta$ for any $s \in [0, t_1]$, and since $y_2(s)$ lies on the boundary of the reachable set, Lemma 2.15 states that $y_2(s)$ does not belong to the interior of the ball centered at $y_1(s) - re^{-\kappa s} p_1(s)/|p_1(s)|$ and of radius $re^{-\kappa s}$. Hence

$$|y_2(s) - y_1(s)|^2 + 2re^{-\kappa s}\langle y_2(s) - y_1(s), p_1(s)/|p_1(s)|\rangle \ \geq \ 0.$$

Reversing the roles of y_1 and y_2 also gives

$$|y_2(s) - y_1(s)|^2 + 2re^{-\kappa s}\langle y_1(s) - y_2(s), p_2(s)/|p_2(s)|\rangle \ \geq \ 0.$$

whence we get

$$\text{(32)} \qquad \left\langle y_1(s) - y_2(s), \frac{p_1(s)}{|p_1(s)|} - \frac{p_2(s)}{|p_2(s)|} \right\rangle \ \leq \ \frac{e^{\kappa T}}{r}|y_1(s) - y_2(s)|^2.$$

Setting $\rho(s) = \frac{1}{2}|y_1(s) - y_2(s)|^2$, we have, for $s \in [t_0, \theta)$,

$$\begin{aligned}
\rho'(s) &= \langle y_1'(s) - y_2'(s), y_1(s) - y_2(s)\rangle \\
&= \left\langle c(s, y_1(s))\tfrac{p_1(s)}{|p_1(s)|} - c(s, y_2(s))\tfrac{p_2(s)}{|p_2(s)|}, y_1(s) - y_2(s)\right\rangle \\
&= \tfrac{1}{2}(c(s, y_1(s)) - c(s, y_2(s)))\left\langle \tfrac{p_2(s)}{|p_2(s)|} + \tfrac{p_1(s)}{|p_1(s)|}, y_1(s) - y_2(s)\right\rangle \\
&\qquad + \tfrac{1}{2}(c(s, y_1(s)) + c(s, y_2(s)))\left\langle \tfrac{p_1(s)}{|p_1(s)|} - \tfrac{p_2(s)}{|p_2(s)|}, y_1(s) - y_2(s)\right\rangle \\
&\leq L_0|y_1(s) - y_2(s)|^2 + \tfrac{Me^{\kappa T}}{r}|y_1(s) - y_2(s)|^2
\end{aligned}$$

(in the last inequality, relations (8) and (32) have been used). Therefore, by the definition of C_2, we have $\rho'(s) \leq 2C_2\rho(s)$ and deduce that

$$|y_1(s) - y_2(s)| \leq |y_1(t_0) - y_2(t_0)|e^{C_2(s - t_0)} \leq \operatorname{diam}(K_i)e^{C_2(s - t_0)} \qquad \forall s \in [t_0, \theta],$$

because $y_1(t_0)$ and $y_2(t_0)$ belong to K_i. So we have proved that, for all $s \in [t_0, \theta]$,

$$\mathrm{diam}(K_i'(s)) \le \mathrm{diam}(K_i) e^{C_2(s-t_0)}.$$

Therefore, because of the definition of θ, we complete the proof of our claim (31) if we show that $\theta = t_1$. To this end, for $s = \theta$, we get

$$\mathrm{diam}(K_i'(\theta)) \le \mathrm{diam}(K_i) e^{C_2(\theta-t_0)} < \eta e^{-L_0 T}$$

from the choice of $\mathrm{diam}(K_i)$ in (29). In particular, since $s \to \mathrm{diam}(K_i'(s))$ is right upper semicontinuous, we necessarily have $\theta = t_1$, and (31) is proved.

We now complete the proof of the lemma: the family $(K_i'(t_1))$ is a covering of $D_{t_1}^{int}(\eta)$, from (30), with

$$\mathrm{diam}(K_i'(t_1)) \le \mathrm{diam}(K_i) e^{C_2(t_1-t_0)} \le \delta$$

from (29). Therefore

$$\mathcal{H}^1_\delta(D_{t_1}^{int}(\eta)) \le \sum_i \mathrm{diam}(K_i'(t_1)) \le e^{C_2(t_1-t_0)}(\mathcal{H}^1(D_{t_0}^{int}(\eta)) + \epsilon).$$

Letting $\delta, \epsilon, \eta \to 0^+$ gives the desired result. ∎

2.5. Proofs of Proposition 2.10 and of Corollary 2.11.

Proof of Proposition 2.10. Taking into account Lemmata 2.14 and 2.16, we obtain

$$\mathcal{H}^1\big(\{\tau = t\}\big) = \mathcal{H}^1\big(D_t^{bnd}\big) + \mathcal{H}^1\big(D_t^{int}\big) \le C_1 \mathcal{H}^1(\partial\Omega) + e^{C_2 T}\mathcal{H}^1(\partial K_0),$$

where $\mathcal{H}^1(\partial K_0)$ is finite because K_0 is compact and satisfies the interior ball condition (see, for instance, [1]). ∎

Proof of Corollary 2.11. Let us recall that τ is Lipschitz continuous and satisfies the Hamilton–Jacobi equation $c(\tau(x), x)|\nabla\tau(x)| = 1$ in the viscosity sense and thus a.e. in $\Omega \setminus K_0$. From assumption (8), we therefore have that

$$\frac{1}{M} \le |\nabla\tau(x)| \le \frac{k_0}{m} \qquad \text{a.e. in } \Omega \setminus K_0. \tag{33}$$

The coarea formula states that

$$\int_{\mathcal{R}(t_1)\setminus\mathcal{R}(t_0)} |\nabla\tau(x)|\, dx = \int_{t_0}^{t_1} \mathcal{H}^1\big(\{\tau = s\}\big)\, ds. \tag{34}$$

By assumption (33) and Proposition 2.10, we deduce, respectively, that

$$\int_{\mathcal{R}(t_1)\setminus\mathcal{R}(t_0)} |\nabla\tau(x)|\, dx \ge \frac{1}{M}|\mathcal{R}(t_1)\setminus\mathcal{R}(t_0)|,$$

$$\int_{t_0}^{t_1} \mathcal{H}^1\big(\{\tau = s\}\big)\, ds \;\le\; C(t_1 - t_0).$$

Substituting the previous two inequalities into (34) completes the proof. ∎

3. Estimate for contraction. Let us consider two velocities c_1 and c_2, both satisfying assumptions (8), and the corresponding reflected control problems (for $j = 1, 2$)

$$\text{(35)} \quad \begin{cases} y'(t) = c_j(t, y(t))\big[b(t) - \langle b(t), \nu(y(t))\rangle \mathbf{1}_{\partial\Omega}(y(t))\nu(y(t))\big] & \text{a.e. in } (0, T), \\ y(0) \in K_j, \end{cases}$$

where $b : \mathbb{R}_+ \to B$ denotes the time measurable control. We assume that the sets K_1 and K_2 fulfill the interior sphere condition of radius $r > 0$ and that Ω satisfies (10)–(11). We denote by $\mathcal{R}_j(t)$ the reachable set of (35) at time t.

The main result of this section is some estimate of the volume of the symmetric difference $\mathcal{R}_1(t)\Delta\mathcal{R}_2(t)$.

PROPOSITION 3.1. *Under the above assumptions on c_j, K_j, and Ω, there exists for any $T > 0$ a constant $N = N(\Omega, r, m, M, L_0, L_1, T)$ such that*

$$|\mathcal{R}_1(t)\Delta\mathcal{R}_2(t)| \le N\gamma(t)$$

with

$$\gamma(t) := d^{\mathcal{H}}(K_1, K_2)e^{(L_0+M/r_1)t} + \int_0^t \|c_1(s, \cdot) - c_2(s, \cdot)\|_\infty e^{(L_0+M/r_1)(t-s)}ds,$$

where r_1 is the constant introduced in (10).

The proof is postponed until the end of the section.

Our starting point is the following lemma.

LEMMA 3.2. *Under the previous hypotheses on c_j, K_j, and Ω, we have*

$$\mathcal{R}_2(t) \subseteq \mathcal{R}_1(t) + \gamma(t)B.$$

In particular, we deduce that

$$\mathcal{R}_2(t) \subseteq \mathcal{R}_1(t + k_0\gamma(t)/m),$$

where k_0 is the constant introduced in inequality (11).

Proof. We set $\gamma_0 := d^{\mathcal{H}}(K_1, K_2)$ and, without loss of generality (thanks to Remark 2.5), consider an extremal trajectory z of (35) for $j = 2$ having the form

$$\begin{cases} z'(t) = c_2(t, z(t))b_2(t), \\ z(0) = z_0 \in K_2. \end{cases}$$

We define y as the reflected trajectory with velocity c_1, driven by the control b_2:

$$\begin{cases} y'(t) = c_1(t, y(t))\big[b_2(t) - \langle b_2(t), \nu(y(t))\rangle \mathbf{1}_{\partial\Omega}(y(t))\nu(y(t))\big], \\ y(0) = y_0, \end{cases}$$

where $y_0 \in K_1$ satisfies $|z_0 - y_0| \le \gamma_0$. For the sake of simplicity, we denote

$$b_1(t) := b_2(t) - \langle b_2(t), \nu(y(t))\rangle \mathbf{1}_{\partial\Omega}(y(t))\nu(y(t)).$$

For $g(s) := |z(s) - y(s)|^2/2$, there holds that

$$\text{(36)} \quad \begin{aligned} g'(s) &= \langle c_2(s, z(s))b_2(s) - c_1(s, y(s))b_1(s), z(s) - y(s)\rangle \\ &= \langle [c_2(s, z(s)) - c_1(s, z(s))]\, b_2(s), z(s) - y(s)\rangle \\ &\qquad + \langle c_1(s, z(s))b_2(s) - c_1(s, y(s))b_1(s), z(s) - y(s)\rangle \\ &\le \|c_1(s, \cdot) - c_2(s, \cdot)\|_\infty |z(s) - y(s)| \\ &\quad + \langle c_1(s, z(s))b_2(s) - c_1(s, y(s))b_1(s), z(s) - y(s)\rangle. \end{aligned}$$

Using assumption (8), we get

$$
\begin{aligned}
&\langle c_1(s,z(s))b_2(s) - c_1(s,y(s))b_1(s), z(s)-y(s)\rangle \\
&= \langle [c_1(s,z(s)) - c_1(s,y(s))]\, b_2(s), z(s)-y(s)\rangle \\
&\qquad - c_1(s,y(s))\langle b_1(s) - b_2(s), z(s)-y(s)\rangle \\
&\le L_0|z(s)-y(s)|^2 \\
&\qquad + c_1(s,y(s))\langle b_2(s), \nu(y(s))\rangle\langle \nu(y(s)), z(s)-y(s)\rangle \mathbf{1}_{\partial\Omega}(y(s)).
\end{aligned}
\tag{37}
$$

Let us recall that $\mathcal{C}\Omega$ fulfills the interior sphere condition with radius r_1. We claim that, for almost every $s \in [0,t]$, there holds that

$$
\langle b_2(s), \nu(y(s))\rangle\langle \nu(y(s)), z(s)-y(s)\rangle \mathbf{1}_{\partial\Omega}(y(s)) \le \frac{1}{r_1}|z(s)-y(s)|^2. \tag{38}
$$

This inequality is obvious if $y \notin \partial\Omega$. Consider $y \in \partial\Omega$, whence we get

$$
|z(s) - y(s) - r_1\nu(y(s))|^2 \ge r_1^2,
$$

from which we deduce that

$$
\langle \nu(y(s)), z(s)-y(s)\rangle \le \frac{1}{2r_1}|z(s)-y(s)|^2.
$$

By the arbitrariness of points $y(s)$ and $z(s)$, we deduce our claim (38).

Substituting relations (37) and (38) into (36), we deduce that

$$
g'(s) \le \|c_1(s,\cdot) - c_2(s,\cdot)\|_\infty |z(s)-y(s)| + (L_0 + M/r_1)|z(s)-y(s)|^2;
$$

in particular, for $\rho(s) := |z(s)-y(s)|$, the previous inequality gives

$$
\rho'(s) \le \|c_1(s,\cdot) - c_2(s,\cdot)\|_\infty + (L_0 + M/r_1)\rho(s).
$$

Applying the Gronwall lemma, we get

$$
\rho(t) \le \rho(0)e^{(L_0+M/r_1)t} + \int_0^t \|c_1(s,\cdot) - c_2(s,\cdot)\|_\infty e^{(L_0+M/r_1)(t-s)}\, ds.
$$

By the arbitrariness of z_0, the first part of the statement is proved.

By assumption (8) and inequality (11), the minimal time for reaching a point $y \in \Omega$ starting from $\bar{y} \in \Omega$ is less than or equal to $k_0|y-\bar{y}|/m$ (this value is the time needed to follow the geodesic between y and $\bar{y}$ at the minimal velocity m). Therefore, the second part of the statement is an immediate consequence of the previous one. □

Proof of Proposition 3.1. We estimate only the measure of $|\mathcal{R}_2(t)\backslash\mathcal{R}_1(t)|$; the other difference can be estimated by similar arguments, so we shall omit it. From Lemma 3.2 and Corollary 2.11 we have

$$
|\mathcal{R}_2(t)\backslash\mathcal{R}_1(t)| \le |\mathcal{R}_1(t + k_0\gamma(t)/m)\backslash\mathcal{R}_1(t)| \le MCk_0\gamma(t)/m.
$$

Setting $N = MCk_0/m$ completes the proof. □

4. Application to dislocation dynamics in a region. The aim of this section is to investigate the existence and uniqueness of the solution to the following nonlocal problem:

$$\left\{\begin{array}{lll} \frac{\partial\rho}{\partial t} = \big(\bar c_1 + I(\rho)\big)|\nabla\rho| & \text{in} & (0,+\infty)\times\Omega, \\ \rho(0,x) = \rho_0(x) \equiv \mathbf{1}_{K_0}(x) & \text{on} & \Omega, \\ \frac{\partial\rho}{\partial\nu} = 0 & \text{on} & (0,+\infty)\times\partial\Omega, \end{array}\right. \tag{39}$$

where ν is the outward unitary normal to $\partial\Omega$ and the map I is defined as follows:

$$I(\rho)(t,x) = \int_\Omega \bar c_2(t,x,y)\rho^*(y)\,dy \qquad \forall\rho\in\mathbb{L}^1(\Omega) \tag{40}$$

for any function $\rho : \Omega \to [0,1]$, where ρ^* denotes the upper semicontinuous envelope of ρ.

Besides assumptions (9)–(11) on the sets Ω and K_0, we require throughout this section that the maps $\bar c_1 : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R}$ and $\bar c_2 : \mathbb{R}^+\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ satisfy the following:

$$\left\{\begin{array}{rl} \text{(i)} & \bar c_1 \text{ and } \bar c_2 \text{ are continuous and derivable w.r.t. the second variable,} \\ \text{(ii)} & |\bar c_1(t,x)| + \|\bar c_2(t,x,\cdot)\|_1 \le M \qquad \forall(t,x)\in\mathbb{R}^+\times\Omega, \\ \text{(iii)} & |\bar c_1(t,x_1) - \bar c_1(t,x_2)| + \|\bar c_2(t,x_1,\cdot) - \bar c_2(t,x_2,\cdot)\|_1 \le L_0|x_1-x_2| \\ & \qquad \forall(t,x_1),(t,x_2)\in\mathbb{R}^+\times\Omega, \\ \text{(iv)} & |D_x\bar c_1(t,x_1) - D_x\bar c_1(t,x_2)| + \|D_x\bar c_2(t,x_1,\cdot) - D_x\bar c_2(t,x_2,\cdot)\|_1 \\ & \qquad\qquad \le L_1|x_1-x_2|, \qquad \forall(t,x_1),(t,x_2)\in\mathbb{R}^+\times\Omega, \\ \text{(v)} & |\bar c_2(t,x,y)| \le M \qquad \forall(t,x,y)\in\mathbb{R}^+\times\Omega\times\Omega \end{array}\right. \tag{41}$$

(where $\|\cdot\|_1$ denotes the $\mathbb{L}^1$ norm in Ω) for some positive constants M, L_0, and L_1, and, for some $m > 0$, the following relation holds:

$$\bar c_1(t,x) - \|\bar c_2(t,x,\cdot)\|_1 \ge m \qquad \forall(t,x)\in[0,+\infty)\times\Omega. \tag{42}$$

This assumption implies that the dislocation is always expanding. We note that under the assumption (41) and (42), the map $c_\rho := \bar c_1 + I(\rho)$ satisfies (8) for any map $\rho \in C^0([0,+\infty),\mathbb{L}^1(\Omega,[0,1]))$. Moreover, the constants entering into relations (8) do not depend on the function ρ.

Let us recall that in the framework of discontinuous *viscosity* solutions (see the monographs by Barles [5] and by Bardi and Capuzzo-Dolcetta [4] for an overview) uniqueness of the solution means that two solutions ρ_1 and ρ_2 have the same upper and lower semicontinuous envelopes: $(\rho_1)^* = (\rho_2)^*$ and $(\rho_1)_* = (\rho_2)_*$. Now we can introduce our definition of viscosity solution.

DEFINITION 4.1. *A function ρ is a viscosity solution to problem* (39) *if $\rho \in C^0([0,+\infty),\mathbb{L}^1(\Omega))$ and if the unique discontinuous solution u to the initial-boundary value problem*

$$\left\{\begin{array}{lll} \frac{\partial u}{\partial t} = c_\rho(t,x)|\nabla u| & \textit{in} & (0,+\infty)\times\Omega, \\ u(0,x) = \rho_0(x) \equiv \mathbf{1}_{K_0}(x) & \textit{on} & \Omega, \\ \frac{\partial u}{\partial\nu} = 0 & \textit{on} & (0,+\infty)\times\partial\Omega \end{array}\right. \tag{43}$$

with $c_\rho(t,x) := \bar c_1(t,x) + I(\rho)(t,x)$ fulfills $u^* = \rho^*$ and $u_* = \rho_*$ in $[0,+\infty)\times\Omega$.

The well-posedness of system (43) is ensured by the following lemma, which also provides a representation formula for this system.

LEMMA 4.2. *Under the above assumptions on Ω, K_0, $\bar{c}_1$, and $\bar{c}_2$, for any $\rho \in C^0([0,+\infty), \mathbb{L}^1(\Omega))$ there exists a unique discontinuous solution u to the initial boundary value problem* (43). *Moreover, for $t > 0$, we have*

$$u(t,x) \in \{0,1\} \qquad \forall x \in \overline{\Omega} \qquad \text{and} \qquad \{x \mid u^*(t,x) = 1\} = \mathcal{R}(t), \tag{44}$$

where $\mathcal{R}(t)$ is the reachable set for the control problem (6) *introduced in section* 2 *with $c(t,x) = c_\rho(t,x)$.*

Proof. In order to show the relation between the level sets of the solution ρ and the reachable set $\mathcal{R}(t)$, let us introduce the following Mayer problem: the value function $\psi(t,x)$ is defined as

$$\psi(t,x) := \max_b \mathbf{1}_{K_0}(y(0)),$$

where y is the solution to the following *backward* reflected control problem:

$$\begin{cases} y'(s) = c(s,y(s))\big[b(s) - \langle b(s), \nu(y(s))\rangle \mathbf{1}_{\partial\Omega}(y(s))\nu(y(s))\big] & \text{a.e. in } [0,t], \\ y(t) = x. \end{cases}$$

Arguing as in [14], one can easily prove that ψ is a solution to (43).

In order to prove the uniqueness of the solution, let us first show that the *fattening* phenomenon (for the precise definition, see [6]) does not occur in $[0,T]$ for every $T \in \mathbb{R}^+$. To this end, we introduce $u_0 \in BUC(\Omega)$, $u_0 \le 1$, such that $\{x \mid u_0(x) \ge 0\} = K_0$ and $\{x \mid u_0(x) > 0\} = \text{Int } K_0$, and we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = c_\rho(t,x)|\nabla u| & \text{in} \quad (0,+\infty)\times\Omega, \\ u(0,x) = u_0(x) & \text{on} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad (0,+\infty)\times\partial\Omega. \end{cases} \tag{45}$$

It is well known (see [6]) that there exists exactly one solution $u \in UC$ of this problem. For every $s \in \mathbb{R}$, we set $K_s := \{u_0 \ge s\}$ and denote $\mathcal{R}_s(t)$ the reachable set in time t when starting from K_s. The function $\mathbf{1}_{\mathcal{R}_s(t)}(x)$ is a (discontinuous) viscosity solution to problem (45) with u_0 replaced by $\mathbf{1}_{K_s}$. The comparison principle (which applies because u is sufficiently regular; see [6]) entails

$$u(t,x) - s - \epsilon \le \mathbf{1}_{\mathcal{R}_s(t)}(x) \qquad \forall s,\epsilon \in \mathbb{R},\ x \in \mathbb{R}^2,\ t \in \mathbb{R}^+$$

and, in particular,

$$\{(t,x) \mid u(t,x) \ge s + \epsilon\} \subseteq \mathcal{R}_s(t) \qquad \forall s,\epsilon \in \mathbb{R},\ t \in \mathbb{R}^+. \tag{46}$$

On the other hand, let v_ϵ be the UC solution to problem (45) with u_0 replaced by $v_{0\epsilon} \in UC$ such that $\{v_{0\epsilon} \ge 0\} = K_{s-\epsilon}$, $\{v_{0\epsilon} > 0\} = \text{Int } K_{s-\epsilon}$ and $v_{0\epsilon} \ge 1$ on K_s. Therefore, again by the comparison principle, we have $\mathbf{1}_{\mathcal{R}_s} \le v_\epsilon$ and also

$$\mathcal{R}_s(t) \subseteq \{v_\epsilon(t,x) \ge 0\} = \{u(t,x) - s + \epsilon \ge 0\} \tag{47}$$

(the equality is due to the *geometricity* of the partial differential equation; see [6]).

By the stability theorem, passing to the limit as $\epsilon \to 0$ in relations (46)–(47), we obtain

$$\overline{\{(t,x) \mid u(t,x) > s\}} \subseteq \cup_t \mathcal{R}_s(t) \subseteq \{(t,x) \mid u(t,x) \ge s\} \qquad \forall s \in \mathbb{R}.$$

Taking into account assumptions (8) and (11), we have

$$\lim_{\eta\to 0^+} \cup_{t\in[0,T]}\mathcal{R}_{s+\eta}(t) = \cup_{t\in[0,T]}\mathcal{R}_s(t).$$

From the last two relations, we deduce that

$$\overline{\{(t,x)\in[0,T]\times\Omega \mid u(t,x)>s\}} = \{(t,x)\in[0,T]\times\Omega \mid u(t,x)\geq s\} \qquad \forall s\in\mathbb{R},$$

whence, by the arbitrariness of T, fattening does not occur. Finally, the uniqueness of the solution follows by an adaptation of the arguments used by Barles, Soner, and Souganidis [6]. □

Let us state the main result of this section.

THEOREM 4.3. *Under the assumptions* (9)–(11) *on the sets* Ω *and* K_0 *and* (41)–(42) *on* $\bar{c}_1$ *and* $\bar{c}_2$*, the initial boundary value problem* (39) *has a unique discontinuous viscosity solution* $\rho\in C^0\left([0,+\infty),\mathbb{L}^1(\Omega)\right)$.

Proof. Let us fix $T_0>0$ and prove the existence and uniqueness of the solution in a strip $[0,T_0]$. For this it is enough to show that there is some positive τ with the following property: for any $T\in[0,T_0]$, if the existence and uniqueness of solutions hold on $[0,T]$, then they hold on $[0,T+\tau]$.

Let us first define τ. For this let C, r_1, and N be the constants given in Propositions 2.10 and 3.1 for the time T_0+1. We fix $\tau\in(0,1]$ and $0<\alpha<1$ with

$$\tau\leq 1/MC, \qquad e^{(L_0+M/r_1)\tau}\leq 1+\alpha\frac{L_0+M/r_1}{NM}.$$

Next, we define the set

$$\mathcal{E}_T := \left\{\rho\in C^0\left([0,T+\tau],\mathbb{L}^1(\Omega)\right) \mid \begin{array}{l}\rho \text{ unique solution to (39) on } [0,T],\\ 0\leq\rho\leq 1,\ \sup\limits_{t\in[T,T+\tau]}\|\rho(t)\|_1\leq\|\rho(T,\cdot)\|_1+1\end{array}\right\}.$$

Let us introduce the mapping $\Phi:\mathcal{E}_T\to C^0\left([0,T+\tau],\mathbb{L}^1(\Omega)\right)$ defined in the following way: for $\rho^0\in\mathcal{E}_T$, $\Phi(\rho^0)$ is the unique viscosity solution to

$$\left\{\begin{array}{lll}\frac{\partial\rho}{\partial t}=\left(\bar{c}_1+I(\rho^0)\right)|\nabla\rho| & \text{in} & (0,T+\tau)\times\Omega,\\ \rho(0,x)=\rho_0(x)\equiv\mathbf{1}_{K_0}(x) & \text{on} & \Omega,\\ \frac{\partial\rho}{\partial\nu}=0 & \text{on} & (0,T+\tau)\times\partial\Omega.\end{array}\right. \tag{48}$$

The well-posedness of Φ is established in Lemma 4.2. We claim that Φ maps $\mathcal{E}_T$ into itself. Combining the representation formula (44) of Lemma 4.2 with Corollary 2.11, we have, for all $t\in[T,T+\tau]$,

$$\|\rho(t,\cdot)\|_1\leq\|\rho(t,\cdot)-\rho(T,\cdot)\|_1+\|\rho(T,\cdot)\|_1\leq\|\rho(T,\cdot)\|_1+1$$

(in the last inequality, relation $\tau\leq 1/MC$ has been used).

Now we claim that Φ is a contraction. Fix $\rho_1^0,\rho_2^0\in\mathcal{E}_T$ and set

$$c_i:=\bar{c}_1+I(\rho_i^0) \qquad \text{and} \qquad \rho_i:=\Phi(\rho_i^0) \qquad (i=1,\,2).$$

We note that since $\rho_1^0=\rho_2^0$ on $[0,T]$, we have $c_1=c_2$ on $[0,T]$. Moreover,

$$\|c_1-c_2\|_\infty=\sup_{t\in[0,T+\tau]}\left\|I(\rho_1^0-\rho_2^0)(t,\cdot)\right\|_\infty\leq M\sup_{t\in[0,T+\tau]}\left\|\rho_1^0(t,\cdot)-\rho_2^0(t,\cdot)\right\|_1.$$

Therefore, using Proposition 3.1 and the fact that $c_1 = c_2$ on $[0,T]$, we have, for $t \in [T, T+\tau]$,

$$\begin{aligned} \|\rho_1(t,\cdot) - \rho_2(t,\cdot)\|_1 &\le N \|c_1 - c_2\|_\infty \int_T^t e^{(L_0+M/r_1)(t-s)} ds \\ &\le NM \frac{e^{(L_0+M/r_1)\tau} - 1}{L_0 + M/r_1} \sup_{t\in[T,T+\tau)} \left\|\rho_1^0(t,\cdot) - \rho_2^0(t,\cdot)\right\|_1 \\ &\le \alpha \sup_{t\in[T,T+\tau)} \left\|\rho_1^0(t,\cdot) - \rho_2^0(t,\cdot)\right\|_1 \end{aligned}$$

(in the last inequality, relation $e^{(L_0+M/r_1)\tau} \le 1 + \alpha \frac{L_0+M/r_1}{NM}$ has been used). In particular, we have proved that Φ is a contraction, and by the Banach fixed point theorem, there exists a unique solution in the strip $[0, T+\tau]$. $\square$

Remark 4.4. This result can be immediately extended to the case of functions $\bar{c}_2$ satisfying only (41)(i)–(iv). Actually, in the proof, one just needs the constant

$$\tilde{M} := \max\{\bar{c}_2(t,x,y) \,|\, t \in [0, T+1],\ x,\ y \in (K_0 + B_{M(T+1)})\},$$

which gives an upper estimate of $\bar{c}_2$ in the reachable set in time $T+1$. In this case, τ needs to be chosen in the following way:

$$\tau \le 1/MC, \qquad e^{(L_0+M/r_1)\tau} \le 1 + \alpha \frac{L_0 + M/r_1}{N\tilde{M}}.$$

5. Appendix: Proof of Lemma 2.3. Without loss of generality, we assume that the function c is defined and fulfills conditions (8) on the whole space $\mathbb{R}^+ \times \mathbb{R}^2$. For $\epsilon > 0$, we introduce the unconstrained controlled problem:

$$x' = c_\epsilon(t,x) b, \qquad |b| \le 1, \qquad \text{where } c_\epsilon(t,x) = \left(c(t,x) - \frac{d_\Omega(x)}{\epsilon}\right)_+,$$

where $(s)_+ = \max(0, s)$. We denote by $\mathcal{R}_\epsilon(t)$ the reachable set at time t starting from K_0 for this control system. Let us point out that it is a subset of $\mathbb{R}^2$ (and not only of $\overline{\Omega}$, as is $\mathcal{R}(t)$). From standard arguments, we have the inclusion $\mathcal{R}(t) \subset \mathcal{R}_\epsilon(t)$ for any $t \ge 0$. We are going to prove that for $\epsilon > 0$ small enough, extremal trajectories for the perturbed problem ending in $\overline{\Omega}$ are actually extremal trajectories for the initial problem.

Let $\epsilon > 0$ small to be chosen later. Also let $x_\epsilon(\cdot)$ be an extremal trajectory of the perturbed problem ending at some $x \in \overline{\Omega}$ at time T. From the definition of c_ϵ, the Pontryagin maximum principle for Lipschitz continuous dynamics (see, for instance, [10]) states that there are $p_\epsilon : [0,T] \to (\mathbb{R}^N \backslash \{0\})$ Lipschitz continuous and $\lambda_\epsilon : [0,T] \to [0,1]$ measurable such that

$$(49) \qquad \begin{cases} x_\epsilon'(t) = c_\epsilon(t, x_\epsilon(t)) \dfrac{p_\epsilon(t)}{|p_\epsilon(t)|} & \forall t \in (0,T), \\ p_\epsilon'(t) = -\left[D_x c(t,x(t)) - \dfrac{\lambda_\epsilon(t)}{\epsilon} Dd(x(t))\right] |p(t)| & \text{for a.e. } t \in (0,T), \\ \lambda_\epsilon(t) = 0 \text{ if } x_\epsilon(t) \in \Omega, & \lambda_\epsilon(t) = 1 \text{ if } x_\epsilon(t) \notin \overline{\Omega}. \end{cases}$$

We claim that $x_\epsilon(t) \in \overline{\Omega}$ for $t \in [0,T]$. Indeed, suppose on the contrary that $x_\epsilon(t) \notin \overline{\Omega}$ for some $t \in [0,T]$. Let (a,b) be the largest interval containing t such that

$x_\epsilon(s) \notin \overline{\Omega}$ for $s \in (a,b)$. We note that $x_\epsilon(a)$ and $x_\epsilon(b)$ belong to $\partial\Omega$ because $x(T) \in \overline{\Omega}$. Moreover, at $s = a$ we have

$$\langle Dd(x_\epsilon(a)), x'_\epsilon(a)\rangle = \left\langle Dd(x_\epsilon(a)), c_\epsilon(a, x_\epsilon(a))\frac{p_\epsilon(a)}{|p_\epsilon(a)|}\right\rangle \geq 0.$$

In particular, $\langle Dd(x_\epsilon(a)), p_\epsilon(a)\rangle \geq 0$. In the same way, since $\langle Dd(x_\epsilon(b)), x'_\epsilon(b)\rangle \leq 0$, we have $\langle Dd(x_\epsilon(b)), p_\epsilon(b)\rangle \leq 0$. Hence

$$\begin{aligned} 0 &\geq \int_a^b \frac{d}{dt}\langle Dd(x_\epsilon(s)), p_\epsilon(s)\rangle\, ds \\ &\geq \int_a^b \left[c_\epsilon(s, x_\epsilon(s)) \left\langle D^2 d(x_\epsilon(s))\frac{p_\epsilon(s)}{|p_\epsilon(s)|}, \frac{p_\epsilon(s)}{|p_\epsilon(s)|}\right\rangle |p_\epsilon(s)| \right. \\ &\qquad \left. - \langle D_x c(s, x_\epsilon(s)), Dd(x_\epsilon(s))\rangle |p_\epsilon(s)| + \tfrac{1}{\epsilon}|p_\epsilon(s)|\right] ds, \end{aligned}$$

where we have used the fact that for $\epsilon > 0$ sufficiently small and $s \in (a,b)$, $x_\epsilon(s)$ is close to $\partial\Omega$, and therefore d is of class $\mathcal{C}^2$ at $x_\epsilon(s)$ with $|Dd(x_\epsilon(s))| = 1$. From our assumptions, c_ϵ, Dc, and D^2d are bounded by some constant M, and therefore

$$0 \geq \int_a^b \left[-M^2 - M + \frac{1}{\epsilon}\right] |p_\epsilon(s)| ds.$$

This leads to a contradiction for $\epsilon < 1/[M + M^2]$ because $p_\epsilon(s) \neq 0$ for $s \in [0,T]$. So we have proved that for $\epsilon > 0$ sufficiently small and $t \in [0,T]$, $x_\epsilon(t) \in \overline{\Omega}$. In particular, x_ϵ is actually an extremal trajectory of the unperturbed problem on $[0,T]$.

We now fix ϵ as above. We now claim that, conversely, any extremal trajectory $x(\cdot)$ of the unperturbed problem on the time interval $[0,T]$ is an extremal trajectory of the perturbed problem. Indeed, let T_ϵ be the minimal time for which there is a solution of the perturbed problem reaching $x(T)$. Obviously, $T_\epsilon \leq T$. Moreover, there is an extremal trajectory $x_\epsilon(\cdot)$ for the perturbed problem reaching $x(T)$ at time T_ϵ. But we have already proved that this trajectory is actually a trajectory of the unperturbed problem. So $T_\epsilon \geq T$, whence we get an equality: $T_\epsilon = T$. This proves that $x(T) \in \partial_{\mathbb{R}^2}\mathcal{R}_\epsilon(T)$, and thus $x(\cdot)$ is actually an extremal trajectory for the perturbed problem.

Now let x be an extremal trajectory of the unperturbed problem on the time interval $[0,T]$. Since it is also extremal for the perturbed problem on $[0,T]$, we can use again the Pontryagin maximum principle: there are $p : [0,T] \to (\mathbb{R}^N\backslash\{0\})$ Lipschitz continuous and $\lambda_\epsilon : [0,T] \to [0,1]$ measurable such that

$$\begin{cases} x'(t) = c_\epsilon(t, x(t))\dfrac{p(t)}{|p(t)|} & \forall t \in (0,T), \\ p'(t) = -\left[D_x c(t, x(t)) - \dfrac{\lambda_\epsilon(t)}{\epsilon} Dd(x(t))\right] |p(t)| & \text{for a.e. } t \in (0,T), \\ \lambda_\epsilon(t) = 0 \text{ if } x(t) \in \Omega, & \lambda_\epsilon(t) = 1 \text{ if } x(t) \notin \overline{\Omega}. \end{cases}$$

Note that $c_\epsilon(t, x(t)) = c(t, x(t))$ because $x(t) \in \overline{\Omega}$ for any t. Let us now identify λ_ϵ. For this we set

$$E = \{t \in [0,T] \mid x(t) \in \partial\Omega\}.$$

Then, for almost every $t \in E$, we have $\langle Dd(x(t)), x'(t)\rangle = 0$, and thus $\langle Dd(x(t)), p(t)\rangle = 0$ thanks to (49). Therefore, for almost every $t \in E$, we also have

$$\left\langle D^2 d(x(t)) c(t,x(t)) \frac{p(t)}{|p(t)|}, p(t) \right\rangle + \left\langle Dd(x(t)), -\left(D_x c(t,x(t)) - \frac{\lambda_\epsilon(t)}{\epsilon} Dd(x(t)) \right) |p(t)| \right\rangle = 0.$$

Since $|Dd| = 1$ and $\lambda_\epsilon \geq 0$, we get

$$\frac{\lambda_\epsilon(t)}{\epsilon} = \Big[-\left\langle D^2 d(x(t)) c(t,x(t)) \frac{p(t)}{|p(t)|}, \frac{p(t)}{|p(t)|} \right\rangle + \langle Dd(x(t)), D_x c(t,x(t))\rangle \Big]_+ \mathbf{1}_{\partial\Omega}(x(t)).$$

Setting $\lambda(t) = \lambda_\epsilon(t)/\epsilon$, we get the desired result.

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