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FINITE FUEL PROBLEM IN NONLINEAR SINGULAR STOCHASTIC CONTROL*

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Abstract. We investigate, via the dynamic programming approach, a finite fuel nonlinear singular stochastic control problem of Bolza type. We prove that the associated value function is continuous and that its continuous extension to the closure of the domain coincides with the value function of a nonsingular control problem, for which we prove the existence of an optimal control. Moreover, such a continuous extension is characterized as the unique viscosity solution of a quasi-variational inequality with suitable boundary conditions of mixed type.

Key words. singular stochastic control problems, degenerate parabolic HJB equations, viscosity solutions, representation formulas

AMS subject classification. 49J20, 93E20, 49L25

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1. Introduction. We study a finite fuel stochastic control problem with finite horizon via the dynamic programming approach. For any initial condition $(\bar{t}, \bar{k}, \bar{x}) \in [0, T[\times[0, K] \times \mathbb{R}^n$ we consider the nonlinear stochastic differential equation

$$x_t = \bar{x} + \int_{\bar{t}}^{t} A(r, x_r)\, dr + \int_{\bar{t}}^{t} B(r, x_r) u_r\, dr + \int_{\bar{t}}^{t} D(r, x_r)\, d\mathcal{W}_r, \tag{1}$$

where the functions A, B, and D are deterministic, $\{\mathcal{W}_t\}$ is a Brownian motion, and $\{u_t\}$ is a control. All the processes are assumed to be defined on a probability space $(\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\})$. Given a closed convex cone $\mathcal{K} \subset \mathbb{R}^m$, the class of admissible controls, denoted by $\mathcal{C}(\bar{t}, \bar{k}, \bar{x})$, is given by the set of $\mathcal{K}$-valued, $\{\mathcal{G}_t\}$-predictable processes verifying the constraint

$$\int_{\bar{t}}^{T} |u_t|\, dt \le K - \bar{k}. \tag{2}$$

For any admissible control u we consider a cost of the form

$$\mathcal{J}(\bar{t}, \bar{k}, \bar{x}, u) = E_Q \left[\int_{\bar{t}}^{T} \left(l_0(r, x_r) + \langle l_1(r, x_r), u_r \rangle \right) dr + g(x_T) \right], \tag{3}$$

where l_0, l_1, and g are deterministic functions. The value function is defined as

$$\mathcal{V}(\bar{t}, \bar{k}, \bar{x}) = \inf_{u \in \mathcal{C}(\bar{t}, \bar{k}, \bar{x})} \mathcal{J}(\bar{t}, \bar{k}, \bar{x}, u). \tag{4}$$

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In this paper we prove the continuity of the value function, and, via a dynamic programming principle, we show that the function V, which is the continuous extension of $\mathcal{V}$ to $[0,T]\times[0,K]\times\mathbb{R}^n$, is a viscosity solution of the following generalized Cauchy problem:

$$\max\left\{-\frac{\partial v}{\partial t}+\mathcal{F}(t,x,Dv,D^2v),-\frac{\partial v}{\partial k}+\mathcal{H}(t,x,Dv)\right\}=0 \quad \text{in }]0,T[\times]0,K[\times\mathbb{R}^n, \tag{5}$$

$$\max\left\{-\frac{\partial v}{\partial t}+\mathcal{F}(t,x,Dv,D^2v),-\frac{\partial v}{\partial k}+\mathcal{H}(t,x,Dv)\right\}\geq 0 \quad \text{on }]0,T[\times\{K\}\times\mathbb{R}^n, \tag{6}$$

$$v\leq g \quad\text{and}\quad \max\left\{-\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+\mathcal{F}(\mathrm{t},\mathrm{x},\mathrm{Dv},\mathrm{D^2v}),-\frac{\partial \mathrm{v}}{\partial \mathrm{k}}+\mathcal{H}(\mathrm{t},\mathrm{x},\mathrm{Dv})\right\}\geq 0 \text{ if } \mathrm{v}<\mathrm{g} \quad \text{on } \{T\}\times]0,K]\times\mathbb{R}^n, \tag{7}$$

where Dv and D^2v denote the gradient and the matrix of the second derivatives of the function $v=v(t,k,x)$ with respect to the x variable,

$$\mathcal{F}(t,x,p,S)\doteq-\langle A(t,x),p\rangle-l_0(t,x)-\frac{1}{2}\mathrm{Tr}\{\tilde{D}(t,x)S\},$$

where $\tilde{D}(t,x)\doteq D(t,x)D(t,x)^T$, and

$$\mathcal{H}(t,x,p)\doteq\max_{w\in\mathcal{K},\ |w|=1}\{-\langle B(t,x)w,p\rangle-\langle l_1(t,x),w\rangle\}$$

for any $(t,x,p,S)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbf{M}(n,n)$, where $\mathbf{M}(n,n)$ denotes the set of $n\times n$ real matrices. A uniqueness theorem proven in [MS2] allows us to characterize V as the unique viscosity solution to the above boundary value problem. V is in fact the value function of a more regular problem for which we can also prove the existence of an optimal control.

This paper presents some new results on the dynamic programming approach of the theory of singular stochastic control problems as well as on its probabilistic aspects.

From the probabilistic point of view, we consider a singular control problem with a dynamic and a cost function in which the terms B and l_1 depend explicitly on the state variable x, an important difference with respect to previous works on singular stochastic controls (see, e.g., [HS1], [BC], [CMR], [SS], and the many references in [FS]). In order to deal with such general dynamics and cost, we introduce an extension of our problem by considering a new set of controls, called auxiliary controls, justified by the observation that optimal controls for the above problem may not exist and in fact quasi-optimal controls may be as close as desired to a control of impulsive type (see, e.g., [FS]), so that discontinuous trajectories should be allowed as solutions. It is known that a measure approach works if the terms B in the dynamics and l_1 in the cost do not depend on the state variable x. Such a special class of *state-independent* problems has been widely investigated in recent years and there are several results on the existence of (generalized) optimal controls, on the regularity, and on the characterization of $\mathcal{V}$ as the unique solution of a Cauchy problem for a suitable nonlinear PDE (see [HS1], [HS2], [SS], [CMR], and the references therein). If, instead, B and l_1 depend on the state variable x, a good definition of the solution to (1) requires a completely different approach in order to guarantee its robustness. In the deterministic context Bressan and Rampazzo [BR] introduced a definition of the generalized solution to (1) based on a time change of the completion of the graphs (t, x_t) (see also [Be], [Se]). Only recently has such a definition been extended to the study of some stochastic control problems by Miller and Runggaldier [MiRu] in 1997, by Dorroh, Ferreyra, and Sundar [DFS] in 1999, and by Miller and Dufour [DM] in 2002. In particular, in [DM] the authors prove the existence of an auxiliary optimal control for a Mayer problem assuming (2). Using their stochastic framework we study the boundary value problem (5)–(7) associated to the minimization problem (1)–(4). In addition, we prove the existence of an optimal control, consistently improving their work since we avoid the convexity assumption, under which their main existence result was proven, but which does not hold in many cases.

This paper is a starting point in the program of extending to stochastic control problems, with nonlinear dynamics and nonlinear costs of the form considered here, several results obtained in singular or coercive linear control problems (that is, with B and l_1 not depending on x) concerning, in particular, the existence of optimal controls and the properties of the associated value functions. Besides the obvious goal of considering nonlinear versions of classical applications such as the finite fuel problem, introduced in [BC] to model an aircraft motion, the study of nonlinear problems is also motivated by some applications to economics for which we refer the interested reader to the recent works [A1], [A2].

From the PDE point of view, we are able to show that the value function is continuous and solves the quasi-variational inequality (5) which can be derived from the dynamic programming principle, either heuristically (as usually done in the literature on singular control), or from an equivalent formulation of the minimum problem that uses compact valued controls (as we do). Here a key tool is the concept of control rules together with the compactification method introduced by El Karoui, Ngoyen, and Jeanblanc-Picqué in [EKNP]. In fact we use an abstract version of the dynamic programming principle (DPP) introduced by Haussmann and Lepeltier [HL] and formulated in terms of control rules, in which, among other things, the terminal time is allowed to be an exit time or even a stopping time chosen by the controller, hard constraints (i.e., state constraints that must be met almost surely) as well as soft constraints (i.e., constraints that must be met in the mean) are considered, and very mild regularity of the data is required.

Thus, the notion of auxiliary controls allows us to reduce the minimization problem to an equivalent one where the controls take values in a compact set. The dynamic programming principle, given in terms of control rules, is the key point for proving that the value function $\mathcal{V}$ defined in (4) is continuous. Both of the concepts are essential in order to write a Hamilton–Jacobi equation like (5) which, a priori, is *not* the formal equation associated to the unbounded control problem, and to show that $\mathcal{V}$ solves (5)–(7) in the viscosity sense. Indeed, the formal equation associated to (1)–(4) involves a different Hamiltonian studied in section 6, which is obtained through a maximization over the unbounded control set $\mathcal{K}$.

A comment about the boundary conditions is in order since conditions (6) and (7) seem original in the setting of singular stochastic control problems. First of all, since we deal with problems of impulsive type, even when considering a finite horizon problem, the limit $\lim_{\bar{t}\to T^-} \mathcal{V}(\bar{t}, \bar{k}, \bar{x})$ does not coincide in general with the final cost $g(\bar{x})$ and, therefore, at time $\bar{t} = T$, we impose (7), which is an alternative between the quasi-variational inequality (5) and $v = g$. Such a generalized boundary condition was introduced in order to characterize continuous value functions of Dirichlet problems in [I], but it also perfectly fits our Cauchy problem. At the boundary $\bar{k} = K$, instead, we introduce the supersolution condition (6) which replaces the Dirichlet condition $v(\bar{t}, K, \bar{x}) = J(\bar{t}, K, \bar{x}, 0)$, usually assumed in finite fuel control problems (see, e.g., [BJM], [FS]). It has the advantage that it does not require the computation of $J(\bar{t}, K, \bar{x}, 0)$. Supersolution type conditions have been considered first by [So] for problems with state constraints, and in fact by considering the fuel consumed at time t as a new variable, in view of (2) such a variable turns out to be constrained in $[0, K]$. Boundary value problems similar to (5)–(7) for first order Hamiltonians were already investigated by the authors in the context of impulsive deterministic control problems when either a constraint on the L^1 norm of the controls or a weak coercivity condition on the Lagrangian is imposed (see, e.g., [MoRa] and [MS1]). In such a context, it is worth mentioning that our approach leads to approximation schemes for the numerical evaluation of the value functions for first order Hamilton–Jacobi equations (see, e.g., [CF]), which for the second order case has not yet been done.

The paper is organized as follows. In section 2 we state the problem precisely and, following [DM], we introduce an auxiliary control problem whose value function V turns out to coincide with $\mathcal{V}$, but with the essential property that the auxiliary controls are *compact-valued*. In section 3 we introduce relaxed controls and control rules and prove, thanks to some technical results contained in the appendix, that there exists an auxiliary optimal control for our problem and that V is in fact the minimum value over relaxed controls. In section 4, using the DPP, we obtain the continuity of $\mathcal{V}$ (see Theorem 4.1). Section 5 is devoted to deducing the boundary value problem (5)–(7) and to showing that $\mathcal{V}$ is a viscosity solution to it. Then we apply a uniqueness theorem proven in [MS2] and prove in Theorem 5.3 that $\mathcal{V}$ is in fact the only solution to (5)–(7) in the class of the bounded functions which are continuous on $\partial(]0, T[\times]0, K[\times\mathbb{R}^n)$. Moreover, in section 6 we show that $\mathcal{V}$ also turns out to represent a solution of a generalized Cauchy problem for a second order semilinear degenerate parabolic PDE involving a noncoercive Hamiltonian defined via maximization over an unbounded set.

Notation. Throughout this paper we shall adopt the following notation. The symbol $|\cdot|$ denotes the norm of vectors and matrices and $\langle\cdot,\cdot\rangle$ denotes the scalar product for vectors. For any positive integer N and any $r > 0$, $B_N(r) = \{v \in \mathbb{R}^N : |v| < r\}$ and $\overline{B}_N(r) = \{v \in \mathbb{R}^N : |v| \le r\}$. $\mathbb{R}_+ = [0, +\infty[$. For arbitrary positive integers N, M, $\mathbf{M}(N, M)$ denotes the set of the $N \times M$ real matrices. $(^T)$ denotes the transposed operator. $\mathcal{C}_b^2(\mathbb{R}^N)$ is the set of the bounded real maps which are continuous on $\mathbb{R}^N$ with their first and second partial derivatives. Given a function $v : E \to \mathbb{R}$, $E \subset \mathbb{R}^N$, the upper and lower semicontinuous envelopes of v are defined by $v^*(x) \doteq \lim_{s\to 0^+} \sup\{v(y) : y \in E, |y - x| \le s\}$, $v_*(x) \doteq \lim_{s\to 0^+} \inf\{v(y) : y \in E, |y - x| \le s\}$ for any $x \in \overline{E}$. Of course, v^* is upper semicontinuous and v_* is lower semicontinuous. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We will use $E_P[\cdot]$ to denote the mathematical expectation on such a space. Given two random variables X, Y, the notation $X = Y$, $X \le Y$ means $P(X = Y) = 1$, $P(X \le Y) = 1$, respectively, $\delta_{\{w\}}$ denotes the Dirac measure at a fixed $w \in \mathcal{K}$, and T and K are fixed positive real numbers.

2. Statement of the problem. In this section we give the precise formulation of the nonlinear singular stochastic control problem described in the introduction, introduce the auxiliary control problem, and prove their equivalence.

2.1. The control problem. Throughout the paper we will use the following hypotheses.

(A0): There are some constants L_1, L_2 such that the deterministic functions $A : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbf{M}(n, m)$, and $D : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbf{M}(n, p)$ verify for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$,

$$\begin{aligned}&|A(t,x)| + |B(t,x)| + |D(t,x)| \le L_1(1 + |x|),\\ &|A(t,x) - A(s,y)| + |B(t,x) - B(s,y)| + |D(t,x) - D(s,y)| \le L_2(|t-s| + |x-y|).\end{aligned}$$

(A1): There are some constants L, L_3 such that the functions $l_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $l_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m$, and $g : \mathbb{R}^n \to \mathbb{R}$ verify for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$,

$$\begin{aligned}&|l_0(t,x) - l_0(s,y)| + |l_1(t,x) - l_1(s,y)| \le L(|t-s| + |x-y|),\\ &|g(x) - g(y)| \le L|x-y|,\end{aligned}$$

and

$$|l_0(t,x)| + |l_1(t,x)| + |g(x)| \le L_3. \tag{8}$$

Definition 2.1. *Given an initial condition $(\bar t, \bar k, \bar x) \in [0, T[\times[0, K] \times \mathbb{R}^n$, a* control *is a term*

$$c = (\Omega, \mathcal{G}, Q, \{\mathcal{G}_t\}, \{u_t\}, \{\mathcal{W}_t\}, \{x_t\}),$$

where

- $(\Omega, \mathcal{G}, Q)$ *is a complete probability space with a right continuous complete filtration* $\{\mathcal{G}_t\}$,
- $\{u_t\}$ *is a $\mathcal{K}$-valued process ($\mathcal{K}$ a closed, convex cone of $\mathbb{R}^m$) defined on $[\bar t, T] \times \Omega$, which is $\{\mathcal{G}_t\}$-predictable,*
- $\{\mathcal{W}_t\}$ *is a standard p-dimensional $\{\mathcal{G}_t\}$-Brownian motion,*
- $\{x_t\}$ *is an $\mathbb{R}^n$-valued process which is $\{\mathcal{G}_t\}$-progressively measurable, with continuous paths, such that*

$$x_t = \bar x + \int_{\bar t}^t A(r, x_r)\,dr + \int_{\bar t}^t B(r, x_r)u_r\,dr + \int_{\bar t}^t D(r, x_r)\,d\mathcal{W}_r \qquad \forall t \in [\bar t, T].$$

A control c is admissible *if*

$$\int_{\bar t}^T |u_t|\,dt \le K - \bar k. \tag{9}$$

The set of admissible controls *will be denoted by $\mathcal{C}(\bar t, \bar k, \bar x)$.*

For any admissible control c we consider a cost of the form

$$\mathcal{J}(\bar t, \bar k, \bar x, c) \doteq E_Q\left[\int_{\bar t}^T \left(l_0(r, x_r) + \langle l_1(r, x_r), u_r\rangle\right) dr + g(x_T)\right]. \tag{10}$$

The value function is defined for $(\bar t, \bar k, \bar x) \in [0, T[\times[0, K] \times \mathbb{R}^n$ by

$$\mathcal{V}(\bar t, \bar k, \bar x) \doteq \inf_{c \in \mathcal{C}(\bar t, \bar k, \bar x)} \mathcal{J}(\bar t, \bar k, \bar x, c). \tag{11}$$

Remark 2.1. If we replace the boundedness hypothesis (8) with

$$|l_0(t,x)| + |l_1(t,x)| + |g(x)| \le L_3(1 + |x|) \qquad \forall t \in \mathbb{R}_+,\ x \in \mathbb{R}^n, \tag{12}$$

the main results of the paper remain true, except that, of course, the value function $\mathcal{V}$ is bounded no more but turns out to verify $|\mathcal{V}(t, k, x)| \le \bar C(1 + |x|)$ for some $\bar C$ and for all $(t, k, x) \in [0, T[\times[0, K] \times \mathbb{R}^n$, as one can deduce from the proof of Theorem 4.1 (see also Corollary 4.2).

2.2. The auxiliary control problem. In this section, following [DM], we introduce an auxiliary control problem, equivalent to the original one, but with the key property that the controls take values in a compact set.

DEFINITION 2.2. *For any $(\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n$ an* auxiliary control *is a term*

$$\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{(t_s, k_s, \xi_s)\}, \theta),$$

where the following **(B1)** *and* **(B2)** *are assumed.*

(B1) • $(\Omega, \mathcal{F}, P)$ *is a complete probability space, with a right continuous complete filtration* $\{\mathcal{F}_s\}$,
- $\{w_s\}$ *is a* $\overline{B}_m(1) \cap \mathcal{K}$*-valued control defined on* $[0, T+K] \times \Omega$ *which is* $\{\mathcal{F}_s\}$*-predictable,*
- θ *is an* $\{\mathcal{F}_s\}$*-stopping time such that* $\theta \le T + K$,

and

(B2) $\{(t_s, k_s, \xi_s)\}$ *is an* $\mathbb{R}^{2+n}$*-valued* $\{\mathcal{F}_s\}$*-progressively measurable process with continuous paths, such that, for* $0 \le s \le T+K$,

$$\begin{cases} t_s = \bar t + \int_0^s w_{0_\sigma}\, d\sigma, \\ k_s = \bar k + \int_0^s |w_\sigma|\, d\sigma, \\ \xi_s = \bar x + \int_0^s (A(t_\sigma, \xi_\sigma) w_{0_\sigma} + B(t_\sigma, \xi_\sigma) w_\sigma)\, d\sigma + \int_0^s D(t_\sigma, \xi_\sigma)\sqrt{w_{0_\sigma}}\, dW_\sigma, \end{cases}$$

where $\{W_s\}$ *is a standard p-dimensional* $\{\mathcal{F}_s\}$*-Brownian motion defined on* $[0, T+K] \times \Omega$ *and where we set* $w_{0_s}(\omega) \doteq 1 - |w_s(\omega)| \forall (s, \omega)$ *just for the sake of notation.*

The cost corresponding to an auxiliary control β *is of the form*

$$J(\bar t, \bar k, \bar x, \beta) \doteq E_P \left[\int_0^\theta (l_0(t_\sigma, \xi_\sigma) w_{0_\sigma} + \langle l_1(t_\sigma, \xi_\sigma), w_\sigma \rangle)\, d\sigma + g(\xi_\theta) + G(t_\theta, k_\theta) \right],$$

where $G(T,k) = 0$ *for all* $k \le K$ *and* $G(t,k) = +\infty$ *otherwise. We use* $\Gamma(\bar t, \bar k, \bar x)$ *to denote the set of auxiliary controls, while*

$$\Gamma^a(\bar t, \bar k, \bar x) \doteq \{\beta \in \Gamma(\bar t, \bar k, \bar x) : \quad J(\bar t, \bar k, \bar x, \beta) < +\infty\} \tag{13}$$

denotes the subset of admissible *auxiliary controls. We define for every* $(\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n$ *the* auxiliary *value function as*

$$V(\bar t, \bar k, \bar x) \doteq \inf_{\beta \in \Gamma^a(\bar t, \bar k, \bar x)} J(\bar t, \bar k, \bar x, \beta). \tag{14}$$

Remark 2.2. The definition of auxiliary controls given in [DM] is slightly different from Definition 2.2. More precisely, fixing an initial condition $(\bar t, \bar k, \bar x)$, the natural extension of [DM], to our setting yields controls $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$ with stopping times θ verifying the constraint

$$\theta \le (T - \bar t) + (K - \bar k), \tag{15}$$

with the cost functional defined by

$$\hat J(\bar t, \bar k, \bar x, \beta) = E_P \left[\int_0^\theta (l_0(t_\sigma, \xi_\sigma) w_{0_\sigma} + \langle l_1(t_\sigma, \xi_\sigma), w_\sigma \rangle)\, d\sigma + g(\xi_\theta) + \hat G(t_\theta) \right], \tag{16}$$

where $\hat{G}(T) = 0$ and $\hat{G}(t) = +\infty$ for all $t \neq T$. Moreover, a control β is admissible if $\hat{J}(\bar{t}, \bar{k}, \bar{x}, \beta) < +\infty$. In fact, we will show, in the proof of Theorem 2.3 below, that the two sets of admissible auxiliary controls coincide and therefore that the two definitions are equivalent. The reason we choose a different formulation of our problem is that Definition 2.2 is better suited to state a dynamic programming principle. It is well known, indeed, that if (t_s, k_s, ξ_s) is a process starting from $(\bar{t}, \bar{k}, \bar{x})$ at $s = 0$, in order to apply the dynamic programming technique, one needs to consider any state (t_s, k_s, ξ_s) for $s > 0$ as the initial condition, that is, to restate the problem with a random variable in place of a deterministic point as initial datum. Following [DM], therefore, one should deal with the hard constraint $\theta \leq (T - t_s) + (K - k_s)$ coming from condition (15).

The problems in Definitions 2.1 and 2.2 are equivalent in the following sense.

THEOREM 2.3. *Assume* **(A0)**, **(A1)**. *Then for any initial condition* $(\bar{t}, \bar{k}, \bar{x}) \in [0, T[\times[0, K] \times \mathbb{R}^n$ *one has*

(i) $\mathcal{C}(\bar{t}, \bar{k}, \bar{x}) \hookrightarrow \Gamma^a(\bar{t}, \bar{k}, \bar{x})$, *that is, for every control* $c \in \mathcal{C}(\bar{t}, \bar{k}, \bar{x})$ *there exists an admissible auxiliary control* $\beta \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})$ *such that* $J(\bar{t}, \bar{k}, \bar{x}, \beta) = \mathcal{J}(\bar{t}, \bar{k}, \bar{x}, c)$;

(ii) *for any admissible auxiliary control* $\beta \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})$ *there is a sequence of controls* $c^n \in \mathcal{C}(\bar{t}, \bar{k}, \bar{x})$ *such that* $\lim_n \mathcal{J}(\bar{t}, \bar{k}, \bar{x}, c^n) = J(\bar{t}, \bar{k}, \bar{x}, \beta)$;

(iii)

$$V(\bar{t}, \bar{k}, \bar{x}) = \mathcal{V}(\bar{t}, \bar{k}, \bar{x}). \tag{17}$$

Proof. Since the proof is based on that given in [DM], we begin by proving that the set $\Gamma^a(\bar{t}, \bar{k}, \bar{x})$ coincides with the set of admissible auxiliary controls introduced in [DM], that is, controls with stopping time θ verifying (15) and with a cost defined by (16), which must be bounded. To prove this claim, let us fix an admissible auxiliary control β in the sense considered in [DM]. First of all, let us notice that condition (15) in fact plays the role of the integral constraint (9) of Definition 2.1, in that it implies

$$\int_0^\theta |w_\sigma| \, d\sigma \leq K - \bar{k}.$$

Indeed, from $\hat{J}(\bar{t}, \bar{k}, \bar{x}, \beta) < +\infty$ it follows that $E_P[\hat{G}(t_\theta)] = 0$, that is, $t_\theta = T$. Moreover, since the stopping time θ verifies (15), one has

$$k_\theta = \bar{k} + \int_0^\theta |w_\sigma| \, d\sigma = \bar{k} + \theta - (t_\theta - \bar{t}) \leq \bar{k} + (K - \bar{k}) + (T - \bar{t}) - (T - \bar{t}) = K.$$

Thus $E_P[G(t_\theta, k_\theta)] = 0$, $\hat{J}(\bar{t}, \bar{k}, \bar{x}, \beta) = J(\bar{t}, \bar{k}, \bar{x}, \beta)$, and β turns out to belong to the set $\Gamma^a(\bar{t}, \bar{k}, \bar{x})$ defined in (13). On the contrary, given a control $\beta \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})$, from $J(\bar{t}, \bar{k}, \bar{x}, \beta) < +\infty$ it follows that $E_P[G(t_\theta, k_\theta)] = 0$, that is, $t_\theta = T$ and $k_\theta \leq K$. Hence $E_P[\hat{G}(t_\theta)] = 0$, $\hat{J}(\bar{t}, \bar{k}, \bar{x}, \beta) = J(\bar{t}, \bar{k}, \bar{x}, \beta)$, and in order to show that β is an admissible auxiliary control in the sense considered in [DM] it remains to prove that θ verifies condition (15). Since by definition

$$k_s + t_s = \bar{k} + \bar{t} + s \qquad s \geq 0,$$

one has that

$$\theta = (k_\theta - \bar{k}) + (t_\theta - \bar{t}) \leq (K - \bar{k}) + (T - \bar{t}),$$

which concludes the proof of the claim.

If the Lagrangian function $l = l_0 + \langle l_1, u\rangle$ is identically zero, statements (i) and (ii) have been proved by Dufour and Miller in Proposition 4.12, and in Proposition 4.8 and Theorem 4.15 of [DM], respectively. This yields the equality $V = \mathcal{V}$ in (iii) for a problem of Mayer type. The extension of these results to a Bolza problem is standard; therefore the proof is concluded. $\square$

In the deterministic case, following the method of the graphs completion, the equivalence between an original singular control problem and a corresponding auxiliary control problem has been proven for several types of problems (see [MoRa], [MS1], and the references therein).

The following simple example taken from [MoRa] shows that at the points of the form $(T, \bar{k}, \bar{x})$, the value function V associated to the auxiliary control problem does not coincide in general with the terminal cost g.

Example 2.1 (see [MoRa]). Let us consider the deterministic control problem

$$x(t) = \bar{x} + \int_{\bar{t}}^{t} (c + u_1(r) + x(r)u_2(r))\, dr \quad \forall t \in [\bar{t}, T],$$

where $\bar{x} \in \mathbb{R}$, c is a positive constant, the control (u_1, u_2) defined on $[\bar{t}, T]$ assumes values on the closed cone

$$\mathcal{K} \doteq \{(w_1, w_2) \in \mathbb{R}^2 : \; w_1 \leq 0,\, w_2 \geq 0\},$$

and it verifies the constraint

$$\int_{\bar{t}}^{T} |(u_1(r), u_2(r))|\, dr \leq K - \bar{k},$$

where $0 \leq \bar{k} \leq K$. Let us minimize the following payoff in Mayer form:

$$\mathcal{J}(\bar{t}, \bar{k}, \bar{x}, u) = \arctan(x(T)).$$

For any $(\bar{t}, \bar{k}, \bar{x}) \in [0, T[\times [0, K] \times \mathbb{R}$, the maximum principle yields the existence of an optimal auxiliary control whose corresponding trajectory has a terminal position given by

$$\xi(T) = \begin{cases} \sinh(\operatorname{arcsinh}(\bar{x}) - (K - \bar{k})) + c(T - \bar{t}), & \bar{x} \leq 0, \\ \sinh(\bar{x} - (K - \bar{k})) + c(T - \bar{t}), & 0 < \bar{x} < K - \bar{k}, \\ \bar{x} - (K - \bar{k}) + c(T - \bar{t}), & \bar{x} \geq K - \bar{k}, \end{cases} \tag{18}$$

so that $\mathcal{V}(\bar{t}, \bar{k}, \bar{x}) = V(\bar{t}, \bar{k}, \bar{x}) = \arctan(\xi(T))$. At the points $(T, \bar{k}, \bar{x}) \in \{T\} \times [0, K] \times \mathbb{R}^n$, V is given again by $\arctan(\xi(T))$ once we put $\bar{t} = T$ and obviously it does not coincide with $g(\bar{x}) = \arctan(\bar{x})$ unless $\bar{k} = K$.

This is a general result: V coincides with the continuous extension to $[0, T] \times [0, K] \times \mathbb{R}^n$ of the original value function $\mathcal{V}$ defined on the set $[0, T[\times [0, K] \times \mathbb{R}^n$ (see Corollary 4.2 in section 4) and in general it does not coincide with g at $\bar{t} = T$. For deterministic control problems, there are well known sufficient conditions under which $V(T, \bar{k}, \bar{x}) = g(\bar{x}) \;\; \forall (T, \bar{k}, \bar{x}) \in \{T\} \times [0, K] \times \mathbb{R}^n$ (see [RS] and also Notes on [CIL, section 7]).

3. Relaxed controls and control rules. We devote this section to the definition of relaxed controls which are needed in order to introduce the concept of control rules and the compactification method, key tools to prove a dynamic programming principle. We follow here the presentation given by Haussman and Lepeltier in [HL], where an earlier work by El Karoui, Ngoyen, and Jeanblanc-Picqué [EKNP] is generalized to the case of unbounded data and controls and no fixed terminal time.

3.1. The martingale model. We introduce the equivalent formulation of the above auxiliary control problem as a martingale problem, where the ambiguous term represented by the Brownian motion, unknown in advance, is removed (see, e.g., Ikeda and Watanabe in[IW]). To this end, we introduce for all $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $(t,k,x) \in \mathbf{R}^{2+n}$, and $w \in \overline{B}_m(1) \cap \mathcal{K}$ the operator $\mathcal{L}$ defined by

$$\begin{aligned}&\mathcal{L}\varphi(t,k,x,w) \doteq \\ &\left[\tfrac{1}{2}\textstyle\sum_{ij} \tilde{D}_{ij}(t,x)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(t,k,x) + \sum_i A_i(t,x)\frac{\partial\varphi}{\partial x_i}(t,k,x) + \frac{\partial\varphi}{\partial t}(t,k,x)\right] w_0 \\ &\qquad + \textstyle\sum_i \langle B_i(t,x), w\rangle \frac{\partial\varphi}{\partial x_i}(t,k,x) + \frac{\partial\varphi}{\partial k}(t,k,x)|w|,\end{aligned} \tag{19}$$

where $w_0 \doteq 1 - |w|$, $\tilde{D}_{ij}$ are the entries of $\tilde{D} = DD^T$, A_i are the components of A, and B_i are the rows of B. Notice that in this formulation the diffusion coefficient D disappears and is replaced by $\tilde{D}$, which, differently from D, is something intrinsic to a process ξ_s as defined in **(B2)**.

The following proposition establishes the correspondence between the martingale model and the control problem with the Brownian motion.

PROPOSITION 3.1 (see [HL, Proposition 3.1]). *Let us assume* **(A0)**, **(A1)**. *Let us fix* $(\bar{t}, \bar{k}, \bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$. *A control* $\beta = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$ *such that*

(B3) • $(\Omega, \mathcal{F}, P)$ *is a probability space, with a filtration* $\{\mathcal{F}_s\}$,
- $\{w_s\}$ *is a* $\overline{B}_m(1) \cap \mathcal{K}$*-valued control, defined on* $[0, T+K] \times \Omega$, $\{\mathcal{F}_s\}$*-progressively measurable,*
- θ *is an* $\{\mathcal{F}_s\}$*-stopping time such that* $\theta \le T + K$

verifies **(B2)** *if and only if it verifies*

(B4) • $\{(t_s, k_s, \xi_s)\}$ *is a* $\mathbb{R}^{2+n}$*-valued,* $\{\mathcal{F}_s\}$*-progressively measurable process for* $s \in [0, T+K]$, *with continuous paths, such that* $(t_s, k_s, \xi_s) = (\bar{t}, \bar{k}, \bar{x})$ *for* $s = 0$, *for any* $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $\mathcal{M}_s(\varphi, \beta)$ *is a* $(P, \{\mathcal{F}_s\})$ *square integrable martingale for* $s \in [0, T+K]$, *where*

$$\mathcal{M}_s(\varphi, \beta) \doteq \varphi(t_s, k_s, \xi_s) - \int_0^s \mathcal{L}\varphi(t_\sigma, k_\sigma, \xi_\sigma, w_\sigma)\, d\sigma.$$

3.2. Relaxed controls. In a relaxed control, the $\overline{B}_m(1) \cap \mathcal{K}$-valued process $\{w_s\}$ is replaced by an $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$-valued process $\{\mu_s\}$, where $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$ is the space of probability measures on $\overline{B}_m(1) \cap \mathcal{K}$. We will extend any bounded measurable map $\psi : \overline{B}_m(1) \cap \mathcal{K} \to \mathbb{R}$ to $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$ by setting

$$\psi(\mu) = \int_{\overline{B}_m(1) \cap \mathcal{K}} \psi(w)\mu(dw).$$

DEFINITION 3.2. *Given* $(\bar{t}, \bar{k}, \bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$ *we say that* $\tilde{\alpha}$ *is a* relaxed control *and write* $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{k}, \bar{x})$ *if*

$$\tilde{\alpha} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_s)\}, \theta),$$

where the following **(B3′)**, **(B4′)** *are assumed.*

(B3′) • $(\Omega, \mathcal{F}, P)$ *is a probability space with a filtration* $\{\mathcal{F}_s\}$,
- $\{\mu_s\}$ *is a* $\mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K})$*-valued process defined on* $[0, T+K] \times \Omega$ *which is* $\{\mathcal{F}_s\}$*-progressively measurable,*
- θ *is an* $\{\mathcal{F}_s\}$*-stopping time such that* $\theta \le T + K$,

(B4′) • *$\{(t_s, k_s, \xi_s)\}$ is an $\mathbb{R}^{2+n}$-valued $\{\mathcal{F}_s\}$-progressively measurable process for $s \in [0, T+K]$, with continuous paths, such that $(t_s, k_s, \xi_s) = (\bar{t}, \bar{k}, \bar{x})$ for $s = 0$, for any $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $\mathcal{M}_s(\varphi, \tilde{\alpha})$ is a $(P, \{\mathcal{F}_s\})$ square integrable martingale for $s \in [0, T+K]$, where*

$$M_s(\varphi, \tilde{\alpha}) \doteq \varphi(t_s, k_s, \xi_s) - \int_0^s \mathcal{L}\varphi(t_\sigma, k_\sigma, \xi_\sigma, \mu_\sigma)\, d\sigma.$$

For any $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{k}, \bar{x})$ we define the cost

$$J(\bar{t}, \bar{k}, \bar{x}, \tilde{\alpha}) = E_P \left[\int_0^\theta \left(l_0(t_\sigma, \xi_\sigma)(1 - |\mu_\sigma|) + \langle l_1(t_\sigma, \xi_\sigma), \mu_\sigma \rangle \right) d\sigma + g(\xi_\theta) + G(t_\theta, k_\theta) \right]. \tag{20}$$

We use $\tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x})$ to denote the subset of admissible *relaxed controls, that is,*

$$\tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x}) \doteq \left\{ \tilde{\alpha} \in \tilde{\Gamma}(\bar{t}, \bar{k}, \bar{x}) : \quad J(\bar{t}, \bar{k}, \bar{\xi}, \tilde{\alpha}) < +\infty \right\}.$$

Remark 3.1. Following [HL], the processes that appear in Definition 3.2 are progressively measurable and the probability space is arbitrary. The processes that appear in the auxiliary controls of Definition 2.2, instead, are predictable processes and the probability space is complete and right continuous. Thus, it is not obvious a priori that the control problem in Definition 3.2 is the relaxed version of our auxiliary control problem. From Lemmatas A1–A3 in [DM], however, it follows that, given an initial condition $(\bar{t}, \bar{k}, \bar{x})$, for any control $\alpha = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{w_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$ verifying **(B3)** and **(B2)** (or, equivalently, **(B3)** and **(B4)**, in view of Proposition 3.1), there exists a new control $\hat{\alpha} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_s\}, \{\hat{w}_s\}, \{(\hat{t}_s, \hat{k}_s, \hat{\xi}_s)\}, \hat{\theta})$, where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ is a suitable modification of $(\Omega, \mathcal{F}, P)$, $\hat{\theta} = \theta$, the process $\{(\hat{t}_s, \hat{k}_s, \hat{\xi}_s)\}$ is indistinguishable from $\{(t_s, k_s, \xi_s)\}$, $\hat{\alpha}$ verifies **(B1)** and **(B2)**, and, moreover, $J(\bar{t}, \bar{k}, \bar{x}, \hat{\alpha}) = J(\bar{t}, \bar{k}, \bar{x}, \alpha)$. Therefore, if $J(\bar{t}, \bar{k}, \bar{x}, \alpha) < +\infty$, then $\hat{\alpha} \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})$.

The set $\Gamma^a(\bar{t}, \bar{k}, \bar{x})$ can be naturally embedded in $\tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x})$; therefore, the inequality

$$\inf_{\tilde{\alpha} \in \tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x})} J(\bar{t}, \bar{k}, \bar{x}, \tilde{\alpha}) \leq \inf_{\alpha \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})} J(\bar{t}, \bar{k}, \bar{x}, \alpha)$$

is trivially verified. In fact, the converse inequality also holds true.

THEOREM 3.3. *Assume* **(A0)**, **(A1)**. *Then for any $(\bar{t}, \bar{k}, \bar{x}) \in [0, T] \times [0, K] \times \mathbb{R}^n$,*

$$V(\bar{t}, \bar{k}, \bar{x}) = \inf_{\alpha \in \Gamma^a(\bar{t}, \bar{k}, \bar{x})} J(\bar{t}, \bar{k}, \bar{x}, \alpha) = \inf_{\tilde{\alpha} \in \tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x})} J(\bar{t}, \bar{k}, \bar{x}, \tilde{\alpha}).$$

Moreover, the infimum over relaxed controls is attained and so is the infimum over auxiliary controls.

Remark 3.2. Dufour and Miller proved in [DM] the existence of an optimal control for the auxiliary problem in the case $l = l_0 + \langle l_1, u \rangle \equiv 0$ and while under the assumption that the set

$$\begin{aligned} \tilde{M}(t, x) \doteq \big\{ &(A(t, x)(1 - |w|) + B(t, x)w, (1 - |w|)D(t, x)D^T(t, x), |w|) : \\ &w \in \overline{B}_m(1) \cap \mathcal{K} \big\} \qquad \text{is convex} \quad \forall (t, x). \end{aligned} \tag{21}$$

It is important to observe that the presence of the terms depending on $|w|$ in (21) implies that such a condition does not hold in most cases. Let us point out that in

Theorem 3.3 the method of the graphs completion yields instead the existence of an optimal control for the auxiliary control problem just under assumptions **(A0)**, **(A1)**, without assumption (21).

Proof of Theorem 3.3. From [HL, Theorem 3.6] it follows straightforwardly that for any initial condition $(\bar{t}, \bar{k}, \bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$ there exists an optimal relaxed control

$$\tilde{\alpha} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_s)\}, \theta) \in \tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x}).$$

Let us set $\mu_{0_s}(\omega) = 1 - |\mu_s(\omega)|$ (pointwise) and let us define γ by

$$\begin{aligned}\gamma(s,\omega) &= \Big(A\mu_{0_s}(\omega) + B\mu_s(\omega), \mu_{0_s}(\omega)\tilde{D}, l_0\mu_{0_s}(\omega) + \langle l_1, \mu_s(\omega)\rangle, \mu_{0_s}(\omega)\Big)\,(t_s(\omega), \xi_s(\omega)) \\ &= \int_{\overline{B}_m(1)\cap\mathcal{K}} \Big(Aw_0 + Bw, w_0\tilde{D}, l_0 w_0 + \langle l_1, w\rangle, w_0\Big)\,(t_s(\omega), \xi_s(\omega))\, \mu_s(\omega, dw),\end{aligned}$$

where $w_0 = 1 - |w|$ for $w \in \overline{B}_m(1) \cap \mathcal{K}$. Since for every (t,x),

$$\begin{aligned}&\Big\{(A(t,x)w_0 + B(t,x)w, w_0\tilde{D}(t,x), z, w_0) : \\ &z \geq l_0(t,x)w_0 + \langle l_1(t,x), w\rangle, \quad (w_0, w) \in \mathbb{R}_+ \times \mathcal{K},\ w_0 + |w| = 1\Big\}, \\ &\subset \Big\{(A(t,x)w_0 + B(t,x)w, w_0\tilde{D}(t,x), z, w_0) : \\ &z \geq l_0(t,x)w_0 + \langle l_1(t,x), w\rangle, \quad (w_0, w) \in \mathbb{R}_+ \times \mathcal{K},\ w_0 + |w| \leq 1\Big\},\end{aligned}$$

where the last set is a compact, convex subset of $\mathbb{R}^n \times \mathbf{M}(n,n) \times \mathbb{R}^2$, arguing as in the proof of Theorem 3.6 in [HL] (see also Theorem A9 in [HL]) one can show that there exist

$$\begin{aligned}&\textit{two } \mathcal{F}_s\text{-progressively measurable processes } \{v_s\},\ \{(w_{0_s}, w_s)\}, \\ &R_+ \text{ and } (\mathbb{R}_+ \times \mathcal{K}) \cap \{(w_0, w) :\ w_0 + |w| \leq 1\}\text{-valued, respectively,}\end{aligned} \tag{22}$$

such that one has

$$\begin{aligned}\gamma(s,\omega) = &\Big(Aw_{0_s}(\omega) + Bw_s(\omega), w_{0_s}(\omega)\tilde{D}, l_0 w_{0_s}(\omega) + \langle l_1, w_s(\omega)\rangle, w_{0_s}(\omega)\Big)\,(t_s(\omega), \xi_s(\omega)) \\ &+(0,0,v_s(\omega),0) \qquad\qquad \text{for almost all } (s,\omega).\end{aligned} \tag{23}$$

Let us define the *noncanonical control* α by

$$\alpha \doteq (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{(w_{0_s}, w_s)\}, \{(t_s, k_s, \xi_s)\}, \theta), \tag{24}$$

where $\{(w_{0_s}, w_s)\}$ is given in (22). In the rest of the proof, with a slight abuse of notation, let us use again the symbols J and $\mathcal{L}$ to denote the cost and the operator in (19) once the scalar process $\{w_{0_s}\}$ in their definitions is not subjected to the constraint $w_{0_s} = 1 - |w_s|$, but is an independent process with values in $[0,1]$. Then by (23) it follows that for any $\varphi \in \mathcal{C}_b^2(\mathbf{R}^{2+n})$, $\mathcal{L}\varphi(t_s, k_s, \xi_s, \mu_s) = \mathcal{L}\varphi(t_s, k_s, \xi_s, (w_{0_s}, w_s))$ except on a (s,ω) null set. Hence for all φ and all $s \in [0, T+K]$,

$$\mathcal{M}_s(\varphi, \tilde{\alpha}) = \mathcal{M}_s(\varphi, \alpha).$$

Moreover, since

$$\begin{aligned}l_0(t_s,\xi_s)\mu_{0_s} + \langle l_1(t_s,\xi_s), \mu_s\rangle &= l_0(t_s,\xi_s)w_{0_s} + \langle l_1(t_s,\xi_s), w_s\rangle + v_s \\ &\geq l_0(t_s,\xi_s)w_{0_s} + \langle l_1(t_s,\xi_s), w_s\rangle,\end{aligned}$$

and $J(\bar{t},\bar{k},\bar{x},\tilde{\alpha}) < +\infty$, then $J(\bar{t},\bar{k},\bar{x},\alpha) \leq J(\bar{t},\bar{k},\bar{x},\tilde{\alpha}) < +\infty$. The noncanonical control α does not belong, however, to the set $\Gamma^a(\bar{t},\bar{k},\bar{x})$ since it does not verify **(B1)** and $w_{0_s} \neq 1 - |w_s|$. Therefore, in order to conclude the proof, it remains to prove the following.

Claim. There exists a control $\check{\alpha} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\check{w}_s\}, \{(\check{t}_s, \check{k}_s, \check{\xi}_s)\}, \check{\theta})$ verifying **(B3)** and **(B4)** and such that $J(\bar{t},\bar{k},\bar{x},\check{\alpha}) = J(\bar{t},\bar{k},\bar{x},\alpha)$.

In view of Remark 3.1, indeed, this is sufficient for the existence of a control $\hat{\alpha} \in \Gamma^a(\bar{t},\bar{k},\bar{x})$ such that $J(\bar{t},\bar{k},\bar{x},\hat{\alpha}) = J(\bar{t},\bar{k},\bar{x},\alpha)$ and $\hat{\alpha}$ is the required optimal auxiliary control. The claim will be proved in appendix (Remark 7.1 and Lemma 7.3). ☐

Remark 3.3. In general, the original control problem described in Definition 2.1 does not have an optimal control while, by Theorem 3.3, the auxiliary control problem does. Thanks to (ii) of Theorem 2.3, this yields a sequence of suboptimal controls $c^n \in \mathcal{C}(\bar{t},\bar{k},\bar{x})$ for the original problem for any $(\bar{t},\bar{k},\bar{x}) \in [0,T[\times[0,K]\times\mathbb{R}^n$.

3.3. Control rules. We are now going to recall very briefly the definition of control rules (for a detailed description, see [HL]). In order to introduce a canonical space for the problem, let us define the following spaces:

$$\mathcal{C}^{2+n} = \{f : [0, T+K] \to \mathbb{R}^{2+n},\ f \text{ continuous}\},$$

endowed with the topology of uniform convergence;

$$\mathcal{U} \doteq \{\nu : [0,T+K] \to \mathbf{M}_1(\overline{B}_m(1) \cap \mathcal{K}),\ \nu \text{ Borel measurable}\},$$

endowed with the stable topology;

$$\mathcal{Z} = \{\zeta : [0,T+K] \to \mathbb{R},\ \zeta = \chi_{s \geq \Delta},\ \Delta \in [0,+\infty]\}, \tag{25}$$

endowed with the topology of weak convergence of the corresponding (point) probability measures. We denote the map $\zeta \to \Delta$ by $\Delta(\cdot)$. Let $\tilde{\mathcal{C}}, \tilde{\mathcal{U}}, \tilde{\mathcal{Z}}$ denote their Borel σ-fields, let $\tilde{\mathcal{C}}_s, \tilde{\mathcal{U}}_s, \tilde{\mathcal{Z}}_s$ denote the σ-fields up to time s (e.g., $\tilde{\mathcal{Z}}_s = \sigma\{\zeta(s') :\ 0 \leq s' \leq s\}$), and let us introduce the canonical setting

$$\Omega = \mathcal{C}^{2+n} \times \mathcal{U} \times \mathcal{Z}, \qquad \mathcal{F} \doteq \tilde{\mathcal{C}} \times \tilde{\mathcal{U}} \times \tilde{\mathcal{Z}}, \qquad \mathcal{F}_s \doteq \tilde{\mathcal{C}}_s \times \tilde{\mathcal{U}}_s \times \tilde{\mathcal{Z}}_s. \tag{26}$$

Notice that Ω is metrizable and separable under the product topology.

Definition 3.4. *Fix $(\bar{t},\bar{k},\bar{x}) \in [0,T] \times [0,K] \times \mathbf{R}^n$, and let Ω, $\mathcal{F}$ and $\{\mathcal{F}_s\}$ be defined by* (26)*. We say that R is a* control rule *and we write that $R \in \mathcal{R}(\bar{t},\bar{k},\bar{x})$ if R is a probability measure on the canonical space $(\Omega,\mathcal{F})$, such that*

$$\tilde{\alpha} = (\Omega, \mathcal{F}, R, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$$

is a relaxed control (i.e., $\tilde{\alpha} \in \tilde{\Gamma}(\bar{t},\bar{k},\bar{x})$), where

$$(t_s, k_s, \xi_s)(\omega) = f_s, \quad \mu_s(\omega) = \nu_s, \quad \theta(\omega) = \Delta(\zeta)$$

for $\omega = (f,\nu,\zeta) \in \Omega$. Finally, we define the cost associated to R as $J(\bar{t},\bar{k},\bar{x},R) \doteq J(\bar{t},\bar{k},\bar{x},\tilde{\alpha})$, where $J(\bar{t},\bar{k},\bar{x},\tilde{\alpha})$ is given in (20)*. The subset $\mathcal{R}^a(\bar{t},\bar{k},\bar{x})$ of the* admissible control rules *can be now defined as follows:*

$$\mathcal{R}^a(\bar{t},\bar{k},\bar{x}) \doteq \left\{R \in \mathcal{R}(\bar{t},\bar{k},\bar{x}) :\quad J(\bar{t},\bar{k},\bar{\xi},R) < +\infty\right\}.$$

Remark 3.4. For the sake of notation, in what follows a given element ω of the canonical space $\mathcal{C}^{2+n} \times \mathcal{U} \times \mathcal{Z}$ will be denoted by $\omega = ((t_\cdot, k_\cdot, \xi_\cdot), \mu_\cdot, \theta)$.

By definition, $\mathcal{R}(\bar t, \bar k, \bar x) \hookrightarrow \tilde\Gamma(\bar t, \bar k, \bar x)$. In fact, the inverse embedding is also valid. In particular, one has the following proposition.

PROPOSITION 3.5. *Fix* $(\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n$ *and assume* **(A0)**, **(A1)**. *Then*

$$V(\bar t, \bar k, \bar x) = \inf_{\tilde\alpha \in \tilde\Gamma^a(\bar t, \bar k, \bar x)} J(\bar t, \bar k, \bar x, \tilde\alpha) = \inf_{R \in \mathcal{R}^a(\bar t, \bar k, \bar x)} J(\bar t, \bar k, \bar x, R). \tag{27}$$

Moreover, the infimum is attained in any one of $\mathcal{R}^a(\bar t, \bar k, \bar x)$, $\tilde\Gamma^a(\bar t, \bar k, \bar x)$, *and* $\Gamma^a(\bar t, \bar k, \bar x)$.

Proof. The first equality in (27) has been obtained in Theorem 3.3, while the second one follows from Theorem 3.13 in [HL]. The minimum for the auxiliary (and hence, for any other) control problem exists in view of Theorem 3.3. $\square$

Let us conclude this subsection by recalling a dynamic programming principle established in [HL]. To this end, let us notice that the auxiliary control problem is in fact an *unconstrained* stopping time control problem. Indeed, from Definition 2.2 it follows that for all $(\bar t, \bar k, \bar x)$ such that either $\bar t > T$ or $\bar k > K$, the set of admissible auxiliary controls $\Gamma^a(\bar t, \bar k, \bar x)$ is empty. Hence the auxiliary value function V might be extended to the whole set $[0, +\infty[\times[0, +\infty[\times\mathbb{R}^n$ in a natural way by setting $V = +\infty$ outside $[0,T] \times [0,K] \times \mathbb{R}^n$.

PROPOSITION 3.6. *Assume* **(A0)**, **(A1)**. *For any* $(\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n$, *one has*

$$V(\bar t, \bar k, \bar x) = \inf \left\{ E_R \left[\int_0^{\rho'} (l_0(t_\sigma, \xi_\sigma)(1 - |\mu_\sigma|) + \langle l_1(t_\sigma, \xi_\sigma), \mu_\sigma \rangle) \, d\sigma + V(t_{\rho'}, k_{\rho'}, \xi_{\rho'}) \right] \right\}, \tag{28}$$

where the infimum is taken over the set $\mathcal{R}^a(\bar t, \bar k, \bar x)$ *and* $\rho' = \rho \wedge \theta$, ρ *being any finite stopping time such that* $0 \le \rho \le \theta$.

4. Continuity of the value function.

THEOREM 4.1. *Let* **(A0)**, **(A1)** *hold. Then the value function* V *is bounded and continuous. More precisely, there exists some* $\bar C > 0$ *such that* V *satisfies the following:*

$$|V(\bar t, \bar k, \bar x)| \le \bar C \qquad \forall (\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n;$$

$$|V(\bar t_1, \bar k_1, \bar x_1) - V(\bar t_2, \bar k_2, \bar x_2)| \le \bar C \left[|\bar x_1 - \bar x_2| + (1 + |\bar x_1| \vee |\bar x_2|) \left(|\bar t_1 - \bar t_2|^{1/2} + |\bar k_1 - \bar k_2| \right) \right]$$

for all $(\bar t_1, \bar k_1, \bar x_1)$, $(\bar t_2, \bar k_2, \bar x_2) \in [0,T] \times [0,K] \times \mathbb{R}^n$.

Proof (*Boundedness*). It is very easy to see that for any initial condition $(\bar t, \bar k, \bar x) \in [0,T] \times [0,K] \times \mathbb{R}^n$ the set of admissible control rules is nonempty. Since the stopping time θ is bounded from above by $T + K$, the boundedness of V follows, therefore, straightforwardly from the boundedness of both the process $\{w_s\}$ and the data l_0, l_1, and g.

Lipschitz continuity in x. Fix $(\bar t, \bar k, \bar x_1), (\bar t, \bar k, \bar x_2) \in [0,T] \times [0,K] \times \mathbb{R}^n$ and assume that $V(\bar t, \bar k, \bar x_1) \ge V(\bar t, \bar k, \bar x_2)$. One has

$$0 \le V(\bar t, \bar k, \bar x_1) - V(\bar t, \bar k, \bar x_2) \le \sup_{P \in \mathcal{R}^a(\bar t, \bar k, \bar x_2)} \big(J(\bar t, \bar k, \bar x_1, Q) - J(\bar t, \bar k, \bar x_2, P) \big)$$

for every $Q \in \mathcal{R}^a(\bar{t}, \bar{k}, \bar{x}_1)$. Take $P \in \mathcal{R}^a(\bar{t}, \bar{k}, \bar{x}_2)$ arbitrary and let

$$(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_{2_s})\}, \theta)$$

be the associated relaxed control. By the definition of control rules, there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$ of $(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\})$, i.e., there exists another probability space $(\Omega', \mathcal{F}', \mathcal{F}'_s, P')$ such that $\tilde{\Omega} = \Omega \times \Omega'$, $\tilde{\mathcal{F}} = \mathcal{F} \times \mathcal{F}'$, $\tilde{\mathcal{F}}_s = \mathcal{F}_s \times \mathcal{F}'_s$, and $\tilde{P} = P \times P'$. We can extend the process $\{((t_\cdot, k_\cdot, \xi_\cdot), \mu_\cdot, \theta)\}$ to $\tilde{\Omega}$ by the following: for $\tilde{\omega} = (\omega, \omega') \in \tilde{\Omega}$,

$$(t_\cdot, k_\cdot, \xi_\cdot)(\tilde{\omega}) = (t_\cdot, k_\cdot, \xi_\cdot)(\omega), \quad \tilde{\mu}_\cdot(\tilde{\omega}) = \mu_\cdot(\omega), \quad \theta(\tilde{\omega}) = \theta(\omega).$$

On $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_s)$ there exists a standard p-dimensional Brownian motion $\{W_s\}$ such that for $s \in [0, T+K]$,

$$\begin{aligned} &t_s = \bar{t} + \textstyle\int_0^s (1 - |\mu|_\sigma)\, d\sigma, \\ &k_s = \bar{k} + \textstyle\int_0^s |\mu_\sigma|\, d\sigma, \\ &\xi_{2s} = \bar{x}_2 + \textstyle\int_0^s (A(t_\sigma, \xi_{2\sigma})(1 - |\mu|_\sigma) + B(t_\sigma, \xi_{2\sigma})\mu_\sigma)\, d\sigma + \int_0^s D(t_\sigma, \xi_{2\sigma})\sqrt{1 - |\mu|_\sigma}\, dW_\sigma, \end{aligned}$$

the control $\tilde{\beta} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_{2_s})\}, \theta) \in \tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x}_2)$, where, by the definition of the set $\mathcal{Z}$ in (25), θ is the first time in which $\chi_{s \geq \theta}$ jumps from 0 to 1 and $J(\bar{t}, \bar{k}, \bar{x}_2, \tilde{\beta}) = J(\bar{t}, \bar{k}, \bar{x}_2, \tilde{P}) = J(\bar{t}, \bar{k}, \bar{x}_2, P)$.

Consider the equations with the initial condition $(\bar{t}, \bar{k}, \bar{x}_1)$, for $s \in [0, T+K]$,

$$\begin{aligned} &t_s = \bar{t} + \textstyle\int_0^s (1 - |\mu|_\sigma)\, d\sigma, \\ &k_s = \bar{k} + \textstyle\int_0^s |\mu_\sigma|\, d\sigma, \\ &\xi_{1s} = \bar{x}_1 + \textstyle\int_0^s (A(t_\sigma, \xi_{1\sigma})(1 - |\mu|_\sigma) + B(t_\sigma, \xi_{1\sigma})\mu_\sigma)\, d\sigma + \int_0^s D(t_\sigma, \xi_{1\sigma})\sqrt{1 - |\mu|_\sigma}\, dW_\sigma \end{aligned} \tag{29}$$

on the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$. Under assumptions **(A0)**, **(A1)**, the strong solution to (29) exists and one can see that $\tilde{\alpha} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_{1s})\}, \theta) \in \tilde{\Gamma}^a(\bar{t}, \bar{k}, \bar{x}_1)$. Therefore, there exists a control rule $Q \in \mathcal{R}^a(\bar{t}, \bar{k}, \bar{x}_1)$ such that

$$J(\bar{t}, \bar{k}, \bar{x}_1, \tilde{\alpha}) = J(\bar{t}, \bar{k}, \bar{x}_1, Q).$$

We have

$$\begin{aligned} &J(\bar{t}, \bar{k}, \bar{x}_1, Q) - J(\bar{t}, \bar{k}, \bar{x}_2, P) = J(\bar{t}, \bar{k}, \bar{x}_1, \tilde{\alpha}) - J(\bar{t}, \bar{k}, \bar{x}_2, \tilde{\beta}) \\ &\leq E_{\tilde{P}} \left[\textstyle\int_0^\theta |l_0(t_\sigma, \xi_{1\sigma}) - l_0(t_\sigma, \xi_{2\sigma})|\, |1 - |\mu_\sigma||\, d\sigma + \int_0^\theta |l_1(t_\sigma, \xi_{1\sigma}) - l_1(t_\sigma, \xi_{2\sigma})||\mu_\sigma|\, d\sigma \right. \\ &\qquad \left. + |g(\xi_{1\theta}) - g(\xi_{2\theta})| \right] \leq L\, E_{\tilde{P}} \left[\textstyle\int_0^\theta |\xi_{1\sigma} - \xi_{2\sigma}|\, d\sigma \right] + L\, E_{\tilde{P}}[|\xi_{1\theta} - \xi_{2\theta}|], \end{aligned}$$

where we have used the Lipschitz continuity of l_0, l_1, and g and L is the same as in **(A1)**. Let us define $\hat{\xi}_{i_s} \doteq \xi_{i_{s \wedge \theta}}$ for all $s \geq 0$ and $i = 1, 2$. By the Burkholder–Gundy and Gronwall inequalities, we obtain that there exists a constant C, depending on the Lipschitz constant L_2 in **(A0)** and on $T+K$, such that, for all $0 \leq \sigma \leq T+K$,

$$E_{\tilde{P}} \left[\sup_{s \leq \sigma} (|\hat{\xi}_{1_s} - \hat{\xi}_{2s}|^2) \right] \leq C |\bar{x}_1 - \bar{x}_2|^2.$$

Since from the definitions of $\{\hat{\xi}_{1_s}\}$ and $\{\hat{\xi}_{2_s}\}$ it follows that

$$\begin{aligned} E_{\tilde{P}} \left[\textstyle\int_0^\theta |\xi_{1\sigma} - \xi_{2\sigma}|\, d\sigma \right] &\leq E_{\tilde{P}} \left[\textstyle\int_0^{T+K} |\hat{\xi}_{1_\sigma} - \hat{\xi}_{2\sigma}|\, d\sigma \right] \\ &\leq \left(\textstyle\int_0^{T+K} E_{\tilde{P}} \left[\sup_{s \leq \sigma} (|\hat{\xi}_{1_s} - \hat{\xi}_{2s}|^2) \right] d\sigma \right)^{1/2}, \end{aligned} \tag{30}$$

in view of the arbitrariness of $P \in \mathcal{R}^a(\bar{t}, \bar{k}, \bar{x}_2)$, the previous estimates yield that

$$0 \le V(\bar{t}, \bar{k}, \bar{x}_1) - V(\bar{t}, \bar{k}, \bar{x}_2) \le \bar{C}|\bar{x}_1 - \bar{x}_2|$$

for a suitable constant $\bar{C}$, depending just on L, L_2, and $T + K$.

Hölder continuity in t. Fix $(\bar{t}_1, \bar{k}, \bar{x})$, $(\bar{t}_2, \bar{k}, \bar{x}) \in [0, T] \times [0, K] \times \mathbb{R}^n$ and assume that $V(\bar{t}_1, \bar{k}, \bar{x}) \ge V(\bar{t}_2, \bar{k}, \bar{x})$.

Case 1. $\bar{t}_1 < \bar{t}_2$. One has

$$0 \le V(\bar{t}_1, \bar{k}, \bar{x}) - V(\bar{t}_2, \bar{k}, \bar{x}) \le \sup_{P \in \mathcal{R}^a(\bar{t}_2, \bar{k}, \bar{x})} \big(J(\bar{t}_1, \bar{k}, \bar{x}, Q) - J(\bar{t}_2, \bar{k}, \bar{x}, P)\big)$$

for every $Q \in \mathcal{R}^a(\bar{t}_1, \bar{k}, \bar{x})$. Take $P \in \mathcal{R}^a(\bar{t}_2, \bar{k}, \bar{x})$ arbitrary and let

$$(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_{2_s}, k_s, \xi_{2_s})\}, \theta_2)$$

be the associated relaxed control. Now, as in the previous step, there exist an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$ of $(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\})$ and a standard Brownian motion $\{W_s\}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$ such that, for $s \in [0, T + K]$,

$$\begin{aligned}
&t_{2_s} = \bar{t}_2 + \textstyle\int_0^s (1 - |\mu_\sigma|)\, d\sigma,\\
&k_s = \bar{k} + \textstyle\int_0^s |\mu_\sigma|\, d\sigma,\\
&\xi_{2_s} = \bar{x} + \textstyle\int_0^s \left(A(t_{2_\sigma}, \xi_{2_\sigma})(1 - |\mu_\sigma|) + B(t_{2_\sigma}, \xi_{2_\sigma})\mu_\sigma\right) d\sigma + \textstyle\int_0^s D(t_{2_\sigma}, \xi_{2_\sigma})\sqrt{1 - |\mu_\sigma|}\, dW_\sigma,
\end{aligned}$$

the control $\tilde{\beta} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_{2_s}, k_s, \xi_{2_s})\}, \theta_2) \in \tilde{\Gamma}^a(\bar{t}_2, \bar{k}, \bar{x})$, and $J(\bar{t}_2, \bar{k}, \bar{x}, \tilde{\beta}) = J(\bar{t}_2, \bar{k}, \bar{x}, P)$. Let us now consider the relaxed control that one obtains from the definition of $\tilde{\beta}$ when μ_s is replaced by $\mu_s \chi_{\{s \le \theta_2\}}$ for $s \ge 0$. It is easy to see that this control is admissible, that is, it belongs to $\tilde{\Gamma}^a(\bar{t}_2, \bar{k}, \bar{x})$ and the corresponding cost coincides with $J(\bar{t}_2, \bar{k}, \bar{x}, P)$. With a small abuse of notation, from now on let us use $\tilde{\beta}$ to denote such control.

Let us introduce the stopping time $\theta_1 \doteq \theta_2 + (\bar{t}_2 - \bar{t}_1)$ and let $\{(t_{1_s}, k_s, \xi_{1_s})\}$ be the strong solution to

$$\begin{aligned}
&t_{1_s} = \bar{t}_1 + \textstyle\int_0^s (1 - |\mu_\sigma|)\, d\sigma,\\
&k_s = \bar{k} + \textstyle\int_0^s |\mu_\sigma|\, d\sigma,\\
&\xi_{1_s} = \bar{x} + \textstyle\int_0^s \left(A(t_{1_\sigma}, \xi_{1_\sigma})(1 - |\mu_\sigma|) + B(t_{1_\sigma}, \xi_{1_\sigma})\mu_\sigma\right) d\sigma + \textstyle\int_0^s D(t_{1_\sigma}, \xi_{1_\sigma})\sqrt{1 - |\mu_\sigma|}\, dW_\sigma
\end{aligned}$$

on the stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$ for $s \in [0, T + K]$. As claimed in Remark 2.2, $\tilde{\beta}$ admissible implies that $\theta_2 \le (T - \bar{t}_2) + (K - \bar{k})$, $t_{2_{\theta_2}} = T$, and $k_{\theta_2} \le K$. Hence one deduces that

$$\theta_1 \le (T - \bar{t}_2) + (K - \bar{k}) + (\bar{t}_2 - \bar{t}_1) = (T - \bar{t}_1) + (K - \bar{k}) \le T + K.$$

Moreover, since we identified μ_s with $\mu_s \chi_{\{s \le \theta_2\}}$ one has

$$t_{1_{\theta_1}} = t_{2_{\theta_2}} + (\theta_1 - \theta_2) - (\bar{t}_2 - \bar{t}_1), \qquad k_{\theta_1} = k_{\theta_2} \le K.$$

Therefore, $t_{1_{\theta_1}} = T$, $k_{\theta_1} \le K$, and the control $\tilde{\alpha} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_{1_s}, k_s, \xi_{1_s})\}, \theta_1)$ is in $\tilde{\Gamma}^a(\bar{t}_1, \bar{k}, \bar{x})$. Thus there exists a control rule $Q \in \mathcal{R}^a(\bar{t}_1, \bar{k}, \bar{x})$ such that

$J(\bar{t}_1, \bar{k}, \bar{x}, \tilde{\alpha}) = J(\bar{t}_1, \bar{k}, \bar{x}, Q)$. We have

$$\begin{aligned}
&J(\bar{t}_1, \bar{k}, \bar{x}, Q) - J(\bar{t}_2, \bar{k}, \bar{x}, P) = J(\bar{t}_1, \bar{k}, \bar{x}, \tilde{\alpha}) - J(\bar{t}_2, \bar{k}, \bar{x}, \tilde{\beta}) \\
&\le E_{\tilde{P}}\left[|g(\xi_{1_{\theta_1}}) - g(\xi_{2_{\theta_2}})|\right] + E_{\tilde{P}}\left[\int_0^{\theta_2} |l_0(t_{1_\sigma}, \xi_{1_\sigma}) - l_0(t_{2_\sigma}, \xi_{2_\sigma})|\, |1 - |\mu_\sigma||\, d\sigma\right] \\
&+E_{\tilde{P}}\left[\int_0^{\theta_2} |l_1(t_{1_\sigma}, \xi_{1_\sigma}) - l_1(t_{2_\sigma}, \xi_{2_\sigma})|\, |\mu_\sigma|\, d\sigma\right] \\
&+E_{\tilde{P}}\left[\int_{\theta_2}^{\theta_2 + (\bar{t}_2 - \bar{t}_1)} |l_0(t_{1_\sigma}, \xi_{1_\sigma})|\, |1 - |\mu_\sigma||\, d\sigma\right] \\
&\le L\left[E_{\tilde{P}}[|\xi_{1_{\theta_1}} - \xi_{2_{\theta_2}}|^2]\right]^{\frac{1}{2}} + L E_{\tilde{P}}\left[\int_0^{\theta_2} (|t_{1_\sigma} - t_{2_\sigma}| + |\xi_{1_\sigma} - \xi_{2_\sigma}|)\, d\sigma\right] + L_3(\bar{t}_2 - \bar{t}_1),
\end{aligned}$$

where the constants L and L_3 are the same as in **(A1)**. In order to conclude the proof, let us introduce for $s \ge 0$ the processes $\hat{\xi}_{i_s} \doteq \xi_{i_{s \wedge \theta_i}}$, for $i = 1, 2$. Since

$$t_{1_{\theta_2}} = t_{2_{\theta_2}} - (\bar{t}_2 - \bar{t}_1),$$

by standard calculations (see, e.g., [F]) one can prove that

$$E_{\tilde{P}}\left[\sup_{s \le \sigma} |\hat{\xi}_{2_s} - \hat{\xi}_{1_s}|^2\right] \le C^2(1 + |\bar{x}|)^2 |\bar{t}_2 - \bar{t}_1|,$$

for every $0 \le \sigma \le T + K$, with C a suitable constant depending on L_1, L_2 in **(A0)** and $T + K$. From the definitions of $\{\hat{\xi}_{2_s}\}$ and $\{\hat{\xi}_{1_s}\}$, this yields

$$E_{\tilde{P}}\left[\left|\xi_{2_{\theta_2}} - \xi_{1_{\theta_1}}\right|^2\right] = E_{\tilde{P}}\left[\left|\hat{\xi}_{2_{T+K}} - \hat{\xi}_{1_{T+K}}\right|^2\right] \le C^2(1 + |\bar{x}|)^2 |\bar{t}_2 - \bar{t}_1|.$$

Therefore, by (30) we obtain

$$J(\bar{t}_1, \bar{k}, \bar{x}, Q) - J(\bar{t}_2, \bar{k}, \bar{x}, P) \le \bar{C}\left[(1 + |\bar{x}|)|\bar{t}_1 - \bar{t}_2|^{\frac{1}{2}} + |\bar{t}_1 - \bar{t}_2|\right],$$

which, by the arbitrariness of P, yields

$$0 \le V(\bar{t}_2, \bar{k}, \bar{x}) - V(\bar{t}_1, \bar{k}, \bar{x}) \le \bar{C}(1 + |\bar{x}|)|\bar{t}_1 - \bar{t}_2|^{\frac{1}{2}}$$

for some constant $\bar{C}$ depending on the constants L, L_2, L_3, and $T + K$ in **(A0)**, **(A1)**.

Case 2. $\bar{t}_1 > \bar{t}_2$. Consider the dynamic programming principle (28) for $V(\bar{t}_2, \bar{k}, \bar{x})$,

$$\begin{aligned}
V(\bar{t}_2, \bar{k}, \bar{x}) = \inf_{R \in \mathcal{R}^a(\bar{t}_2, \bar{k}, \bar{x})} \Bigg\{ E_R\Bigg[\int_0^{r \wedge \theta} &(l_0(t_\sigma, \xi_\sigma)(1 - |\mu_\sigma|) \\
&+ \langle l_1(t_\sigma, \xi_\sigma), \mu_\sigma\rangle)\, d\sigma + V(t_{r \wedge \theta}, k_{r \wedge \theta}, \xi_{r \wedge \theta})\Bigg]\Bigg\},
\end{aligned} \tag{31}$$

where we choose the (deterministic) time $r = \bar{t}_1 - \bar{t}_2$. It is easy to see that there exists an admissible control rule $P \in \mathcal{R}^a(\bar{t}_2, \bar{k}, \bar{x})$ associated to a relaxed control

$$(\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$$

and such that

$$P(\mu_s = \delta_{\{0\}} \quad 0 \le s \le \theta, \quad \theta = T - \bar{t}_2) = 1.$$

Then $P(\theta \geq r) = 1$; by the boundedness of l_0 one has

$$V(\bar{t}_2, \bar{k}, \bar{x}) \leq E_P\left[\int_0^r |l_0(t_\sigma, \xi_\sigma)|\, d\sigma + V(t_r, k_r, \xi_r)\right] \leq L_3 r + E_P[V(t_r, k_r, \xi_r)],$$

and by the Lipschitz continuity of the value function in x,

$$V(t_r, k_r, \xi_r) \leq V(t_r, k_r, \bar{x}) + C|\xi_r - \bar{x}|.$$

Hence

$$V(\bar{t}_2, \bar{k}, \bar{x}) - E_P[V(t_r, k_r, \bar{x})] \leq L_3 r + C E_P[|\xi_r - \bar{x}|] \leq L_3 r + C(E_P[|\xi_r - \bar{x}|^2])^{\frac{1}{2}}. \tag{32}$$

From the definition of control rules, we know that under P,

$$\begin{aligned} t_r &= \bar{t}_2 + r = \bar{t}_1, \\ k_r &= \bar{k}, \\ \xi_r &= \bar{x} + \textstyle\int_0^r A(t_\sigma, \xi_\sigma)\, d\sigma + M_r, \end{aligned} \tag{33}$$

where $\{M_r\}$ is a continuous square integrable martingale with

$$\langle M \rangle_r = \int_0^r \mathrm{Tr}\{\tilde{D}(t_\sigma, \xi_\sigma)\}\, d\sigma.$$

Therefore, by the Burkholder–Davis–Gundy inequality there exists a constant C, depending on L_1 in **(A0)**, such that

$$E_P[|\xi_r - \bar{x}|^2] \leq C^2(1 + |\bar{x}|)^2(r^2 + r). \tag{34}$$

Therefore, (32), (33), and (34) yield

$$0 \leq V(\bar{t}_2, \bar{k}, \bar{x}) - V(\bar{t}_1, \bar{k}, \bar{x}) \leq \bar{C}(1 + |\bar{x}|)|\bar{t}_2 - \bar{t}_1|^{\frac{1}{2}}$$

for some constant $\bar{C}$ depending on the constants introduced in **(A0)**, **(A1)**.

Lipschitz continuity in k. Fix $(\bar{t}, \bar{k}_1, \bar{x}), (\bar{t}, \bar{k}_2, \bar{x}) \in [0, T] \times [0, K] \times \mathbb{R}^n$, and assume that $V(\bar{t}, \bar{k}_1, \bar{x}) \geq V(\bar{t}, \bar{k}_2, \bar{x})$.

Case 1. $\bar{k}_1 < \bar{k}_2$. One has

$$0 \leq V(\bar{t}, \bar{k}_1, \bar{x}) - V(\bar{t}, \bar{k}_2, \bar{x}) \leq \sup_{P \in \mathcal{R}^a(\bar{t}, \bar{k}_2, \bar{x})} \big(J(\bar{t}, \bar{k}_1, \bar{x}, Q) - J(\bar{t}, \bar{k}_2, \bar{x}, P)\big)$$

for every $Q \in \mathcal{R}^a(\bar{t}, \bar{k}_1, \bar{x})$. As in the previous step, take $P \in \mathcal{R}^a(\bar{t}, \bar{k}_2, \bar{x})$ arbitrary and let $\{W_s\}$ be a standard Brownian motion on a suitable $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\})$ such that

$$\tilde{\beta} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_s, k_{2_s}, \xi_s)\}, \theta_2) \in \tilde{\Gamma}^a(\bar{t}, \bar{k}_2, \bar{x})$$

is a relaxed control, where, for $s \in [0, T + K]$,

$$\begin{aligned} &t_s = \bar{t} + \textstyle\int_0^s (1 - |\mu_\sigma|)\, d\sigma, \\ &k_{2_s} = \bar{k}_2 + \textstyle\int_0^s |\mu_\sigma|\, d\sigma, \\ &\xi_s = \bar{x} + \textstyle\int_0^s \left(A(t_\sigma, \xi_\sigma)(1 - |\mu_\sigma|) + B(t_\sigma, \xi_\sigma)\mu_\sigma\right)\, d\sigma + \int_0^s D(t_\sigma, \xi_\sigma)\sqrt{1 - |\mu_\sigma|}\, dW_\sigma, \end{aligned}$$

and $J(\bar{t}, \bar{k}_2, \bar{x}, \tilde{\beta}) = J(\bar{t}, \bar{k}_2, \bar{x}, P)$. Moreover, setting for $s \geq 0$,

$$k_{1_s} \doteq \bar{k}_1 + \int_0^s |\mu_\sigma|\, d\sigma = k_{2_s} - (\bar{k}_2 - \bar{k}_1),$$

one easily sees that the control

$$\tilde{\alpha} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_s\}, \{\mu_s\}, \{(t_s, k_{1_s}, \xi_s)\}, \theta_2)$$

is in $\tilde{\Gamma}^a(\bar{t}, \bar{k}_1, \bar{x})$. As before, there exists a control rule $Q \in \mathcal{R}^a(\bar{t}, \bar{k}_1, \bar{x})$ such that $J(\bar{t}, \bar{k}_1, \bar{x}, \tilde{\alpha}) = J(\bar{t}, \bar{k}_1, \bar{x}, Q)$. Since the cost functional J and the state process $\{\xi_s\}$ do not depend explicitly on the k variable, one has that

$$J(\bar{t}, \bar{k}_2, \bar{x}, P) = J(\bar{t}, \bar{k}_2, \bar{x}, \tilde{\beta}) = J(\bar{t}, \bar{k}_1, \bar{x}, \tilde{\alpha}) = J(\bar{t}, \bar{k}_1, \bar{x}, Q).$$

As a consequence, in this case,

$$V(\bar{t}, \bar{k}_1, \bar{x}) = V(\bar{t}, \bar{k}_2, \bar{x}).$$

Case 2. $\bar{k}_1 > \bar{k}_2$. Consider the dynamic programming principle (28) for $V(\bar{t}, \bar{k}_2, \bar{x})$,

$$\begin{aligned} V(\bar{t}, \bar{k}_2, \bar{x}) = \inf_{R \in \mathcal{R}^a(\bar{t}, \bar{k}_2, \bar{x})} \Bigg\{ E_R \Bigg[\int_0^{r \wedge \theta} &(l_0(t_\sigma, \xi_\sigma)(1 - |\mu_\sigma|) \\ &+ \langle l_1(t_\sigma, \xi_\sigma), \mu_\sigma \rangle)\, d\sigma + V(t_{r\wedge\theta}, k_{r\wedge\theta}, \xi_{r\wedge\theta}) \Bigg] \Bigg\}, \end{aligned}$$

where we choose the (deterministic) time $r = \bar{k}_1 - \bar{k}_2$. Let us fix an arbitrary $w \in \mathcal{K}$ with $|w| = 1$. Then there exists a control rule $P \in \mathcal{R}^a(\bar{t}, \bar{k}_2, \bar{x})$ associated to a relaxed control

$$\tilde{\beta} = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{\mu_s\}, \{(t_s, k_s, \xi_s)\}, \theta)$$

such that

$$P(\mu_s = \delta_{\{w\}} \quad 0 \le s \le \theta, \quad \theta = K - \bar{k}_2) = 1,$$

and $J(\bar{t}, \bar{k}_2, \bar{x}, \tilde{\beta}) = J(\bar{t}, \bar{k}_2, \bar{x}, P)$. Then, arguing as in the case "$\bar{t}_1 > \bar{t}_2$" of the proof of the continuity in t, we can deduce that an estimate analogous to (32) is still verified, that is

$$V(\bar{t}, \bar{k}_2, \bar{x}) - E_P[V(t_r, k_r, \bar{x})] \le L_3 r + C E_P[|\xi_r - \bar{x}|] \le L_3 r + C(E_P[|\xi_r - \bar{x}|^2])^{\frac{1}{2}}. \tag{35}$$

Now, under P we have

$$\begin{aligned} t_r &= \bar{t}, \\ k_r &= \bar{k}_2 + r = \bar{k}_1, \\ \xi_r &= \bar{x} + \textstyle\int_0^r B(t_\sigma, \xi_\sigma)\, d\sigma. \end{aligned} \tag{36}$$

Therefore, since $E_P[|B(t_s, \xi_s)|^2] \le E_P[[L_1(1 + |\xi_s|)]^2] \le C^2(1 + |\bar{x}|)^2$, we deduce that for $0 \le r \le \theta$,

$$E_P[|\xi_r - \bar{x}|^2] \le C^2(1 + |\bar{x}|)^2 r^2. \tag{37}$$

Then (35), (36), and (37) yield

$$0 \le V(\bar{t}, \bar{k}_2, \bar{x}) - V(\bar{t}, \bar{k}_1, \bar{x}) \le \bar{C}(1 + |\bar{x}|)|\bar{k}_2 - \bar{k}_1|.$$

The proof of Theorem 4.1 is thus concluded. $\square$

By Theorem 2.3, as a straightforward consequence of Theorem 4.1 one has the following corollary.

COROLLARY 4.2. *Assume* **(A0)**, **(A1)**. *Then there exists a unique, bounded, continuous extension of the value function* $\mathcal{V} : [0,T[\times[0,K]\times\mathbb{R}^n \to \mathbb{R}$, *still denoted by* $\mathcal{V}$, *to the closed set* $[0,T]\times[0,K]\times\mathbb{R}^n$ *which coincides with the auxiliary value function* V. *Hence there exists some* $\bar{C} > 0$ *such that*

$$|\mathcal{V}(\bar{t},\bar{k},\bar{x})| \le \bar{C} \qquad \forall(\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbb{R}^n,$$

$$|\mathcal{V}(\bar{t}_1,\bar{k}_1,\bar{x}_1) - \mathcal{V}(\bar{t}_2,\bar{k}_2,\bar{x}_2)| \le \bar{C}[|\bar{x}_1-\bar{x}_2| + (1+|\bar{x}_1|\vee|\bar{x}_2|)(|\bar{t}_1-\bar{t}_2|^{1/2} + |\bar{k}_1-\bar{k}_2|)]$$

for all $(\bar{t}_1,\bar{k}_1,\bar{x}_1)$, $(\bar{t}_2,\bar{k}_2,\bar{x}_2) \in [0,T]\times[0,K]\times\mathbb{R}^n$.

Therefore, from now on $\mathcal{V}$ will denote the extension of $\mathcal{V}$ to $[0,T]\times[0,K]\times\mathbb{R}^n$, which exists being equal to V.

5. Dynamic programming equation and boundary conditions. This section is devoted to showing that the value function $\mathcal{V}$ is a viscosity solution of (5)–(7). To this end, we will recall below the definition of viscosity sub- and supersolutions with generalized boundary conditions (see, e.g., [CIL]). A formal derivation of the boundary value problem described in the introduction is given in the following subsection.

5.1. Heuristic derivation of the quasi-variational inequality and of the boundary conditions. It is quite easy to deduce heuristically the boundary value problem (5)–(7) once we consider the value function V of the auxiliary optimization control problem defined in Definition 2.2 to which our original control problem, introduced in Definition 2.1, is equivalent. The auxiliary control problem is indeed formulated as an *unconstrained stopping time problem*, with bounded controls and discontinuous final cost given by

$$\tilde{G}(t,k,x) \doteq g(x) - G(t,k) \qquad \forall(t,k,x) \in \mathbb{R}^{2+n}.$$

Therefore, assuming V of class $C^{1,2}$, using Ito's formula and arguing as usual (see, e.g., [FS]), we can deduce from the dynamic programming principle (28) that V verifies the following equation:

$$\tilde{\mathcal{F}}\left(x, DV, \frac{\partial V}{\partial t}, \frac{\partial V}{\partial k}, D^2V\right) = 0 \qquad \text{in } \ ]0,T[\times]0,K[\times\mathbb{R}^n,$$

where

$$\tilde{\mathcal{F}}(x,p_x,p_t,p_k,S) \doteq \max_{\{(w_0,w):\ w_0\ge 0,\ \ w\in\mathcal{K},\ w_0+|w|=1\}}\{-\tfrac{1}{2}w_0\mathrm{Tr}\{\tilde{D}(t,x)S\}$$
$$-\langle A(t,x)w_0 + B(t,x)w, p_x\rangle - l_0(t,x)w_0 - \langle l_1(t,x), w\rangle - p_t\,w_0 - p_k|w|\},$$

which is, in turn, equivalent to the quasi-variational inequality (5), as shown in [MS2].

More precisely, one can show that the value function of an optimal stopping time problem verifies

$$\max\left\{\tilde{\mathcal{F}}\left(x, DV, \frac{\partial V}{\partial t}, \frac{\partial V}{\partial k}, D^2V\right); V - \tilde{G}\right\} = 0 \qquad \text{in } \ \mathbb{R}^{2+n}, \tag{38}$$

due to the fact that the controller can decide to stop as soon as it is convenient (for the derivation of (38) in a viscosity framework we refer to [BP] and [BCD]). Since

$V(t,k,x) = +\infty$ outside $[0,T] \times [0,K] \times \mathbb{R}^n$ and the lower semicontinuous exit cost $\tilde{G}(t,k,x)$ is equal to $g(x)$ for $(t,x,k) \in \{T\} \times [0,K] \times \mathbb{R}^n$ and to $+\infty$ otherwise, by (38) it follows easily that (7) holds for every $(t,x,k) \in \{T\} \times [0,K] \times \mathbb{R}^n$ and that (6) holds for $(t,x,k) \in [0,T[\times[0,K] \times \mathbb{R}^n$.

We underline that, as far as we know, there is not in the literature a dynamic programming principle for the problem of Definition 2.1; hence, even assuming the value function $\mathcal{V}$ to be regular enough, there is no way to deduce the equation and the boundary conditions (5)–(7) directly for the original control problem.

5.2. Viscosity solution.

DEFINITION 5.1. *A locally bounded function v defined on $[0,T] \times [0,K] \times \mathbb{R}^n$ is a* viscosity subsolution *of* (5)–(7) *if for every point $\bar{z} = (\bar{t}, \bar{k}, \bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$ and for every map $\phi \in C_b^2([0,T] \times [0,K] \times \mathbb{R}^n)$ such that $v^* - \phi$ has a local maximum at $\bar{z}$ one has*

$$\max\left\{-\frac{\partial \phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})), -\frac{\partial \phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z}))\right\} \leq 0$$

if $\bar{z} \in]0,T[\times]0,K[\times\mathbb{R}^n$, and

$$v^*(\bar{z}) \leq g(\bar{x})$$

if $\bar{z} \in \{T\}\times]0,K] \times \mathbb{R}^n$.

A locally bounded function v defined on $[0,T] \times [0,K] \times \mathbb{R}^n$ is a viscosity supersolution *of* (5)–(7) *if for every point $\bar{z} = (\bar{t}, \bar{k}, \bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$ and for every map $\phi \in C_b^2([0,T] \times [0,K] \times \mathbb{R}^n)$ such that $v_* - \phi$ has a local minimum at $\bar{z}$ one has*

$$\max\left\{-\frac{\partial \phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})), -\frac{\partial \phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z}))\right\} \geq 0$$

if $\bar{z} \in]0,T[\times]0,K] \times \mathbb{R}^n$, and

$$\max\left\{-\frac{\partial \phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})), -\frac{\partial \phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z}))\right\} \geq 0 \quad \text{or} \quad v_*(\bar{z}) \geq g(\bar{x})$$

if $\bar{z} \in \{T\}\times]0,K] \times \mathbb{R}^n$.

A locally bounded function v defined on $[0,T] \times [0,K] \times \mathbb{R}^n$ is called a viscosity solution *of* (5)–(7) *if it is both a viscosity sub- and supersolution of* (5)–(7).

Example 5.1. Consider the control problem introduced in Example 2.1 and the following boundary value problem:

$$\max\left\{-\frac{\partial v}{\partial t} - cDv, -\frac{\partial v}{\partial k} + \mathcal{H}(x, Dv)\right\} = 0 \quad \text{in }]0,T[\times]0,K[\times\mathbb{R}, \tag{39}$$

$$\max\left\{-\frac{\partial v}{\partial t} - cDv, -\frac{\partial v}{\partial k} + \mathcal{H}(x, Dv)\right\} \geq 0 \quad \text{on }]0,T[\times\{K\} \times \mathbb{R}, \tag{40}$$

$$\begin{aligned} v(T,k,x) \leq \arctan(x) \quad &\text{and} \quad \max\left\{-\tfrac{\partial v}{\partial t} - cDv, -\tfrac{\partial v}{\partial k} + \mathcal{H}(x, Dv)\right\} \geq 0 \\ &\text{if} \quad v(T,k,x) < \arctan(x) \quad \text{on } \{T\}\times]0,K] \times \mathbb{R}, \end{aligned} \tag{41}$$

where

$$\mathcal{H}(x,p) = \max_{(w_1, w_2) \in \mathcal{K},\ |(w_1,w_2)| = 1} \left\{-(w_1 + xw_2)p\right\}.$$

It is not difficult to prove that the value function $\mathcal{V}$ is a classical solution of (39, while we refer to [MoRa] to show that it satisfies (40)–(41) in the viscosity sense.

THEOREM 5.2. *Assume* **(A0)**, **(A1)**. *Then the value function* $\mathcal{V} : [0,T]\times[0,K]\times\mathbb{R}^n \to \mathbb{R}$ *solves the boundary value problem* (5)–(7) *in the viscosity sense.*

Proof. We postpone the proof of this theorem to the end of this section. ∎

THEOREM 5.3. *Assume* **(A0)**, **(A1)**. *Then the value function* $\mathcal{V} : [0,T]\times[0,K]\times\mathbb{R}^n \to \mathbb{R}$ *is the unique viscosity solution of* (5)–(7) *among the bounded functions defined on* $[0,T]\times[0,K]\times\mathbb{R}^n$ *which are continuous on* $\partial(]0,T[\times]0,K[\times\mathbb{R}^n)$.

Proof. This result follows straightforwardly from Corollary 6.1 of Theorem 3.1 in [MS2] in view of Theorem 5.2 and Corollary 4.2. ∎

Remark 5.1. When the boundedness assumption (8) in **(A1)** is weakened in the linear growth condition (12) introduced in Remark 2.1, in order to apply Corollary 6.1 in [MS2] we have to introduce the following stronger growth hypothesis on A, B, and $\tilde{D}$:

(A2) for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ for which

$$|A(t,x)| + |B(t,x)| \le C_\varepsilon + \varepsilon|x|, \quad |\tilde{D}(t,x)| \le C_\varepsilon + \varepsilon|x|^2 \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n.$$

Then under hypotheses **(A0)**, **(A1)** with (8) replaced by (12) and **(A2)**, we obtain the uniqueness for the viscosity solution to (5)–(7) among the functions v which are continuous on $\partial(]0,T[\times]0,K]\times\mathbb{R}^n)$ and such that

$$\sup_{x\in\mathbb{R}^n}\frac{|v(t,k,x)|}{1+|x|} < +\infty \qquad \text{uniformly for } \ (t,k)\in[0,T]\times[0,K].$$

Proof of Theorem 5.2. Since $\mathcal{V} = V$, let us prove the theorem for V. Owing to Theorem 4.1, V is continuous, so that $V^* = V_* = V$. In what follows, for any $\bar{z} = (\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbb{R}^n$ and any $r' > 0$ let us set $\Theta_{r'} \doteq [0\vee(\bar{t}-r'), T\wedge(\bar{t}+r')[\times[0\vee(\bar{k}-r'), K\wedge(\bar{k}+r')[\times B_n(r')$.

Step 1. Let us start by showing that V is a viscosity subsolution of (5)–(7). Since at any point of the form $(T,\bar{k},\bar{x})$ (with $\bar{k} < K$ and $\bar{x}\in\mathbb{R}^n$) it is clear that there exists a control rule $P \in \mathcal{R}^a(T,\bar{k},\bar{x})$ such that

$$P\,(\theta = 0) = 1,$$

from the very definition of V it follows that $V(T,\bar{k},\bar{x}) \le g(\bar{x})$. Hence it remains to prove that V is a viscosity subsolution of (5). We argue by contradiction. If this fails to hold, then there is a point $\bar{z} = (\bar{t},\bar{k},\bar{x}) \in]0,T[\times]0,K[\times\mathbb{R}^n$, a test function $\phi \in C_b^2([0,T]\times[0,K]\times\mathbb{R}^n)$, and a constant $r_1 > 0$ such that $\bar{z}$ is a local maximum point for $V - \phi$, that is,

$$V(z) - \phi(z) \le V(\bar{z}) - \phi(\bar{z}) \quad \forall z = (t,k,x) \in \overline{\Theta}_{r_1},$$

and

$$\max\left\{-\frac{\partial\phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})), -\frac{\partial\phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z}))\right\} > 0.$$

Here, either

$$-\frac{\partial\phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})) > 0 \tag{42}$$

or

$$-\frac{\partial\phi}{\partial k}(\bar{z})+\mathcal{H}(\bar{z},D\phi(\bar{z}))>0 \tag{43}$$

is verified. Since $\bar{t}<T$ and $\bar{k}<K$, in the definition of Θ_{r_1}, reducing r_1 if necessary, we can always assume that $\bar{t}+r_1<T$ and $\bar{k}+r_1<K$. If (42) is true, from the definition of $\mathcal{F}$, from the regularity hypotheses in **(A0)**, **(A1)** and from the fact that $\phi\in C^2$, it then follows that there exists some positive constant $r_2\leq r_1$ such that

$$-\frac{\partial\phi}{\partial t}(t,k,x)-\langle A(t,x),D\phi(t,k,x)\rangle-l_0(t,x)-\frac{1}{2}\mathrm{Tr}\{\tilde{D}(t,x)D^2\phi(t,k,x)\}>0$$

for all $z=(t,k,x)\in\overline{\Theta}_{r_2}$, and $\bar{t}+r_2<T$, $\bar{k}+r_2<K$. Take a control rule $P\in\mathcal{R}^a(\bar{t},\bar{k},\bar{x})$ such that

$$P\left(\mu_s=\delta_{\{0\}}\quad 0\leq s\leq\theta,\quad \theta=T-\bar{t}\right)=1.$$

It is easy to see that such a control rule exists; that, setting

$$\rho\doteq\inf\left\{s\in]0,T+K]:\ (t_s,k_s,\xi_s)\notin\Theta_{r_2}\right\},$$

by the continuity of the state process (t_s,k_s,ξ_s) one gets

$$P\left(T-\bar{t}>\rho\right)=1,\qquad P\left(\rho>0\right)=1;$$

and that, for $0\leq s<\rho$,

$$-\frac{\partial\phi}{\partial t}(t_s,k_s,\xi_s)-\langle A(t_s,\xi_s),D\phi(t_s,k_s,\xi_s)\rangle-\frac{1}{2}\mathrm{Tr}\{\tilde{D}(t_s,\xi_s)D^2\phi(t_s,k_s,\xi_s)\}-l_0(t_s,\xi_s)>0.$$

Since $\mu_s=\delta_{\{0\}}$, this yields that

$$E_P\left[\int_0^\rho\left(-\mathcal{L}\phi(t_s,k_s,\xi_s,\mu_s)-l_0(t_s,\xi_s)(1-|\mu_s|)-\langle l_1(t_s,\xi_s),\mu_s\rangle\right)ds\right]>0. \tag{44}$$

By the definition of control rule one has

$$\phi(t_\rho,k_\rho,\xi_\rho)=\phi(\bar{t},\bar{k},\bar{x})+\int_0^\rho\mathcal{L}\phi(t_s,k_s,\xi_s,\mu_s)\,ds+\mathcal{M}_\rho\phi,$$

where $\mathcal{M}_\rho\phi$ is a continuous square-integrable martingale with respect to P. Hence,

$$E_P\left[\phi(t_\rho,k_\rho,\xi_\rho)-\phi(\bar{t},\bar{k},\bar{x})\right]=E_P\left[\int_0^\rho\mathcal{L}\phi(t_s,k_s,\xi_s,\mu_s)\,ds\right].$$

Since $(t_\rho,k_\rho,\xi_\rho)\in\overline{\Theta}_{r_2}$, one has

$$\begin{aligned}&E_P\left[V(t_\rho,k_\rho,\xi_\rho)-V(\bar{t},\bar{k},\bar{x})\right]\leq E_P\left[\phi(t_\rho,k_\rho,\xi_\rho)-\phi(\bar{t},\bar{k},\bar{x})\right]\\&=E_P\left[\int_0^\rho\mathcal{L}\phi(t_s,k_s,\xi_s,\mu_s)\,ds\right]<-E_P\left[\int_0^\rho\left(l_0(t_s,\xi_s)(1-|\mu_s|)+\langle l_1(t_s,\xi_s),\mu_s\rangle\right)ds\right],\end{aligned}$$

where the last inequality follows from (44). This can be rewritten as

$$V(\bar{t},\bar{k},\bar{x})>E_P\left[\int_0^\rho\left(l_0(t_s,\xi_s)(1-|\mu_s|)+\langle l_1(t_s,\xi_s),\mu_s\rangle\right)+V(t_\rho,k_\rho,\xi_\rho)\right],$$

in contradiction with the dynamic programming principle (28).

If (43) is true, reasoning as before one can deduce that there exist some positive constant $r_2 \le r_1$ and a vector $\bar{w} \in \mathcal{K}$ with $|\bar{w}| = 1$ such that

$$-\frac{\partial \phi}{\partial k}(t,k,x) - \langle B(t,x)\bar{w}, p\rangle - \langle l_1(t,x), \bar{w}\rangle > 0 \quad \forall z = (t,k,x) \in \overline{\Theta}_{r_2}.$$

Then, let us introduce a control rule $P \in \mathcal{R}^a(\bar{t},\bar{k},\bar{x})$ such that

$$P\left(\mu_s = \delta_{\{\bar{w}\}} \quad 0 \le s \le \theta, \quad \theta = K - \bar{k}\right) = 1.$$

From now on, the proof proceeds, with obvious changes, as in the previous case. The proof that $\mathcal{V}$ is a viscosity subsolution of (5)–(7) is therefore concluded.

Step 2. Let us assume by contradiction that V fails to be a viscosity supersolution of (5)–(7). Thus there is a point $\bar{z} = (\bar{t},\bar{k},\bar{x}) \in [0,T] \times [0,K] \times \mathbb{R}^n$, a test function $\phi \in C_b^2([0,T] \times [0,K] \times \mathbb{R}^n)$, and a constant $r_1 > 0$ such that $\bar{z}$ is a local minimum point for $V - \phi$, that is,

$$V(z) - \phi(z) \ge V(\bar{z}) - \phi(\bar{z}) \quad \forall z = (t,k,x) \in \overline{\Theta}_{r_1},$$

and either of the following cases hold.

Case 1. $(\bar{t},\bar{k},\bar{x})$ is such that $\bar{t} < T$, $\bar{k} \le K$, and

$$-\frac{\partial \phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})) < 0 \;\text{ and }\; -\frac{\partial \phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z})) < 0.$$

Case 2. $(\bar{t},\bar{k},\bar{x})$ is such that $\bar{t} = T$, $\bar{k} < K$, and

$$\max\left\{-\frac{\partial \phi}{\partial t}(\bar{z}) + \mathcal{F}(\bar{z}, D\phi(\bar{z}), D^2\phi(\bar{z})), -\frac{\partial \phi}{\partial k}(\bar{z}) + \mathcal{H}(\bar{z}, D\phi(\bar{z}))\right\} < 0 \;\text{ and }\; V(\bar{z}) < g(\bar{x}).$$

Let us first consider Case 1. Since $\bar{t} < T$, in the definition of Θ_{r_1}, reducing r_1 if necessary, we can always assume that $\bar{t} + r_1 < T$. From the regularity hypotheses in **(A0)**, **(A1)** and from the fact that $\phi \in C^2$, it follows that there exist some positive constants $r_2 \le r_1$ and $\varepsilon > 0$ such that

$$(45) \quad -\frac{\partial \phi}{\partial t}(z) + \mathcal{F}(z, D\phi(z), D^2\phi(z)) < -\varepsilon \;\text{ and }\; -\frac{\partial \phi}{\partial k}(z) + \mathcal{H}(z, D\phi(z)) < -\varepsilon$$

for all $z = (t,k,x) \in \overline{\Theta}_{r_2}$, and $\bar{t} + r_2 \le \bar{t} + r_1 < T$. Take an optimal control rule $P \in \mathcal{R}^a(\bar{t},\bar{k},\bar{x})$. Such a control rule exists in view of Proposition 3.5. Let us notice that at any point $(\bar{t},\bar{k},\bar{x})$ with $\bar{t} < T$, every control rule $Q \in \mathcal{R}^a(\bar{t},\bar{k},\bar{x})$ verifies

$$(46) \qquad Q\left(\theta \ge T - \bar{t} > 0\right) = 1.$$

Moreover (see Remark 2.2), in both cases, $\bar{k} < K$ and $\bar{k} = K$, one has

$$(47) \qquad Q\left(k_s \le K \quad 0 \le s \le \theta\right) = 1.$$

Equations (46) and (47) hold, in particular, for $Q = P$. Let us define the exit time

$$\rho \doteq \inf\{s \in]0, T+K] : \\ (t_s, k_s, \xi_s) \notin [0 \vee (\bar{t} - r_2), \bar{t} + r_2[\times [0 \vee (\bar{k} - r_2), \bar{k} + r_2[\times B_n(r_2)\}.$$

Observe that, if $\bar{k} < K$, then it is not restrictive to assume that $\bar{k}+r_2 < K$, so that ρ coincides with the first exit time from the set Θ_{r_2}. In case $\bar{k} = K$, instead, $\bar{k}+r_2 > K$ and ρ may be greater than the first exit time from Θ_{r_2}. However, taking into account (47), it is not difficult to see that in both cases $\bar{k} < K$ and $\bar{k} = K$, one has

$$(t_s, k_s, \xi_s) \in \overline{\Theta}_{r_2} \qquad 0 \le s \le \rho \wedge \theta.$$

Now, from the continuity of the process (t_s, k_s, ξ_s) it follows that

$$P\left(\rho > 0\right) = 1,$$

which together with (46) yields that the stopping time $\rho' \doteq \rho \wedge \theta$ verifies

$$P\left(\rho' > 0\right) = 1.$$

Therefore by (45) it follows that

$$E_P\left[\int_0^{\rho'} \left(\mathcal{L}\phi(t_s, k_s, \xi_s, \mu_s) + l_0(t_s, \xi_s)(1 - |\mu_s|) + \langle l_1(t_s, \xi_s), \mu_s\rangle\right) ds\right] \ge \varepsilon E_P[\rho'].$$

Applying Ito's formula, we have

$$E_P[\phi(t_{\rho'}, k_{\rho'}, \xi_{\rho'})] = \phi(\bar{t}, \bar{k}, \bar{x}) + E_P\left[\int_0^{\rho'} \mathcal{L}\phi(t_s, k_s, \xi_s, \mu_s)\, ds\right]$$

which yields

$$\begin{aligned} &E_P\left[\phi(t_{\rho'}, k_{\rho'}, \xi_{\rho'}) - \phi(\bar{t}, \bar{k}, \bar{x})\right] = E_P\left[\textstyle\int_0^{\rho'} \mathcal{L}\phi(t_s, k_s, \xi_s, \mu_s)\, ds\right] \\ &\ge E_P\left[\textstyle\int_0^{\rho'} \left(-l_0(t_s, \xi_s)(1 - |\mu_s|) - \langle l_1(t_s, \xi_s), \mu_s\rangle\right) ds\right] + \varepsilon E_P[\rho']. \end{aligned}$$

Since $(t_{\rho'}, k_{\rho'}, \xi_{\rho'}) \in \overline{\Theta}_{r_2}$ and $E_P[\rho'] > 0$, setting $\varepsilon' \doteq \varepsilon E_P[\rho']$, $\varepsilon' > 0$ one has

$$\begin{aligned} &E_P\left[V(t_{\rho'}, k_{\rho'}, \xi_{\rho'}) - V(\bar{t}, \bar{k}, \bar{x})\right] \ge E_P\left[\phi(t_{\rho'}, k_{\rho'}, \xi_{\rho'}) - \phi(\bar{t}, \bar{k}, \bar{x})\right] \\ &\ge E_P\left[\textstyle\int_0^{\rho'} \left(-l_0(t_s, \xi_s)(1 - |\mu_s|) - \langle l_1(t_s, \xi_s), \mu_s\rangle\right) ds\right] + \varepsilon', \end{aligned}$$

which, rewritten as

$$V(\bar{t}, \bar{k}, \bar{x}) \le E_P\left[\int_0^{\rho'} \left(l_0(t_s, \xi_s)(1 - |\mu_s|) + \langle l_1(t_s, \xi_s), \mu_s\rangle\right) ds + V(t_{\rho'}, k_{\rho'}, \xi_{\rho'})\right] - \varepsilon',$$

contradicts the dynamic programming principle (28).

Let $(T, \bar{k}, \bar{x})$ be some point satisfying the assumptions of Case 2. Since by definition $V(T, K, \bar{x}) = g(\bar{x})\ \ \forall \bar{x} \in \mathbb{R}^n$, it must be that $\bar{k} < K$. Hence in the definition of Θ_{r_1}, reducing r_1 if necessary, we can always assume that $\bar{k} + r_1 < K$, $r_1 < T$ (while $(T+r_1) \wedge T = T$), and from the regularity hypotheses in **(A0)**, **(A1)** and from $\phi \in C^2$, it follows that there exist some positive constants $r_2 \le r_1$ and $\varepsilon > 0$ such that

$$-\frac{\partial \phi}{\partial t}(z) + \mathcal{F}(z, D\phi(z), D^2\phi(z)) < -\varepsilon, \quad -\frac{\partial \phi}{\partial k}(z) + \mathcal{H}(z, D\phi(z)) < -\varepsilon, \quad V(z) < g(x) - \varepsilon$$

for all $z=(t,k,x)\in\overline{\Theta}_{r_2}$, where now $\Theta_{r_2}=[T-r_2,T[\times[0\vee(\bar{k}-r_2),\bar{k}+r_2[\times B_n(r_2)$. Let $P\in\mathcal{R}^a(T,\bar{k},\bar{x})$ be an optimal control rule, which exists in view of Proposition 3.5. It is not difficult to see that every control rule $Q\in\mathcal{R}^a(T,\bar{k},\bar{x})$ is such that

$$Q\left(t_s=T,\quad 0\le s\le\theta\right)=1, \tag{48}$$

and, if in addition $J(T,\bar{k},\bar{x},Q)<g(\bar{x})$, then ($\bar{k}<K$ and)

$$Q\left(\theta>0\right)=1. \tag{49}$$

Since $V(T,\bar{k},\bar{x})<g(\bar{x})$ by hypothesis, (48) and (49) hold, in particular, for $Q=P$. Let us define

$$\rho\doteq\inf\left\{s\in]0,T+K]:\ (t_s,k_s,\xi_s)\notin[T-r_2,T+r_2[\times[0\vee(\bar{k}-r_2),\bar{k}+r_2[\times B_n(r_2)\right\}.$$

The exit time ρ may be greater than the first exit time from Θ_{r_2}, but from (48) it follows that

$$(t_s,k_s,\xi_s)\in\overline{\Theta}_{r_2},\qquad 0\le s\le\rho\wedge\theta.$$

Now the continuity of the process (t_s,k_s,ξ_s) implies that

$$P\left(\rho>0\right)=1.$$

Owing to (49), this yields that the stopping time $\rho'\doteq\rho\wedge\theta$ verifies

$$P\left(\rho'>0\right)=1.$$

From now on, the proof is analogous to the proof of Case 1, so we omit it. This proves that $\mathcal{V}$ is a viscosity supersolution of (5)–(7).

The proof that $\mathcal{V}$ is a viscosity solution of (5)–(7) is therefore concluded. □

6. Existence of solutions for generalized Cauchy problems with discontinuous Hamiltonians. The results of the previous sections allow us to prove the existence of a viscosity solution to a boundary value problem involving a second order semilinear Hamilton–Jacobi–Bellman equation such as

$$-\frac{\partial v}{\partial t}-\frac{1}{2}\mathrm{Tr}\{\tilde{D}(t,x)D^2v\}+H\left(t,x,\frac{\partial v}{\partial k},Dv\right)=0\qquad\text{on}\quad]0,T[\times]0,K[\times\mathbb{R}^n, \tag{50}$$

and mixed boundary conditions such as

$$-\frac{\partial v}{\partial t}-\frac{1}{2}\mathrm{Tr}\{\tilde{D}(t,x)D^2v\}+H\left(t,x,\frac{\partial v}{\partial k},Dv\right)\ge 0\quad\text{on }]0,T[\times\{K\}\times\mathbb{R}^n, \tag{51}$$

$$\begin{aligned} v\le g\ \text{ and }\ -\tfrac{\partial v}{\partial t}-\tfrac{1}{2}\mathrm{Tr}\{\tilde{D}(t,x)D^2v\}+H(t,x,\tfrac{\partial v}{\partial k},Dv)\ge 0 \text{ if } v<g\\ \text{on } \{T\}\times]0,K]\times\mathbb{R}^n,\end{aligned} \tag{52}$$

for a (possibly discontinuous) Hamiltonian of the form

$$H(t,x,p_k,p)\doteq\sup_{w\in\mathcal{K}}\left\{-\langle A(t,x)+B(t,x)w,p\rangle-l_0(t,x)-\langle l_1(t,x),w\rangle-p_k|w|\right\} \tag{53}$$

$\forall(t,x,p_k,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. We will refer to such a problem as a *generalized Cauchy problem with discontinuous Hamiltonian.*

We point out that the Hamiltonian H in (53), defined via a maximization over the unbounded control set $\mathcal{K}$, is the natural Hamiltonian related to the original minimization problem (4). In other words, at least formally, one expects that the value function $\mathcal{V}$ is a viscosity solution to the generalized Cauchy problem (50)–(52) rather than to (5)–(7). Such an observation motivates the study of such a boundary value problem (also in more general form, as in [MS2]), mainly dealing with existence and uniqueness of solutions.

Since H in (53) is in general discontinuous and equal to $+\infty$ in many points, we interpret solutions to (50)–(52) in the sense of the definition, due to Ishii [I], of discontinuous viscosity solutions for discontinuous Hamiltonians which we recall in the definition below.

DEFINITION 6.1. *A locally bounded function v defined on $[0,T[\times[0,K]\times\mathbb{R}^n$ is a* viscosity subsolution *of* (50)–(52) *if for every point $\bar{z} = (\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbb{R}^n$ and for every map $\phi \in C_b^2([0,T]\times[0,K]\times\mathbb{R}^n)$ such that $v^* - \phi$ has a local maximum at $\bar{z}$ one has*

$$-\frac{\partial\phi}{\partial t}(\bar{z}) - \mathrm{Tr}\{\tilde{D}(\bar{z})D^2\phi(\bar{z})\} + H_*\left(\bar{z}, \frac{\partial\phi}{\partial k}(\bar{z}), D\phi(\bar{z})\right) \le 0$$

if $\bar{z} \in]0,T[\times]0,K[\times\mathbb{R}^n$, and

$$v^*(\bar{z}) \le g(\bar{x})$$

if $\bar{z} \in \{T\}\times]0,K]\times\mathbb{R}^n$.

A locally bounded function v defined on $[0,T[\times[0,K]\times\mathbb{R}^n$ is a viscosity supersolution *of* (50)–(52) *if for every point $\bar{z} = (\bar{t},\bar{k},\bar{x}) \in [0,T]\times[0,K]\times\mathbb{R}^n$ and for every map $\phi \in C_b^2([0,T]\times[0,K]\times\mathbb{R}^n)$ such that $v_* - \phi$ has a local minimum at $\bar{z}$ one has*

$$-\frac{\partial\phi}{\partial t}(\bar{z}) - \mathrm{Tr}\{\tilde{D}(\bar{z})D^2\phi(\bar{z})\} + H^*\left(\bar{z}, \frac{\partial\phi}{\partial k}(\bar{z}), D\phi(\bar{z})\right) \ge 0$$

if $\bar{z} \in]0,T[\times]0,K]\times\mathbb{R}^n$, and

$$-\frac{\partial\phi}{\partial t}(\bar{z}) - \mathrm{Tr}\{\tilde{D}(\bar{z})D^2\phi(\bar{z})\} + H^*\left(\bar{z}, \frac{\partial\phi}{\partial k}(\bar{z}), D\phi(\bar{z})\right) \ge 0 \quad \text{or} \quad v_*(\bar{z}) \ge g(\bar{x})$$

if $\bar{z} \in \{T\}\times]0,K]\times\mathbb{R}^n$.

A locally bounded function v defined on $[0,T[\times[0,K]\times\mathbb{R}^n$ is a called a viscosity solution *of* (50)–(52) *if it is both a viscosity sub- and supersolution of* (50)–(52).

We can show that $\mathcal{V}$ is a viscosity solution to the generalized Cauchy problem (50)–(52) since there exists a one-to-one correspondence among solutions to (50)–(52) and solutions to (5)–(7), as specified by the following theorem.

THEOREM 6.2 (see [MS2, Theorem 3.4]). *Assume* **(A0)***,* **(A1)***. Let* v *(resp., v):* $[0,T]\times[0,K]\times\mathbb{R}^n \to \mathbb{R}$ *be a upper (resp., lower) semicontinuous locally bounded function. Then*

(a) v *is a viscosity subsolution to* (50)–(52) *if and only if it is a viscosity subsolution to* (5)–(7)*,*

(b) *v is a viscosity supersolution to* (50)–(52) *if and only if it is a viscosity supersolution to* (5)–(7).

Therefore we can state the following existence (and uniqueness) theorem whose proof is a consequence of Theorems 6.2 and 5.3.

THEOREM 6.3. *Assume* **(A0)**, **(A1)**. *Then the value function* $\mathcal{V}$ *solves the boundary value problem* (50)–(52) *in the viscosity sense. Moreover, its continuous extension to* $[0,T]\times[0,K]\times\mathbb{R}^n$ *is the unique viscosity solution of* (50)–(52) *among the bounded functions defined on* $[0,T]\times[0,K]\times\mathbb{R}^n$ *which are continuous on* $\partial(]0,T[\times]0,K[\times\mathbb{R}^n)$.

7. Appendix. Before proving Lemma 7.3 on the claim stated in Theorem 3.3, we need to introduce some definitions and prove some technical results in Lemmas 7.1 and 7.2.

Let us consider the noncanonical control $\alpha = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{(w_{0_s},w_s)\}, \{(t_s,k_s,\xi_s)\}, \theta)$ given by (24), whose existence is proved in Theorem 3.3. Let us understand that whenever the operators J and $\mathcal{L}$ have to be evaluated on such α, the constraint $w_{0_s} = 1-|w_s|$ in their definition must be dropped. Let us notice that since we are only interested with the (random) time interval $0\le s\le\theta$, with a small abuse of notation, we will denote still by α the control in which (w_{0_s},w_s) is replaced by $(w_{0_s},w_s)\chi_{\{s\le\theta\}} + (1,0)\chi_{\{s>\theta\}}$.

Remark 7.1. Given the control $\alpha = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{(w_{0_s},w_s)\}, \{(t_s,k_s,\xi_s)\}, \theta)$, by (24), in Case 1 has

$$\int_0^\theta (w_{0_s}+|w_s|)\,ds = 0$$

(such an eventuality can happen only if $\bar t = T$), it is easy to check that any control $\check\alpha$, $\check\alpha = (\Omega,\{\mathcal{F}\},P,\{\mathcal{F}_s\},\check w_s,(\check t_s,\check k_s,\check\xi_s),\check\theta)\in\Gamma^a(\bar t,\bar k,\bar x)$, such that

$$P\left(\check\theta = 0\right) = 1,$$

verifies the claim in Theorem 3.3 and $J(\bar t,\bar k,\bar x,\check\alpha) = J(\bar t,\bar k,\bar x,\alpha)$.

LEMMA 7.1. *Assume* **(A0)**, **(A1)**. *Let us consider the noncanonical control* $\alpha = (\Omega,\mathcal{F},P,\{\mathcal{F}_s\},\{(w_{0_s},w_s)\}, \{(t_s,k_s,\xi_s)\},\theta)$, *given by* (24). *Assume that* $\int_0^\theta(w_{0_s}+|w_s|)\,ds>0$. *Let us define*

$$\Phi_s \doteq \int_0^s (w_{0_r}+|w_r|)\,dr,$$

for $0\le s\le T+K$. *Let us denote by* $\{\Psi_\sigma\}$ *the right inverse of* Φ*:*

$$\Psi_\sigma \doteq \inf\{s\ge 0:\ \Phi_s>\sigma\},$$

for $0\le\sigma\le\Phi_{T+K}$. *Then* $\Phi_{T+K}>0$ *and* $\{\Psi_\sigma\}$ *is a right continuous time change satisfying the following properties:*

(i) $\Psi_{\Phi_s}\ge s\ \forall s\ge 0$, $\Phi_{\Psi_\sigma}=\sigma\ \forall\sigma\ge 0$;

(ii) *let*

$$\check{\mathcal{F}}_\sigma \doteq \mathcal{F}_{\Psi_\sigma}\forall\sigma>0; \tag{54}$$

then $\check{\mathcal{F}}_\sigma$ *is a filtration on the probability space* $(\Omega,\mathcal{F},P)$*;*

(iii) Φ_θ *is a* $\check{\mathcal{F}}_\sigma$*-stopping time such that* $\Phi_\theta\le T+K$.

Proof. The proof follows from the definition of time change and right inverse and from Proposition 1.1, Chapter V, in [RY]. $\Phi_\theta\le T+K$ since $w_{0_s}+|w_s|\le 1$ for $s\ge0$ by definition. ∎

LEMMA 7.2. *Assume* **(A0)**, **(A1)**. *Let us consider the noncanonical control* $\alpha = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{(w_{0_s}, w_s)\}, \{(t_s, k_s, \xi_s)\}, \theta)$, *given by* (24). *Assume that* $\int_0^\theta (w_{0_s} + |w_s|)\, ds > 0$. *On the probability space* $(\Omega, \mathcal{F}, P)$ *let us consider the filtration* $\check{\mathcal{F}}_\sigma$ *given by* (54). *Let us define the processes*

$$\check{t}_\sigma \doteq t_{\Psi_\sigma}, \quad \check{u}_\sigma \doteq \int_0^{\Psi_\sigma} w_r\, dr, \quad \check{k}_\sigma \doteq k_{\Psi_\sigma},$$

for $0 \le \sigma \le \Phi_{T+K}$. *Then there exists a* $\check{\mathcal{F}}_\sigma$*-progressively measurable process* $\{\check{w}_\sigma\}$, $\overline{B}_m(1) \cap \mathcal{K}$ *valued, such that, for* $0 \le \sigma \le \Phi_{T+K}$,

$$\begin{aligned} \check{t}_\sigma &= \bar{t} + \textstyle\int_0^\sigma (1 - |\check{w}_r|)\, dr \left(= \bar{t} + \int_0^{\Psi_\sigma} w_{0_r}\, dr\right), \\ \check{u}_\sigma &= \textstyle\int_0^\sigma \check{w}_r\, dr \left(= \int_0^{\Psi_\sigma} w_r\, dr\right), \\ \check{k}_\sigma &= \bar{k} + \textstyle\int_0^\sigma |\check{w}_r|\, dr \left(= \bar{k} + \int_0^{\Psi_\sigma} |w_r|\, dr\right). \end{aligned} \tag{55}$$

Proof. Let $\{u_s\}$ denote the (strong) solution to

$$u_s = \int_0^s w_r\, dr. \tag{56}$$

Since $\sigma = \Phi_s$ is the arclength parameter of both the processes (t_s, u_s) and (t_s, k_s), we know that $(\check{t}_\sigma, \check{u}_\sigma, \check{k}_\sigma)$ is absolutely continuous for $0 \le \sigma \le \Phi_{T+K}$. Consequently, from Proposition 3.13, Chapter I, in [JS], it follows that there exists a $\check{\mathcal{F}}_\sigma$-progressively measurable process $(\tilde{w}_{0_\sigma}, \tilde{w}_\sigma, z_\sigma)$, $\mathbb{R}_+ \times \mathcal{K} \times \mathbb{R}_+$-valued, such that, for $0 \le \sigma \le \Phi_{T+K}$,

$$\check{t}_\sigma = \bar{t} + \int_0^\sigma \tilde{w}_{0_r}\, dr, \quad \check{u}_\sigma = \int_0^\sigma \tilde{w}_r\, dr, \quad \check{k}_\sigma = \bar{k} + \int_0^\sigma z_r\, dr.$$

Moreover, by the properties of the arclength parameter for almost all $\omega \in \Omega$ there exists a set of measure zero $\mathcal{N}_\omega$ such that $\check{t}'_\sigma(\omega) + |\check{u}'_\sigma(\omega)| = 1$ and $\check{t}'_\sigma(\omega) + \check{k}'_\sigma(\omega) = 1$ for every $\sigma \notin \mathcal{N}_\omega$. This implies that $\tilde{w}_{0_\sigma}(\omega) + |\tilde{w}_\sigma(\omega)| = 1$ and $z_\sigma(\omega) = |\tilde{w}_\sigma(\omega)|$ for $\sigma \notin \mathcal{N}_\omega$. Let us define for every $\sigma \ge 0$ and for $i = 1, \dots, m$ the process

$$\check{w}^i_\sigma \doteq (-1 \vee \tilde{w}^i_\sigma) \wedge 1.$$

$\check{w}$ is $\check{\mathcal{F}}_\sigma$-progressively measurable, $\overline{B}_m(1) \cap \mathcal{K}$-valued. Moreover, $(\check{t}_\sigma, \check{u}_\sigma, \check{k}_\sigma)$ is indistinguishable from $(\bar{t} + \int_0^\sigma (1 - |\check{w}_r|)\, dr, \int_0^\sigma \check{w}_r\, dr, \bar{k} + \int_0^\sigma |\check{w}_r|\, dr)$. Indeed for almost all $\omega \in \Omega$ we have, for $0 \le \sigma \le \Phi_{T+K}$,

$$\check{t}_\sigma(\omega) = \bar{t} + \textstyle\int_{[0,\sigma]} \tilde{w}_{0_r}(\omega)\, dr = \bar{t} + \int_{[0,\sigma]\setminus\mathcal{N}_\omega} (1 - |\tilde{w}_r(\omega)|)\, dr = \bar{t} + \int_0^\sigma (1 - |\check{w}_r(\omega)|)\, dr,$$

$$\check{u}_\sigma(\omega) = \textstyle\int_{[0,\sigma]} \tilde{w}_r(\omega)\, dr = \int_{[0,\sigma]\setminus\mathcal{N}_\omega} \tilde{w}_r(\omega)\, dr = \int_0^\sigma \check{w}_r(\omega)\, dr,$$

$$\check{k}_\sigma(\omega) = \bar{k} + \textstyle\int_{[0,\sigma]} z_r(\omega)\, dr = \bar{k} + \int_{[0,\sigma]\setminus\mathcal{N}_\omega} z_r(\omega)\, dr = \bar{k} + \int_0^\sigma |\check{w}_r(\omega)|\, dr. \qquad \square$$

LEMMA 7.3. *Assume* **(A0)**, **(A1)**. *Let us consider the noncanonical control* $\alpha = (\Omega, \mathcal{F}, P, \{\mathcal{F}_s\}, \{(w_{0_s}, w_s)\}, \{(t_s, k_s, \xi_s)\}, \theta)$, *given by* (24). *Assume that* $\int_0^\theta (w_{0_s} + |w_s|)\, ds > 0$ *and* $J(\bar{t}, \bar{k}, \bar{x}, \alpha) < +\infty$. *Then the control*

$$\check{\alpha} \doteq (\Omega, \mathcal{F}, P, \check{\mathcal{F}}_\sigma, \{\check{w}_\sigma\}, \{(\check{t}_\sigma, \check{k}_\sigma, \check{\xi}_\sigma)\}, \check{\theta}),$$

where $\check\theta \doteq \Phi_\theta$, $\{\check w_\sigma\}$ *is the process whose existence is proved in* Lemma 7.2, $\{\check{\mathcal{F}}_\sigma\}$ *is given by* (54), $\{(\check t_\sigma, \check k_\sigma)\}$ *are given by* (55), $\check\xi_\sigma \doteq \xi_{\Psi_\sigma}$, *for* $0 \le \sigma \le \Phi_{T+K}$, *verifies* **(B3)**, **(B4)**, *and is such that*

$$J(\bar t, \bar k, \bar x, \check\alpha) = J(\bar t, \bar k, \bar x, \alpha).$$

Proof. We already proved that $\check\alpha$ verifies **(B3)** in Lemmas 7.1 and 7.2; therefore, in order to prove that condition **(B4)** holds, one has to show that for every $\varphi \in \mathcal{C}_b^2(\mathbb{R}^{2+n})$ $\check{\mathcal{M}}_\sigma(\varphi, \check\alpha)$ is a $(P, \{\check{\mathcal{F}}_\sigma\})$ square integrable martingale for $\sigma \in [0, \Phi_{T+K}]$, where

$$\check{\mathcal{M}}_\sigma(\varphi, \check\alpha) \doteq \varphi(\check t_\sigma, \check k_\sigma, \check\xi_\sigma) - \int_0^\sigma \mathcal{L}\varphi(\check t_r, \check k_r, \check\xi_r, \check w_r)\, dr.$$

To this end let us consider the square integrable martingale $\mathcal{M}_s(\varphi, \alpha)$ associated to the control α and let us notice that it is Ψ-continuous (that is, it is constant on each stochastic interval where $(w_{0_s}, w_s) = (0,0)$; see Definition 1.3 Chapter V, [RY]) and that by (56) and (55) one has

$$\begin{array}{ll} dt_s = w_{0_s} d_s, & dt_{\Psi_\sigma} = d\check t_\sigma = (1 - |\check w_\sigma|)d\sigma, \\ du_s = w_s ds, & du_{\psi_\sigma} = \check w_\sigma d\sigma, \\ dk_s = |w_s| ds, & dk_{\Psi_\sigma} = d\check k_\sigma = |\check w_\sigma| d\sigma. \end{array}$$

By Proposition 1.4, Chapter V of [RY], for every process H which is $\mathcal{F}_s$-progressively measurable, if we denote by $\check H_\sigma \doteq H_{\psi_\sigma}$, since the process (t_s, u_s, k_s) is Ψ-continuous, one has that

$$\begin{array}{l} \int_0^{\Psi_\sigma} H_s w_{0_s} ds = \int_0^{\Psi_\sigma} H_s\, dt_s = \int_0^\sigma \check H_\sigma d\check t_\sigma = \int_0^\sigma \check H_\sigma (1 - |\check w_\sigma|) d\sigma, \\ \int_0^{\Psi_\sigma} H_s w_s ds = \int_0^{\Psi_\sigma} H_s\, du_s = \int_0^\sigma \check H_\sigma d\check u_\sigma = \int_0^\sigma \check H_\sigma \check w_\sigma\, d\sigma, \\ \int_0^{\Psi_\sigma} H_s |w_s| ds = \int_0^{\Psi_\sigma} H_s\, dk_s = \int_0^\sigma \check H_\sigma d\check k_\sigma = \int_0^\sigma \check H_\sigma |\check w_\sigma| d\sigma. \end{array} \tag{57}$$

Therefore, for any $\varphi \in \mathcal{C}_b^2(\mathbb{R}^{2+n})$, by applying the first equation of (57) with $H_s = \frac{1}{2}\sum_{ij} \tilde D_{ij}(t_s, \xi_s)\, \frac{\partial^2\varphi}{\partial x_i \partial x_j}(t_s, k_s, \xi_s) + \sum_i A_i(t_s, \xi_s) \frac{\partial\varphi}{\partial \xi_i}(t_s, k_s, \xi_s) + \frac{\partial\varphi}{\partial t}(t_s, k_s, \xi_s)$, the second one with $H_s = (B_1(t_s, \xi_s) \frac{\partial\varphi}{\partial x_1}(t_s, k_s, \xi_s), \dots, B_n(t_s, \xi_s) \frac{\partial\varphi}{\partial x_n}(t_s, k_s, \xi_s))^T$, the third one with $H_s = \frac{\partial\varphi}{\partial k}(t_s, k_s, \xi_s)$, by Proposition 1.5, Chapter V of [RY] one concludes that $\check{\mathcal{M}}_\sigma(\varphi, \check\alpha)$ is a martingale since it coincides with $\mathcal{M}_{\psi_\sigma}(\varphi, \alpha)$. Finally, it is also easy to see that $\check\alpha$ is admissible. Indeed since $J(\bar t, \bar k, \bar x, \alpha) < +\infty$, then $t_\theta = T$ and $k_\theta \le K$, which implies by Proposition 1.4, Chapter V of [RY] that

$$\check t_{\Phi_\theta} = \bar t + \int_0^{\Phi_\theta} (1 - |\check w_r|)\, dr = t_\theta = T \quad \text{and} \quad \check k_{\Phi_\theta} = \bar k + \int_0^{\Phi_\theta} |\check w_r|\, dr = k_\theta \le K$$

and

$$J(\bar t, \bar k, \bar k, \check\alpha) = E_P\left[\int_0^{\Phi_\theta} (l_0(\check t_\sigma, \check\xi_\sigma)(1 - |\check w_\sigma|) + \langle l_1(\check t_\sigma, \check\xi_\sigma), \check w_\sigma\rangle)\, d\sigma \right. \\ \left. + g(\check\xi_{\Phi_\theta}) + G(\check t_{\Phi_\theta}, \check k_{\Phi_\theta}) \right]. \tag{58}$$

By the first and second equations in (57) with $H_s = l_0(t_s, \xi_s)$ and $H_s = l_1(t_s, \xi_s)$, respectively, and by the fact that, by Proposition 3.1, $\check\xi_{\Phi_\theta} = \xi_\theta$, one has $J(\bar t, \bar k, \bar k, \check\alpha) = J(\bar t, \bar k, \bar x, \alpha)$. $\square$

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