# Generalized rank-constrained matrix approximations

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#### Abstract

In this paper we give an explicit solution to the rank constrained matrix approximation in Frobenius norm, which is a generalization of the classical approximation of an  $m \times n$  matrix A by a matrix of rank k at most.

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# 1 Introduction

Let  $\mathbb{C}^{m \times n}$  be set of  $m \times n$  complex valued matrices, and denote by  $\mathcal{R}(m, n, k) \subseteq \mathbb{C}^{m \times n}$ the variety of all  $m \times n$  matrices of rank k at most. Fix  $A = [a_{ij}]_{i,j=1}^{m,n} \in \mathbb{C}^{m \times n}$ . Then  $A^* \in \mathbb{C}^{n \times m}$  is the conjugate transpose of A, and  $||A||_F := \sqrt{\sum_{i,j=1}^{m,n} |a_{ij}|^2}$ is the Frobenius norm of A. Recall that the singular value decomposition of A, abbreviated here as SVD, is given by  $A = U_A \Sigma_A V_A^*$ , where  $U_A \in \mathbb{C}^{m \times m}, V_A \in \mathbb{C}^{n \times n}$ are unitary matrices,  $\Sigma_A := \text{diag}(\sigma_1(A), \ldots, \sigma_{\min(m,n)}(A)) \in \mathbb{C}^{m \times n}$  is a generalized diagonal matrix, with the singular values  $\sigma_1(A) \ge \sigma_2(A) \ge \ldots \ge 0$  on the main diagonal. The number of positive singular values of A is r, which is equal to the rank of A, denoted by rank A. Let  $U_A = [\mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_m], V_A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n]$  be the representations of U, V in terms of their m, n columns respectively. Then  $\mathbf{u}_i$  and  $\mathbf{v}_i$ are called the *left* and the *right* singular vectors of A, respectively, that correspond to the singular value  $\sigma_i(A)$ . Let

$$P_{A,L} := \sum_{i=1}^{\operatorname{rank} A} \mathbf{u}_i \mathbf{u}_i^* \in \mathbb{C}^{m \times m}, \quad P_{A,R} := \sum_{i=1}^{\operatorname{rank} A} \mathbf{v}_i \mathbf{v}_i^* \in \mathbb{C}^{n \times n},$$
(1.1)

be the orthogonal projections on the range of A and  $A^*$  respectively. Denote by

$$A_k := \sum_{i=1}^k \sigma_i(A) \mathbf{u}_i \mathbf{v}_i^* \in \mathbb{C}^{m \times m}$$

for k = 1, ..., rank A. For k > rank A we define  $A_k := A \ (= A_{\text{rank } A})$ . For  $1 \le k < \text{rank } A$ , the matrix  $A_k$  is uniquely defined if and only if  $\sigma_k(A) > \sigma_{k+1}(A)$ .

The enormous application of SVD decomposition of A in pure and applied mathematics, is derived from the following approximation property:

$$\min_{X \in \mathcal{R}(m,n,k)} ||A - X||_F = ||A - A_k||_F, \quad k = 1, \dots$$
(1.2)

The latter is known as the Eckart-Young theorem [2]. We note that the work [2] implied a number of extensions. We cite [4, 5, 7, 8] as some recent references. Another application of SVD is a formula for the Moore-Penrose inverse  $A^{\dagger} := V_A \Sigma_A^{\dagger} U_A^* \in \mathbb{C}^{n \times m}$  of A, where  $\Sigma_A^{\dagger} := \text{diag}(\frac{1}{\sigma_1(A)}, \dots, \frac{1}{\sigma_{\text{rank } A}(A)}, 0, \dots, 0) \in \mathbb{C}^{n \times m}$ . See for example [1].

### 2 Main Result

Below, we provide generalizations of the classical minimal problem given in (1.2).

**Theorem 2.1** Let matrices  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{m \times p}$  and  $C \in \mathbb{C}^{q \times n}$  be given. Then

$$X = B^{\dagger} (P_{B,L} A P_{C,R})_k C^{\dagger} \tag{2.1}$$

is a solution to the minimal problem

$$\min_{X \in \mathcal{R}(p,q,k)} ||A - BXC||_F,$$
(2.2)

having the minimal  $||X||_F$ . This solution is unique if and only if either

$$k \ge \operatorname{rank} P_{B,L}AP_{C,R}$$
 or  $1 \le k < \operatorname{rank} P_{B,L}AP_{C,R}$ 

and

$$\sigma_k(P_{B,L}AP_{C,R}) > \sigma_{k+1}(P_{B,L}AP_{C,R})$$

**Proof of Theorem 2.1** Recall that the Frobenius norm is invariant under the multiplication from the left and the right by the corresponding unitary matrices. Hence  $||A - BXC||_F = ||\widetilde{A} - \Sigma_B \widetilde{X} \Sigma_C||$ , where  $\widetilde{A} := U_B^* A V_C$  and  $\widetilde{X} := V_B^* X U_C$ . Clearly, X and  $\widetilde{X}$  have the same rank and the same Frobenius norm. Thus, it is enough to consider the minimal problem  $\min_{\widetilde{X} \in \mathcal{R}(p,q,k)} ||\widetilde{A} - \Sigma_B \widetilde{X} \Sigma_C||_F$ .

Let  $s = \operatorname{rank} B$  and  $t = \operatorname{rank} C$ . Clearly if B or C is a zero matrix, then  $X = \mathbf{0}$  is the solution to the minimal problem (2.2). In this case either  $P_{B,L}$  or  $P_{C,R}$  are zero matrices, and the theorem holds trivially in this case.

Let us consider the case  $1 \leq s, 1 \leq t$ . Define  $B_1 := \operatorname{diag}(\sigma_1(B), \ldots, \sigma_s(B)) \in \mathbb{C}^{s \times s}, C_1 := \operatorname{diag}(\sigma_1(C), \ldots, \sigma_t(C)) \in \mathbb{C}^{t \times t}$ . Partition  $\widetilde{A}$  and  $\widetilde{X}$  into four block matrices  $A_{ij}$  and  $X_{ij}$  with i, j = 1, 2 so that  $\widetilde{A} = [A_{ij}]_{i,j=1}^2$  and  $\widetilde{X} = [X_{ij}]_{i,j=1}^2$ , where  $A_{11}, X_{11} \in \mathbb{C}^{s \times t}$ . (For certain values of s and t, we may have to partition  $\widetilde{A}$  or  $\widetilde{X}$  to less than four block matrices.) Next, observe that  $Z := \Sigma_B \widetilde{X} \Sigma_C = [Z_{ij}]_{i,j=1}^2$ , where  $Z_{11} = B_1 X_{11} C_1$  and all other blocks  $Z_{ij}$  are zero matrices. Since  $B_1$  and  $C_1$  are invertible we deduce

$$\operatorname{rank} Z = \operatorname{rank} Z_{11} = \operatorname{rank} X_{11} \le \operatorname{rank} X \le k$$

The approximation property of  $(A_{11})_k$  yields the inequality  $||A_{11} - Z_{11}||_F \ge ||A_{11} - (A_{11})_k||_F$  for any  $Z_{11}$  of rank k at most. Hence for any Z of the above form,

$$||\widetilde{A} - Z||_F^2 = ||A_{11} - Z_{11}||_F^2 + \sum_{2 < i+j \le 4} ||A_{ij}||_F^2 \ge ||A_{11} - (A_{11})_k||_F^2 + \sum_{2 < i+j \le 4} ||A_{ij}||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{11} - (A_{11})_k||_F^2 + \sum_{1 \le i+j \le 4} ||A_{ij}||_F^2 \le ||A_{ij} - A_{ij}||_F^2 \le ||A_{ij} -$$

Thus  $\widehat{X} = [X_{ij}]_{i,j=1}^2$ , where  $X_{11} = B_1^{-1}(A_{11})_k C_1^{-1}$  and  $X_{ij} = \mathbf{0}$  for all  $(i, j) \neq (1, 1)$ is a solution to the problem  $\min_{\widetilde{X} \in \mathcal{R}(p,q,k)} ||\widetilde{A} - \Sigma_B \widetilde{X} \Sigma_C||_F$  with the minimal Frobenius form. This solution is unique if and only if the solution  $Z_{11} = (A_{11})_k$  is the unique solution to the problem  $\min_{Z_{11} \in \mathcal{R}(s,t,k)} ||A_{11} - Z_{11}||_F$ . This happens if either  $k \geq \operatorname{rank} A_{11}$  or  $1 \leq k < \operatorname{rank} A_{11}$  and  $\sigma_k(A_{11}) > \sigma_{k+1}(A_{11})$ . A straightforward calculation shows that  $\widehat{X} = \Sigma_B^{\dagger}(P_{\Sigma_B,L} \widetilde{A} P_{\Sigma_C,R})_k \Sigma_C^{\dagger}$ . Thus, a solution of (2.2) with the minimal Frobenius norm is given by

$$X = B^{\dagger} U_B (P_{\Sigma_B, L} U_B^* A V_C P_{\Sigma_C, R})_k V_C^* C^{\dagger}$$
  
=  $B^{\dagger} U_B (U_B^* P_{B, L} A P_{C, R} V_C)_k V_C^* C^{\dagger}$   
=  $B^{\dagger} (P_{B, L} A P_{C, R})_k C^{\dagger}.$ 

This solution is unique if and only if either  $k \geq \operatorname{rank} P_{B,L}AP_{C,R}$  or  $1 \leq k < \operatorname{rank} P_{B,L}AP_{C,R}$  and  $\sigma_k(P_{B,L}AP_{C,R}) > \sigma_{k+1}(P_{B,L}AP_{C,R})$ .

A special case of the minimal problem (2.2), where X is a rank one matrix and C the identity matrix, was considered by Michael Elad [3] in the context of image processing.

## 3 Examples

First observe that the classical approximation problem given by (1.2) is equivalent to the case  $m = p, n = q, B = I_m, C = I_n$ . (Here,  $I_m$  is the  $m \times m$  identity matrix.) Clearly  $P_{I_m,L} = I_m, P_{I_n,R} = I_n, I_m^{\dagger} = I_m, I_n^{\dagger} = I_n$ . In this case we obtain the classical solution  $B^{\dagger}(P_{B,L}AP_{C,R})_k C^{\dagger} = A_k$ .

Second, if p = m, q = n and B, C are non-singular, then rank  $(BXC) = \operatorname{rank} X$ . In this case,  $P_{B,L} = I_m$  and  $P_{C,R} = I_n$ , and the solution to (2.2) is given by  $X = B^{-1}A_kC^{-1}$ .

Next, a particular case of the problem (2.2) occurs in study of a random vector estimation (see, for example, [9, 6]) as follows. Let  $(\Omega, \Sigma, \mu)$  be a probability space, where  $\Omega$  is the set of outcomes,  $\Sigma$  a  $\sigma$ -field of measurable subsets  $\Delta \subset \Omega$  and  $\mu : \Sigma \mapsto [0, 1]$  an associated probability measure on  $\Sigma$  with  $\mu(\Omega) = 1$ . Suppose that  $\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)$  and  $\mathbf{y} \in L^2(\Omega, \mathbb{R}^n)$  are random vectors such that  $\mathbf{x} = (x_1, \ldots, x_m)^T$  and  $\mathbf{y} = (y_1, \ldots, y_n)^T$  with  $x_i, y_j \in L^2(\Omega, \mathbb{R})$  for  $i = 1, \ldots, m$  and  $j = 1, \ldots, n$ , respectively. Let  $E_{xy} = [e_{ij,xy}] \in \mathbb{R}^{m \times n}, E_{yy} = [e_{jk,yy}] \in \mathbb{R}^{n \times n}$  be correlation matrices with entries

$$e_{ij,xy} = \int_{\Omega} x_i(\omega) y_j(\omega) d\mu(\omega), \quad e_{jk,yy} = \int_{\Omega} y_j(\omega) y_k(\omega) d\mu(\omega),$$
$$i = 1, \dots, m, \quad j, k = 1, \dots, n, \quad \omega \in \Omega.$$

The problems considered in [9, 6] are reduced to finding a solution to the problem (2.2) with  $A = E_{xy}E_{yy}^{1/2\dagger}$ ,  $B = I_n$  and  $C = E_{yy}^{1/2}$  where we write  $E_{yy}^{1/2\dagger} = (E_{yy}^{1/2})^{\dagger}$ . Let the SVD of  $E_{yy}^{1/2}$  be given by  $E_{yy}^{1/2} = V_n \Sigma V_n^*$  and let rank  $E_{yy}^{1/2} = r$ . Here,  $V_n = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$  with  $\mathbf{v}_i$  the *i*-th column of  $V_n$ . By Theorem 2.1, the solution to this particular case of the problem (2.2) having the minimal Frobenius norm is given by  $X = (E_{xy}E_{yy}^{1/2\dagger}V_rV_r^*)_k E_{yy}^{1/2\dagger}$ , where  $E_{yy}^{1/2\dagger}V_rV_r^* = E_{yy}^{1/2\dagger}$ . Therefore,  $X = (E_{xy}E_{yy}^{1/2\dagger})_k E_{yy}^{1/2\dagger}$ . The conditions for the uniqueness follow directly from Theorem 2.1.

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