# Generalized rank-constrained matrix approximations 

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#### Abstract

In this paper we give an explicit solution to the rank constrained matrix approximation in Frobenius norm, which is a generalization of the classical approximation of an $m \times n$ matrix $A$ by a matrix of rank $k$ at most.


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## 1 Introduction

Let $\mathbb{C}^{m \times n}$ be set of $m \times n$ complex valued matrices, and denote by $\mathcal{R}(m, n, k) \subseteq \mathbb{C}^{m \times n}$ the variety of all $m \times n$ matrices of rank $k$ at most. Fix $A=\left[a_{i j}\right]_{i, j=1}^{m, n} \in \mathbb{C}^{m \times n}$. Then $A^{*} \in \mathbb{C}^{n \times m}$ is the conjugate transpose of $A$, and $\|A\|_{F}:=\sqrt{\sum_{i, j=1}^{m, n}\left|a_{i j}\right|^{2}}$ is the Frobenius norm of $A$. Recall that the singular value decomposition of $A$, abbreviated here as $S V D$, is given by $A=U_{A} \Sigma_{A} V_{A}^{*}$, where $U_{A} \in \mathbb{C}^{m \times m}, V_{A} \in \mathbb{C}^{n \times n}$ are unitary matrices, $\Sigma_{A}:=\operatorname{diag}\left(\sigma_{1}(A), \ldots, \sigma_{\min (m, n)}(A)\right) \in \mathbb{C}^{m \times n}$ is a generalized diagonal matrix, with the singular values $\sigma_{1}(A) \geq \sigma_{2}(A) \geq \ldots \geq 0$ on the main diagonal. The number of positive singular values of $A$ is $r$, which is equal to the rank of $A$, denoted by rank $A$. Let $U_{A}=\left[\begin{array}{llll}\mathbf{u}_{1} & \mathbf{u}_{2} & \ldots & \mathbf{u}_{m}\end{array}\right], V_{A}=\left[\begin{array}{llll}\mathbf{v}_{1} & \mathbf{v}_{2} & \ldots & \mathbf{v}_{n}\end{array}\right]$ be the representations of $U, V$ in terms of their $m, n$ columns respectively. Then $\mathbf{u}_{i}$ and $\mathbf{v}_{i}$ are called the left and the right singular vectors of $A$, respectively, that correspond to the singular value $\sigma_{i}(A)$. Let

$$
\begin{equation*}
P_{A, L}:=\sum_{i=1}^{\text {rank } A} \mathbf{u}_{i} \mathbf{u}_{i}^{*} \in \mathbb{C}^{m \times m}, \quad P_{A, R}:=\sum_{i=1}^{\text {rank } A} \mathbf{v}_{i} \mathbf{v}_{i}^{*} \in \mathbb{C}^{n \times n} \tag{1.1}
\end{equation*}
$$

be the orthogonal projections on the range of $A$ and $A^{*}$ respectively. Denote by

$$
A_{k}:=\sum_{i=1}^{k} \sigma_{i}(A) \mathbf{u}_{i} \mathbf{v}_{i}^{*} \in \mathbb{C}^{m \times n}
$$

for $k=1, \ldots, \operatorname{rank} A$. For $k>\operatorname{rank} A$ we define $A_{k}:=A\left(=A_{\operatorname{rank} A}\right)$. For $1 \leq k<\operatorname{rank} A$, the matrix $A_{k}$ is uniquely defined if and only if $\sigma_{k}(A)>\sigma_{k+1}(A)$.

The enormous application of SVD decomposition of $A$ in pure and applied mathematics, is derived from the following approximation property:

$$
\begin{equation*}
\min _{X \in \mathcal{R}(m, n, k)}\|A-X\|_{F}=\left\|A-A_{k}\right\|_{F}, \quad k=1, \ldots \tag{1.2}
\end{equation*}
$$

The latter is known as the Eckart-Young theorem [2]. We note that the work [2] implied a number of extensions. We cite $[4,5,7,8]$ as some recent references. Another application of SVD is a formula for the Moore-Penrose inverse $A^{\dagger}:=V_{A} \Sigma_{A}^{\dagger} U_{A}^{*} \in \mathbb{C}^{n \times m}$ of $A$, where $\Sigma_{A}^{\dagger}:=\operatorname{diag}\left(\frac{1}{\sigma_{1}(A)}, \ldots, \frac{1}{\sigma_{\text {rank } A}(A)}, 0, \ldots, 0\right) \in \mathbb{C}^{n \times m}$. See for example [1].

## 2 Main Result

Below, we provide generalizations of the classical minimal problem given in (1.2).
Theorem 2.1 Let matrices $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$ be given. Then

$$
\begin{equation*}
X=B^{\dagger}\left(P_{B, L} A P_{C, R}\right)_{k} C^{\dagger} \tag{2.1}
\end{equation*}
$$

is a solution to the minimal problem

$$
\begin{equation*}
\min _{X \in \mathcal{R}(p, q, k)}\|A-B X C\|_{F}, \tag{2.2}
\end{equation*}
$$

having the minimal $\|X\|_{F}$. This solution is unique if and only if either

$$
k \geq \operatorname{rank} P_{B, L} A P_{C, R} \quad \text { or } \quad 1 \leq k<\operatorname{rank} P_{B, L} A P_{C, R}
$$

and

$$
\sigma_{k}\left(P_{B, L} A P_{C, R}\right)>\sigma_{k+1}\left(P_{B, L} A P_{C, R}\right)
$$

Proof of Theorem 2.1 Recall that the Frobenius norm is invariant under the multiplication from the left and the right by the corresponding unitary matrices. Hence $\|A-B X C\|_{F}=\left\|\widetilde{A}-\Sigma_{B} \widetilde{X} \Sigma_{C}\right\|$, where $\widetilde{A}:=U_{B}^{*} A V_{C}$ and $\widetilde{X}:=V_{B}^{*} X U_{C}$. Clearly, $X$ and $\widetilde{X}$ have the same rank and the same Frobenius norm. Thus, it is enough to consider the minimal problem $\min _{\tilde{X} \in \mathcal{R}(p, q, k)}\left\|\widetilde{A}-\Sigma_{B} \widetilde{X} \Sigma_{C}\right\|_{F}$.

Let $s=\operatorname{rank} B$ and $t=\operatorname{rank} C$. Clearly if $B$ or $C$ is a zero matrix, then $X=\mathbf{0}$ is the solution to the minimal problem (2.2). In this case either $P_{B, L}$ or $P_{C, R}$ are zero matrices, and the theorem holds trivially in this case.

Let us consider the case $1 \leq s, 1 \leq t$. Define $B_{1}:=\operatorname{diag}\left(\sigma_{1}(B), \ldots, \sigma_{s}(B)\right) \in$ $\mathbb{C}^{s \times s}, C_{1}:=\operatorname{diag}\left(\sigma_{1}(C), \ldots, \sigma_{t}(C)\right) \in \mathbb{C}^{t \times t}$. Partition $\widetilde{A}$ and $\widetilde{X}$ into four block matrices $A_{i j}$ and $X_{i j}$ with $i, j=1,2$ so that $\widetilde{A}=\left[A_{i j}\right]_{i, j=1}^{2}$ and $\widetilde{X}=\left[X_{i j}\right]_{i, j=1}^{2}$, where $A_{11}, X_{11} \in \mathbb{C}^{s \times t}$. (For certain values of $s$ and $t$, we may have to partition $\widetilde{A}$ or $\widetilde{X}$ to less than four block matrices.) Next, observe that $Z:=\Sigma_{B} \tilde{X} \Sigma_{C}=\left[Z_{i j}\right]_{i, j=1}^{2}$, where $Z_{11}=B_{1} X_{11} C_{1}$ and all other blocks $Z_{i j}$ are zero matrices. Since $B_{1}$ and $C_{1}$ are invertible we deduce

$$
\operatorname{rank} Z=\operatorname{rank} Z_{11}=\operatorname{rank} X_{11} \leq \operatorname{rank} \widetilde{X} \leq k
$$

The approximation property of $\left(A_{11}\right)_{k}$ yields the inequality $\left\|A_{11}-Z_{11}\right\|_{F} \geq \| A_{11}-$ $\left(A_{11}\right)_{k} \|_{F}$ for any $Z_{11}$ of rank $k$ at most. Hence for any $Z$ of the above form,

$$
\|\widetilde{A}-Z\|_{F}^{2}=\left\|A_{11}-Z_{11}\right\|_{F}^{2}+\sum_{2<i+j \leq 4}\left\|A_{i j}\right\|_{F}^{2} \geq\left\|A_{11}-\left(A_{11}\right)_{k}\right\|_{F}^{2}+\sum_{2<i+j \leq 4}\left\|A_{i j}\right\|_{F}^{2}
$$

Thus $\widehat{X}=\left[X_{i j}\right]_{i, j=1}^{2}$, where $X_{11}=B_{1}^{-1}\left(A_{11}\right)_{k} C_{1}^{-1}$ and $X_{i j}=\mathbf{0}$ for all $(i, j) \neq(1,1)$ is a solution to the problem $\min _{\tilde{X} \in \mathcal{R}(p, q, k)}\left\|\widetilde{A}-\Sigma_{B} \widetilde{X} \Sigma_{C}\right\|_{F}$ with the minimal Frobenius form. This solution is unique if and only if the solution $Z_{11}=\left(A_{11}\right)_{k}$ is the unique solution to the problem $\min _{Z_{11} \in \mathcal{R}(s, t, k)}\left\|A_{11}-Z_{11}\right\|_{F}$. This happens if either $k \geq \operatorname{rank} A_{11}$ or $1 \leq k<\operatorname{rank} A_{11}$ and $\sigma_{k}\left(A_{11}\right)>\sigma_{k+1}\left(A_{11}\right)$. A straightforward calculation shows that $\widehat{X}=\Sigma_{B}^{\dagger}\left(P_{\Sigma_{B}, L} \widetilde{A} P_{\Sigma_{C}, R}\right)_{k} \Sigma_{C}^{\dagger}$. Thus, a solution of (2.2) with the minimal Frobenius norm is given by

$$
\begin{aligned}
X & =B^{\dagger} U_{B}\left(P_{\Sigma_{B}, L} U_{B}^{*} A V_{C} P_{\Sigma_{C}, R}\right)_{k} V_{C}^{*} C^{\dagger} \\
& =B^{\dagger} U_{B}\left(U_{B}^{*} P_{B, L} A P_{C, R} V_{C}\right)_{k} V_{C}^{*} C^{\dagger} \\
& =B^{\dagger}\left(P_{B, L} A P_{C, R}\right)_{k} C^{\dagger} .
\end{aligned}
$$

This solution is unique if and only if either $k \geq \operatorname{rank} P_{B, L} A P_{C, R}$ or $1 \leq k<$ $\operatorname{rank} P_{B, L} A P_{C, R}$ and $\sigma_{k}\left(P_{B, L} A P_{C, R}\right)>\sigma_{k+1}\left(P_{B, L} A P_{C, R}\right)$.

A special case of the minimal problem (2.2), where $X$ is a rank one matrix and $C$ the identity matrix, was considered by Michael Elad [3] in the context of image processing.

## 3 Examples

First observe that the classical approximation problem given by (1.2) is equivalent to the case $m=p, n=q, B=I_{m}, C=I_{n}$. (Here, $I_{m}$ is the $m \times m$ identity matrix.) Clearly $P_{I_{m}, L}=I_{m}, P_{I_{n}, R}=I_{n}, I_{m}^{\dagger}=I_{m}, I_{n}^{\dagger}=I_{n}$. In this case we obtain the classical solution $B^{\dagger}\left(P_{B, L} A P_{C, R}\right)_{k} C^{\dagger}=A_{k}$.

Second, if $p=m, q=n$ and $B, C$ are non-singular, then $\operatorname{rank}(B X C)=\operatorname{rank} X$. In this case, $P_{B, L}=I_{m}$ and $P_{C, R}=I_{n}$, and the solution to (2.2) is given by $X=B^{-1} A_{k} C^{-1}$.

Next, a particular case of the problem (2.2) occurs in study of a random vector estimation (see, for example, $[9,6])$ as follows. Let $(\Omega, \Sigma, \mu)$ be a probability space, where $\Omega$ is the set of outcomes, $\Sigma$ a $\sigma$-field of measurable subsets $\Delta \subset \Omega$ and $\mu: \Sigma \mapsto[0,1]$ an associated probability measure on $\Sigma$ with $\mu(\Omega)=1$. Suppose that $\mathbf{x} \in L^{2}\left(\Omega, \mathbb{R}^{m}\right)$ and $\mathbf{y} \in L^{2}\left(\Omega, \mathbb{R}^{n}\right)$ are random vectors such that $\mathbf{x}=\left(x_{1}, \ldots\right.$, $\left.x_{m}\right)^{T}$ and $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right)^{T}$ with $x_{i}, y_{j} \in L^{2}(\Omega, \mathbb{R})$ for $i=1, \ldots, m$ and $j=1$, $\ldots, n$, respectively. Let $E_{x y}=\left[e_{i j, x y}\right] \in \mathbb{R}^{m \times n}, E_{y y}=\left[e_{j k, y y}\right] \in \mathbb{R}^{n \times n}$ be correlation matrices with entries

$$
\begin{array}{r}
e_{i j, x y}=\int_{\Omega} x_{i}(\omega) y_{j}(\omega) d \mu(\omega), \quad e_{j k, y y}=\int_{\Omega} y_{j}(\omega) y_{k}(\omega) d \mu(\omega), \\
i=1, \ldots, m, \quad j, k=1, \ldots, n, \quad \omega \in \Omega .
\end{array}
$$

The problems considered in [9, 6] are reduced to finding a solution to the problem (2.2) with $A=E_{x y} E_{y y}^{1 / 2 \dagger}, B=I_{n}$ and $C=E_{y y}^{1 / 2}$ where we write $E_{y y}^{1 / 2 \dagger}=\left(E_{y y}^{1 / 2}\right)^{\dagger}$. Let the SVD of $E_{y y}^{1 / 2}$ be given by $E_{y y}^{1 / 2}=V_{n} \Sigma V_{n}^{*}$ and let rank $E_{y y}^{1 / 2}=r$. Here, $V_{n}=\left[\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}\right]$ with $\mathbf{v}_{i}$ the $i$-th column of $V_{n}$. By Theorem 2.1, the solution to this particular case of the problem (2.2) having the minimal Frobenius norm is given by $X=\left(E_{x y} E_{y y}^{1 / 2 \dagger} V_{r} V_{r}^{*}\right)_{k} E_{y y}^{1 / 2 \dagger}$, where $E_{y y}^{1 / 2 \dagger} V_{r} V_{r}^{*}=E_{y y}^{1 / 2 \dagger}$. Therefore, $X=$ $\left(E_{x y} E_{y y}^{1 / 2 \dagger}\right)_{k} E_{y y}^{1 / 2 \dagger}$. The conditions for the uniqueness follow directly from Theorem 2.1.

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