The Inverse Source Problem in Non-homogeneous Background Media

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Abstract

The scalar wave inverse source problem (ISP) is investigated for the case where the source is embedded in a non-homogeneous medium with known index of refraction profile $n(\mathbf{r})$. It is shown that the solution to the ISP having minimum energy (so-called minimum energy source) can be obtained via a simple method of constrained optimization. This method is applied to the special case when the non-homogeneous background is spherically symmetric ($n(\mathbf{r}) = n(r)$) and yields the minimum energy source in terms of a series of spherical harmonics and radial wave functions that are solutions to a Sturm-Liouville problem. The special case of a source embedded in a spherical region of constant index that differs from the background is treated in detail and results from computer simulations are presented for this case.

1 Introduction

The inverse source problem (ISP) consists of determining a source ρ to the inhomogeneous Helmholtz equation

$$[\nabla^2 + k_0^2 n^2(\mathbf{r})]U(\mathbf{r}) = -\rho(\mathbf{r}) \tag{1}$$

that radiates a specified field U everywhere *outside* the support volume τ of the source. In this equation k_0 is a constant wavenumber and $n(\mathbf{r})$ is an index of refraction distribution that depends on position $\mathbf{r}$ and that is assumed to go to unity for sufficiently large r. We will assume throughout this paper that the source volume τ is a sphere, centered at the origin, and having a radius a. The ISP then consists of computing a source ρ that generates a *prescribed* field U everywhere outside this sphere.

There are a number of treatments of the ISP for the free space case where the index distribution $n(\mathbf{r})$ is constant (equal to unity) throughtout space [1, 2, 3, 4, 5, 6]. Most of these treatments make use of the fact that the source's *radiation pattern* (see below) determines, in principle, the field everywhere outside the source volume [7]. Using this fact the ISP can be cast in terms of the radiation pattern: determine a source ρ that generates a prescribed radiation pattern. It is also well known [8, 9, 10] that there exist an infinity of sources that radiate fields that vanish identically outside their support volumes so that the ISP does not possess a unique solution; i.e., an infinity of solutions can be obtained by adding any one of these *non-radiating* sources to any given solution [11]. Thus, in order to obtain a unique solution to the ISP it is necessary to add constraints that the source must satisfy in addition to yielding a specified radiation pattern. The natural choice of constraint is source energy $\mathcal{E}$ defined to be the L^2 norm of the source over the source volume τ:

$$\mathcal{E} = \int d^3r\, |\rho(\mathbf{r})|^2. \tag{2}$$

The solution to the ISP that minimizes the source energy as defined in Eq.(2) is called the *minimum energy solution* (ME solution) and denoted by ρ_{ME}. It can be shown that the minimum energy solution is orthogonal to the non-radiating sources [3] and is the *pseudo-inverse* of the ISP [5, 6].

The source energy as defined in Eq.(2) is an important measure of the realizability of the source to achieve a given radiation pattern. For the free space case one finds that the energy of the minimum energy source (ME source) depends critically on the product $x = k_0 a$ of the (free space) wavenumber k_0 and the source radius a [2, 6]. For a given radiation pattern this energy $\mathcal{E}(x)$ is small for $x > l_0$ where l_0 is a parameter that characterizes the radiation pattern and that increases with increasing fine detail in the pattern. However, it is found that $\mathcal{E}(x)$ increases exponentially with decreasing x below the critical value l_0. This exponential increase of source energy indicates that the given radiation pattern cannot be physically realized by *any* source having that specific $k_0 a$ product: it is necessary to either decrease the wavelength (increase k_0) or increase the source radius. This result is equivalent to the well known result in antenna theory that states that reactive energy and the quality Q of an antenna increase exponentially with decreasing $k_0 a$ if one attempts to achieve super-resolution [12].

As far as the authors of this paper know there is only one treatment of the ISP for the case where the background index distribution $n(\mathbf{r})$ in which the source is embedded is non-homogeneous [13]. In that paper it is shown that the ME solution satisfies an integral equation whose kernel is the imaginary part of the outgoing wave Green function of the inhomogeneous Helmholtz equation (1). Using this fact it is shown in [13] that the ME solution can be expanded into a series of eigenfunctions of this integral equation with expansion coefficients that can be determined from the radiation pattern. The paper shows that the formalism reduces to the known theory of the ISP when the index $n = 1$ (homogeneous medium case) but does not include any examples of the general theory.

In this paper we revisit the case of a source embedded in a known non-homogeneous background index and solve the minimum energy ISP using a simple method of constrained optimization. Besides providing a simpler formulation of the problem than was used in [13] the method yields a solution that is directly implemented without the need of first computing a Green function for the Helmholtz equation (1) and then computing the eigenfunctions of the imaginary part of this quantity. The special case of a spherically symmetric index $n(\mathbf{r}) = n(r)$ is treated in detail and it is shown that the ME solution to the ISP has exactly the same mathematical form as the solution for the constant index case but with the spherical Bessel functions employed in the constant index solution replaced by radial wave functions that are solutions to a Sturm-Liouville problem. In particular, these radial wave functions are the radial wave scattering functions obtained in the scattering of an incident plane wave from the spherically symmetric index distribution $n(r)$. This latter problem has been studied extensively in both quantum mechanical scattering [14] and in optical [15] and electromagnetic [16] scattering and there exists a number of index distributions for which the scattering wave functions have been computed and that can be used to compute the ME solution to the ISP.

The motivation for the research presented in this paper is the possibility of optimally selecting the source region index of refraction distribution $n(\mathbf{r})$ to achieve some specified radiation pattern that would otherwise not be possible for a source embedded in free space. As mentioned above and reviewed in section 3, in the free space case the ME source energy increases exponentially with decreasing $x = k_0 a$ below a critical point that is determined by the fine detail that is desired in the radiation pattern. The question then is whether this limitation can be removed by embedding the source in a non-homogeneous background medium. In effect we create a new "effective source" that consists of the actual physical source interacting with the non-homogeneous background. In the simplest case we can consider a source embedded in a cavity with partially reflecting walls. This cavity will, of course, have a pronounced effect on the radiated field and, possibly, can aid in achieving desired properties of the radiation pattern.

In this paper we limit our attention, for the most part, to source regions characterized by a spherically symmetric index of refraction distribution $n(\mathbf{r}) = n(r)$ although many of our results can be generalized to sources embedded in cavities and non-spherically symmetric index distributions.

The realizability of a given radiation pattern is investigated in some detail by examining the energy of the ME source (as defined by Eq.(2)) on the index of refraction profile of the source region. It is found that this energy depends critically on the (weighted) L^2 norm of the radial wavefunctions taken over the source region. This fact suggests that by proper choice of the index of refraction profile $n(r)$ that the energy can be minimized for any given radiation pattern; i.e., an index distribution can be selected that results in a source having minimum energy for a given prescribed radiation pattern.

The L^2 norms of the radial wave functions are found to be dependent on "resonant" properties of the source index distribution and are also related in a one-to-one fashion with the different angular modes of the radiation pattern. These facts suggest the interesting possibility of exploiting the "resonances" of the source index distribution to selectively control the shape and form of the radiation pattern. This possibility is briefly considered in the computer simulation study.

The final section of the paper treats the simple example of a source embedded in a homogeneous sphere whose constant index of refraction differs from that of the background medium. This is the simplest example of a spherically symmetric index of refraction distribution and the scattering wave functions are well known (the so-called MIE scattering problem [15, 16]) and easily computed. The minimum energy source is computed for this case and results from a computer simulation study that examines the dependence of the energy of the ME source on the index of refraction of the source region is presented. It is found that the source energy depends in a non-linear manner on the value of the source region index and that generally, the larger the index the lower the source energy for any given radiation pattern. These results suggest that more efficient sources can be obtained by simply embedding a given radiator (e.g., antenna) in a homogeneous background medium.

2 Problem Formulation

We introduce the scattering potential defined according to the equation

$$V(\mathbf{r}) = k_0^2[1 - n^2(\mathbf{r})]$$

and rewrite Eq.(1) in a form that we will use in the development to follow. In particular we find that

$$[\nabla^2 + k_0^2 - V(\mathbf{r})]U(\mathbf{r}) = -\rho(\mathbf{r}), \tag{3}$$

where the scattering potential V and source ρ both vanish outside the source region τ.

The outgoing wave solution to Eq.(3) is the field radiated by the source and has the asymptotic form

$$U(r\mathbf{s}) = f(\mathbf{s})\frac{e^{ik_0 r}}{r} \quad k_0 r \to \infty \tag{4}$$

as $k_0 r \to \infty$ in the direction specified by the unit vector $\mathbf{s}$. In the above equation f is the source's *radiation pattern*. It is well known that the radiation pattern specified for all directions $\mathbf{s}$ uniquely determines the field U everywhere outside the source region τ [7]; i.e., knowledge of the radiated field everywhere outside τ is equivalent to knowledge of the radiation pattern $f(\mathbf{s})$ specified for all directions $\mathbf{s}$.

The inverse source problem (ISP) consists of determining a source distribution $\rho(\mathbf{r})$ that radiates a specifed field everywhere *outside* the source region τ. Because the ISP requires only that the field radiated by the source be specified outside τ the problem does not possess a unique solution because of the possible presence of *non-radiating sources* [8, 9, 10] within τ. A non-radiating source generates a field that vanishes identically outside τ and, hence, can be added to any given solution to the ISP to yield a different solution. Also, because the radiation pattern uniquely determines the field everywhere outside τ, the ISP is equivalent to the problem of determining a source that generates a prescribed radiation pattern $f(\mathbf{s})$ for all observation directions $\mathbf{s}$.

Most treatments of the ISP cast the problem in terms of the radiation pattern but only require that the source generate the radiation pattern to within a given accuracy defined by the integral squared error

$$E = \int_{4\pi} d\Omega \, |\hat{f}(\mathbf{s}) - f(\mathbf{s})|^2 \tag{5}$$

where f is the prescribed radiation pattern and $\hat{f}$ the radiation pattern actually generated by the source. More precisely, the desired radiation pattern is approximated by a finite series of spherical harmonics $Y_l^m(\mathbf{s})$

$$f(\mathbf{s}) \approx \hat{f}(\mathbf{s}) = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_l^m(\mathbf{s}) \tag{6}$$

and the source is required only to generate the approximate radiation pattern $\hat{f}$. Here, we have used the unit vector $\mathbf{s}$ having polar angle θ and azimuthal angle ϕ to denote the θ, ϕ arguments of the spherical harmonics. Because the spherical harmonics are orthonormal and complete over the unit sphere the approximated radiation pattern satisfies Eq.(5) with an error E given by

$$E = \sum_{l=L+1}^{\infty} \sum_{m=-l}^{l} |a_{l,m}|^2$$

where the expansion coefficients (multipole moments) $a_{l,m}$, $l > L$ are the higher order (neglected) expansion coefficients of the ideal radiation pattern.

Besides requiring only that the source generate the radiation pattern within a finite error most treatments of the ISP also require that the source minimize the *source energy* defined by

$$\mathcal{E} = \int d^3r \, |\rho(\mathbf{r})|^2. \tag{7}$$

We will show (see also [3]) that minimizing the source energy leads to a unique solution of the ISP which we refer to as the *minimum energy source* and designate by ρ_{ME}. This solution has the distinct advantage of being the most efficient source that solves the ISP for a given scattering potential (background index of refraction) $V(\mathbf{r})$. Since the minimum energy source and, hence, source energy depend on the scattering potential, an interesting question arises as to the dependence of the source energy on the background index distribution and, in particular, on which index distributions lead to lowest source energies. This question provides much of the motivation for studying the ISP in non-homogeneous backgrounds since it leads to the possibility of designing extremely efficient sources (e.g., antennas) that are embedded in such backgrounds.

Our goal in this paper is to develop the formalism for solving the ISP as defined above and to test and evaluate the formalism in a set of computer simulations. We will first treat the case of a source embedded in free space and then extend the free space theory to the general case of a source embedded in an inhomogeneous background medium. The free space case is important in that it provides a benchmark of performance as well as a frame of reference for the general theory.

3 Free Space Case

The minimum energy ISP as defined above has been solved within both the scalar wave formulation under consideration here [1, 2, 3, 4, 5] and for the electromagnetic wave formulation [6] in the special case where the scattering potential $V(\mathbf{r})$ vanishes; i.e., when the source is embedded in free space. We will review the scalar wave free space case here where, however, we will employ a somewhat different solution methodology to find the minimum energy source than has been used in earlier work. We will use this same procedure for the general case of non-vanishing scattering potentials later in the paper.

The outgoing wave solution to Eq.(3) is given in terms of an outgoing wave Green function G by the expression

$$U(\mathbf{r}) = -\int_\tau d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}'), \tag{8}$$

where τ is the source volume which is a sphere of radius a centered at the origin. In the free space case where the scattering potential $V = 0$ the Green function is given by

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{9}$$

from which it is easy to show that

$$G(r\mathbf{s},\mathbf{r}') \sim -\frac{1}{4\pi}e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{e^{ik_0r}}{r} \tag{10}$$

as $r \to \infty$ in the direction $\mathbf{s}$. Using the above result we conclude from Eq.(8) that the radiation pattern is given by

$$f(\mathbf{s}) = \frac{1}{4\pi}\int_\tau d^3r'\,\rho(\mathbf{r}')e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}. \tag{11}$$

We can obtain an expansion of the radiation pattern in a series of spherical harmonics by using the well known expansion

$$e^{-ik_0\mathbf{s}\cdot\mathbf{r}'} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l j_l(k_0r')Y_l^{m*}(\hat{\mathbf{r}}')Y_l^m(\mathbf{s}) \tag{12}$$

where j_l denotes the spherical Bessel function of the first kind of order l and Y_l^m are the spherical harmonics of degree l and order m and $\hat{\mathbf{r}}'$ denotes the unit vector in the $\mathbf{r}'$ direction. Upon substituting Eq.(12) into Eq.(11) we find that

$$f(\mathbf{s}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{l,m}Y_l^m(\mathbf{s}) \tag{13}$$

where the expansion coefficients (multipole moments) $a_{l,m}$ are given by

$$\begin{aligned} a_{l,m} &= \int_{4\pi} d\Omega_s f(\mathbf{s})Y_l^{m*}(\mathbf{s}) \\ &= (-i)^l\int_\tau d^3r'\,\rho(\mathbf{r}')j_l(k_0r')Y_l^{m*}(\hat{\mathbf{r}}'). \end{aligned} \tag{14}$$

3.1 Minimum Energy Source

The minimum energy solution to the ISP is required to satisfy Eq.(14) for some given set of multipole moments $a_{l,m}$, $l = 0, 1, \ldots, L$ and also to minimize the source energy defined according to Eq.(7). Computing the minimum energy source can be cast as one of constrained minimization where the generalized Lagrangian is given by

$$\mathcal{L} = \mathcal{E} + \sum_{l=0}^{L}\sum_{m=-l}^{l}C_{l,m}[a_{l,m}^* - i^l\int_\tau d^3r\,\rho^*(\mathbf{r})j_l(k_0r)Y_l^m(\hat{\mathbf{r}})] + \text{c.c.}$$

where $\mathcal{E}$ is the source energy defined in Eq.(7) and c.c. stands for the complex conjugate of the second term on the r.h.s. of the equation and the $C_{l,m}$ are a set of Lagrange multipliers to be determined. On expressing the source energy in terms of ρ and ρ^* and taking the first variation of the above Lagrangian we obtain

$$\delta\mathcal{L} = \int_\tau d^3r\,\delta\rho^*(\mathbf{r})\left[\rho(\mathbf{r}) - \sum_{l=0}^{L}\sum_{m=-l}^{l} C_{l,m} i^l j_l(k_0 r) Y_l^m(\hat{\mathbf{r}})\right] + \text{c.c.}$$

which, when set equal to zero, yields the solution

$$\rho_{ME}(\mathbf{r}) = \begin{cases} \sum_{l=0}^{L}\sum_{m=-l}^{l} C_{l,m} i^l j_l(k_0 r) Y_l^m(\hat{\mathbf{r}}) & \text{if } r < a \\ 0 & \text{if } r > a. \end{cases}$$

The Lagrange multipliers $C_{l,m}$ are determined from the condition that the source generate the multipole moments according to Eq.(14). We find that

$$C_{l,m} = \frac{a_{l,m}}{\sigma_l^2} \tag{15}$$

where

$$\sigma_l^2 = \int_0^a r^2 dr\, j_l^2(k_0 r). \tag{16}$$

On making use of the above expression for the Lagrange multipliers we finally conclude that the minimum energy solution to the free space ISP is given by

$$\rho_{ME}(\mathbf{r}) = \begin{cases} \sum_{l=0}^{L}\sum_{m=-l}^{l} i^l \frac{a_{l,m}}{\sigma_l^2} j_l(k_0 r) Y_l^m(\hat{\mathbf{r}}) & \text{if } r < a \\ 0 & \text{if } r > a. \end{cases} \tag{17}$$

3.2 Source Energy

The source energy $\mathcal{E}$ is readily computed using the minimum energy source given in Eq.(17). We find that

$$\mathcal{E}_{ME} = \int_\tau d^3r\,|\rho_{ME}(\mathbf{r})|^2 = \sum_{l=0}^{L}\sum_{m=-l}^{l} \frac{|a_{l,m}|^2}{\sigma_l^2} \tag{18}$$

where we have added the subscript "ME" to denote the energy of the minimum energy source. Now it is easy to show that the quantities σ_l^2 depend critically on the product k_0a of the free space wavenumber with the source radius a. In particular these quantites can be shown to be given by

$$\sigma_l^2 = \int_0^a r^2 dr\,|j_l(k_0 r)|^2 = \frac{a^3}{2}[{j_l}^2(k_0 a) - j_{l-1}(k_0 a) j_{l+1}(k_0 a)]. \tag{19}$$

and to decrease exponentially to zero for $l > k_0a$. It then follows that the largest value L of the index l allowed in the approximation Eq.(6) is $L = k_0a$ if we want to maintain low source energy. Values of $L >> k_0a$ will lead to extremely high source energy and unstable source distributions.

To illustrate the remarks made above concerning the behavior of the quantities σ_l^2 on the index l and the product k_0a we show in Fig. 1 semilog plots of σ_l^2 as a function the index l for various values of $x = k_0a$. It is seen from these plots that these quantites decay exponentially to zero for $l >> x$ so that at wavenumber k_0 a source of radius a can only efficiently radiate a radiation pattern whose maximum l value is $L = k_0a$. Similar behaviour is exhibited in Fig.2 which shows semilog plots of $\sigma_l^2(x)$ as a function of x for values of $l = 10, 20$, and 30.

We computed the energy of the minimum energy source for a model radiation pattern $f(\theta)$ having multipole coefficients a_l^m given by

$$a_l^m = \begin{cases} \frac{1}{\sqrt{L+1}} & \text{if } m = 0 \text{ and } l \le L \\ 0 & \text{else} \end{cases} \tag{20}$$

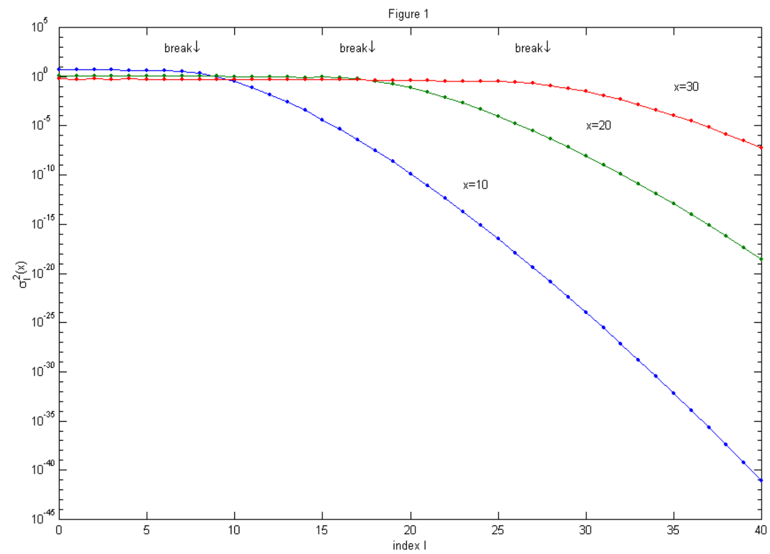

Figure 1: Behavior of σ_l^2 as a function of index l for three different values of $x = k_0 a$ equal to 10, 20, and 30. Plots indicate an exponetial decay of these quantities for $l >> x$. The break points are seen to occur when $x \approx l$.

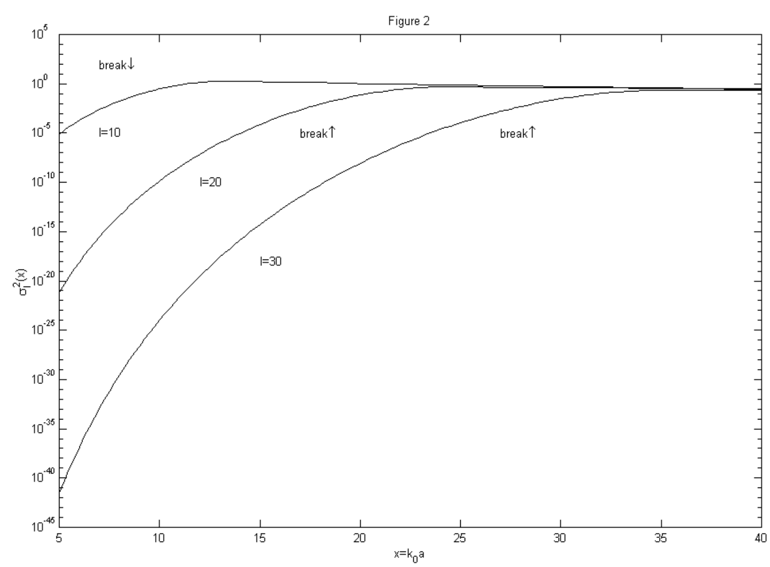

Figure 2: σ_l^2 as a function of $x = k_0 a$ for values of $l_0 = 10$ (.), $l_0 = 20$ (o) and $l_0 = 30$ (x). Plots indicate an exponetial growth of these quantities for $x << l_0$. The break points are seen to occur when $x \approx l$.

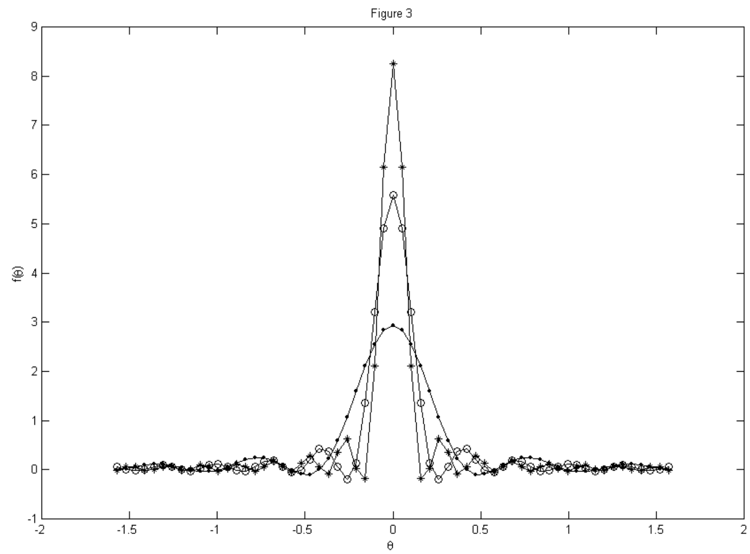

Figure 3: Plots of the model radiation pattern for $L = 10$ (.), $L = 20$ (o) and $L = 30$ (x).

This radiation pattern is circularly symmetric about the z axis (is independent of ϕ since $m = 0$, $\forall l$), has an effective beam width inversely related to the cut-off value L and has unit energy; i.e.,

$$\int_{4\pi} \sin\theta d\theta d\phi\, |f(\theta)|^2 = \sum_{l=0}^{L} \frac{1}{L+1} = 1.$$

We show plots of the model radiation pattern as a function of angle θ in Fig. 3 for values of the parameter L equal to $L = 10$, $L = 20$ and $L = 30$. It is clear from these plots that the larger the value of L the narrower is the radiation pattern and, hence, higher is the directivity of the source.

Using the coefficients given in Eq.(20) we computed the source energy using Eq.(18) with the σ_l^2 given by Eq.(19) and with three different L values of $L = 10, 20$ and $L = 30$. We show in Fig. **??** plots of the source energies as a function of $x = k_0 a$ for the three different values of the parameter L. It is seen that as expected the source energy becomes extremely large if we try to achieve an L value that exceeds the critical value $L = k_0 a$. This is, of course, due to the fact that the quantities σ_l^2 become extremely small when $k_0 a << l$ as is indicated in Fig.2.

4 Non-homogeneous Backgrounds

The solution of Eq.(3) for a non-homogeneous index distribution $n(\mathbf{r})$ can be expressed in terms of a Green function via Eq.(8) where, however, the Green function is no longer the free space Green function defined in Eq.(9). More important for our purposes is the fact that the asymptotic behavior of the Green function for the variable index case is also of the general free space form Eq.(10) where, however, the plane wave $\exp(-ik_0\mathbf{s}\cdot\mathbf{r}')$ is replaced by the *scattering wave function* $\psi^+(\mathbf{r}'; -k_0\mathbf{s})$. The scattering wave functions $\psi^+(\mathbf{r}; -k_0\mathbf{s})$ are solutions to Eq.(3) for the special case where the source ρ is a delta function located at infinity along the direction defined by the unit vector $\mathbf{s}$. Equivalently, these wave functions are solutions to the homogeneous Helmholtz equation

$$[\nabla^2 + k_0^2 - V(\mathbf{r})]\psi^+(\mathbf{r}; -k_0\mathbf{s}) = 0, \tag{21}$$

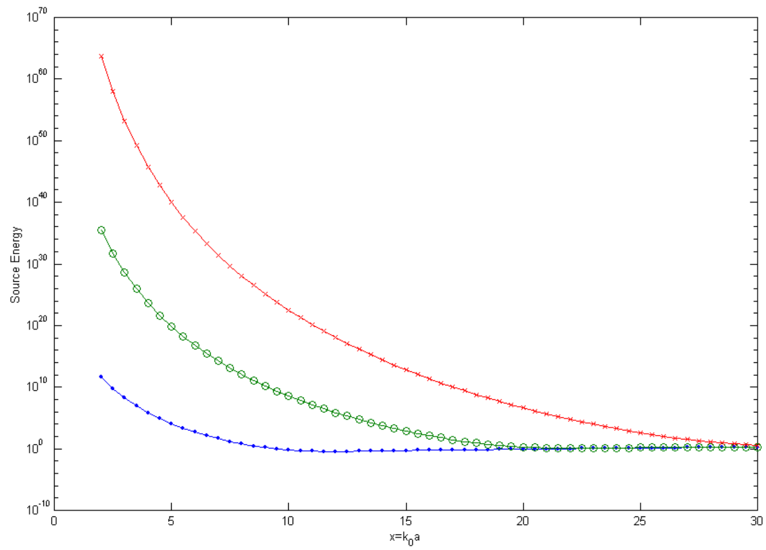

Figure 4: Source energy as a function of $x = k_0 a$ for values of $L = 10$ (.), $L = 20$ (o) and $L = 30$ (x). Plots indicate an exponetial growth of these quantities for $x << L$.
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which obey the asymptotic condition

$$\psi^+(r\hat{\mathbf{r}}; -k_0\mathbf{s}) \sim e^{-ik_0\mathbf{s}\cdot\mathbf{r}} + g(\hat{\mathbf{r}}; -k_0\mathbf{s})\frac{e^{ik_0 r}}{r} \tag{22}$$

as $k_0 r \to \infty$ in the $\hat{\mathbf{r}}$ direction. In the above equation g is the so-called scattering amplitude associated with the scattering potential V and plays a similar role in scattering problems as does the source radiation pattern f in radiation problems. The scattering wave functions correspond physically to the wavefields that result when an incident plane wave propagating in the $-\mathbf{s}$ direction scatters off the inhomogeneous index of refraction distribution $n(\mathbf{r})$. Note that in the limit when the scattering potential vanishes these wavefunctions simply reduce to the incident plane wave.

In terms of the scattering wave functions the radiation pattern of the source in the non-homogeneous index case is given by (see appendix)

$$f(\mathbf{s}) = \frac{1}{4\pi}\int_\tau d^3r'\,\rho(\mathbf{r}')\psi^+(\mathbf{r}'; -k_0\mathbf{s}), \tag{23}$$

which is simply the free space result Eq.(10) with the plane wave $\exp(-ik_0\mathbf{s}\cdot\mathbf{r}')$ replaced by the *scattering wave function* $\psi^+(\mathbf{r}'; -k_0\mathbf{s})$. The ISP for non-homogeneous media then reduces to determining a source ρ that satisfies Eq.(23) for all observation directions $\mathbf{s}$. Clearly, in the special case when $V = 0$ the scattering wave function reduces to the plane wave and Eq.(23) reduces to the free space result Eq.(11).

4.1 Spherically Symmetric Backgrounds

In this paper we will restrict our attention to the case of spherically symmetric index distributions $n(\mathbf{r}) = n(r)$. The scattering wave functions then satisfy (cf., Eq.(21))

$$[\nabla^2 + k_0^2 - V(r)]\psi^+(\mathbf{r}; -k_0\mathbf{s}) = 0, \tag{24}$$

where the eigenfunctions ψ^+ are required to satisfy the boundary condition Eq.(22). Because of the spherical symmetry of the scattering potential V, the wavefield ψ^+ can only depend on the magnitude r of the field point vector $\mathbf{r}$ and the polar angle γ formed between the direction of propagation $-\mathbf{s}$ of the incident plane wave and $\mathbf{r}$. If we then take the incident wave direction to be the positive z axis and express the Helmholtz operator in spherical polar coordinates Eq.(24) can then be written in the form

$$[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\gamma}\frac{\partial}{\partial\gamma}(\sin\gamma\frac{\partial}{\partial\gamma}) - V(r) + k_0^2]\psi^+(r,\gamma) = 0 \tag{25}$$

where γ is the polar angle formed between the positive z axis and the field point vector $\mathbf{r}$ and where we have used the fact that the field must be independent of the azimuthal angle ϕ. The boundary condition Eq.(22) becomes

$$\psi^+(r,\gamma) \sim e^{ik_0 r\cos\gamma} + g(\gamma)\frac{e^{ik_0 r}}{r} \tag{26}$$

where we have set $-k_0\mathbf{s}\cdot\mathbf{r} = k_0 z = k_0 r\cos\gamma$ and where $g(\gamma)$ is the scattering amplitude.

We can expand the scattering wave function $\psi^+(r,\gamma)$, the incident plane wave $\exp(ik_0 r\cos\gamma)$ and the scattering amplitude $g(\gamma)$ into series of Legendre polynomials as follows

$$\begin{aligned}
\psi^+(r,\gamma) &= \sum_l i^l(2l+1)\psi_l(r)P_l(\cos\gamma), \\
e^{ik_0 r\cos\gamma} &= \sum_l i^l(2l+1)j_l(k_0 r)P_l(\cos\gamma), \\
g(\gamma) &= \sum_l i^l(2l+1)A_l P_l(\cos\gamma),
\end{aligned}$$

where we have introduced the factors $i^l(2l+1)$ into the expansions for the scattering wave function and the scattering amplitude for later notational convienence. In these equations j_l is the spherical Bessel function of the first kind of order l and the A_l are expansion coefficients of the scattering amplitude that depend on the specific form of the scattering potential $V(r)$. On substituting the first of these equations into Eq.(25) we find that the radial dependent coefficients $\psi_l(r)$ satisfy the equation

$$[\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - \frac{l(l+1)}{r^2} - V(r) + k_0^2]\psi_l(r) = 0 \tag{27}$$

where we have used the fact that

$$\frac{1}{\sin\gamma}\frac{\partial}{\partial\gamma}(\sin\gamma\frac{\partial}{\partial\gamma})P_l(\cos\gamma) = -l(l+1)P_l(\cos\gamma).$$

The asymptotic behavior of the radial functions $\psi_l(r)$ is obtained by substituting the expansions for the scattering wave function, the incident plane wave, and the scattering amplitude into Eq.(26). We find that

$$\psi_l(r) \sim j_l(k_0 r) + A_l\frac{e^{ik_0 r}}{r}. \tag{28}$$

Besides satisfying the bounary condition Eq.(28) we also require that the radial functions be everywhere continuous with continuous first derivatives.

Once the radial functions $\psi_l(r)$ are computed the scattering wave functions $\psi^+(r,\gamma)$ corresponding to an incident plane wave propagating along the z axis is given by the expansion in Legendre polynomials above. However, the defining equation for the radiation pattern Eq.(23) requires that we have the scattering wave functions for all directions $-\mathbf{s}$ of the incident plane wave. This can be easily accomplished by using the addition theorem for spherical harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}(-1)^l Y_l^{m*}(\hat{\mathbf{r}})Y_l^m(\mathbf{s}),$$

where now γ is the angle formed between the *arbitrary* incident wave direction $-\mathbf{s}$ and the field direction $\hat{\mathbf{r}}$. On using the addition theorem we obtain

$$\psi^{+}(\mathbf{r};-k_0\mathbf{s}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l \psi_l(r) Y_l^{m*}(\hat{\mathbf{r}}) Y_l^m(\mathbf{s}), \tag{29}$$

which is the generalization of Eq.(12) to spherically symmetric non-homogeneous index of refraction distributions.

4.2 Minimum Energy Source

Upon substituting the expansion Eq.(29) into Eq.(23) we obtain Eq.(13) where, however, the multipole moments are now given by

$$\begin{aligned} a_{l,m} &= \int_{4\pi} d\Omega_s f(\mathbf{s}) Y_l^{m*}(\mathbf{s}) \\ &= (-i)^l \int_{\tau} d^3r\, \rho(\mathbf{r}) \psi_l(r) Y_l^{m*}(\hat{\mathbf{r}}) \end{aligned} \tag{30}$$

which is the generalization of Eq.(14) to the case where the source is embedded in a non-homogeneous but spherically symmetric index of refraction profile. The generalized Eq.(30) is seen to result from Eq.(14) under the replacement of the spherical Bessel functions j_l by the radial functions ψ_l. The minimum energy solution to the ISP is required to satisfy Eq.(30) for some given set of multipole moments $a_{l,m}$, $l = 0, 1, \ldots, L$ and also to minimize the source energy defined according to Eq.(7).

As in the free space case computing the minimum energy source can be cast as one of constrained minimization where the generalized Lagrangian is now given by

$$\mathcal{L} = \mathcal{E} + \sum_{l=0}^{L} \sum_{m=-l}^{l} C_{l,m} [a_{l,m}^* - i^l \int_{\tau} d^3r\, \rho^*(\mathbf{r}) \psi_l^*(r) Y_l^m(\hat{\mathbf{r}})] + \text{c.c.}$$

where, as before, $\mathcal{E}$ is the source energy defined in Eq.(7) and c.c. stands for the complex conjugate of the second term on the r.h.s. of the equation and the $C_{l,m}$ are a set of Lagrange multipliers to be determined. On expressing the source energy in terms of ρ and ρ^* and taking the first variation of the above Lagrangian we obtain

$$\delta\mathcal{L} = \int_{\tau} d^3r\, \delta\rho^*(\mathbf{r}) \left[\rho(\mathbf{r}) - \sum_{l=0}^{L} \sum_{m=-l}^{l} C_{l,m} i^l \psi_l^*(r) Y_l^m(\hat{\mathbf{r}}) \right] + \text{c.c.}$$

which, when set equal to zero, yields the solution

$$\rho_{ME}(\mathbf{r}) = \begin{cases} \sum_{l=0}^{L} \sum_{m=-l}^{l} C_{l,m} i^l \psi_l^*(r) Y_l^m(\hat{\mathbf{r}}) & \text{if } r < a \\ 0 & \text{if } r > a. \end{cases}$$

The Lagrange multipliers $C_{l,m}$ are determined from the condition that the source generate the multipole moments according to Eq.(30). We find that

$$C_{l,m} = \frac{a_{l,m}}{\Sigma_l^2} \tag{31}$$

where

$$\Sigma_l^2 = \int_0^a r^2 dr\, |\psi_l(r)|^2. \tag{32}$$

On making use of the above expression for the Lagrange multipliers we finally conclude that the minimum energy solution to the ISP for spherically symmetric background index distributions is given by

$$\rho_{ME}(\mathbf{r}) = \begin{cases} \sum_{l=0}^{L} \sum_{m=-l}^{l} (i)^l \frac{a_{l,m}}{\Sigma_l^2} \psi_l^*(r) Y_l^m(\hat{\mathbf{r}}) & \text{if } r < a \\ 0 & \text{if } r > a. \end{cases} \tag{33}$$

The source energy is found to be given by the free space formula Eq.(18) where, however, the σ_l^2 are replaced by the Σ_l^2 defined in Eq.(32). As in the free space case the source energy is seen to depend inversely on the Σ_l^2. Although these quantities are strictly positive they can become extremely small leading to extremely high source energy and associated instability in the minimum energy source. Thus, it is of interest to maximize these quantities especially for large values of the index l which is associated with fine detail (high resolution) in the radiation pattern.

The energy of the ME source is obtained by substituting Eq.(33) into the source energy definition given in Eq.(2). We obtain the same expression as was obtained in the free space case Eq.(18) where, however, the σ_l^2 are replaced by the Σ_l^2. It is clear that the source energy is minimized by maximizing the Σ_l^2 which, in turn, is equivalent to maximizing the weighted L^2 norm of the radial functions ψ_l over the interval $[0, a]$. Since the radial functions ψ_l are solutions to a Sturm-Liouville problem the energy minimization problem reduces to finding scattering potentials $V(r)$ whose corresponding Sturm-Liouville problem has solutions with maximum norm over this interval. This problem, although simple to state, appears to be non-trivial and the authors' offer no simple recipe for computing optimum potentials at this time. However, in the following section we will treat a simple class of potentials that illustrate the dependence of source energy on selection of V.

5 Piecewise Constant Backgrounds

In this section we consider the special case where the scattering potential $V(r)$ is constant throughout the source region; i.e.,

$$V(r) = \begin{cases} k_0^2 - k^2 & \text{if } r < a \\ 0 & \text{if } r > a, \end{cases} \tag{34}$$

This is certainly a spherically symmetric scattering potential so that the scattering wave functions can be expanded in the form of Eq.(29) where the radial functions $\psi_l(r)$ satisfy Eq.(27) with $V(r)$ given in Eq.(34) above. Thus we find that

$$[\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - \frac{l(l+1)}{r^2} + k^2]\psi_l(r) = 0 \quad \text{if } 0 < r < a$$
$$[\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) - \frac{l(l+1)}{r^2} + k_0^2]\psi_l(r) = 0 \quad \text{if } a < r < \infty$$

together with the boundary condition from Eq.(28):

$$\psi_l(r) \sim j_l(k_0 r) + A_l \frac{e^{ik_0 r}}{r} \quad \text{if } r > a. \tag{35}$$

We also require that the radial functions be finite and continuous with a continuous first derivative. The set of differential equations together with the boundary and continuity conditions allow us to obtain a unique solution for the radial function.

5.0.1 Radial Function

The radial function has the general form

$$\psi_l(r) = Aj_l(kr) + Bh_l(kr) \quad \text{if } 0 < r < a$$
$$\psi_l(r) = Cj_l(k_0 r) + Dh_l(k_0 r) \quad \text{if } a < r < \infty$$

where j_l is the spherical Bessel function of the first kind and h_l the spherical Hankel function of the first kind. The requirement that the radial function be finite at the origin $r = 0$ requires that $B = 0$ while the boundary condition Eq.(35) requires that the constant $C = 1$. The remaining constants A and D are determined by the continuity requirements applied at the boundary $r = a$. These conditions are

$$A j_l(ka) = j_l(k_0 a) + D h_l(k_0 a)$$
$$A {j_l}'(ka) = \frac{k_0}{k}[{j_l}'(k_0 a) + D {h_l}'(k_0 a)]$$

from which we obtain the solution

$$A = \frac{k_0}{k}\frac{{j_l}'(k_0 a)h_l(k_0 a) - j_l(k_0 a){h_l}'(k_0 a)}{{j_l}'(ka)h_l(k_0 a) - \frac{k_0}{k} j_l(ka){h_l}'(k_0 a)}$$
$$D = -\frac{{j_l}'(ka)j_l(k_0 a) - \frac{k_0}{k} j_l(ka){j_l}'(k_0 a)}{{j_l}'(ka)h_l(k_0 a) - \frac{k_0}{k} j_l(ka){h_l}'(k_0 a)}$$

The expression for the constant A can be further simplified by using the Wronskian relation for spherical Bessel functions

$$j_l'(k_0 a)h_l(k_0 a) - j_l(k_0 a)h_l'(k_0 a) = \frac{-i}{k_0^2 a^2}.$$

We find that

$$A = \frac{\frac{-i}{k_0 k a^2}}{{j_l}'(ka)h_l(k_0 a) - \frac{k_0}{k} j_l(ka){h_l}'(k_0 a)}. \tag{36}$$

5.1 Source Energy

The quantities Σ_l^2 are found using Eq.(32) to be given by

$$\begin{aligned} {\Sigma_l}^2 &= \int_0^a r^2 dr\, |\psi_l(r)|^2 \\ &= |A|^2 \int_0^a r^2 dr\, |j_l(kr)|^2 \\ &= T_l(k, k_0)\sigma_l^2(k) \end{aligned} \tag{37}$$

where $\sigma_l^2(k)$ is the free space quantity defined in Eq.(19) *but with k_0 replaced by k* and

$$T_l(k, k_0) = |A|^2 = \frac{1}{k_0^2 a^4 |k j_l'(ka)h_l(k_0 a) - k_0 j_l(ka)h_l'(k_0 a)|^2}. \tag{38}$$

In the limit when $k \to k_0$ we have that

$$T_l(k, k_0) \to T(k_0, k_0) = \frac{1}{k_0^4 a^4 |j_l'(k_0 a)h_l(k_0 a) - j_l(k_0 a)h_l'(k_0 a)|^2} = 1$$

where we have used the Wronskian between the spherical Bessel and spherical Hankel functions. It then follows that $\Sigma_l^2 \to \sigma_l^2(k_0)$ in this limit as required.

The quantities $T_l(k, k_0)$ appearing in the expression for the Σ_l^2 have a simple interpretation: they are the magnitude square of the transmission coefficients relating the amplitudes of the outgoing multipole fields radiated by the source ρ evaluated on the exterior of the source region to the amplitude of the outgoing wave multipole fields radiated by the source on the interior of the source region. In particular, at the interior of the boundary at $r = a$ we can express the field radiated by the source in the form of a superposition of outgoing and standing wave solutions to the homogeneous Helmholtz equation with wavenumber $k = nk_0$ while outside this sphere the field is a superposition of outgoing wave solutions to the Helmholtz equation with wavenumber k_0. Because the total field and normal derivative must be continuous across the boundary we find that for each multipole mode we require

$$\begin{aligned} h_l(ka) + r_l j_l(ka) &= t_l h_l(k_0 a) \\ h'_l(ka) + r_l j'_l(ka) &= \frac{k_0}{k} t_l h'_l(k_0 a) \end{aligned}$$

where r_l and t_l are reflection and transmission coefficients and the primes denote derivatives. Solving for the transmission coefficients t_l we find that

$$\begin{aligned} t_l &= \frac{j'_l(ka)h_l(ka) - j_l(ka)h'_l(ka)}{j'_l(ka)h_l(k_0a) - \frac{k_0}{k} j_l(ka)h'_l(k_0a)} \\ &= \frac{\frac{-i}{k^2a^2}}{j'_l(ka)h_l(k_0a) - \frac{k_0}{k} j_l(ka)h'_l(k_0a)} \end{aligned} \quad (39)$$

which is seen to be identical to the coefficient A obtained earlier so that $|t_l|^2 = |A|^2 = T_l$ as indicated.

The interpretation of the quantities T_l as being the magnitude square of the transmission coefficients from the field modes in the interior of the source region to the field modes outside this region makes perfect sense in view of the formula Eq.(37) for the quantities Σ_l^2. In particular, to minimize source energy $\mathcal{E}$ (i.e., to obtain an efficient source) we wish to maximize the Σ_l^2 which, in turn, requires us to maximize the T_l or, equivalently, maximize the amount of energy transmitted from the source interior to the source exterior. As we will find in our simulations presented below the $T_l(k, k_0)$ vary *inversely* with index value n so that low source energy is obtained by selecting n to be small. On the other-hand the Σ_l^2 and, hence, the source energy also depend on the free space quantities σ_l^2 evaluated at the source region wavenumber k and, as is easily confirmed from the results presented in section 3, the free space quantities $\sigma_l^2(k)$ *increase* with k and, hence, index n, at fixed source radius a. Thus, the two quantities entering into the expression Eq.(37) for Σ_l^2 have opposite dependences on source index n and it is necessary to carefully evaluate the relative importance of each quantity in order to select an index value that leads to small source energy.

We show in Fig. 5-7 plots of the free space quantities $\sigma_l^2(k)$, the modulus square of the transmission coefficient $T_l(k, k_0)$ and, finally, the $\Sigma_l^2 = T_l(k, k_0)\sigma_l^2(k)$ plotted as a function of the product $x = k_0 a$ of the free space wavenumber with the source radius $a = 10$ and for two values of the source region index n ($n = .5$, $n = 1.5$) and for l values of $l = 10, 20$ and $l = 30$. The following conclusions can be drawn from these plots:

- For fixed a and fixed l the free space quantities $\sigma_l^2(k = nk_0)$ *increase* with increasing index n for any given free space wavenumber k_0.
- For fixed a and fixed l the quantities $T_l(k = nk_0, k_0)$ oscillate with k_0. The oscillations indicate the presence of resonances of the scattering functions within the source volume. The T_l are *decreasing* functions of index n at any given free space wavenumber k_0.
- The $\Sigma_l^2 = T_l\sigma_l^2$ oscillate with respect to k_0 due to the resonances of the scattering states in the source region and are also dependent on the source region index n. *For k_0 values below the critical point $k_0 a = l$, the growth of the free space quantity $\sigma_l^2(k = nk_0)$ with respect to index n tends to outweigh the decay of $T_l(k = nk_0, k_0)$ with respect to n with the result that the product Σ_l^2 is an increasing function of n.*

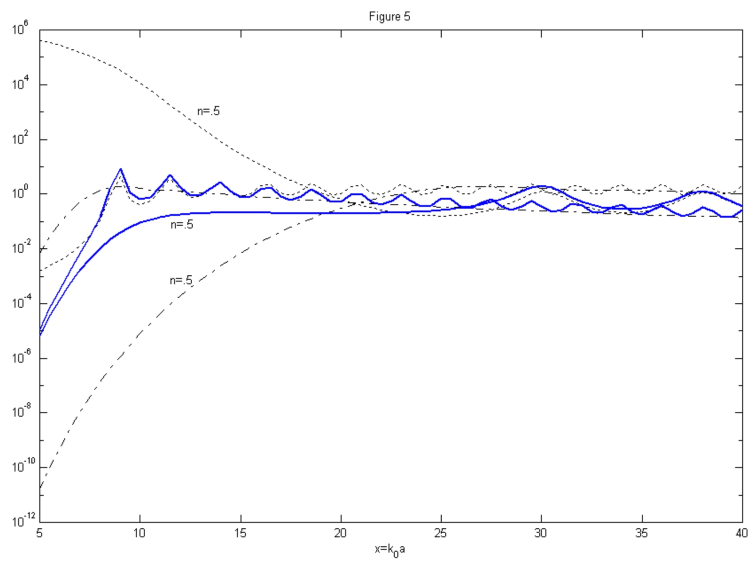

Figure 5: Plots of $\sigma_l^2(k = nk_0)$ (dash-dot), $T_l(k = nk_0, k_0)$ (dotted), and $\Sigma_l^2 = T_l\sigma_l^2$ (solid) for $l = 10$ and $n = .5$ and $n = 1.5$. It is seen from the plots that the larger n value yields larger Σ_l^2.

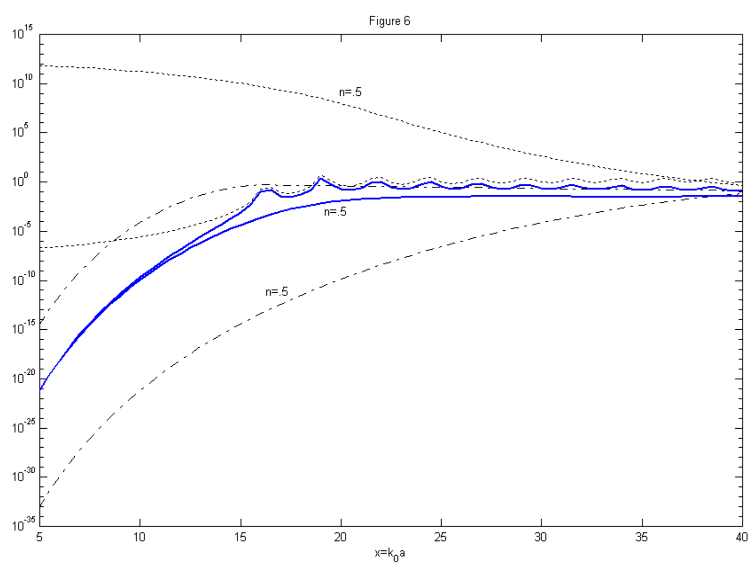

Figure 6: Plots of $\sigma_l^2(k = nk_0)$ (dash-dot), $T_l(k = nk_0, k_0)$ (dotted), and $\Sigma_l^2 = T_l\sigma_l^2$ (solid) for $l = 20$ and $n = .5$ and $n = 1.5$. It is seen from the plots that the larger n value yields larger Σ_l^2.

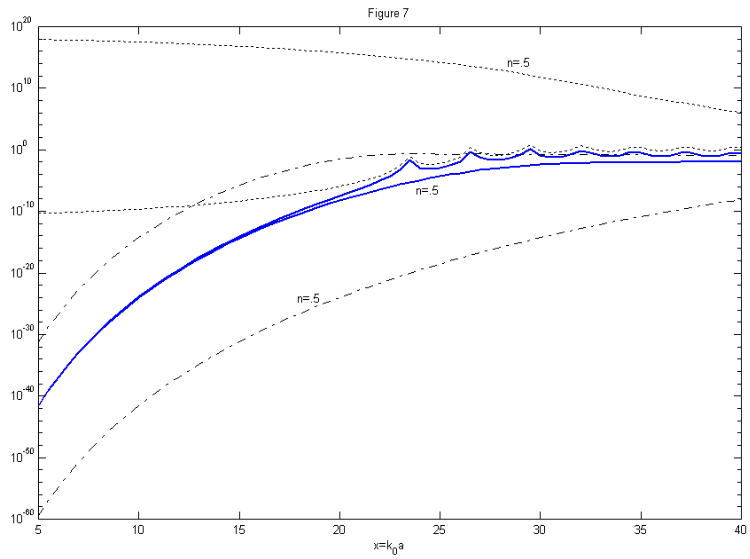

Figure 7: Plots of $\sigma_l^2(k = nk_0)$ (dash-dot), $T_l(k = nk_0, k_0)$ (dotted), and $\Sigma_l^2 = T_l\sigma_l^2$ (solid) for $l = 30$ and $n = .5$ and $n = 1.5$. It is seen from the plots that the larger n value yields larger Σ_l^2.

A more indepth look at the behavior of the Σ_l^2 as a function of k_0, n and l can be obtained from Figs. 8-10. These figures show composite plots of Σ_l^2 for $n = .5, 1$ and $n = 1.5$ for three different l values ($l = 10, 20, 30$). It is clear from these plots that by making the source region index $n > 1$ it is possible to increase the Σ_l^2 beyond their free space values $\sigma_l^2(k_0)$ and thereby obtain sources which have lower energy than those embedded in free space.

We conclude from the above results that like the free space quantities $\sigma_l^2(k_0)$, the Σ_l^2 decay exponentially to zero for $l >> k_0a$. However, by proper selection of the index n of the source region it is possible to obtain higher values of these quantites in the vicinity of the critical point $k_0a = l$ and, hence, lower source energy than can be obtained in the free space case for the same radiation pattern. This result follows from the fact that the radiation pattern expansion coefficients a_l^m *are independent of the source region index distribution* so that minimum source energy is obtained by simply maximizing the Σ_l^2.

We computed the source energy for the model radiation pattern employed in the free space examples of section 3.2. Using the coefficients given in Eq.(20) we computed the source energy using Eq.(18) with the Σ_l^2 given by Eq.(37). We show in Fig. 11 plots of the source energies as a function of $x = k_0a$ for three different values of the cut-off parameter L and for a source radius of $a = 10$ and and index value of $n = 1.5$. We also show for comparison the plots of the source energy for a source embedded in free space. It is seen that as expected the source energy becomes extremely large if we try to achieve an L value that exceeds the critical value $L = k_0a$. This is, of course, due to the fact that the quantities Σ_l^2 become extremely small when $l > k_0a$.

Finally, in Fig. 12 we show plots of the "gain" defined as the ratio of source energy for a source in a constant index sphere to that of a source embedded in free space. We show three plots corresponding to the three case illustrated in Fig. 11.

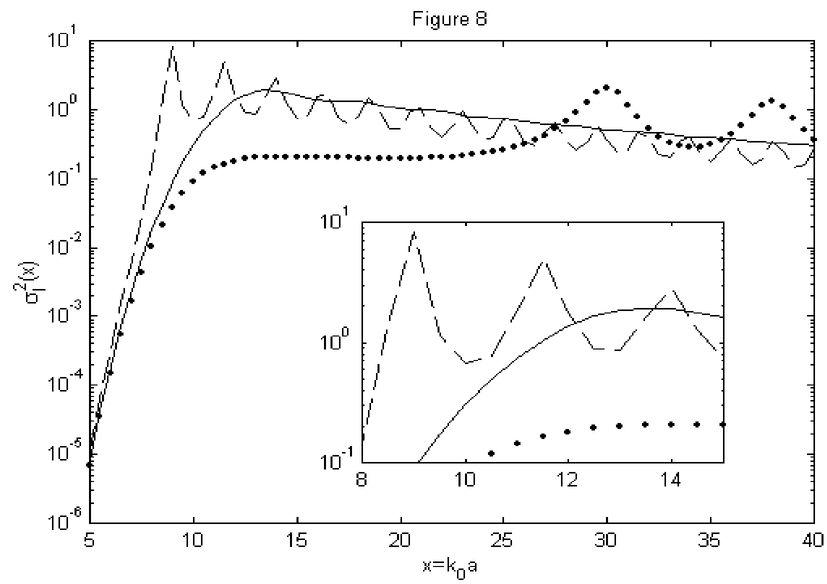

Figure 8: Plots of $\Sigma_l^2 = T_l\sigma_l^2$ for $l = 10$ and $n = .5$ (xx), $n = 1$ (xx) and $n = 1.5$ (xx). It is seen from the plots that the larger n value yields larger Σ_l^2 in the vicinity of the critical point $k_0a = l$.

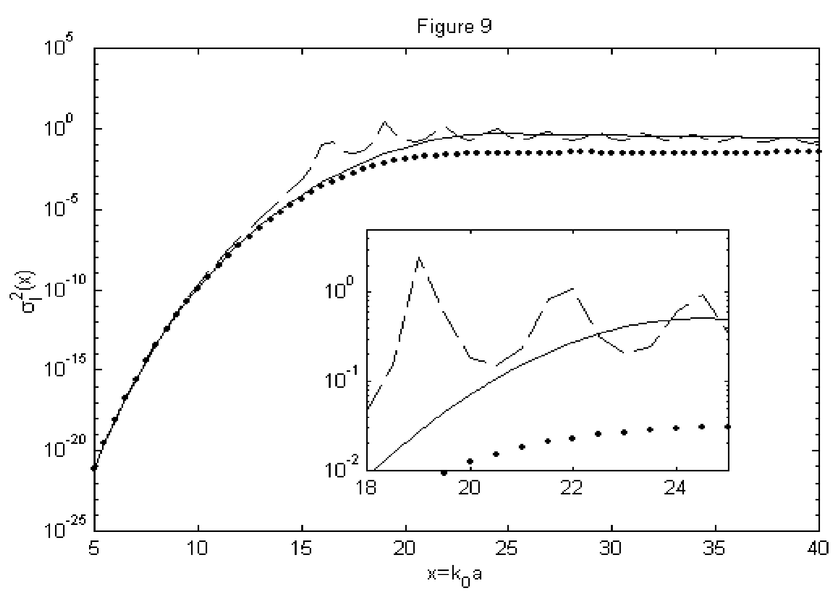

Figure 9: Plots of $\Sigma_l^2 = T_l\sigma_l^2$ for $l = 20$ and $n = .5$ (xx), $n = 1$ (xx) and $n = 1.5$ (xx). It is seen from the plots that the larger n value yields larger Σ_l^2 in the vicinity of the critical point $k_0a = l$.

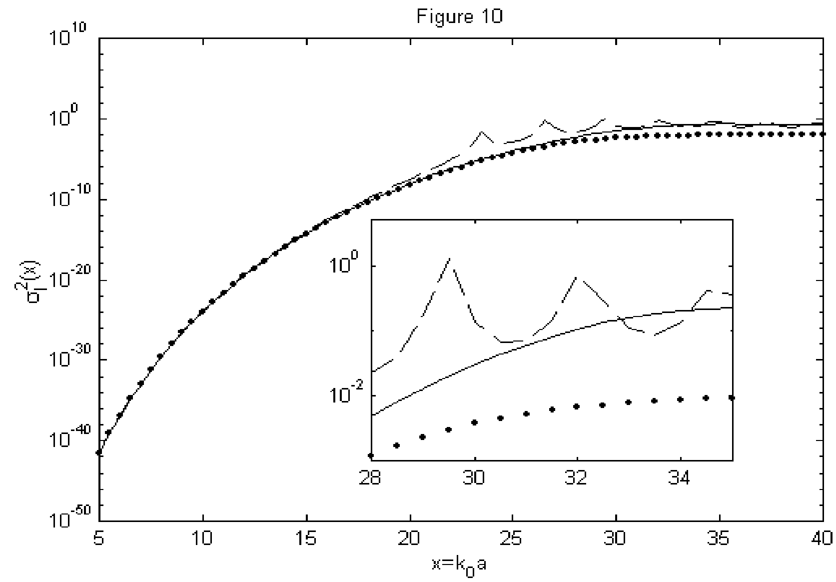

Figure 10: Plots of $\Sigma_l^2 = T_l \sigma_l^2$ for $l = 30$ and $n = .5$ (xx), $n = 1$ (xx) and $n = 1.5$ (xx). It is seen from the plots that the larger n value yields larger Σ_l^2 in the vicinity of the critical point $k_0 a = l$.

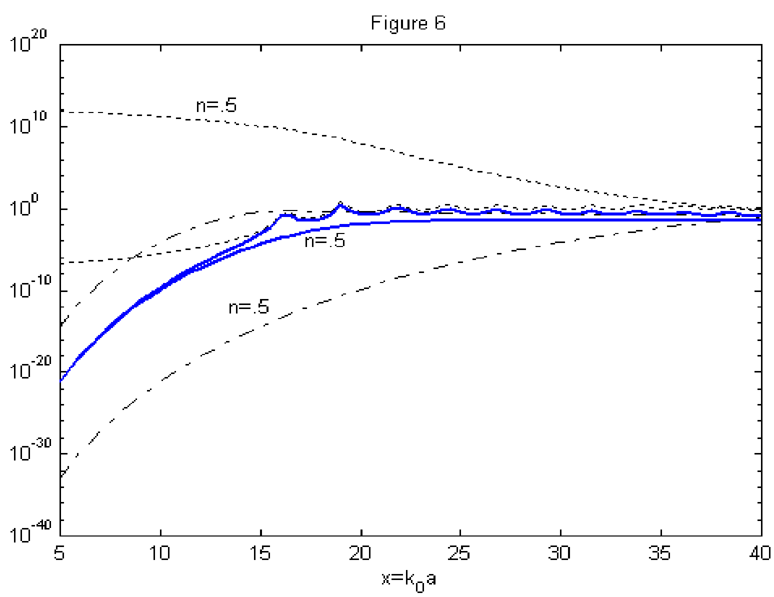

Figure 11: Plots of source energy for $a = 10$, $n = 1$ (solid) and $n = 1.5$ (with asterisks) and for $L = 10, 20$, and $L = 30$. It is seen from the plots that the larger n value yields smaller energy and, hence, a more efficient source in the vicinity of the critical point $k_0 a = L$.

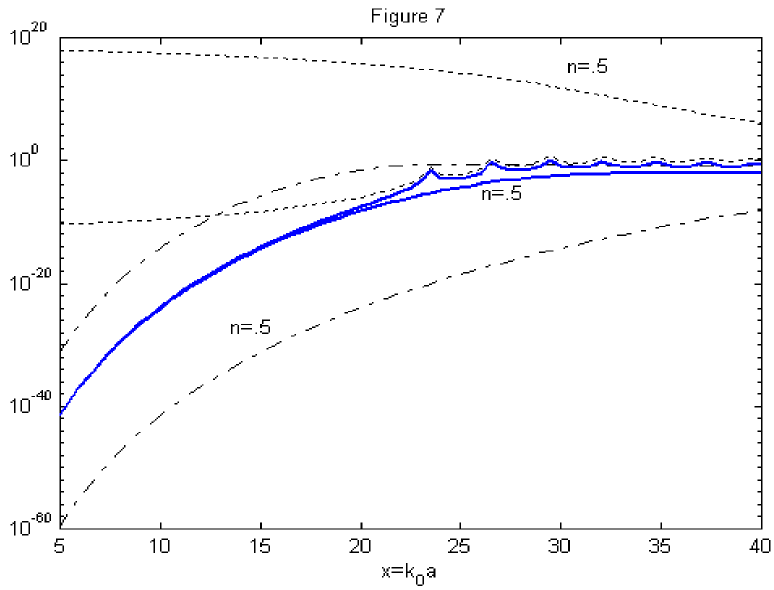

Figure 12: Plots of source energy "gain" defined to be the ratio of source energy when the source is embedded in an index distribution to that of the source energy when embedded in vacumn. The three sets of curves are for $a = 10$, $n = 1$ (solid) and $n = 1.5$ (with asterisks) and for $L = 10, 20$, and $L = 30$. It is seen from the plots that the larger n value yields smaller gain and, hence, a more efficient source in the vicinity of the critical point $k_0a = L$.

6 Summary and Conclusions

We have developed the basic theory of the inverse source problem for compactly supported sources embedded in an inhomogeneous index profile $n(\mathbf{r})$. Most of our results pertain, in particular, to spherically symmetric index distributions $n(\mathbf{r}) = n(r)$ although the underlying formalism is applicable to general, non-symmetric distributions. For the class of spherically symmetric index profiles we showed how to construct the so-called *minimum energy source* that generates a given radiation pattern subject to the constraint that the sources L^2 norm over the source region is minimum. It was found that the energy of the minimum energy source depended on the index profile $n(r)$ and we examined this dependence using computer simulations for the case of piece-wise constant valued profiles that are unity outside the (spherical) source volume and constant within the source volume and for a "model" radiation pattern characterized by a "resolution parameter" L that was inversely related to the effective angular width of the radiation pattern. The simulations showed that, in general, the source energy increases exponentially when the wavenumber source radius product $k_0a >> L$ independent of whether or not the source is embedded in a background medium or not. However, it was found that by embedding the source in a spherical region having constant index $n > 1$ that the source energy was smaller than that obtained for a source in vacumn over moderate ranges of the wavenumber source radius product k_0a in the immediate vicinity of the critical value $k_0a = L$. This suggests that embedding sources in "designer" background distributions may lead to significant improvement in source efficiency.

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