# EXTREMAL FIRST DIRICHLET EIGENVALUE OF DOUBLY CONNECTED PLANE DOMAINS AND DIHEDRAL SYMMETRY 

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#### Abstract

We deal with the following eigenvalue optimization problem: Given a bounded domain $D \subset \mathbb{R}^{2}$, how to place an obstacle $B$ of fixed shape within $D$ so as to maximize or minimize the fundamental eigenvalue $\lambda_{1}$ of the Dirichlet Laplacian on $D \backslash B$. This means that we want to extremize the function $\rho \mapsto \lambda_{1}(D \backslash \rho(B))$, where $\rho$ runs over the set of rigid motions such that $\rho(B) \subset D$. We answer this problem in the case where both $D$ and $B$ are invariant under the action of a dihedral group $\mathbb{D}_{n}, n \geq 2$, and where the distance from the origin to the boundary is monotonous as a function of the argument between two axes of symmetry. The extremal configurations correspond to the cases where the axes of symmetry of $B$ coincide with those of $D$.


## 1. Introduction and Statement of the main Result

The relations between the shape of a domain and the eigenvalues of its Dirichlet or Neumann Laplacian, have been intensively investigated since the 1920's when Faber [5] and Krahn [12] have proved independently the famous eigenvalue isoperimetric inequality first conjectured by Rayleigh (1877): the first Dirichlet eigenvalue $\lambda_{1}(\Omega)$ of any bounded domain $\Omega \subset \mathbb{R}^{n}$ satisfies

$$
\lambda_{1}(\Omega) \geq \lambda_{1}\left(\Omega^{*}\right)
$$

where $\Omega^{*}$ is a ball having the same volume as $\Omega$. We refer to the review papers of Ashbaugh [1, 2] and Henrot [9] for a survey of recent results on optimization problems involving eigenvalues.

The present work deals with the following eigenvalue optimization problem: Given a bounded domain $D$, we want to place an obstacle (or a hole) $B$, of fixed shape, inside $D$ so as to maximize or minimize the fundamental eigenvalue $\lambda_{1}$ of the Laplacian or Schrödinger operator on $D \backslash B$ with Zero Dirichlet conditions on the boundary.

In other words, the problem is to optimize the principal eigenvalue function $\rho \mapsto \lambda_{1}(D \backslash \rho(B))$, where $\rho$ runs over the set of rigid motions such that $\rho(B) \subset D$.

The first result obtained in this direction concerned the case where both $D$ and $B$ are disks of given radii. Indeed, it follows from Hersch's work [10 that the maximum of $\lambda_{1}$ is achieved when the disks are concentric (see also [14]). This result has been extended to any dimension by several authors (Harrell, Kröger and Kurata [8], Kesavan [11, ...). Actually, Harrell, Kröger and Kurata [8] gave a more general result showing that, if the domain $D$ satisfies an interior symmetry property with respect to a hyperplane $P$ passing through the center of the spherical obstacle $B$

[^0](which means that the image by the reflection with respect to $P$ of one component of $D \backslash P$ is contained in $D)$, then the Dirichlet fundamental eigenvalue $\lambda_{1}(D \backslash B)$ decreases when the center of $B$ moves perpendicularly to $P$ in the direction of the boundary of $D$. In the particular case where both the domain $D$ and the obstacle $B$ are balls, this implies that the minimum of $\lambda_{1}(D \backslash B)$ corresponds to the limit case where $B$ touches the boundary of $D$.

Notice that when the obstacle $B$ is a disk, only translations of $B$ may affect the $\lambda_{1}$ of $D \backslash B$ and the optimal placement problem reduces to the choice of the center of $B$ inside $D$.

In the present work we investigate a kind of dual problem in the sense that we consider a nonspherical obstacle $B$ whose center of mass is fixed inside $D$, and seek the optimal positions while turning $B$ around its center.

It is of course hopeless to expect a universal solution to this problem. In fact, we will restrict our investigation to a class of domains satisfying a dihedral symmetry and a monotonicity conditions.

Thus, let $D$ be a simply-connected plane domain and assume that the following conditions are satisfied:
(i) ( $\mathbb{D}_{n}$-symmetry) for an integer $n \geq 2, D$ is invariant under the action of the dihedral group $\mathbb{D}_{n}$ of order $2 n$ generated by the rotation $\rho_{\frac{2 \pi}{n}}$ of angle $\frac{2 \pi}{n}$ and a reflection $S$. Such a domain admits $n$ axes of symmetry passing through the origin and such that the angle between 2 consecutive axes is $\frac{\pi}{n}$.
(ii) (monotonicity of the boundary) the distance $d(O, x)$ from the origin to a point $x$ of the boundary of $D$ is monotonous as a function of the argument of $x$, in a sector delimited by two consecutive symmetry axes.

Notice that assumption (i) guarantees that the center of mass of $D$ is at the origin. Regular $n$-gones centered at the origin are the simplest examples of domains satisfying these assumptions. More generally, if $g$ is any positive even $\frac{2 \pi}{n}$-periodic continuous function that is monotonous on the interval $\left(0, \frac{\pi}{n}\right)$, then the domain

$$
D=\left\{r e^{\mathbf{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<g(\theta)\right\}
$$

satisfies assumptions (i) and (ii). Actually, up to a rigid motion, any domain satisfying assumptions (i) and (ii) can be parametrized in such a manner.

It is worth noticing that, due to the monotonicity condition, the "distance to the origin" function on the boundary of $D$ achieves its maximum and its minimum alternatively at the intersection points of $\partial D$ with the $2 n$ half-axes of symmetry. The $n$ points of $\partial D$ at maximal (resp. minimal) distance from the origin will be called "outer vertices" (resp. "inner vertices") of D.

Our main result is the following

Theorem 1. Let $D$ and $B$ be two plane domains satisfying the assumptions of $\mathbb{D}_{n}$-symmetry and monotonicity (i) and (ii) above for an integer $n \geq 2$. Assume furthermore that $B$ has $C^{2}$ boundary and that $\rho(B) \subset D$ for all $\rho \in S O(2)$. Then, the fundamental Dirichlet eigenvalue $\lambda_{1}(D \backslash B)$ of $D \backslash B$ is optimized exactly when the axes of symmetry of $B$ coincide with those of $D$.

The maximizing configuration corresponds to the case where the outer vertices of $B$ and $D$ lie on the same half-axes of symmetry (we will then say that $B$ occupies the "ON" position in $D$ ).

The minimizing configuration corresponds to the case where the outer vertices of $B$ lie on the half-axes of symmetry passing through the inner vertices of $D$ (this is what will be called the "OFF" position).

Actually, we will prove that, except for the trivial case where $D$ or $B$ is a disk, the fundamental Dirichlet eigenvalue of $D \backslash B$ decreases gradually when $B$ switches from "ON" to "OFF".

The main ingredients of the proof of Theorem 1are Hadamard's variation formula for $\lambda_{1}$ and the technique of domain reflection initiated by Serrin [17] in PDE's setting.


Examples of maximal (left) and minimal (right) configurations with $n=2,3$ and 4 respectively
Extensions of Theorem 1 to the following situations can be obtained up to slight changes in the proof (indeed, only the Hadamard formula should be replaced by the variation formula corresponding to the new functional):
(1) Soft obstacles: instead considering the Dirichlet Laplacian on $D \backslash B$, we consider the Schrödinger type operator

$$
H(\alpha, B):=\Delta-\alpha \chi_{B}
$$

acting on $H_{0}^{1}(D)$, where $\alpha>0$ and $\chi_{B}$ is the indicator function of $B$. Optimization problems related to the fundamental eigenvalue of operators of this kind have been investigated in particular in [8] and [3]. Under the assumptions of Theorem 1 on $D$ and $B, \forall \alpha>0$, the fundamental eigenvalue of $H(\alpha, B)$ achieves its maximum at the "ON" position and its minimum at the "OFF" position.
(2) Wells: this case corresponds to the operator $H(\alpha, B)$ with $\alpha<0$. Under the circumstances of Theorem [1, $\forall \alpha<0$, the first eigenvalue of $H(\alpha, B)$ achieves its maximum at the "OFF" position and its minimum at the "ON" position.
(3) Stationary problem : the problem now is to optimize the Dirichlet energy $J(D \backslash B):=\int_{D \backslash B}|\nabla u|^{2} d x$ of the unique solution $u$ of the problem

$$
\left\{\begin{aligned}
\Delta u & =-1 & & \text { in } D \backslash B \\
u & =0 & & \text { on } \partial(D \backslash B),
\end{aligned}\right.
$$

This problem was treated in [11, Section 2] in the case where both $D$ and $B$ are balls. Under the assumptions of Theorem 1 on $D$ and $B$, one can prove that $J(D \backslash B)$ achieves its maximum when $B$ is at the "ON" position and its minimum when $B$ is at the "OFF" position.

## 2. Proof of the main result

Without loss of generality, we may assume that the domain $D$ and the obstacle $B$ are centered at the origin and are both symmetric with respect to the $x_{1}$-axis so that they can be parametrized in polar coordinates by

$$
\begin{aligned}
& D=\left\{r e^{\mathbf{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<g(\theta)\right\} \\
& B=\left\{r e^{\mathbf{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<f(\theta)\right\}
\end{aligned}
$$

where $f$ and $g$ are two positive even $\frac{2 \pi}{n}$-periodic functions which are nondecreasing on $\left(0, \frac{\pi}{n}\right)$. To avoid technicalities, we suppose throughout that $g$ is continuous and $f$ is $C^{2}$. Extensions of our result to a wider class of domains would certainly be possible up to some additional technical difficulties.

The condition that the obstacle $B$ can freely rotate around his center inside $D$, that is $\rho(\bar{B}) \subset D$ for all $\rho \in S O(2)$, amounts to the following:

$$
f\left(\frac{\pi}{n}\right)=\max _{0 \leq \theta \leq 2 \pi} f(\theta)<\min _{0 \leq \theta \leq 2 \pi} g(\theta)=g(0)
$$

Let us denote, for all $t \in \mathbb{R}$, by $\rho_{t}$ the rotation of angle $t$, that is, $\forall \zeta \in \mathbb{R}^{2} \cong \mathbb{C}$, $\rho_{t}(\zeta)=e^{\mathbf{i} t} \zeta$, and set

$$
B_{t}:=\rho_{t}(B) \text { and } \Omega(t):=D \backslash B_{t} .
$$

Let $\lambda(t)$ be the fundamental eigenvalue of the Dirichlet Laplacian on $\Omega(t)$. It is well known that, since it is simple, the first Dirichlet eigenvalue $\lambda(t)$ is a differentiable function of $t$ (see [6, 15] ). We denote by $u(t)$ the one parameter family of nonnegative first eigenfunctions satisfying, $\forall t \in \mathbb{R}$,

$$
\left\{\begin{aligned}
\Delta u(t) & =-\lambda(t) u(t) & & \text { in } \Omega(t) \\
u(t) & =0 & & \text { on } \partial \Omega(t) \\
\int_{\Omega(t)} u^{2}(t) & =1 & &
\end{aligned}\right.
$$

The derivative of $\lambda(t)$ is then given by the following so-called Hadamard formula (see [4, 6, 7, 16]):

$$
\begin{equation*}
\lambda^{\prime}(t)=\int_{\partial B_{t}}\left|\frac{\partial u(t)}{\partial \eta_{t}}\right|^{2} \eta_{t} \cdot v d \sigma \tag{1}
\end{equation*}
$$

where $\eta_{t}$ is the inward unit normal vector field of $\partial \Omega(t)$ (hence, along $\partial B_{t}$ the vector $\eta_{t}$ is outward with respect to $\left.B_{t}\right)$ and $v$ denotes the restriction to $\partial \Omega(t)=\partial D \cup \partial B_{t}$ of the deformation vector field. In our case, the vector $v$ vanishes on $\partial D$ and is given by $v(\zeta)=\mathbf{i} \zeta$ for all $\zeta \in \partial B_{t}$.

Since both $\Omega$ and $B$ are invariant by the dihedral group $\mathbb{D}_{n}$, it follows that, $\forall t \in \mathbb{R}, \Omega\left(t+\frac{2 \pi}{n}\right)=\Omega_{t}$. Moreover, if we denote by $S_{0}$ the reflection with respect to the $x_{1}$-axis, then we clearly have $\rho_{-t}=S_{0} \circ \rho_{t} \circ S_{0}$ which gives $B_{-t}=S_{0}\left(B_{t}\right)$ and $\Omega_{-t}=S_{0}\left(\Omega_{t}\right)$. Hence, as a function of $t$, the first Dirichlet eigenvalue of $\Omega_{t}$ is even and periodic of period $\frac{2 \pi}{n}$, that is, $\forall t \in \mathbb{R}$,

$$
\lambda\left(t+\frac{2 \pi}{n}\right)=\lambda(t) \text { and } \lambda(-t)=\lambda(t)
$$

Therefore, it suffices to investigate the variations of $\lambda(t)$ on the interval $\left[0, \frac{\pi}{n}\right]$ and Theorem 1 is a consequence of the following:

Theorem 2. Assume that neither $D$ nor $B$ is a disk.
(i) $\forall t \in\left(0, \frac{\pi}{n}\right), \lambda^{\prime}(t)<0$. Hence, $\lambda(t)$ is strictly decreasing on $\left(0, \frac{\pi}{n}\right)$.
(ii) $\forall k \in \mathbb{Z}, \lambda^{\prime}\left(k \frac{\pi}{n}\right)=0$ and $k \frac{\pi}{n}, k \in \mathbb{Z}$, are the only critical points of $\lambda$ on $\mathbb{R}$.

Hence, $\lambda(t)$ achieves its maximum for $t=0 \bmod \frac{2 \pi}{n}$ which corresponds to the "ON" position, and its minimum for $t=\frac{\pi}{n} \bmod \frac{2 \pi}{n}$ which corresponds to the "OFF" position. Of course, if $D$ or $B$ is a disk, then the function $\lambda(t)$ is constant.

In what follows we will denote, for any $\alpha \in \mathbb{R}$, by $z_{\alpha}$ the $\theta=\alpha$ axis, that is $z_{\alpha}:=\left\{r e^{\mathbf{i} \alpha} ; r \in \mathbb{R}\right\}$, and by $z_{\alpha}^{+}$the half-axis $\left\{r e^{\mathbf{i} \alpha} ; r \geq 0\right\}$.

We start the proof with the following elementary lemma.
Lemma 1. Let $K$ be a plane domain defined in polar coordinates by $K=\left\{r e^{\mathbf{i} \theta} ; \theta \in\right.$ $[0,2 \pi), 0 \leq r<h(\theta)\}$, where $h$ is a positive $2 \pi$-periodic function of classe $C^{1}$, and let $v$ be a vector field whose restriction to $\partial K$ is given by

$$
v(\theta):=v\left(h(\theta) e^{\mathbf{i} \theta}\right)=\mathbf{i} h(\theta) e^{\mathbf{i} \theta}=h(\theta) e^{\mathbf{i}\left(\theta+\frac{\pi}{2}\right)} .
$$

We denote by $\eta$ the unit outward normal vector field of $\partial K$. One has, at any point $h(\theta) e^{\mathbf{i} \theta}$ of $\partial K$ where $\eta$ is defined,
(i) $\eta(\theta):=\eta\left(h(\theta) e^{\mathbf{i} \theta}\right)=\frac{h(\theta) e^{\mathbf{i} \theta}-\mathbf{i} h^{\prime}(\theta) e^{\mathbf{i} \theta}}{\sqrt{h^{2}(\theta)+h^{\prime 2}(\theta)}}$
(ii) $\eta \cdot v(\theta)=\frac{-h(\theta) h^{\prime}(\theta)}{\sqrt{h^{2}(\theta)+h^{\prime 2}(\theta)}}$. Hence, $\eta \cdot v(\theta)$ has constant sign on an interval $I$ if and only if $h$ is monotonous in $I$.
(iii) if for some $\alpha>0$, the domain $K$ is symmetric with respect to the axis $z_{\alpha}$, then the function $\eta \cdot v$ is antisymmetric w.r.t this axis, that is

$$
\eta \cdot v(\alpha+\theta)=-\eta \cdot v(\alpha-\theta)
$$

Proof. Assertions (i) and (ii) are direct consequences from the definition of $K$. The fact that $K$ is symmetric with respect to the axis $z_{\alpha}$ implies that the function $h$ satisfies $h(\alpha+\theta)=h(\alpha-\theta)$. Therefore, (iii) follows immediately from (ii).

We will denote by $S_{\alpha}$ the symmetry with respect to the axis $z_{\alpha}$. We will also denote, for $\alpha<\beta$, by $\sigma(\alpha, \beta)$ the sector delimited by $z_{\alpha}^{+}$and $z_{\beta}^{+}$, that is

$$
\sigma(\alpha, \beta)=\left\{r e^{\mathbf{i} \theta} ; r>0 \text { and } \alpha<\theta<\beta\right\}
$$

Lemma 2. Let $D$ be as above. For all $t \in\left(0, \frac{\pi}{n}\right)$, we have:

$$
S_{\frac{\pi}{n}+t}\left(D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)\right) \subseteq D \cap \sigma\left(t, \frac{\pi}{n}+t\right) .
$$

Moreover, if $D$ is not a disk, then

$$
S_{\frac{\pi}{n}+t}\left(\partial D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)\right) \cap D \neq \emptyset
$$

Proof. The action of the symmetry $S_{\frac{\pi}{n}+t}$ is given in polar coordinates by $S_{\frac{\pi}{n}+t}\left(r e^{\mathbf{i} \theta}\right)=$ $r e^{\mathbf{i}\left(2\left(\frac{\pi}{n}+t\right)-\theta\right)}$. Hence,

$$
S_{\frac{\pi}{n}+t}\left(D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)\right)=S_{\frac{\pi}{n}+t}(D) \cap \sigma\left(t, \frac{\pi}{n}+t\right)
$$

Moreover, the domain $D$ being parametrized by a positive even $\frac{2 \pi}{n}$-periodic function $g(\theta)$, that is $D=\left\{r e^{\mathbf{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<g(\theta)\right\}$, its image $S_{\frac{\pi}{n}+t}(D)$ can be parametrized in the same manner by the function $g^{*}(\theta)=g(\theta-2 t)$. Thus

$$
S_{\frac{\pi}{n}+t}(D) \cap \sigma\left(t, \frac{\pi}{n}+t\right)=\left\{r e^{\mathbf{i} \theta} ; \theta \in\left(t, \frac{\pi}{n}+t\right), 0 \leq r<g(\theta-2 t)\right\} .
$$

Therefore, we need to prove that $F(\theta)=g(\theta)-g^{*}(\theta)$ is nonnegative for every $\theta$ in the interval $\left(t, \frac{\pi}{n}+t\right)$. This will be possible thanks to the assumptions of symmetry (that is $g$ is even and $\frac{2 \pi}{n}$-periodic) and monotonicity (that is $g$ is nondecreasing on $\left.\left[0, \frac{\pi}{n}\right]\right)$. Indeed, these properties imply that on the interval $\left(t, \frac{\pi}{n}+t\right)$,

- $g$ achieves its maximum at $\theta=\frac{\pi}{n}$,
- $g^{*}$ achieves its minimum at $\theta=2 t$.

case $2 t<\frac{\pi}{n}$

case $2 t>\frac{\pi}{n}$

Four cases must be considered separately:

- If $t<\theta \leq \min \left\{2 t, \frac{\pi}{n}\right\}$, we may write, since $g$ is even, $F(\theta)=g(\theta)-g(2 t-\theta)$, with $0 \leq 2 t-\theta<\theta \leq \frac{\pi}{n}$. Since $g$ is nondecreasing on $\left[0, \frac{\pi}{n}\right]$, we get $F(\theta) \geq 0$.
- If $\max \left\{2 t, \frac{\pi}{n}\right\} \leq \theta<\frac{\pi}{n}+t$, we may write, since $g$ is even and $\frac{2 \pi}{n}$-periodic, $F(\theta)=g\left(2 \frac{\pi}{n}-\theta\right)-g(\theta-2 t)$ with $0 \leq \theta-2 t<2 \frac{\pi}{n}-\theta \leq \frac{\pi}{n}$. Hence, $F(\theta) \geq 0$.
- If $2 t<\frac{\pi}{n}$ and $2 t \leq \theta \leq \frac{\pi}{n}$, then $0 \leq \theta-2 t<\theta \leq \frac{\pi}{n}$ and, then, $F(\theta)=$ $g(\theta)-g(\theta-2 t) \geq 0$.
- If $2 t>\frac{\pi}{n}$ and $\frac{\pi}{n} \leq \theta \leq 2 t$, then $0 \leq 2 t-\theta<2 \frac{\pi}{n}-\theta \leq \frac{\pi}{n}$ and, then, $F(\theta)=g\left(2 \frac{\pi}{n}-\theta\right)-g(2 t-\theta) \geq 0$.
Hence, $F(\theta)$ is nonnegative for all $\theta$ in $\left(t, \frac{\pi}{n}+t\right)$.
Now, if $D$ is not a disk, then $g$ is nonconstant on $\left[0, \frac{\pi}{n}\right]$. Following the arguments above, we deduce that the function $F(\theta)$ is positive somewhere on $\left(t, \frac{\pi}{n}+t\right)$ which means that $S_{\frac{\pi}{n}+t}\left(\partial D \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)\right)$ meets the interior of $D$.

Proof of Theorem 2. Notice first that, since $\lambda$ is an even and $\frac{2 \pi}{n}$-periodic function of $t$, one immediately gets, $\forall k \in \mathbb{Z}, \lambda\left(k \frac{\pi}{n}-t\right)=\lambda\left(k \frac{\pi}{n}+t\right)$ and, then,

$$
\lambda^{\prime}\left(k \frac{\pi}{n}\right)=0
$$

Alternatively, one can deduce that $\lambda^{\prime}\left(k \frac{\pi}{n}\right)=0$ from Hadamard's variation formula (11) after noticing that the domain $\Omega\left(k \frac{\pi}{n}\right)$ is symmetric with respect to the $x_{1}$-axis and that the first Dirichlet eigenfunction $u\left(k \frac{\pi}{n}\right)$ satisfies $u \circ S_{0}=u$, where $S_{0}$ is the symmetry with respect to the $x_{1}$-axis.

Let us fix a $t$ in $\left(0, \frac{\pi}{n}\right)$ and denote by $u$ the nonnegative first Dirichlet eigenfunction of $\Omega(t)$ satisfying $\int_{\Omega(t)} u^{2}=1$. The domain $\Omega(t)$ is clearly invariant by the rotation $\rho_{\frac{2 \pi}{n}}$ of angle $\frac{2 \pi}{n}$, hence $u \circ \rho_{\frac{2 \pi}{n}}=u$. On the other hand, the domain $B$ being parametrized by a positive even $\frac{2 \pi}{n}$-periodic function $f(\theta)$, that is $B=\left\{r e^{\mathrm{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<f(\theta)\right\}$, one has

$$
B_{t}=\left\{r e^{\mathbf{i} \theta} ; \theta \in[0,2 \pi), 0 \leq r<h(\theta)\right\},
$$

with $h(\theta)=f(\theta-t)$. Hence, the function $\eta_{t} \cdot v$ is invariant by $\rho_{\frac{2 \pi}{n}}$ (Lemma 11) and we have (Hadamard formula (11))

$$
\lambda^{\prime}(t)=\int_{\partial B_{t}}\left|\frac{\partial u}{\partial \eta_{t}}\right|^{2} \eta_{t} \cdot v d \sigma=n \int_{\partial B_{t} \cap \sigma\left(t, \frac{2 \pi}{n}+t\right)}\left|\frac{\partial u}{\partial \eta_{t}}\right|^{2} \eta_{t} \cdot v d \sigma
$$

Since $B_{t}$ is symmetric with respect to the axis $z_{\frac{\pi}{n}+t}$, we have (Lemma 11), $\eta_{t} \cdot v\left(\frac{\pi}{n}+\right.$ $t+\theta)=-\eta_{t} \cdot v\left(\frac{\pi}{n}+t-\theta\right)$ or, equivalently, $\eta_{t} \cdot v(x)=-\eta_{t} \cdot v\left(x^{*}\right)$, where $x^{*}$ denotes the symmetric of $x$ with respect to $z_{\frac{\pi}{n}+t}$. This yields

$$
\lambda^{\prime}(t)=n \int_{\partial B_{t} \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)}\left(\left|\frac{\partial u}{\partial \eta_{t}}(x)\right|^{2}-\left|\frac{\partial u}{\partial \eta_{t}}\left(x^{*}\right)\right|^{2}\right) \eta_{t} \cdot v(x) d \sigma
$$

Notice that the function $h(\theta)$ is decreasing between $\frac{\pi}{n}+t$ and $\frac{2 \pi}{n}+t$ and, then, $\eta_{t} \cdot v$ is nonnegative on $\partial B_{t} \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)$ (Lemma (1).

Let $H(t):=\Omega(t) \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)$. Applying Lemma 2, and since $B_{t}$ is symmetric with respect to the axis $z_{\frac{\pi}{n}+t}$, one gets

$$
S_{\frac{\pi}{n}+t}(H(t)) \subset \Omega(t) \cap \sigma\left(t, \frac{\pi}{n}+t\right)
$$

Hence, the function $w(x)=u(x)-u\left(x^{*}\right)$ is well defined on $H(t)$ and satisfies $w(x)=0$ for all $x$ in $\partial H(t) \cap\left(\partial B_{t} \cup z_{\frac{\pi}{n}+t} \cup z_{\frac{2 \pi}{n}+t}\right)$. Moreover, since $u$ vanishes on $\partial D$ and is positive inside $\Omega(t), w(x) \leq 0$ for all $x$ in $\partial H(t) \cap \partial D$ and $w(x)<0$ for certain $x$ in $\partial H(t) \cap \partial D$ (recall that $D$ is not a disk and apply the second part of Lemma (2).

Therefore, the nonconstant function $w$ satisfies the following:

$$
\left\{\begin{aligned}
\Delta w & =-\lambda(t) w & & \text { in } H(t) \\
w & \leq 0 & & \text { on } \partial H(t) .
\end{aligned}\right.
$$

Hence, $w$ must be nonpositive on the whole of $H(t)$. Otherwise, a nodal domain $V \subset H(t)$ of $w$ would have the same first Dirichlet eigenvalue as $\Omega(t)$. But, due to the invariance of $\Omega(t)$ by $\rho_{\frac{2 \pi}{n}}$, the domain $\Omega(t)$ would contain $n$ copies of $V$ leading to a strong contradiction with the domain monotonicity theorem for eigenvalues. Therefore, $\Delta w \geq 0$ in $H(t)$ and $w$ achieves its maximal value (i.e. zero) on $\partial B_{t} \cap$ $\sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right) \subset \partial H(t)$. The Hopf maximum principle (see [13, Theorem 7, ch.2]) then implies that, at any regular point $x$ of $\partial B_{t} \cap \sigma\left(\frac{\pi}{n}+t, \frac{2 \pi}{n}+t\right)$, one has

$$
\frac{\partial w}{\partial \eta_{t}}(x)=\frac{\partial u}{\partial \eta_{t}}(x)-\frac{\partial u}{\partial \eta_{t}}\left(x^{*}\right)<0
$$

It follows that $\lambda^{\prime}(t) \leq 0$ and that the equality holds if and only if $\eta_{t} \cdot v \equiv 0$. By Lemma 1, this last equality occurs if and only if $f$ is constant which means that $B$ is a disk.

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