

Limit Time Optimal Synthesis for a Control-Affine System on S^2

P. Mason*, R. Salmoni†, U. Boscain‡, Y. Chitour§

March 3, 2018

Abstract For $\alpha \in]0, \pi/2[$, let $(\Sigma)_\alpha$ be the control system $\dot{x} = (F + uG)x$, where x belongs to the two-dimensional unit sphere S^2 , $u \in [-1, 1]$ and F, G are 3×3 skew-symmetric matrices generating rotations with perpendicular axes of respective length $\cos(\alpha)$ and $\sin(\alpha)$. In this paper, we study the time optimal synthesis (TOS) from the north pole $(0, 0, 1)^T$ associated to $(\Sigma)_\alpha$, as the parameter α tends to zero. We first prove that the TOS is characterized by a “two-snakes” configuration on the whole S^2 , except for a neighborhood U_α of the south pole $(0, 0, -1)^T$ of diameter at most $\mathcal{O}(\alpha)$. We next show that, inside U_α , the TOS depends on the relationship between $r(\alpha) := \pi/2\alpha - [\pi/2\alpha]$ and α . More precisely, we characterize three main relationships, by considering sequences $(\alpha_k)_{k \geq 0}$ satisfying (a) $r(\alpha_k) = \bar{r}$; (b) $r(\alpha_k) = C\alpha_k$ and (c) $r(\alpha_k) = 0$, where $\bar{r} \in (0, 1)$ and $C > 0$. In each case, we describe the TOS and provide, after a suitable rescaling, the limiting behavior, as α tends to zero, of the corresponding TOS inside U_α .

Keywords: control-affine systems, optimal synthesis, minimum time, asymptotics

AMS subject classifications: 49J15

*Institut Elie Cartan UMR 7502, Nancy-Université/CNRS/INRIA POB 239, 54506 Vandoeuvre-lès-Nancy, France
Paolo.Mason@iecn.u-nancy.fr

†Laboratoire des signaux et systèmes, Université Paris-Sud, CNRS, Supélec, 91192 Gif-Sur-Yvette, France
rebecca.salmoni@lss.supelec.fr

‡SISSA, via Beirut 2-4 34014 Trieste, Italy boscain@sissa.it and Le2i, CNRS UMR 5158, Université de Bourgogne, 9, avenue
Alain Savary- BP 47870, 21078 Dijon, France

§Laboratoire des signaux et systèmes, Université Paris-Sud, CNRS, Supélec, 91192 Gif-Sur-Yvette, France
yacine.chitour@lss.supelec.fr

The first author was (partially) supported by IdF–Aide au partage des projets européens

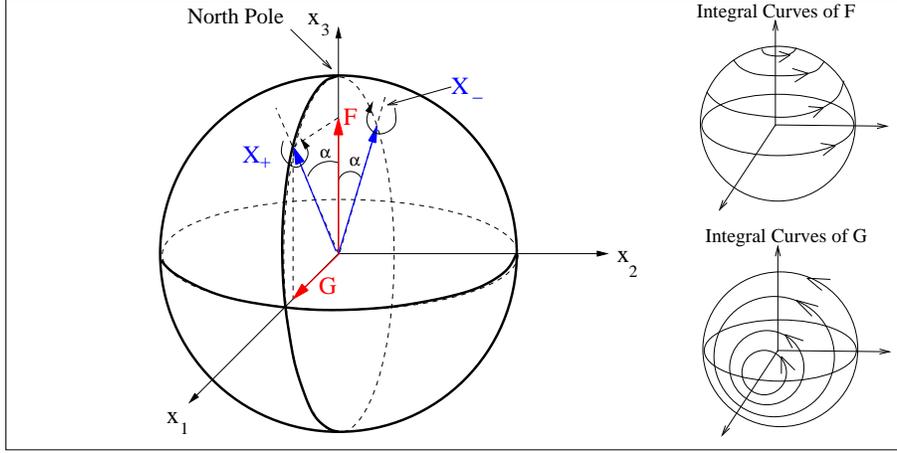


Figure 1: Geometric interpretation of the system $(\Sigma)_\alpha$. The vector fields $X_+ := F + G$ and $X_- := F - G$ are two rotations of norm one making an angle α with the axis x_3 .

1 Introduction

Let $\alpha \in]0, \pi/2[$. On the unit sphere $S^2 \subset \mathbb{R}^3$, consider the control system $(\Sigma)_\alpha$ defined by

$$(\Sigma)_\alpha \quad \dot{x} = (F + uG)x, \quad x = (x_1, x_2, x_3)^T, \quad \|x\|^2 = 1, \quad |u| \leq 1, \quad (1)$$

where F and G are two 3×3 skew-symmetric matrices representing two orthogonal rotations with axes of length respectively $\cos(\alpha)$ and $\sin(\alpha)$, $\alpha \in]0, \pi/2[$ (for the precise meaning of length, see Section 2.3). With no loss of generality, we assume that

$$F := \begin{pmatrix} 0 & -\cos(\alpha) & 0 \\ \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha) \\ 0 & \sin(\alpha) & 0 \end{pmatrix}. \quad (2)$$

In this paper, we aim at describing the time optimal synthesis (TOS for short) from the north pole $N := (0, 0, 1)^T$ for $(\Sigma)_\alpha$, i.e. for every $\bar{x} \in S^2$ we want to find the time optimal trajectory steering N to \bar{x} in minimum time (see Figure 1).

In particular *we are interested in the qualitative shape of the time optimal synthesis in a neighborhood of the south pole $S = (0, 0, -1)^T$, in the limit $\alpha \rightarrow 0$* . The interest for that problem stems from motion planning issues in aeronautics and quantum control, see [5, 7] for instance.

The present paper is actually a continuation of [5] in the sense that it answers questions raised in the latter paper. There, the purpose was to provide a lower and an upper bound for $N(\alpha)$, the maximum number of switchings for time optimal trajectories for the left invariant control system

$$(S)_\alpha \quad \dot{g} = g(F + uG), \quad g \in SO(3), \quad |u| \leq 1, \quad (3)$$

where F and G are defined in (2). Recall that, for such control systems, it is known (cf. for instance [5, 7]) that every time optimal trajectory is a finite concatenation of bang arcs (i.e. $u \equiv \pm 1$) or singular arcs ($u = 0$). A bang arc is an integral trajectory corresponding to the rotations

$$X_+ := F + G, \quad X_- := F - G, \quad (4)$$

and is denoted by $e^{tX_\varepsilon}x$, $t \in [0, T]$, where $\varepsilon = \pm$, x is the starting point of the bang arc and T is its time duration. Moreover, a switching time – or simply a switching – along a time optimal trajectory is a time t_0 so that the control u is not constant in any open neighborhood of t_0 .

To estimate $N(\alpha)$, a suitable Hopf map $\Pi : SO(3) \rightarrow S^2$ was introduced to project $(S)_\alpha$ onto $(\Sigma)_\alpha$. In particular, every time optimal trajectory of $(\Sigma)_\alpha$ is the projection by Π of a time optimal trajectory of $(S)_\alpha$. It results that, if a time optimal trajectory on S^2 has a certain number of switchings, then this number is lower than or equal to the maximum number of switchings for the optimal problem on $SO(3)$. The construction of time optimal trajectories of $(\Sigma)_\alpha$ was performed according to the general theory of time optimal synthesis on 2-D manifolds developed in [4, 3, 8, 9, 10, 11, 14, 15], and recently gathered in the book [6].

The question of studying $N(\alpha)$ was first addressed in [1] where, using the index theory developed by Agrachev, the authors proved that $N(\alpha) \leq [\pi/\alpha]$, where $[\cdot]$ stands for the integer part. That result was not only an indirect indication that $N(\alpha)$ would tend to infinity as α tends to zero, but it also provided a hint on the asymptotic of $N(\alpha)$ as α tends to zero. Notice that for $\alpha = 0$ the systems (1) and (3) are not controllable. With the techniques developed in [5], enough properties for the TOS associated to $(\Sigma)_\alpha$, $\alpha < \pi/4$, were identified in order to improve the upper bound of [1] and to actually show that, for α small

$$N(\alpha) \leq k_M + 5, \quad \text{where} \quad k_M := \left\lfloor \frac{\pi}{2\alpha} \right\rfloor.$$

In [5], it is proved that, for $\alpha < \pi/4$, the extremals associated to $(\Sigma)_\alpha$ (i.e. the trajectories candidate for time optimality obtained after using the Pontryagin Maximum Principle –PMP for short–), starting from the north pole N are bang-bang trajectories, i.e. finite concatenations of bang arcs of the type

$$e^{s_f X_{-\varepsilon'}} e^{v(s_i) X_{\varepsilon'}} \dots e^{v(s_i) X_{-\varepsilon}} e^{s_i X_\varepsilon} N$$

, where the initial time duration s_i verifies $s_i \in (0, \pi]$, all the time durations of the interior bang arcs are equal to $v(s_i)$, where the function v is defined in Eq. (12) below, and the final time duration s_f verifies $s_f \leq v(s_i)$. Of particular importance for the construction of the TOS, are the switching curves, i.e. the curves made by points where the control switches from $+1$ to -1 or viceversa and defined inductively by

$$C_1^\varepsilon(s) = e^{X_\varepsilon v(s)} e^{X_{-\varepsilon} s} N, \quad C_k^\varepsilon(s) = e^{X_\varepsilon v(s)} C_{k-1}^{-\varepsilon}(s), \quad (\text{where } \varepsilon = \pm 1 \text{ and } k = 2, \dots, k_M). \quad (5)$$

Since the PMP gives just a necessary condition for optimality, it is crucial to determine the time after which an extremal is no more optimal. In [5], we showed that the number of bangs must be lower than or equal to $k_M + 1$ and the extremals cover the sphere S^2 according to the "two-snakes" configuration as depicted in Figure 2. The two "snakes" correspond to extremal trajectories starting respectively with control $+1$ and -1 . For more details, see [5].

However, in [5], we were not able to construct the complete TOS associated to $(\Sigma)_\alpha$. In particular, we could not show the optimality of all the extremals up to $k_M - 1$ bangs arcs and we could not complete analytically the construction of the synthesis in a neighborhood of the south pole S . There, the minimum time front develops singularities due to the compactness of S^2 . We only provided numerical simulations describing the evolution of the extremal front in a neighborhood of the south pole. As $\alpha \rightarrow 0$, these numerical simulations suggested the emergence of an interesting phenomenon (see Fig. 3) : define the remainder

$$r(\alpha) := \pi/2\alpha - [\pi/2\alpha]. \quad (6)$$

Then, there are three possible patterns of TOS in the neighborhood of the south pole S , each of them depending on a relation between $r(\alpha)$ and α . See Section 2.2 below, where these relations are formulated as well as conjectures.

In [7], the TOS for (Σ_α) was studied in the context of quantum control. The control system (Σ_α) describes the population transfer problem for the x_3 -component of the spin of a (spin 1/2) particle, driven by a magnetic field, which is constant along the x_3 -axis and controlled along the x_1 -axis, with bounded amplitude. In that paper, the TOS, for $\alpha \geq \pi/4$ was completed and, in the case $\alpha < \pi/4$, further information was obtained, for what concerns time optimal trajectories steering the north to the south pole (in fact the most interesting trajectories for the quantum mechanical problem). Such optimal trajectories belong to a set Ξ containing at most 8 trajectories, half of them starting with control $+1$ and the other half starting with control -1 , and

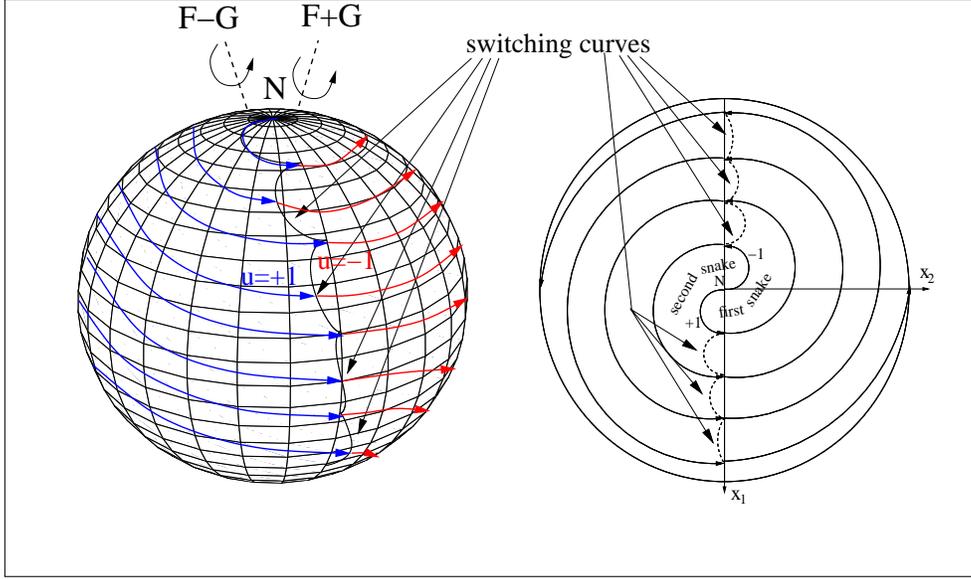


Figure 2: The “two-snakes” configuration defined by the extremal flow. Notice that this set of trajectories covers the whole sphere, but in principle not all extremals are optimal and a point can be reached by more than one trajectory at the same or at different times.

switching exactly at the same times. It was also proved that the cardinality of Ξ depends on the remainder $r(\alpha)$ defined in Eq. (6). For instance, for α and $r(\alpha)$ small enough, then Ξ contains exactly 8 trajectories (four of them are optimal) while if $r(\alpha)$ is close to 1, then Ξ contains only 4 trajectories (two of them are optimal).

The purpose of the present paper consists in studying the TOS associated to $(\Sigma)_\alpha$ as α tends to zero, focusing in particular on its behavior inside a neighborhood of the south pole. Roughly speaking, we want to determine, as α tends to zero, what could be a possible limit for the TOS associated to $(\Sigma)_\alpha$ (as suggested for instance by the patterns depicted in Fig. 3) and then to prove the convergence (in some suitable sense) of the TOS associated to $(\Sigma)_\alpha$ to that limit. To proceed, we embark on the study of a geometric object $\mathcal{F}(\alpha, T)$ called the *extremal front at time T* along $(\Sigma)_\alpha$ and defined as the set of points reached at time T by extremal trajectories starting from N (see Section 3.1 for a precise definition). The extremal front $\mathcal{F}(\alpha, T)$ contains the *minimum time front* $OF(\alpha, T)$, i.e. the set of points reached at time T by time optimal trajectories. When $\mathcal{F}(\alpha, T) = OF(\alpha, T)$, we say that $\mathcal{F}(\alpha, T)$ is *optimal*.

We first prove, in the case in which k_M is odd (being the other case analogous), that the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is made up of the union of two curves $\mathcal{E}^\varepsilon(\alpha, \cdot) : (0, \pi] \rightarrow S^2$, $\varepsilon = \pm$, with $\mathcal{E}^\varepsilon(\alpha, \cdot) = \Pi_{x_3}\mathcal{E}^{-\varepsilon}(\alpha, \cdot)$, where Π_{x_3} is the orthogonal symmetry with respect to the x_3 -axis. Moreover, for α small enough, $\mathcal{E}^\varepsilon(\alpha, \cdot)$ admits a convergent power series of the type $\sum_{l \geq 0} f_l^\varepsilon(s, r(\alpha))\alpha^l$, where the $f_l^\varepsilon(s, r)$ are real-analytic functions of $(s, r) \in \mathbb{R}^2$, 2π -periodic in s with

$$f_0^+(s, r) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad f_1^+(s, r) = \begin{pmatrix} -2rc_s \\ 2rs_s \\ 0 \end{pmatrix}, \quad f_2^+(s, r) = \begin{pmatrix} \frac{\pi}{2}(4r + c_s)s_s^2 \\ \frac{\pi}{4}(3 + 8rc_s + c_2s)s_s \\ 2r^2 \end{pmatrix}. \quad (7)$$

As a trivial consequence, we deduce that for $r \in [0, 1]$, $s \in \mathbb{R}$ and α small enough, we have

$$\mathcal{E}^\varepsilon(\alpha, s) = f_0^\varepsilon(s, r(\alpha)) + f_1^\varepsilon(s, r(\alpha))\alpha + f_2^\varepsilon(s, r(\alpha))\alpha^2 + \mathcal{O}(\alpha^3). \quad (8)$$

and

$$\frac{\partial}{\partial s}\mathcal{E}^\varepsilon(\alpha, s) = \frac{\partial}{\partial s}f_1^\varepsilon(s, r(\alpha))\alpha + \frac{\partial}{\partial s}f_2^\varepsilon(s, r(\alpha))\alpha^2 + \mathcal{O}(\alpha^3). \quad (9)$$

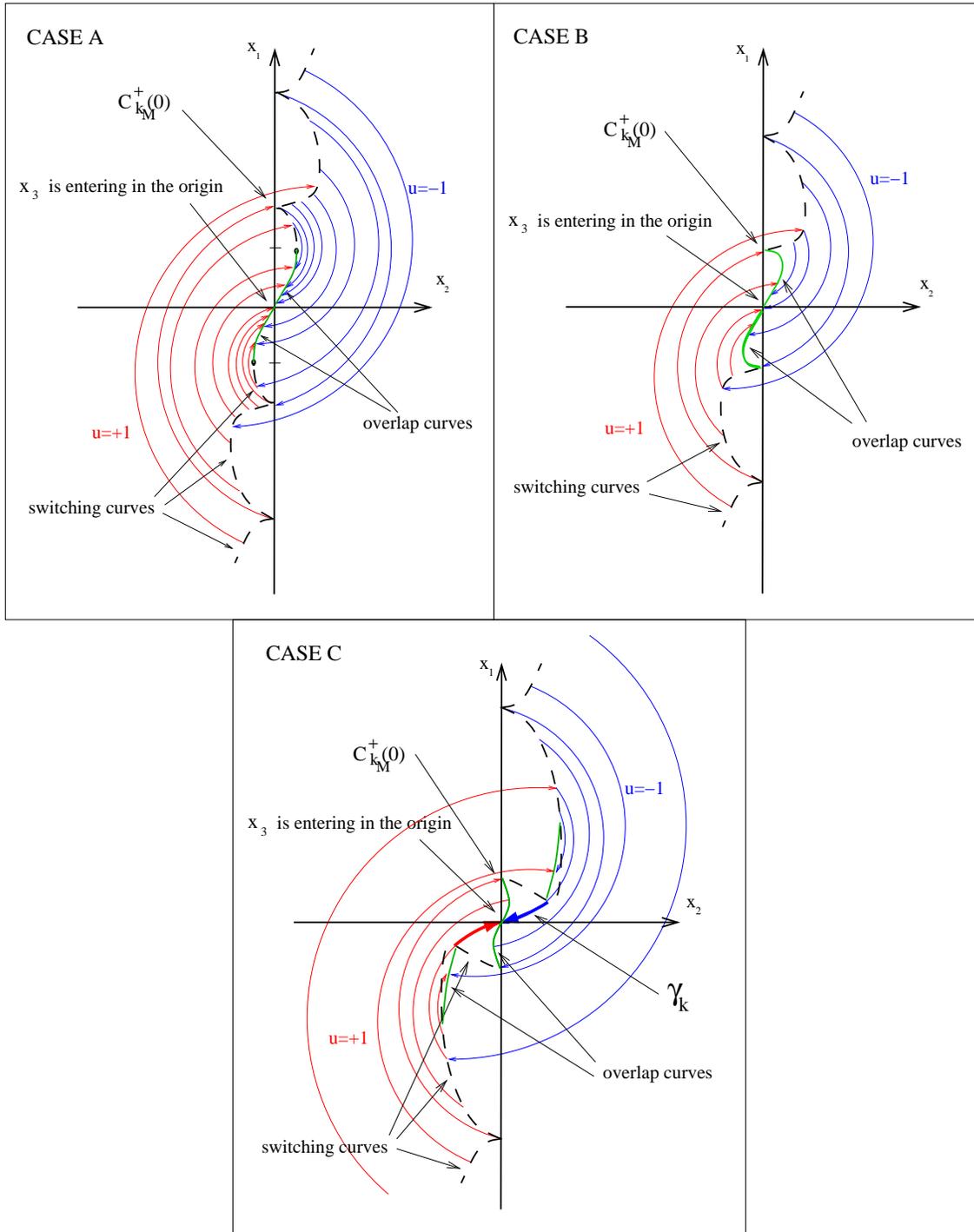


Figure 3: Conjectured shapes of the synthesis in a neighborhood of the south pole. Switching curves are \mathcal{C}^1 curves made by points in which the control switches from $+1$ to -1 or viceversa. Overlap curves are \mathcal{C}^1 curves made by points reached optimally by more than one trajectory. The curve γ_k is a bang arc that is also an overlap curve since trajectories having a different history travel on it at the same time.

where $|\mathcal{O}(\alpha^3)| \leq \bar{C}|\alpha|^3$ with $\bar{C} > 0$ constant independent of (r, s, α) .

Then we show that $\mathcal{F}(\alpha, T)$ is actually optimal for $T \leq (k_M - 1)\pi$ and α small enough (see Remark 7 below). Moreover, we show that $\mathcal{F}(\alpha, (k_M - 1)\pi)$ is a circle of radius $2(1 + r(\alpha))\alpha$ up to order α^2 (see Remark 6 below). As a consequence of the optimality of $\mathcal{F}(\alpha, (k_M - 1)\pi)$, we get that all the extremals of the “two-snakes” configuration depicted in Figure 2 are optimal up to time $(k_M - 1)\pi$. In other words, if U_α is the connected component of $S^2 \setminus \mathcal{F}(\alpha, (k_M - 1)\pi)$ containing the south pole, we obtain the optimal synthesis on $S^2 \setminus U_\alpha$. Notice that U_α is a neighborhood of the south pole of size proportional to α .

The expressions (8)–(9) are central tools to understand the possible asymptotic behaviors of the TOS associated to $(\Sigma)_\alpha$, as α tends to zero.

For this purpose we observe that the expressions of f_1^+ and f_2^+ in (7), depend explicitly on the remainder $r(\alpha)$. This fact suggests the need to impose particular relationships between α and $r(\alpha)$ in order to define any asymptotic behavior. In other words we must let α goes to zero, only along certain subsequences $(\alpha_k)_{k \geq 0}$ where a specific relationship holds between α_k and $r(\alpha_k)$. The analysis of Eq. (7) will help us to determine such relationships and to prove that the conjectures made in [5] about the qualitative shape of the synthesis near the south pole were true (see Section 2.2 and Figure 3). In particular we will see that there are exactly three qualitatively different asymptotic behaviours of the synthesis as α goes to zero, described by the following cases.

First, we analyze the case in which α is arbitrarily small, with $r(\alpha) \in (0, 1)$ uniformly far from 0 and 1. To simplify further the discussion, it is reasonable to consider the following.

$$(C1) \quad \text{For } \bar{r} \in (0, 1), \text{ let } \alpha \text{ tend to zero along the subsequence } \alpha_k := \frac{\pi}{2(k + \bar{r})}, \text{ so that } r(\alpha_k) = \bar{r}.$$

In this case $\mathcal{E}^\varepsilon(\alpha, \cdot)$ is approximated, up to order α^2 , by the expression $S + f_1^\varepsilon(\cdot, \bar{r})$. As a consequence $\mathcal{F}(\alpha, k_M\pi)$ is approximately a circle of radius $2\bar{r}\alpha$ centered at the south pole. We are then able to give a qualitative description of the optimal synthesis, as stated below in Theorem 1. This synthesis turns out to be exactly the one described in Figure 3 (case B), as predicted in [5].

It remains then to study the cases in which $r(\alpha)$ can be arbitrarily close to 0 or 1. For this purpose we first consider the case in which $r(\alpha)/\alpha$ remains bounded above and below by positive constants as α tends to zero. From Eq. (7) it is clear that this is equivalent to say that $f_1^\varepsilon(\cdot, r)\alpha$ is comparable to $f_2^\varepsilon(\cdot, r)\alpha^2$. For simplicity we consider the following.

$$(C2) \quad \text{For } C > 0, \text{ let } \alpha \text{ tend to zero along a subsequence } (\alpha_k)_{k \geq 0} \text{ such that } r(\alpha_k) = C\alpha_k.$$

In this case $\mathcal{E}^\varepsilon(\alpha, \cdot)$ is well approximated by $S + (f_1^\varepsilon(\cdot, C) + f_2^\varepsilon(\cdot, 0))\alpha^2$. If $C > \pi/4$, the synthesis is equivalent the one of the previous case. On the other hand if $C < \pi/4$ the synthesis is more complicated (see Section 5) and it turns out to be exactly the one described in Figure 3 (case C), as predicted in [5].

If α and $r(\alpha)$ tend to zero with $r(\alpha)/\alpha$ tending to infinity (resp. to zero) it is possible to see that the synthesis is qualitatively equivalent to the one of case (C1) (resp. (C2)).

The third interesting case is the following.

$$(C3) \quad \text{Let } \alpha \text{ tend to zero along the subsequence } \alpha_k := \frac{\pi}{2k}, \text{ so that } r(\alpha_k) = 0.$$

In this case the extremal front at time $k_M\pi$ contains the south pole and the corresponding optimal front reduces to that point. The optimal synthesis is then described starting from the extremal front $\mathcal{F}(\alpha, (k_M - 1)\pi) = OF(\alpha, (k_M - 1)\pi)$, and it corresponds to the one described in Figure 3 (case A), as predicted in [5].

With similar arguments, one can see that in the case in which α is small and $r(\alpha)$ is close to 1, the optimal synthesis is qualitatively equivalent either to that of Case (C1) or to that of Case (C3), and this concludes the description of the possible asymptotic behaviors as α tends to 0.

Remark 1 It is interesting to notice that numerical simulations show that for α decreasing to zero continuously, the qualitative shape of the optimal synthesis described in Figure 3 alternates cyclically in the order BCABCA....

Let us describe the results obtained in the case (C1) in more details. Since $\mathcal{F}(\alpha, k_M\pi)$ is approximated, up to $\mathcal{O}(\alpha^2)$, by a circle of center S and radius $2\bar{r}\alpha$, we are able to show that it is optimal, so that all the extremals of the “two-snakes” configuration depicted in Figure 2 are optimal up to time $k_M\pi$. In other words,

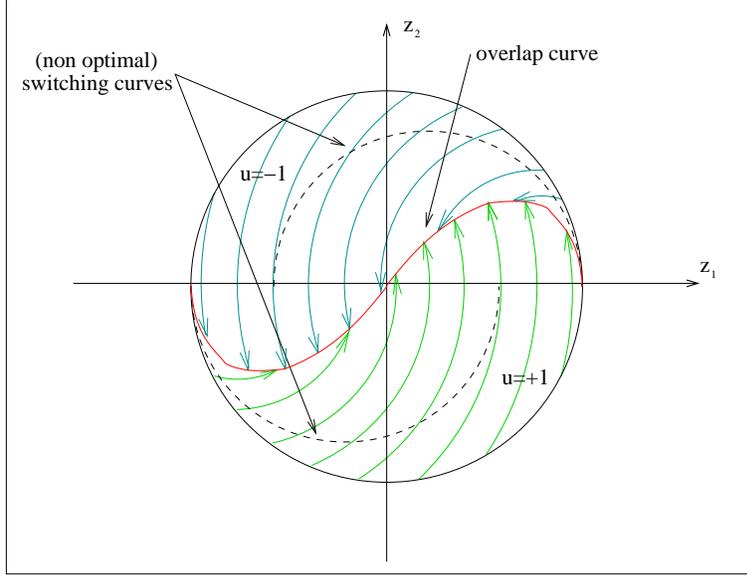


Figure 4: Optimal synthesis for the linear pendulum

if V_α is the connected component of $S^2 \setminus \mathcal{F}(\alpha, k_M \pi)$ containing the south pole, we obtain the optimal synthesis on $S^2 \setminus V_\alpha$.

As α tends to zero, V_α collapses on S . Hence one must rescale the problem by a factor $1/\alpha$, in order to describe the TOS inside V_α . Also notice that since we are in a neighborhood of the south pole we can project the problem on the plane (x_1, x_2) . We are now in a position to define a possible limit behavior for the TOS inside V_α . Let M_α be the linear mapping from \mathbb{R}^3 onto \mathbb{R}^2 defined as the composition of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha$. Denote by $(\tilde{\Sigma})_\alpha$ (resp. $\widetilde{OF}(\alpha, k_M \pi)$) the image by M_α of $(\Sigma)_\alpha$ (resp. $OF(\alpha, k_M \pi)$). Then, $(\tilde{\Sigma})_\alpha$ is a perturbation by $\mathcal{O}(\alpha^2)$ of the standard linearized pendulum

$$(Pen) : \begin{cases} \dot{z}_1 = -z_2, \\ \dot{z}_2 = z_1 + u, \end{cases} \quad (z_1, z_2) \in \mathbb{R}^2, \quad |u| \leq 1 \quad (10)$$

while $\widetilde{OF}(\alpha, k_M \pi)$ is a perturbation by $\mathcal{O}(\alpha^2)$ of $C(0, 2\bar{r})$, the planar circle of center $(0, 0)$ and radius $2\bar{r}$. As a consequence, the candidate limit TOS inside V_α is the one associated to the problem of reaching in minimum time every point of the ball $B(0, 2\bar{r})$ starting from $C(0, 2\bar{r})$, along the dynamics of the standard linearized pendulum. To prove such a result, we first study the above mentioned optimal control problem and show that the corresponding TOS is characterized by an overlap curve γ_{pen}^o , which is the set of points $z \in \mathbb{R}^2$ with $z_1 z_2 \geq 0$ and belonging to the locus (see Figure 7)

$$z_1^4 + z_2^4 + 2z_1^2 z_2^2 - 4\bar{r}^2 z_1^2 + (4 - 4\bar{r}^2) z_2^2 = 0.$$

The optimal synthesis inside $C(0, 2\bar{r})$ is then described by the following feedback, defined on $B(0, 2\bar{r}) \setminus \gamma_{pen}^o$: “above” γ_{pen}^o , the control u is constantly equal to -1 and “below” γ_{pen}^o , it is constantly equal to 1 (see Fig. 4). Finally, the asymptotic result we prove in Section 4.2 is the following.

Theorem 1 For $\bar{r} \in (0, 1)$, let $(\alpha_k)_{k \geq 1}$ be the sequence defined by $\alpha_k := \frac{\pi}{2(k+\bar{r})}$ for $k \geq 1$. Consider γ_{pen}^o , the overlap curve of the TOS for the optimal control problem consisting of starting from $C(0, 2\bar{r})$, the planar circle of center $(0, 0)$ and radius $2\bar{r}$, and reaching in minimum time every point of $B(0, 2\bar{r})$ along the control system (10). Then, for k large enough, the TOS associated to $(\tilde{\Sigma})_{\alpha_k}$ inside $\widetilde{OF}(\alpha_k, k_M \pi)$ is characterized by an overlap curve $\gamma_{\alpha_k}^o$ so that the optimal feedback takes the value -1 “above” $\gamma_{\alpha_k}^o$, and the value 1 “below”

$\gamma_{\alpha_k}^o$. Moreover, $\gamma_{\alpha_k}^o$ converges to γ_{pen}^o in the C^0 topology, uniformly with respect to \bar{r} in any compact interval of $(0, 1)$.

The results in the cases (C2) and (C3) are described in more details in Sections 5 and 6.

Remark 2 Notice that the sequence $(\alpha_k)_{k \geq 1}$ defined above has been chosen in order to simplify the previous statement. Indeed the same result could be restated in a more general way by taking an arbitrary sequence $(\tilde{\alpha}_k)_{k \geq 1}$ converging to zero and such that $r(\tilde{\alpha}_k)$ converges to \bar{r} , or letting the remainder vary on a compact subinterval of $(0, 1)$.

The paper is organized as follows. In the second section, we collect basic facts, notations, results, and conjectures of [5]. The third section gathers the detailed description of the extremal front and the proof of Eq. (7). Sections 4, 5 and 6 treat respectively the cases (C1), (C2) and (C3). In the appendix, we finally prove a technical result needed throughout the paper.

2 Notations and previous results

2.1 Basic Facts

Definition 1 An admissible control $u(\cdot)$ for the system (1)–(2) is a measurable function $u(\cdot) : [a, b] \rightarrow [-1, 1]$, while an admissible trajectory is an absolutely continuous function $x(\cdot) : [a, b] \rightarrow S^2$ satisfying (1) a.e. for some admissible control $u(\cdot)$. If $x(\cdot)$ is an admissible trajectory and $u(\cdot)$ the corresponding control, we say that $(x(\cdot), u(\cdot))$ is an admissible pair.

For every $\bar{x} \in S^2$, the minimization problem consists of determining an admissible pair steering the north pole to \bar{x} in minimum time. More precisely

Problem (P) Consider the control system (1)–(2). For every $\bar{x} \in S^2$, find an admissible pair $(x(\cdot), u(\cdot))$ defined on $[0, T]$ such that $x(0) = N$, $x(T) = \bar{x}$ and $x(\cdot)$ is time optimal.

An optimal synthesis from the north pole (in the following optimal synthesis, for short) is the collection of all the solutions to the problem (P). More precisely

Definition 2 (Optimal Synthesis) An optimal synthesis for the problem (P) is the collection of all time optimal trajectories $\Gamma = \{x_{\bar{x}}(\cdot) : [0, b_{\bar{x}}] \mapsto S^2, \bar{x} \in S^2 : x_{\bar{x}}(0) = N, x_{\bar{x}}(b_{\bar{x}}) = \bar{x}\}$.

For more elaborated definitions of optimal synthesis see [6, 12] and references therein. The standard tool to look for optimal trajectories is a first order necessary condition for optimality known as the Pontryagin Maximum Principle (PMP for short), (cf. [2, 13]) as stated below for our minimum time problem on S^2 .

Define the following real-valued map on $T^*S^2 \times [-1, 1]$, called Hamiltonian,

$$\mathcal{H}(\lambda, x, u) := \langle \lambda, (F + uG)x \rangle .$$

Set:

$$H(\lambda, x) := \max_{v \in [-1, 1]} \mathcal{H}(\lambda, x, v). \quad (11)$$

The PMP asserts that, if $\gamma : [a, b] \rightarrow S^2$ is a time optimal trajectory corresponding to a control $u : [a, b] \rightarrow [-1, 1]$, then there exists a nontrivial field of covectors along γ , that is a never vanishing absolutely continuous function $\lambda : t \in [a, b] \mapsto \lambda(t) \in T_{\gamma(t)}^*S^2$ and a constant $\lambda_0 \leq 0$ such that, for a.e. $t \in \text{Dom}(\gamma)$, we have:

i) $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(\lambda(t), \gamma(t), u(t)) = -\lambda(t)(F + u(t)G),$

ii) $\mathcal{H}(\lambda(t), \gamma(t), u(t)) + \lambda_0 = 0,$

iii) $\mathcal{H}(\lambda(t), \gamma(t), u(t)) = H(\gamma(t), \lambda(t))$.

In the more general case in which the target and the initial datum (also called *source*) are two smooth manifolds \mathcal{N}_0 and \mathcal{N}_1 the previous statement must be modified by adding the so-called *transversality conditions*:

iv) $\langle \lambda(a), v \rangle = 0 \quad \forall v \in T_{\gamma(a)}\mathcal{N}_0, \quad \langle \lambda(b), w \rangle = 0 \quad \forall w \in T_{\gamma(b)}\mathcal{N}_1$.

Remark 3 A trajectory γ (resp. a couple (γ, λ)) satisfying the conditions given by the PMP is said to be an *extremal* (resp. an *extremal pair*). An extremal corresponding to $\lambda_0 = 0$ is said to be an *abnormal extremal*, otherwise we call it a *normal extremal*.

Definition 3 (bang, singular for the problem (1)-(2)) A control $u(\cdot) : [a, b] \rightarrow [-1, 1]$ is said to be a *bang control* if $u(t) = +1$ a.e. in $[a, b]$ or $u(t) = -1$ a.e. in $[a, b]$. A control $u(\cdot) : [a, b] \rightarrow [-1, 1]$ is said to be a *singular control* if $u(t) = 0$, a.e. in $[a, b]$. A finite concatenation of bang controls is called a *bang-bang control*. A *switching time* of $u(\cdot)$ is a time $\bar{t} \in [a, b]$ such that, for every $\varepsilon > 0$, u is not bang or singular on $(\bar{t} - \varepsilon, \bar{t} + \varepsilon) \cap [a, b]$. A trajectory of the control system (1), (2) is said to be a *bang trajectory* (or *arc*), *singular trajectory* (or *arc*), *bang-bang trajectory*, if it corresponds respectively to a bang control, singular control, bang-bang control. If \bar{t} is a switching time, the corresponding point on the trajectory $x(\bar{t})$ is called a *switching point*.

2.2 Description of Previous Results

In [5, 7] it was proved that, for every couple of points, there exists a time optimal trajectory joining them. Moreover it was proved that every time optimal trajectory is a finite concatenation of bang and singular trajectories. More precisely we have:

Proposition 1 For the minimum time problem associated to (1)-(2), for each pair of points p and q belonging to S^2 , there exists a time optimal trajectory joining p to q . Moreover every time optimal trajectory for (1)-(2) is a finite concatenation of bang and singular trajectories.

Notice that the previous proposition does not apply if $\alpha = 0$ or $\alpha = \pi/2$, since in these cases the controllability property is lost.

In [7] it has been proved that $\alpha = \pi/4$ is a bifurcation for the qualitative shape of the time optimal synthesis, for instance the time optimal synthesis contains a singular arc if and only if $\alpha > \pi/4$. Since in this paper we are interested in the limit $\alpha \rightarrow 0$, in the following we always assume $\alpha < \pi/4$. In this case, using the PMP, the following properties characterizing the optimal trajectories were established in [5]:

- i) $x(\cdot)$ is bang bang;
- ii) the duration s_i of the first bang arc satisfies $s_i \in (0, \pi]$,
- iii) the time duration between two consecutive switchings is the same for all interior bang arcs (i.e. excluding the first and the last bang) and it is equal to $v(s_i)$, where $v(\cdot)$ is the following function

$$v(s) = \pi + 2 \arctan \left(\frac{\sin(s)}{\cos(s) + \cot^2(\alpha)} \right). \quad (12)$$

One can immediately check that this function satisfies $v(0) = v(\pi) = \pi$ and $v(s) > \pi$ for every $s \in (0, \pi)$,

- iv) the time duration of the last arc is $s_f \in (0, v(s_i)]$,

Moreover, thanks to the analysis given in [5], one easily gets (always in the case $\alpha < \pi/4$):

- v) the number of switchings N_x of $x(\cdot)$ satisfies the following inequality

$$N_x \leq \left\lceil \frac{\pi}{2\alpha} \right\rceil + 1. \quad (13)$$

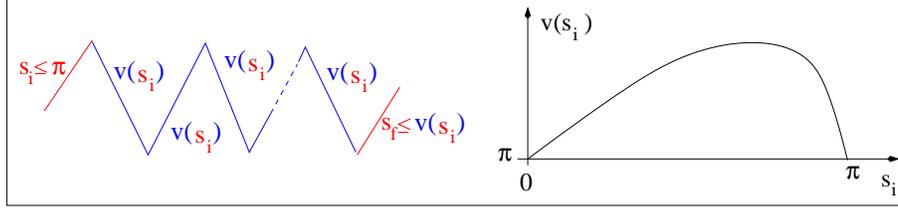


Figure 5: Time optimal trajectories for $\alpha < \pi/4$

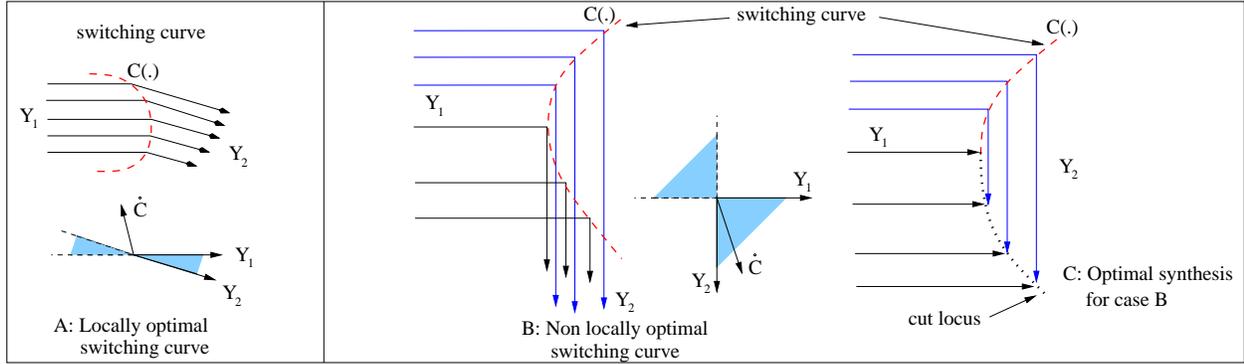


Figure 6: Locally optimal switching curves and non locally optimal switching curves with the corresponding synthesis

Conditions **i)-v)** define a set of candidate optimal trajectories. Notice that conditions **i)-v)** are just necessary conditions for optimality and one is faced with the problem of selecting, among them, those that are really optimal. In particular, given a trajectory satisfying conditions **i)-v)**, one would like to find the time after which it is no more optimal.

Some questions remained unsolved, in particular questions relative to local optimality of the switching curves defined in Eq. (5). Roughly speaking we say that a switching curve is locally optimal if it never “reflects” the trajectories (see Fig. 6 A).¹ When a family of trajectories is reflected by a switching curve then local optimality is lost and some cut locus appear in the optimal synthesis.

Definition 4 A *cut locus* is a set of points reached at the same time by two (or more) time optimal trajectories. A subset of a cut locus that is a connected C^1 manifold is called overlap curve.

An example showing how a “reflection” on a switching curve generates a cut locus is portrayed in Fig. 6 B and C. More details are given later. More precisely the following questions were formulated in [5]:

Question 1 Are the switching curves $C_k^\varepsilon(s)$, $s \in (0, \pi]$, locally optimal? More precisely, one would like to understand how the candidate optimal trajectories described above lose optimality.

Question 2 What is the shape of the optimal synthesis in a neighborhood of the south pole?

Numerical simulations suggested some conjectures regarding the above questions. More precisely, in [5] the following conjectures were made:

¹ More precisely consider a smooth switching curve C between two smooth vector field Y_1 and Y_2 on a smooth two dimensional manifold. Let $C(s)$ be a smooth parametrization of C . We say that C is locally optimal if, for every $s \in \text{Dom}(C)$, we have $\dot{C}(s) \neq \alpha_1 Y_1(C(s)) + \alpha_2 Y_2(C(s))$, for every α_1, α_2 s.t. $\alpha_1 \alpha_2 \geq 0$. The points of a switching curve on which this relation is not satisfied are usually called “conjugate points”. See Fig. 6.

C1 The curves $C_k^\varepsilon(s)$, ($k = 1, \dots, k_M$) are locally optimal if and only if $k \leq \lceil \frac{\pi-\alpha}{2\alpha} \rceil - 1$.

Analyzing the evolution of the minimum time wave front in a neighborhood of the south-pole, it is reasonable to conjecture that:

C2 The shape of the optimal synthesis in a neighborhood of the south pole depends on the remainder $r(\alpha)$ defined in Eq. (6). Notice that $r(\alpha)$ belongs to the interval $[0, 1)$. More precisely, it was conjectured in [5] that for $\alpha \in (0, \pi/4)$, there exist two positive numbers α_1 and α_2 such that $0 < \alpha_1 < \alpha < \alpha_2 < 2\alpha$ and:

CASE A: $r(\alpha) \in (\frac{\alpha_2}{2\alpha}, 1)$. The switching curve $C_{k_M}^\varepsilon$ glues to an overlap curve that passes through the origin (Fig. 3, Case A).

CASE B: $r(\alpha) \in [\frac{\alpha_1}{2\alpha}, \frac{\alpha_2}{2\alpha}]$. The switching curve $C_{k_M}^\varepsilon$ is not reached by optimal trajectories in the interval $]0, \pi]$. At the point $C_{k_M}^\varepsilon(0)$, an overlap curve starts and passes through the origin (Fig. 3, Case B).

CASE C: $r(\alpha) \in (0, \frac{\alpha_1}{2\alpha})$. The situation is more complicated and it is depicted in the bottom of Fig. 3, Case C.

For $r = 0$, the situation is the same as in CASE A, but for the switching curve starting at $C_{k_M-1}^\varepsilon(0)$.

As explained in the introduction, the presence of several cyclically alternating patterns of optimal synthesis, each of them depending on an arithmetic property of α , was already confirmed in [7], by counting the number of optimal trajectories reaching the south pole.

Remark 4 The first conjecture is implicitly disproved by the results of this paper. More precisely an immediate consequence of our results is that the switching curve $C_{k_M-2}^\varepsilon$ is always locally optimal, while $C_{k_M-1}^\varepsilon$ is not, in general. However, for every fixed $\bar{r} < \frac{1}{2}$ there exists α small enough with $\bar{r} \leq r(\alpha) < \frac{1}{2}$ such that $C_{k_M-1}^\varepsilon$ is locally optimal too, which contradicts the conjecture. On the other hand Conjecture **C2** is correct and, at the light of our main results, is completely proved and clarified.

2.3 Notations

All along the paper we use the notation $\varepsilon = \pm 1$. The set $so(3)$ of 3×3 skew-symmetric matrices is a three-dimensional vector space on which the following bilinear map

$$\langle A, B \rangle = -Tr(AB), \quad A, B \in so(3),$$

is an inner product. For $A \in so(3)$, $\|A\| := \sqrt{\langle A, A \rangle}$ is the norm (or length) of A . With the above notations, F and G are perpendicular and normalized so that $\|F\| = \cos(\alpha)$ and $\|G\| = \sin(\alpha)$.

Let Id be the 3×3 identity matrix. We recall that $N = (0, 0, 1)^T$ and denote the south pole as $S = (0, 0, -1)^T$. Set $c_t := \cos(t)$ and $s_t := \sin(t)$ for $t \in [0, 2\pi)$. Recall that $X_+ := F + G$ and $X_- := F - G$ and we have

$$X_+ = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & -s_\alpha \\ 0 & s_\alpha & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -c_\alpha & 0 \\ c_\alpha & 0 & s_\alpha \\ 0 & -s_\alpha & 0 \end{pmatrix}.$$

Let Π_{x_3} be the orthogonal symmetry with respect to the x_3 -axis, i.e. Π_{x_3} is represented in the canonical basis by $Diag(-1, -1, 1)$. Then, we have the following trivial but useful property.

$$\Pi_{x_3} X_\varepsilon = X_{-\varepsilon} \Pi_{x_3}. \quad (14)$$

We next recall standard formulas for a rotation e^{tY} of $SO(3)$ in terms of its axis Y (whose length is equal to one) and its angle t . We have

$$e^{tY} = Id + s_t Y + (1 - c_t) Y^2. \quad (15)$$

Moreover, for $t \in [0, 2\pi)$, we have

$$e^{\Theta(t)Z_-(t)} := e^{tX_+} e^{tX_-} \quad e^{\Theta(t)Z_+(t)} := e^{tX_-} e^{tX_+},$$

where the unit vectors $Z_+(t), Z_-(t)$ are defined by

$$Z_+(t) = \begin{pmatrix} 0 & -C(t) & -B(t) \\ C(t) & 0 & 0 \\ B(t) & 0 & 0 \end{pmatrix}, \quad Z_-(t) = \begin{pmatrix} 0 & -C(t) & B(t) \\ C(t) & 0 & 0 \\ -B(t) & 0 & 0 \end{pmatrix}, \quad (16)$$

with $B(t) := \frac{s_\alpha s_{t/2}}{\sqrt{s_{t/2}^2 s_\alpha^2 + c_{t/2}^2}}$, $C(t) := -\frac{c_{t/2}}{\sqrt{s_{t/2}^2 s_\alpha^2 + c_{t/2}^2}}$ and the angle $\Theta(t)$ by

$$\Theta(t) = 2 \arccos(s_{t/2}^2 c_{2\alpha} - c_{t/2}^2). \quad (17)$$

3 The Extremal Front

3.1 Definition and description

As said in the introduction, $\mathcal{F}(\alpha, T)$ the extremal front along $(\Sigma)_\alpha$ at time T is the set of points reached at time T by extremal trajectories starting from N , i.e.

$$\mathcal{F}(\alpha, T) := \{\bar{x} \in S^2 : \exists \text{ an extremal pair } (x(\cdot), \lambda(\cdot)) \text{ such that } x(0) = N, \quad x(T) = \bar{x}\}. \quad (18)$$

Such extremals are parametrized by the length of the first bang arc, the one of the last bang arc and the number of arcs:

$$\Xi^+(s, t) = \overbrace{e^{X_\varepsilon t} e^{X_{-\varepsilon} v(s)} \dots e^{X_{-v(s)}} e^{X_{+s}}}_n N, \quad (19)$$

$$\Xi^-(s, t) = \overbrace{e^{X_{\varepsilon'} t} e^{X_{-\varepsilon'} v(s)} \dots e^{X_{+v(s)}} e^{X_{-s}}}_{n'} N, \quad (20)$$

where $s \in (0, \pi]$, $t \in (0, v(s)]$, the number of bang arcs (n and n' respectively) is an integer and

- (-) $\varepsilon = +1$ (resp. $\varepsilon = -1$), if n is odd (resp. even),
- (-) $\varepsilon' = +1$ (resp. $\varepsilon' = -1$), if n' is even (resp. odd).

Roughly speaking, we would like to compute the limit, as $\alpha \rightarrow 0$, of $\mathcal{F}(\alpha, T)$, when T is such that the extremal front reaches a neighborhood of the south pole.

The idea is that, once one knows the extremal front $\mathcal{F}(\alpha, T)$ and if it is optimal, then one can continue to build the synthesis for times bigger than T using $\mathcal{F}(\alpha, T)$ as a source for the minimization problem.

The identification of the front $\mathcal{F}(\alpha, T)$ is not easy since it requires the computation of the product of several exponentials of matrices. Moreover, if $\mathcal{F}(\alpha, T)$ crosses some switching curve, then the number of exponentials in general depend on the point.

This problem is overcome by considering $\mathcal{F}(\alpha, T)$ only at times equal to multiples of π . Indeed, first notice that, for $T = \pi \left[\frac{\pi}{2\alpha} \right]$, the extremal front reaches the points $C_{k_M}^\pm(0)$, i.e. the points where the last switching curves $C_{k_M}^\pm$ start. Thanks to Proposition 2 below, at these times, every extremal trajectory has the same number of switchings. The extremal front at times that are not multiple of π can be obtained *a posteriori*, continuing the extremal front, as explained above.

From the structure of the extremal trajectories it follows that the time at which the point $C_k^\pm(s)$ is reached is $T_k(s) = s + kv(s)$.

Lemma 1 *Let k be an integer satisfying $1 \leq k \leq \mathcal{N}_{mon} := \left\lceil \frac{(\cot(\alpha)^2 - 1)^2}{2\cot(\alpha)^2 - 1} \right\rceil$, then $T_k(s)$ is a strictly increasing function of s .*

Proof. It holds

$$\frac{d}{ds}T_k(s) = \frac{1 + 2c_s \cot(\alpha)^2 + \cot(\alpha)^4 + k \left(2 + 2c_s \cot(\alpha)^2\right)}{1 + 2c_s \cot(\alpha)^2 + \cot(\alpha)^4}. \quad (21)$$

It is clear that the denominator of the above fraction is never vanishing on $[0, \pi]$ if $\alpha < \pi/4$. On the other hand the numerator, as a function of s , reaches its minimum at $s = \pi$, where it is equal to $(\cot(\alpha)^2 - 1)^2 - k(2\cot(\alpha)^2 - 1)$, and then the conclusion follows easily. \blacksquare

As a consequence, we obtain the following important corollary.

Corollary 1 *Let k be an integer satisfying $1 \leq k \leq \mathcal{N}_{\text{mon}}$. If an extremal trajectory is switching at time $T = k\pi$, then the length s of the first bang arc satisfies $s = \pi$.*

Since for α small $k_M \leq \mathcal{N}_{\text{mon}}$, then, for $T = k\pi$ where k is a positive integer such that $k \leq \lceil \pi/(2\alpha) \rceil$, we have that all the extremal trajectories switch exactly k times (except the trajectories with length of the first switching equal to π that switch $k - 1$ times). Therefore, the extremal front $\mathcal{F}(\alpha, k\pi)$ is described by the next proposition.

Proposition 2 *Let k be a positive integer such that $1 \leq k \leq \lceil \pi/(2\alpha) \rceil$. Then, if α is small enough, we have*

$$\mathcal{F}(\alpha, k\pi) = \{\mathcal{E}^+(\alpha, k, s), \quad s \in (0, \pi]\} \cup \{\mathcal{E}^-(\alpha, k, s), \quad s \in (0, \pi]\}, \quad \text{where :} \quad (22)$$

$$\mathcal{E}^+(\alpha, k, s) := \begin{cases} e^{(k\pi - (k-1)v(s) - s)X_-} e^{\frac{k-1}{2}\Theta(v(s))Z_-} e^{sX_+} e^N & \text{for } k \text{ odd,} \\ e^{(k(\pi - v(s)) - s)X_+} e^{\frac{k}{2}\Theta(v(s))Z_-} e^{sX_+} e^N & \text{for } k \text{ even.} \end{cases} \quad (23)$$

The expression for \mathcal{E}^- is the same as the expression for \mathcal{E}^+ after exchanging the subscripts $+$ and $-$. As a consequence, $\mathcal{E}^{-\varepsilon} = \Pi_{x_3} \mathcal{E}^\varepsilon$, where Π_{x_3} is the orthogonal symmetry with respect to the x_3 -axis.

Remark 5 Notice that $\mathcal{E}^\varepsilon(\alpha, k, 0) = \mathcal{E}^{-\varepsilon}(\alpha, k, \pi)$, $\varepsilon = \pm$, so that \mathcal{F} is described by a continuous closed curve.

3.2 Description of the extremal front $\mathcal{F}(\alpha, k_M\pi)$ and consequences

As sketched in the introduction, we must describe the optimal synthesis on S^2 deprived of a neighborhood of the south pole. For that purpose, we will provide the precise asymptotics of $\mathcal{F}(\alpha, k_M\pi)$, as α tends to zero, and derive, from its topological nature, the minimum time front at time $k_M\pi$.

From now on, for simplicity, we drop the dependence of \mathcal{E}^ε on k_M , i.e. we set $\mathcal{E}^\varepsilon(\alpha, s) := \mathcal{E}^\varepsilon(\alpha, k_M, s)$, and we assume that k_M is odd.

In the following, it will be useful to think of α and r as two independent variables. For this purpose, define

$$\begin{aligned} \psi(\alpha, r, s) &:= \left(\frac{\pi}{2\alpha} - r\right) (\pi - v(s)) + v(s) - s, \\ \theta(\alpha, r, s) &:= \left(\frac{\pi}{4\alpha} - \frac{1+r}{2}\right) \Theta(v(s)), \\ \chi^\varepsilon(\alpha, r, s) &:= e^{\psi(\alpha, r, s)X_{-\varepsilon}} e^{\theta(\alpha, r, s)Z_{-\varepsilon}} e^{sX_\varepsilon} e^N. \end{aligned}$$

It is clear from (23) that

$$\mathcal{E}^\varepsilon(\alpha, s) = \chi^\varepsilon(\alpha, r(\alpha), s).$$

The following result is the key point in order to describe the extremal front at time $k_M\pi$.

Lemma 2 *There exists $\alpha_0 > 0$ such that the function χ^ε , $\varepsilon = \pm$, defined above, is real-analytic for $(r, s, \alpha) \in \mathbb{R}^2 \times I$, where $I = (-\alpha_0, \alpha_0)$. Moreover, it admits a convergent power series*

$$\chi^\varepsilon(\alpha, r, s) = \sum_{l \geq 0} f_l^\varepsilon(s, r) \alpha^l, \quad (24)$$

where the $f_l^\varepsilon(s, r)$ are real-analytic functions of $(s, r) \in \mathbb{R}^2$, 2π -periodic in s (therefore they are bounded over $\mathbb{R} \times [0, 1]$).

As a consequence, the extremal front $\mathcal{F}(\alpha, k_M\pi)$, which is a continuous closed curve, is piecewise analytic with discontinuities at $s = 0, \pi$ for derivatives of order greater than or equal to one.

Proof of Lemma 2.

We will prove the proposition only for χ^+ . Since χ^+ is 2π -periodic in s and r enters in an affine way in ψ and θ , the real issue of analyticity revolves around the variable α . First of all, it is clear that $v(s)$ is actually a real-analytic function for $(s, \alpha) \in \mathbb{R} \times I$, where $I = (-\alpha_0, \alpha_0)$ with $\alpha_0 > 0$ small enough. Therefore, one has only to prove the real-analyticity of $\tilde{\psi}(\alpha, s) := \frac{v(s) - \pi}{\alpha}$ and $\frac{\beta(s, \alpha)}{\alpha}$, where $\beta(s, \alpha) := \Theta(v(s))$, for $(s, \alpha) \in \mathbb{R} \times I$, where $I = (-\alpha_0, \alpha_0)$, for some $\alpha_0 > 0$.

Note that

$$\tilde{\psi}(\alpha, s) = \frac{2}{\alpha} \arctan(s_\alpha^2 \mu(s)), \quad \mu(s) := \frac{s_s}{c_\alpha^2 + s_\alpha^2 c_s}.$$

The function μ is real-analytic for $(s, \alpha) \in \mathbb{R} \times I$, with I open neighborhood of zero, and thus uniformly bounded over $\mathbb{R} \times I$. In addition, $\arctan(\cdot)$ is real analytic in a neighborhood of zero. Hence the conclusion for $\tilde{\psi}(\alpha, s)$.

As for $\frac{\beta(s, \alpha)}{\alpha}$, rewrite first Eq. (17) as

$$\cos(\beta(s)) = 1 - G(s, \alpha),$$

with

$$G(s, \alpha) := 2s_\alpha^2 \left[1 + \frac{c_\alpha^2 s_\alpha^2 \mu^2(s)}{1 + s_\alpha^4 \mu^2(s)} + 2s_\alpha^2 \left(1 + \frac{c_\alpha^2 s_\alpha^2 \mu^2(s)}{1 + s_\alpha^4 \mu^2(s)} \right)^2 \right]. \quad (25)$$

We first need to determine a convergent power series for β from the expression

$$\beta = \arccos(1 - G). \quad (26)$$

Note that $|G(s, \alpha)| \leq 5\alpha^2$ for α small enough. We first expand $\arccos(1 - G)$ in a power series in G . Starting from the power series

$$(1 - t)^{-1/2} = 1 + \sum_{m \geq 1} s_m t^m,$$

with radius of convergence equal to 1 we get

$$\frac{d}{dG}(\arccos(1 - G)) = -\frac{1}{\sqrt{2G}} \frac{1}{\sqrt{1 - G/2}},$$

and, after simple integration,

$$\arccos(1 - G) = -\sqrt{2G} \left(1 + \sum_{m \geq 1} \frac{s_m}{2^{m+1}(m+1/2)} G^m \right), \quad (27)$$

Finally, from Eq. (25), G can be written as $2s_\alpha^2(1 + s_\alpha^2 H(s, \alpha))$ with $H(s, \alpha)$ uniformly bounded by 3. Then,

$$\sqrt{2G(s, \alpha)} = 2s_\alpha (1 + s_\alpha^2 H(s, \alpha))^{1/2}. \quad (28)$$

Gathering Eqs. (26),(27),(28), we get the real-analyticity of $\frac{\beta(s, \alpha)}{\alpha}$ for $(s, \alpha) \in \mathbb{R} \times I$, where $I = (-\alpha_0, \alpha_0)$, for some $\alpha_0 > 0$ small enough. ■

We next compute f_0, f_1, f_2 and obtain the following proposition.

Proposition 3 *For α small enough, the function χ^ε , $\varepsilon = \pm$ defined above and its derivative with respect to s have the following expansion*

$$\chi^\varepsilon(\alpha, r, s) = f_0^\varepsilon(s, r) + f_1^\varepsilon(s, r)\alpha + f_2^\varepsilon(s, r)\alpha^2 + \mathcal{O}(\alpha^3), \quad (29)$$

$$\frac{\partial}{\partial s} \chi^\varepsilon(\alpha, r, s) = \frac{\partial}{\partial s} f_0^\varepsilon(s, r) + \frac{\partial}{\partial s} f_1^\varepsilon(s, r)\alpha + \frac{\partial}{\partial s} f_2^\varepsilon(s, r)\alpha^2 + \mathcal{O}(\alpha^3) \quad (30)$$

where f_l^ε , $l = 0, 1, 2$ are defined as in (7) and $|\mathcal{O}(\alpha^3)| \leq C|\alpha^3|$, with the constant C independent of $s \in \mathbb{R}$ and $r \in [0, 1]$.

Proof of Proposition 3. We will prove the proposition only for χ^+ . To proceed, we list the expansions of the form (29) for several quantities, obtained after elementary computations.

$$\psi(\alpha, r, s) = \pi - s - \pi s_s \alpha + 2(1-r)s_s \alpha^2 + \mathcal{O}(\alpha^3) \quad (31)$$

$$\theta(\alpha, r, s) = \pi - 2\alpha(1+r) + \frac{\pi s_s^2}{2} \alpha^2 + \mathcal{O}(\alpha^3) \quad (32)$$

$$Z_-(v(s)) = \begin{pmatrix} 0 & -\alpha s_s & 1 - \frac{s_s^2}{2} \alpha^2 \\ \alpha s_s & 0 & 0 \\ -1 + \frac{s_s^2}{2} \alpha^2 & 0 & 0 \end{pmatrix} + \mathcal{O}(\alpha^3). \quad (33)$$

Using Eqs. (31),(32), we get that

$$\sin(\psi(\alpha, r, s)) = s_s + \pi s_s c_s \alpha - (2(1-r)s_s c_s + \frac{\pi}{2} s_s^3) \alpha^2 + \mathcal{O}(\alpha^3) \quad (34)$$

$$\cos(\psi(\alpha, r, s)) = c_s - \pi s_s^2 \alpha + (2(1-r)s_s^2 - \frac{\pi^2}{2} s_s^2 c_s) \alpha^2 + \mathcal{O}(\alpha^3) \quad (35)$$

$$\sin(\theta(\alpha, r, s)) = 2\alpha(1+r) - \frac{\pi s_s^2}{2} \alpha^2 + \mathcal{O}(\alpha^3) \quad (36)$$

$$\cos(\theta(\alpha, r, s)) = -1 + 2\alpha^2(1+r)^2 + \mathcal{O}(\alpha^3). \quad (37)$$

Using Eqs. (15), (34) and (35) we obtain

$$e^{\psi(\alpha, r, s)X_-} = \begin{pmatrix} -c_s + \pi s_s^2 \alpha & -s_s - \pi c_s s_s \alpha & -(1+c_s)\alpha \\ s_s + \pi s_s c_s \alpha & c_s + \pi s_s^2 \alpha & s_s \alpha \\ -(1+c_s)\alpha & -s_s \alpha & 1 \end{pmatrix} + \mathcal{R}(s)\alpha^2 + \mathcal{O}(\alpha^3), \quad (38)$$

where

$$\mathcal{R}(s) = \begin{pmatrix} c_s + \frac{\pi^2}{2} c_s s_s^2 + 1 - 2(1+r)s_s^2 & \frac{s_s}{2} + 2c_s s_s(1+r) + \frac{\pi^2}{2} s_s^3 & \pi s_s^2 \\ -\frac{s_s}{2} - 2c_s s_s(1+r) - \frac{\pi^2}{2} s_s^3 & -2(1+r)s_s^2 + \frac{\pi^2}{2} c_s s_s^2 & \pi s_s c_s \\ \pi s_s^2 & -\pi s_s c_s & -1 - c_s \end{pmatrix},$$

and using Eq. (15), we have

$$e^{sX_+ N} = \begin{pmatrix} s_\alpha c_\alpha (1 - c_s) \\ -s_\alpha s_s \\ 1 - s_\alpha^2 (1 - c_s) \end{pmatrix} = \begin{pmatrix} \alpha(1 - c_s) \\ -\alpha s_s \\ 1 - \alpha^2 (1 - c_s) \end{pmatrix} + \mathcal{O}(\alpha^3). \quad (39)$$

An easy computation yields

$$Z_-^2(v(s)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -c^2(s) & b(s)c(s) \\ 0 & b(s)c(s) & -b^2(s) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\alpha^2 s_s^2 & \alpha s_s \\ 0 & \alpha s_s & -1 + \alpha^2 s_s^2 \end{pmatrix} + \mathcal{O}(\alpha^3). \quad (40)$$

Using Eqs. (15), (36) (37) and (39) and the previous equation, we get

$$e^{\theta(\alpha, r, s)Z_-(v(s))} e^{sX_+ N} = \begin{pmatrix} \alpha(1 + 2r + c_s) - \alpha^2 \frac{\pi}{2} s_s^2 \\ \alpha s_s \\ -1 + \alpha^2 (1 + 2r + 2r^2 + c_s + 2r c_s) \end{pmatrix} + \mathcal{O}(\alpha^3). \quad (41)$$

Applying $e^{\psi(s)X_-}$ to the previous equation and using Eq. (38), we finally get Eq. (29) for $\varepsilon = +$. The expression of the derivative (30) is then an immediate consequence of the analyticity of the function χ^+ .

The results for χ^- and $\frac{\partial}{\partial s} \chi^-$ are obtained similarly together with Eq. (14). ■

Since the quantities of the form $\mathcal{O}(\alpha^3)$ in Proposition 3 satisfy $|\mathcal{O}(\alpha^3)| \leq C|\alpha^3|$ for some C independent of r , the expressions (8)–(9) are straightforward consequences. Hence the shape of the extremal front at time

$T = k_M\pi$ is known for α small. In particular its image with respect to the map M_α defined in Section 1 is approximated, in the \mathcal{C}^1 sense, by a circle of radius $2r(\alpha)$ centered at the origin.

We finally note that, for k_M even, Lemma 2 is still valid, while, with computations similar to those made in the proof of Proposition 3, it is easy to see that the formulas for f_k^ε , $k = 0, 1, 2$ simply differ, with respect to (7), for the sign of the first two components.

Remark 6 Repeating the previous computations, we also obtain series expansions for $\mathcal{E}^\varepsilon(s, k_M - 1, \alpha)$ and $\frac{\partial}{\partial s}\mathcal{E}^\varepsilon(s, k_M - 1, \alpha)$. Indeed, we just have to replace r by $1 + r$. In that case the shape of the extremal front $\mathcal{F}(\alpha, (k_M - 1)\pi)$, after applying the map M_α , is approximated, in the \mathcal{C}^1 sense, by a circle of radius $2(1 + r(\alpha))$ centered at the origin.

4 Case $r(\alpha) = \bar{r} \in (0, 1)$

In this section, we study the case in which α tends to zero with $r(\alpha) = \bar{r}$, for a constant $\bar{r} \in (0, 1)$. More precisely we consider the decreasing sequence $\alpha_k = \frac{\pi}{2(k+\bar{r})}$, for $k \geq 1$. We first describe the minimum time front at $T = k_M\pi$, then we identify and study the candidate for the limit synthesis and finally we prove Theorem 1.

4.1 Description of the minimum time front at $T = k_M\pi$

The purpose of the paragraph is to prove the following proposition.

Proposition 4 *Fix $\delta > 0$ small. For α small enough with $r(\alpha) > \delta$, the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is homeomorphic to a circle. As a consequence, the switching curves defined inductively in Eq. (5) are optimal up to $k = k_M$ and $OF(\alpha, k_M\pi)$, the minimum time front at time $k_M\pi$ coincides with $\mathcal{F}(\alpha, k_M\pi)$.*

Proof of Proposition 4. From Proposition 3 we get that the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is the union of two arcs, $\mathcal{E}^+(\alpha, s)$, $s \in [0, \pi]$ and $\mathcal{E}^-(\alpha, s)$, $s \in [0, \pi]$ so that, for $\varepsilon = \pm$ and $s \in [0, \pi]$,

$$\mathcal{E}^\varepsilon(\alpha, s) = \begin{pmatrix} -2r(\alpha)\varepsilon\alpha c_s \\ 2r(\alpha)\varepsilon\alpha s_s \\ -1 \end{pmatrix} + \mathcal{O}(\alpha^2), \quad (42)$$

and

$$\frac{\partial}{\partial s}\mathcal{E}^\varepsilon(\alpha, s) = 2r(\alpha)\varepsilon\alpha \begin{pmatrix} s_s \\ c_s \\ 0 \end{pmatrix} + \mathcal{O}(\alpha^2). \quad (43)$$

Moreover, at $s = 0$ and $s = \pi$, the derivatives of $\mathcal{E}^\varepsilon(\alpha, s)$ are only one-sided, i.e. as $s > 0$ tends to zero and $s < \pi$ tends to π . By a trivial continuity argument, one can parameterize $\mathcal{F}(\alpha, k_M\pi)$ as a closed continuous curve γ defined on $[0, 2\pi]$ so that $\gamma(s) = \mathcal{E}^+(\alpha, s)$ for $s \in (0, \pi]$ and $\gamma(s) = \mathcal{E}^-(\alpha, s - \pi)$ for $s \in (\pi, 2\pi]$. Moreover, with the previous computations, it is immediate that γ is in fact piecewise \mathcal{C}^1 with possible discontinuity jumps for $\frac{d}{ds}\gamma$ at $s = 0$ and $s = \pi$.

Since the curve γ is in a neighborhood of the south pole of size proportional to α (thanks to Eq. (42)), it is enough to prove that the orthogonal projection γ_1 of γ on the (x_1, x_2) -plane is homeomorphic to the circle e^{is} , $s \in [0, 2\pi]$. Using Eq. (42), we see that $\|\gamma_1(s)\| = 2r(\alpha)\alpha + \mathcal{O}(\alpha^2)$ on $[0, 2\pi]$, which implies that the continuous function $\|\gamma_1(s)\|$ is always strictly positive for α small enough. We can therefore parameterize γ_1 using polar coordinates (ρ, β) , i.e., for $s \in [0, 2\pi]$,

$$\gamma_1(s) = \rho(s)e^{i\beta(s)},$$

where $\rho(\cdot) := \|\gamma_1(\cdot)\|$ and the function $\beta(\cdot)$ are defined on $[0, 2\pi]$, continuous and piecewise \mathcal{C}^1 , with possible jumps of discontinuity for their derivatives at $s = 0$ and $s = \pi$.

In addition $\rho(0) = \rho(2\pi)$, $\beta(0) \equiv \beta(2\pi) \equiv \pi \pmod{2\pi}$ and, from Eq. (42), $\beta(s) = \pi - s + \mathcal{O}(\alpha)$. To prove Proposition 4, it suffices now to prove that β is a monotone bijection from $[0, 2\pi]$ to $[-\pi, \pi]$. The latter simply results from Eq. (43). Indeed, from that equation, we get that $\frac{d}{ds}\beta(s) = -1 + \mathcal{O}(\alpha)$ where β is differentiable and the one-sided derivatives at $s = 0$ and $s = \pi$ verify the same equation. We deduce that β is strictly decreasing for α small enough.

We next show that $OF(\alpha, k_M\pi)$, the minimum time front at time $k_M\pi$ coincides with $\mathcal{F}(\alpha, k_M\pi)$. By the results of [7], we first notice that any time minimal trajectory starting at the north pole reaches the south pole in time $T > k_M\pi$. Therefore $OF(\alpha, k_M\pi)$ is not empty and is included in $\mathcal{F}(\alpha, k_M\pi)$ according to the PMP. According to Theorem 27 of [6], $OF(\alpha, k_M\pi)$ is a one-dimensional piecewise C^1 compact embedded submanifold of S^2 . By an easy topological argument, we deduce from the above that $OF(\alpha, k_M\pi)$ coincides with $\mathcal{F}(\alpha, k_M\pi)$. \blacksquare

Remark 7 Thanks to Remark 6, and with arguments similar to those of the previous proof, one can prove that $\mathcal{F}(\alpha, (k_M - 1)\pi)$ is optimal for α small enough, with no assumptions on the remainder r .

4.2 Optimal synthesis for the linear pendulum control problem

Recall that $M_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the composition of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha$. With the results of the previous subsection, it is clear that the original control problem on S^2 can be reduced, near the south pole, to a planar control problem on the neighborhood of the south pole delimited by $\widetilde{OF}(\alpha, k_M\pi) := M_\alpha(OF(\alpha, k_M\pi))$ along $(\widetilde{\Sigma})_\alpha$, the control system obtained as the image of $(\Sigma)_\alpha$ by M_α , i.e.

$$(\widetilde{\Sigma})_\alpha : \begin{cases} \dot{z}_1 = -\cos(\alpha)z_2, \\ \dot{z}_2 = \cos(\alpha)z_1 + u \frac{\sin(\alpha)}{\alpha} \sqrt{1 - (\alpha z_1)^2 - (\alpha z_2)^2}, \end{cases} \quad (z_1, z_2) \in \mathbb{R}^2, \quad |u| \leq 1. \quad (44)$$

It is therefore natural to conjecture (simply set $\alpha = 0$ in $\widetilde{OF}(\alpha, k_M\pi)$ and $(\widetilde{\Sigma})_\alpha$) that the limit synthesis should be that of connecting the circle of radius $2r(\alpha)$, $C(0, 2r(\alpha))$, to every point of the disk $B(0, 2r(\alpha))$ along the control system (Pen) given by Eq. (10), which we rewrite as

$$(Pen) \quad \dot{z} = A_0 z + u b_0, \quad \text{with} \quad A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (45)$$

where $z \in \mathbb{R}^2$ and $u \in [-1, 1]$. The control system (Pen) corresponds to a linear pendulum with a forcing term.

Theorem 1 simply states that the conjecture is correct and, as a first step for an argument, we describe, in more details in this subsection, the conjectured limit synthesis. Hence we focus on the following problem.

(P) Fixed $\rho \in]0, 2]$, for any given $\bar{y} \in B(0, \rho)$ find a time optimal trajectory connecting the circle of radius ρ centered at the origin to \bar{y} along the control system (Pen) .

Remark 8 The problem of computing the TOS for the linear pendulum, taking the origin as a source, is studied in any textbook of optimal control. Here as source we take the circle of radius ρ centered at the origin, which is a level set of the Hamiltonian $H = \frac{1}{2}(z_1^2 + z_2^2)$ associated to the uncontrolled system.

It is easy to see that the solutions of problem **(P)** must be bang-bang trajectories. Indeed since (Pen) is a bidimensional linear control system it is well known that this property is guaranteed by the Kalman controllability condition $\det(b_0, A_0 b_0) \neq 0$, which is satisfied by (Pen) . To determine the TOS, we first look for the switching curves. We know that every extremal trajectory for the problem **(P)** must satisfy the transversality condition of the PMP stated in Section 2.1. Here the source manifold is the circle $C(0, \rho)$, and the transversality condition essentially translates into the property that the vector $\lambda(0)$ (that, without loss of generality, we will assume unitary) is proportional to $z(0) \in C(0, \rho)$ (identifying the cotangent space with the plane \mathbb{R}^2). To determine completely $\lambda(0)$, it is enough to observe that a necessary condition for $z(\cdot)$ to be optimal is that $\dot{z}(0)$ points inside the disk $B(0, \rho)$, i.e., if we denote by u_{opt} the corresponding control, then

$$\langle z(0), \dot{z}(0) \rangle \geq 0 \iff \langle z(0), A_0 z(0) + u_{opt} b_0 \rangle \geq 0 \iff \langle z(0), u_{opt} b_0 \rangle \geq 0.$$

Therefore, $u_{opt} = -\text{sgn} \langle z(0), b_0 \rangle$. On the other hand, from the maximality condition of the PMP, we have $u_{opt} = \text{sgn} \langle \lambda(0), b_0 \rangle$ and, therefore, one can define $\lambda(0) := -z(0)/\rho$. Finally $u_{opt} = -\text{sgn}(z_2(0))$ (except at the points $\pm(\rho, 0)$), while the switching time t_{sw} must satisfy the condition $\langle \lambda(t_{sw}), b_0 \rangle = \lambda_2(t_{sw}) = 0$.

Consider now the adjoint system

$$\begin{cases} \dot{\lambda}_1 = -\lambda_2, \\ \dot{\lambda}_2 = \lambda_1. \end{cases} \quad (46)$$

If we identify \mathbb{R}^2 with the complex plane, so that $z = z_1 + iz_2$ and $\lambda = \lambda_1 + i\lambda_2$, then the equations (45), (46) become

$$\dot{z} = i(z + u) \quad \text{and} \quad \dot{\lambda} = i\lambda.$$

Moreover we can set $z(0) = -\rho e^{-i\theta}$ and $\lambda(0) = e^{-i\theta}$ for some $\theta \in [0, 2\pi[$ and the corresponding solutions are:

$$\begin{cases} z(t) = (z(0) + u_{opt})e^{it} - u_{opt} = -\rho e^{i(t-\theta)} + u_{opt}(e^{it} - 1), \\ \lambda(t) = \lambda(0)e^{it} = e^{i(t-\theta)}. \end{cases}$$

The switching curves are determined by the relation $t_{sw} \equiv \theta \pmod{\pi}$ and this allows to conclude that the switching curves are the following two semicircles of radius 1:

$$\begin{cases} z(\theta) = 1 - \rho - e^{i\theta} & \theta \in [0, \pi[, \\ z(\theta) = \rho - 1 - e^{i\theta} & \theta \in [\pi, 2\pi[. \end{cases}$$

These switching curves cannot be optimal for $\rho < 2$ since they are not locally optimal, as can be easily checked using the definition given in Section 2.2. We conclude that the optimal trajectories are bang arcs and the corresponding control depends on the sign of the component $z_2(0)$ of the starting point.

To conclude the description of the synthesis, it is enough to determine the cut locus, i.e. the set of points that are reached by two or more optimal trajectories at the same time. Assume that $z \in \mathbb{C}$ belongs to the cut locus. Then, there exist $s \in [0, \pi)$, $s' \in [\pi, 2\pi)$ and t such that

$$\begin{cases} z = -\rho e^{i(t-s)} + 1 - e^{it}, \\ z = -\rho e^{i(t-s')} - 1 + e^{it}. \end{cases} \quad (47)$$

Therefore $|z - 1 + e^{it}| = |z + 1 - e^{it}| = \rho$. In particular, denoting by \bar{z} the complex conjugate to z , we have

$$(z - 1 + e^{it})(\bar{z} - 1 + e^{-it}) - (z + 1 - e^{it})(\bar{z} + 1 - e^{-it}) = -4z_1 + 4z_1 \cos t + 4z_2 \sin t = 0, \quad (48)$$

$$(z - 1 + e^{it})(\bar{z} - 1 + e^{-it}) + (z + 1 - e^{it})(\bar{z} + 1 - e^{-it}) = 2z_1^2 + 2z_2^2 + 4 - 4 \cos t = 2\rho^2. \quad (49)$$

From (48) we have that $\cos t = \frac{z_1^2 - z_2^2}{z_1^2 + z_2^2}$, and, substituting in (49), we find that z must satisfy the equation

$$z_1^4 + z_2^4 + 2z_1^2 z_2^2 - \rho^2 z_1^2 + (4 - \rho^2) z_2^2 = 0. \quad (50)$$

The previous computation show that the cut locus is a subset of the set of points belonging to the locus defined by (50). Actually it is easy to see that this is the proper subset obtained with the additional condition $z_1 z_2 \geq 0$, that corresponds to $t \leq \pi$. The precise shape of the optimal synthesis, which is now clear, is portrayed in Fig. 4 for a particular value of $\rho < 2$. Notice that, from the previous computations, we have $\rho e^{is'} = \rho e^{is} + 2 - 2e^{it}$ and, since $\rho e^{is'} + \rho e^{is} = 2\rho e^{is'} - 2 + 2e^{it}$ and $\rho e^{is'} - \rho e^{is} = 2 - 2e^{it}$ are orthogonal in the complex plane, we find easily the following equation:

$$(2 - \rho \cos s')(\cos t - 1) - \rho \sin s' \sin t = 0.$$

Consequently, for $t \in [0, 2\pi[$ and $s' \in [\pi, 2\pi[$, one has, along the overlap curve

$$t = t(s') = -2 \arctan \frac{\rho \sin s'}{2 - \rho \cos s'}. \quad (51)$$

This expression will be useful in the following. Also, notice that combining (47) and (51) one easily finds a parametrization of the overlap curve in terms of s' and that in an analogous way it is possible to parameterize it by means of s . From now on we will denote by $\gamma_{pen}^o(\cdot)$ the parameterization of the overlap curve with respect to the parameter s .

Remark 9 If $\rho = 2$ the previous reasoning does not apply and indeed the synthesis is different. In this case the overlap curve coincides with the switching curves and with the trajectories reaching the origin corresponding to $u = \pm 1$. A simple way to prove this fact is to study the optimal synthesis starting from the origin with vector fields with opposite signs, and observe that the extremal front at time π is a circle of radius 2.

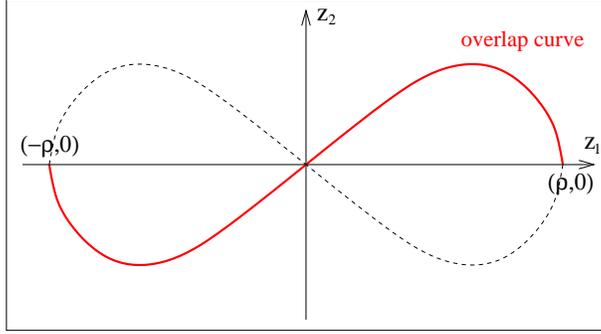


Figure 7: The overlap curve for the pendulum problem

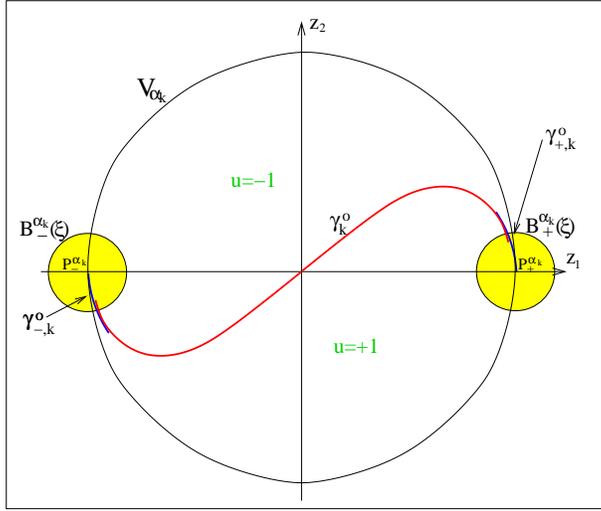


Figure 8: Propositions 5 and 6

4.3 Proof of Theorem 1

The proof of Theorem 1 is divided in two parts. Roughly speaking, defining $P_\varepsilon^\alpha := M_\alpha(C_{k_M}^\varepsilon(0))$ for $\varepsilon = \pm$, we will look separately at the shape of the synthesis far from P_ε^α and inside neighborhoods of P_ε^α , $\varepsilon = \pm$. Let us call V_α the image with respect to M_α of the neighborhood of the south pole enclosed by $OF(\alpha, k_M\pi)$ and $B_\varepsilon^\alpha(\xi)$, the ball of center P_ε^α and radius ξ .

Then, the previous cases correspond to the following two propositions whose meaning is clarified by Figure 8.

Proposition 5 *Let $\bar{r} \in (0, 1)$ and $\alpha_k := \frac{\pi}{2(k+\bar{r})}$. Then, for any $\xi > 0$ there exist a positive integer \bar{k} and a compact interval $I \subset (0, \pi)$ such that it is possible to find a curve γ_k^o , defined on I for $k \geq \bar{k}$, verifying the following: γ_k^o divides $V_{\alpha_k} \setminus (B_+^{\alpha_k}(\xi) \cup B_-^{\alpha_k}(\xi))$ in two connected components, such that “above” γ_k^o the optimal feedback associated to the synthesis for $\alpha = \alpha_k$ takes the value -1 and, below γ_k^o , it is equal to 1 , and in particular γ_k^o is an overlap curve for $\alpha = \alpha_k$. Moreover, γ_k^o converges to γ_{pen}^o in the C^0 topology of I .*

Proposition 6 *Consider the notations defined above. Then there exist $\xi > 0$, τ_ε , $\varepsilon = \pm$ with $0 < \tau_- < \tau_+ < \pi$ and a positive integer \bar{k} such that, for every $k \geq \bar{k}$, it is possible to find two curves $\gamma_{\varepsilon,k}^o$ and $\gamma_{\pm,k}^o$, defined respectively on $[0, \tau_-]$ and $[\tau_+, \pi]$, verifying the following: $\gamma_{\varepsilon,k}^o$ divides $V_{\alpha_k} \cap B_\varepsilon^{\alpha_k}(\xi)$ in two connected components, such that “above” $\gamma_{\varepsilon,k}^o$ the optimal feedback associated to the synthesis for $\alpha = \alpha_k$ takes the value -1 and, below*

$\gamma_{\varepsilon,k}^o$, it is equal to 1, and in particular the $\gamma_{\varepsilon,k}^o$ are overlap curves for $\alpha = \alpha_k$. Moreover, $\gamma_{-,k}^o$ and $\gamma_{+,k}^o$ converge to γ_{pen}^o in the C^0 topology respectively of $[0, \tau_-]$ and $[\tau_+, \pi]$.

The choice of studying the synthesis separately in neighborhoods of P_ε^α and far from P_ε^α is justified by the fact that the proofs of the previous propositions rely on different implicit function arguments.

It is clear that, combining Proposition 5, for an appropriate choice of ξ , with Proposition 6, one almost completes the proof of Theorem 1. We will not prove explicitly that the convergence of γ_k^o to γ_{pen}^o with respect to the parameter \bar{r} is uniform in any closed interval of $(0, 1)$. As explained in Remark 10, this can be done with the same methods used in the proofs of Propositions 5,6, but with much more computational efforts.

We will therefore only provide the complete proofs of the propositions. For this purpose, the first step consists in checking whether the switching curves $C_{k_M}^\varepsilon$, $\varepsilon = \pm 1$ are optimal or not. In that regard and similarly to the case of the linear pendulum, we have the following result:

Lemma 3 *Let $\bar{r} \in]0, 1[$. Then, if α is small enough and $r(\alpha) = \bar{r}$, the switching curve $C_{k_M}^\varepsilon$ is nowhere locally optimal i.e. all the extremal trajectories switching on $C_{k_M}^\varepsilon$ lose optimality before reaching it.*

Proof of Lemma 3. For simplicity we define $S(s) := C_{k_M}^+(s)$ and we assume k_M odd. As in the proof of Lemma 6, we get the following asymptotic expansions, after applying the map M_α :

$$S(s) = \begin{pmatrix} 2r - 1 + c_s \\ s_s \end{pmatrix} + \mathcal{O}(\alpha), \quad S(0) = \begin{pmatrix} 2r + \mathcal{O}(\alpha) \\ 0 \end{pmatrix}, \quad (52)$$

$$S'(s) = \begin{pmatrix} -s_s \\ c_s \end{pmatrix} + \mathcal{O}(\alpha), \quad S'(0) = \begin{pmatrix} 0 \\ 1 + \mathcal{O}(\alpha) \end{pmatrix}, \quad (53)$$

$$S''(s) = \begin{pmatrix} -c_s \\ -s_s \end{pmatrix} + \mathcal{O}(\alpha). \quad (54)$$

Integrating the above equation, we have

$$S'(s) = S'(0) + \int_0^s S''(\tau) d\tau = \begin{pmatrix} -s_s + \mathcal{O}(s\alpha) \\ c_s + \mathcal{O}(\alpha) \end{pmatrix}, \quad (55)$$

$$S(s) = S(0) + \int_0^s S'(\tau) d\tau = \begin{pmatrix} 2r - 1 + c_s + \mathcal{O}(\alpha) \\ s_s + \mathcal{O}(s\alpha) \end{pmatrix} \quad (56)$$

and therefore

$$\frac{1}{c_\alpha} X_\pm(S(s)) = \begin{pmatrix} -S_2(s) \\ S_1(s) \pm \frac{\tan \alpha}{\alpha} \sqrt{1 - \alpha^2 S_1(s)^2 - \alpha^2 S_2(s)^2} \end{pmatrix} = \begin{pmatrix} -s_s + \mathcal{O}(s\alpha) \\ 2r - 1 + c_s + \mathcal{O}(\alpha) \pm (1 + \mathcal{O}(\alpha^2)) \end{pmatrix}.$$

Here S_i , $i = 1, 2$, denotes the i -th component of S . Dividing the above equation by $1 + \mathcal{O}(\alpha)$, we can assume that the first component is identically equal to $-s_s$. The same can be done with the expression (55), so that it is possible to compare the three vectors obtained in this way simply by looking at the second components, which are equal respectively to $2r - 1 + c_s \pm 1 + \mathcal{O}(\alpha)$ and $c_s + \mathcal{O}(\alpha)$. In particular, the fact that $S(\cdot)$ is nowhere locally optimal if α is small enough follows from the inequalities $2r - 2 + c_s + \mathcal{O}(\alpha) < c_s + \mathcal{O}(\alpha) < 2r + c_s + \mathcal{O}(\alpha)$. \blacksquare

A straightforward consequence of the previous result is the presence of a non trivial cut locus in the neighborhood of the south pole enclosed by $F(\alpha, k_M \pi)$. It remains to clearly define that cut-locus, which is the purpose of Propositions 5 and 6.

4.3.1 Proof of Proposition 5

As usual, we only provide an argument in the case k_M odd and we fix the remainder equal to $\bar{r} \in (0, 1)$.

Recall that, according to Section 4.1, $OF(\alpha, k_M \pi)$ is approximately (up to order α^2) a circle of radius $2\bar{r}\alpha$. To describe the synthesis inside the neighborhood of the south pole enclosed by $OF(\alpha, k_M \pi)$, it is more convenient to use the two dimensional control system $(\tilde{\Sigma})_\alpha$, which is rewritten as follows by using Eq. (44),

$$\dot{z} = c_\alpha A_0 z + u \frac{s_\alpha}{\alpha} \sqrt{1 - \alpha^2 \|z\|^2} b_0.$$

We set $\tilde{X}_\varepsilon^\alpha(z) := c_\alpha A_0 z + \varepsilon \frac{s_\alpha}{\alpha} \sqrt{1 - \alpha^2} \|z\|^2 b_0$ and $\tilde{X}_\varepsilon^{pen}(z) := A_0 z + \varepsilon b_0$, for $\varepsilon = \pm$, and we define $\widetilde{OF}(\alpha, k_M \pi)$ as the image by M_α of $OF(\alpha, k_M \pi)$. Then we know that, up to order α , $\widetilde{OF}(\alpha, k_M \pi)$ is a circle of radius $2\bar{r}$. In particular, as in the proof of Proposition 4, one can construct a piecewise smooth parameterization $\sigma_\alpha : [0, 2\pi] \rightarrow \tilde{F}(\alpha, k_M \pi)$ so that $\sigma_\alpha(0) = P_-^\alpha$, $\sigma_\alpha(\pi) = P_+^\alpha$ with a loss of regularity only occurring at $s = 0, \pi$ (with two-sided differentials at any order). In particular $\sigma_\alpha(\cdot)$ approximates in the \mathcal{C}^0 sense the function $\sigma : [0, 2\pi] \rightarrow \mathbb{C} \sim \mathbb{R}^2$, defined as $\sigma(s) = 2\bar{r} e^{i(\pi-s)}$, which is a parameterization of the circle of radius $2\bar{r}$.

Taking into account Lemma 3, the cut-locus in V_α is contained inside the set of points $Q \in \mathbb{R}^2$, besides P_ε^α , such that there exists $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, 2\pi)$ for which $Q = e^{t\tilde{X}_+^\alpha} \sigma_\alpha(s') = e^{t\tilde{X}_-^\alpha} \sigma_\alpha(s)$.

In view of applying an inverse function result for characterizing this set, we consider the map Φ defined on $[0, \pi] \times [\pi, 2\pi] \times [0, \pi]$ by

$$\Phi(s, s', t) := (s, e^{t\tilde{X}_+^{pen}} \sigma(s') - e^{t\tilde{X}_-^{pen}} \sigma(s)),$$

which takes values in \mathbb{R}^3 . Similarly, for $k \geq 1$ and α_k as in the proposition, we consider the map Φ_k defined on $[0, \pi] \times [\pi, 2\pi] \times [0, \pi]$ by

$$\Phi_k(s, s', t) := (s, e^{t\tilde{X}_+^{\alpha_k}} \sigma_{\alpha_k}(s') - e^{t\tilde{X}_-^{\alpha_k}} \sigma_{\alpha_k}(s)).$$

Note that, since the vector fields $\tilde{X}_\varepsilon^{\alpha_k}$ converge uniformly to $\tilde{X}_\varepsilon^{pen}$ on V_α , it is easy to see that Φ_k converges to Φ in the \mathcal{C}^1 norm.

For (Pen) , a point of the overlap curve, besides P_ε^α , is then identified with a triple $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, \pi)$ such that $\Phi(s, s', t) = (s, 0, 0)$. In other words, the overlap curve can be parameterized by means of the map $w : [0, \pi] \rightarrow \mathbb{R}^3$ defined implicitly by $\Phi(w(s)) = (s, 0, 0)$, while γ_{pen}^o can be obtained as the composition of the two maps $e^{t\tilde{X}_-^{pen}} \sigma(s)$ and $w(s)$.

Similarly, we would like to define the overlap curve corresponding to $(\Sigma)_{\alpha_k}$, for k large enough, by means of the function w_k defined by $\Phi_k(w_k(s)) = (s, 0, 0)$. To proceed, we will apply Theorem 3. The first task consists of computing $\det D\Phi$ along the overlap curve.

Lemma 4 *Along the set of triples $(s, s', t) \in (0, \pi) \times (\pi, 2\pi) \times (0, \pi)$ for which $e^{t\tilde{X}_+^{pen}} \sigma(s') = e^{t\tilde{X}_-^{pen}} \sigma(s)$, we have*

$$\det D\Phi(s, s', t) = \frac{4\bar{r}(1 - \bar{r}^2) \sin s'}{(1 - \bar{r} \cos s')^2 + (\bar{r} \sin s')^2}.$$

Proof of Lemma 4. One has

$$\det D\Phi(s, s', t) = \det \left((e^{t\tilde{X}_+^{pen}})_* \frac{d\sigma}{ds'}, \tilde{X}_+^{pen} e^{t\tilde{X}_+^{pen}} \sigma(s') - \tilde{X}_-^{pen} e^{t\tilde{X}_-^{pen}} \sigma(s) \right).$$

By taking into account that $\Phi(s, s', t) = 0$, the previous determinant is equal to twice the first component of $(e^{t\tilde{X}_+^{pen}})_* \frac{d\sigma}{ds'}$, i.e., $\det D\Phi(s, s', t) = 4\bar{r} \sin(s' - t)$. Using Eq. (51), one concludes. \blacksquare

Observe that $\det D\Phi \neq 0$ if $s' \neq 0, \pi$. In particular, if we consider a closed interval $I \subset (0, \pi)$, then the set $\text{Im}(w)|_I$ plays the role of the compact set \mathcal{K} in Theorem 3. All the assumptions of the theorem are then verified, and therefore we have proved the existence of a map w_k defined on I , satisfying $\Phi_k(w_k(s)) = (s, 0, 0)$ and converging uniformly to w . If we define γ_k^o as the composition of the two maps $e^{t\tilde{X}_-^{\alpha_k}} \sigma(s)$ and $w(s)$, then, since I was chosen arbitrarily, the proof of the theorem is complete. \blacksquare

4.3.2 Proof of Proposition 6

With the previous notations, let φ_k be the smooth map defined on $[0, \pi] \times [\pi, 2\pi] \times [0, 2\pi]$ by

$$\varphi_k(s, s', t) = e^{t\tilde{X}_+^{\alpha_k}} \sigma_k(s') - e^{t\tilde{X}_-^{\alpha_k}} \sigma_k(s).$$

For the rest of this paragraph, we drop the index k to get lighter notations.

From the Taylor expansion of φ around the points $(0, 2\pi, 0)$ and $(\pi, \pi, 0)$, we derive the asymptotic behaviors of the cut locus close to the points P_ε^α , $\varepsilon = \pm$, since that cut locus belongs to the level set $\varphi = 0$. We will only perform computations at $(0, 2\pi, 0)$ since they are entirely similar at $(\pi, \pi, 0)$.

Let us call $\varphi^{(1)}$ $\varphi^{(2)}$ the two components of φ . We use $\varphi_s^{(i)}$ to denote the partial derivative of the component $\varphi^{(i)}$ with respect to s evaluated in $(0, 2\pi, 0)$ and we define in an analogous way all the (multiple) partial derivatives evaluated in $(0, 2\pi, 0)$. Set $\tilde{s} := s' - 2\pi$. Then, after computations, we have $\varphi_s^{(1)} = \varphi_{\tilde{s}}^{(1)} = \varphi_t^{(1)} = 0$ and

$$\begin{aligned} \varphi_{ss}^{(1)} &= -2\bar{r} + \mathcal{O}(\alpha) & \varphi_{\tilde{s}\tilde{s}}^{(1)} &= 2\bar{r} + \mathcal{O}(\alpha), & \varphi_{tt}^{(1)} &= 2 + \mathcal{O}(\alpha), \\ \varphi_{s\tilde{s}}^{(1)} &= 0, & \varphi_{st}^{(1)} &= -2\bar{r} + \mathcal{O}(\alpha), & \varphi_{\tilde{s}t}^{(1)} &= 2\bar{r} + \mathcal{O}(\alpha), \\ \varphi_s^{(2)} &= 2\bar{r} + \mathcal{O}(\alpha), & \varphi_{\tilde{s}}^{(2)} &= -2\bar{r} + \mathcal{O}(\alpha), & \varphi_t^{(2)} &= 2\bar{r} + \mathcal{O}(\alpha). \end{aligned}$$

We thus get

$$\begin{aligned} \varphi^{(1)}(s, \tilde{s}, t) &= \varphi_{ss}^{(1)}s^2 + \varphi_{\tilde{s}\tilde{s}}^{(1)}\tilde{s}^2 + \varphi_{tt}^{(1)}t^2 + 2\varphi_{st}^{(1)}st + 2\varphi_{\tilde{s}t}^{(1)}\tilde{s}t + \mathcal{O}(|(s, \tilde{s}, t)|^3) \\ &= -2\bar{r}s^2 + 2r\tilde{s}^2 + 2t^2 - 4\bar{r}st + 4r\tilde{s}t + \mathcal{O}(\alpha|(s, \tilde{s}, t)|^2) + \mathcal{O}(|(s, \tilde{s}, t)|^3), \end{aligned} \quad (57)$$

and

$$\varphi^{(2)}(s, \tilde{s}, t) = \varphi_s^{(2)}s + \varphi_{\tilde{s}}^{(2)}\tilde{s} + \varphi_t^{(2)}t + \mathcal{O}(|(s, \tilde{s}, t)|^2) = 2\bar{r}s - 2r\tilde{s} - 2t + \mathcal{O}(\alpha|(s, \tilde{s}, t)|) + \mathcal{O}(|(s, \tilde{s}, t)|^2), \quad (58)$$

where, here, $\mathcal{O}(\cdot)$ is uniform with respect to α .

Fix $\xi_0 > 0$ small. We are looking at the cut locus in a neighborhood of P_ε^α , and thus, we can assume $|(s, \tilde{s}, t)| < \xi_0$ for some $\xi_0 > 0$. The purpose of subsequent computations consists of expressing $\tilde{s} < 0$ and $t > 0$ as functions of s , for $0 \leq s \leq \xi_0$, by using the equations $\varphi^{(1)} = 0$ and $\varphi^{(2)} = 0$.

From $\varphi^{(2)} = 0$, by applying the implicit function theorem, for ξ_0 small enough and $|(s, \tilde{s})| < \xi_0$, we get $t = h(s, \tilde{s})$ with $h \in \mathcal{C}^1$. Moreover, since $h(s, \tilde{s}) = \mathcal{O}(|(s, \tilde{s})|)$ we have

$$h(s, \tilde{s}) = \bar{r}s - r\tilde{s} + \mathcal{O}(\alpha|(s, \tilde{s})|) + \mathcal{O}(|(s, \tilde{s})|^2). \quad (59)$$

Consider now the map

$$\phi(s, \tilde{s}) = \frac{\varphi^{(1)}(s, \tilde{s}, h(s, \tilde{s}))}{s - \tilde{s}},$$

which is well defined and \mathcal{C}^1 for $s > 0, \tilde{s} < 0$. Again, it is possible to apply the implicit function theorem to the equation $\phi = 0$, so that we get \tilde{s} as a \mathcal{C}^1 function of s and this gives the existence of the overlap curve. Moreover, by combining Eqs. (58) and (59), we get the following

$$\phi(s, \tilde{s}) = s(1 + \bar{r}) + \tilde{s}(1 - \bar{r}) + \mathcal{O}(\alpha|(s, \tilde{s})|) + \mathcal{O}(|(s, \tilde{s})|^2) = 0, \quad (60)$$

and then, $|\tilde{s}| = \mathcal{O}(|s|)$. Therefore, from this estimate and the above ones, we immediately obtain that

$$\tilde{s} = -\left(\frac{1 + \bar{r}}{1 - \bar{r}} + \mathcal{O}(\alpha)\right)s + \mathcal{O}(s^2), \quad t = \left(\frac{2\bar{r}}{1 - \bar{r}} + \mathcal{O}(\alpha)\right)s + \mathcal{O}(s^2), \quad (61)$$

from which we get that the overlap curve converges uniformly to γ_{pen}^o .

The proof of Proposition 6 is now complete. ■

Remark 10 In order to prove that the overlap curve γ_k^o converges to γ_{pen}^o uniformly with respect to \bar{r} in any closed interval $I \subset (0, 1)$, it is enough to follow the lines of the proofs of Propositions 5, 6 by considering \bar{r} as an additional variable. For instance, for the Proposition 5, one needs to define the maps $\tilde{\Phi} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $\tilde{\Phi}(\bar{r}, s, s', t) := (\bar{r}, s, e^{t\tilde{X}_+^{pen}} \sigma(s') - e^{t\tilde{X}_-^{pen}} \sigma(s))$ (recall that $\sigma(\cdot)$ depends on \bar{r}) and $\tilde{\Phi}_k : \mathbb{R}^4 \rightarrow \mathbb{R}^4$, $\tilde{\Phi}_k(\bar{r}, s, s', t) := (\bar{r}, s, e^{t\tilde{X}_+^{\alpha_k}} \sigma_k(s') - e^{t\tilde{X}_-^{\alpha_k}} \sigma_k(s))$ (where $\alpha_k = \pi/(2(\bar{r} + k))$ and $\sigma_k(\cdot)$ depends on \bar{r}). The uniformity with respect to \bar{r} is then proved by applying Theorem 3 to $\tilde{\Phi}, \tilde{\Phi}_k$ with $\mathcal{K} = \{\tilde{\Phi}^{-1}(\bar{r}, s, 0, 0) : (\bar{r}, s) \in I \times J\}$ where $I \times J$ is a compact subset of $(0, 1) \times (0, \pi)$.

5 Case $r = C\alpha$

5.1 Description of the minimum time front at time $k_M\pi$

Fix $C > 0$ and consider the sequence (α_k) such that $r(\alpha_k) = C\alpha_k$, $k \geq 0$. As before, we drop the index k when possible. For α_k small enough, one deduces, from the analysis of [7], that the south pole is not reached at time $k_m\pi = \lfloor \frac{\pi}{2\alpha} \rfloor \pi$, so that the optimal front at time $k_M\pi$ is not empty. The next result provides a description of the extremal front at time $k_M\pi$.

Lemma 5 Define the planar curve $\mathcal{L} : [0, 2\pi] \rightarrow \mathbb{R}^2$ by

$$\mathcal{L}(s) = \begin{pmatrix} c_s(-2C + \pi s_s^2/2) \\ s_s(\pi + 2C - \pi s_s^2/2) \end{pmatrix}. \quad (62)$$

Then, for $s \in [0, \pi]$, we have

$$\mathcal{E}^+(\alpha, s) = (\alpha^2 \mathcal{L}(s), -1)^T + \mathcal{O}(\alpha^3), \quad (63)$$

and

$$\frac{\partial}{\partial s} \mathcal{E}^+(\alpha, s) = (\alpha^2 \frac{d}{ds} \mathcal{L}(s), 0)^T + \mathcal{O}(\alpha^3). \quad (64)$$

At $s = 0$ and $s = \pi$, the derivatives are only one-sided, i.e. as $s > 0$ tends to zero and $s < \pi$ tends to π .

Similarly, we have, for $s \in [0, \pi]$,

$$\mathcal{E}^-(\alpha, k_M\pi, s) = (\alpha^2 \mathcal{L}(s + \pi), -1)^T + \mathcal{O}(\alpha^3), \quad (65)$$

and

$$\frac{\partial}{\partial s} \mathcal{E}^-(\alpha, k_M\pi, s) = (\alpha^2 \frac{d}{ds} \mathcal{L}(s + \pi), 0)^T + \mathcal{O}(\alpha^3), \quad (66)$$

with one-sided derivatives at $s = 0$ and $s = \pi$.

Proof of Lemma 5. This is immediate from Proposition 3 applied in the case $r(\alpha) = C\alpha$. ■

For $C < \pi/4$, consider $\theta_d \in (0, \pi/2)$ with $\sin(\theta_d) = 2\sqrt{C/\pi}$. The curve $\mathcal{L}(s)$ has two double points $D^+ = \mathcal{L}(s_1^+) = \mathcal{L}(s_2^+)$, with $s_1^+ = \theta_d$ and $s_2^+ = \pi - \theta_d$, and $D^- = \mathcal{L}(s_1^-) = \mathcal{L}(s_2^-)$, with $s_1^- = \pi + \theta_d$ and $s_2^- = 2\pi - \theta_d$. It also has four cuspidal points Cp_i^ε , $i = 1, 2$ and $\varepsilon = \pm$, corresponding to the values $s = s_{cusp,i}^\varepsilon$, where $\sin^2 s = \frac{2+4C/\pi}{3}$.

Finally, let σ be the closed Jordan curve defined as the restriction of $\mathcal{L}(s)$ to $[0, s_1^+] \cup [s_2^+, s_1^-] \cup [s_2^-, 2\pi]$. If $C > \pi/4$, we simply define σ to be \mathcal{L} .

At the light of the previous result, we get that $\mathcal{F}(\alpha, k_M\pi)$, the extremal front at time $k_M\pi$, is contained inside a neighborhood W_α of the south pole of order $\mathcal{O}(\alpha^2)$ neighborhood of the south pole. Therefore, in order to understand the shape of the optimal synthesis inside W_α , we must rescale the whole problem by N_α , the linear mapping from \mathbb{R}^3 onto \mathbb{R}^2 defined as the composition of the orthogonal projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha^2$.

For $x \in W_\alpha$, we first consider $(\Lambda)_\alpha$, the image of (Σ) by N_α , i.e. $(\Lambda)_\alpha$ is the planar control system given by

$$(\Lambda)_\alpha : \begin{cases} \dot{z}_1 = -c_\alpha z_2, \\ \dot{z}_2 = c_\alpha z_1 + u \frac{s_\alpha}{\alpha^2} \sqrt{1 - \alpha^4 \|z\|^2}. \end{cases} \quad (67)$$

Let \mathcal{L}_α be the image of $\mathcal{F}(\alpha, k_M\pi)$ by N_α . From Lemma 5, \mathcal{L}_α converges to \mathcal{L} in the C^1 topology. It is clear that, for $C > \pi/4$, $\mathcal{L}_\alpha : [0, 2\pi] \rightarrow \mathbb{R}^2$ is homeomorphic to e^{is} , $s \in [0, 2\pi]$. In the case where $C < \pi/4$, the next lemma shows that, for α small enough, \mathcal{L}_α has the same shape as \mathcal{L} .

Lemma 6 If $C < \pi/4$, then \mathcal{L}_α is described by the following picture, where $Cp_i^\varepsilon(\alpha) = \mathcal{L}_\alpha(s_{cusp,i}^\varepsilon(\alpha))$, $i = 1, 2$ and $\varepsilon = \pm$, are cuspidal points and $D^\varepsilon(\alpha)$ are double points with

$$D^+(\alpha) = \mathcal{L}_\alpha(s_1^+(\alpha)) = \mathcal{L}_\alpha(s_2^+(\alpha)), \quad D^-(\alpha) = \mathcal{L}_\alpha(s_1^-(\alpha)) = \mathcal{L}_\alpha(s_2^-(\alpha)), \quad (68)$$

where $s_{cusp,i}^\varepsilon(\alpha)$ and $s_i^\varepsilon(\alpha)$ tend respectively to $s_{cusp,i}^\varepsilon$ and s_i^ε as α tends to zero, for $i = 1, 2$ and $\varepsilon = \pm$. For α small enough, set σ_α , the closed curve defined as the restriction of $\mathcal{L}_\alpha(s)$ to $[0, s_1^+(\alpha)] \cup [s_2^+(\alpha), s_1^-(\alpha)] \cup [s_2^-(\alpha), 2\pi]$. Then, it is a Jordan curve.

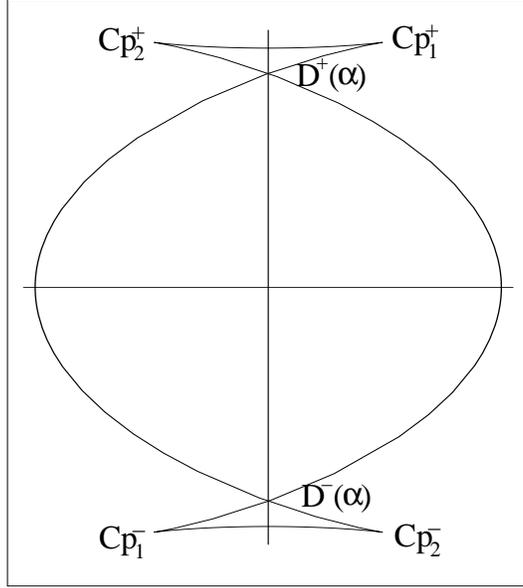


Figure 9: Graph of the function \mathcal{L}_α for $C < \pi/4$

Proof of Lemma 6. For $i = 1, 2$ and $\varepsilon = \pm$, the existence of the cuspidal points $Cp_i^\varepsilon(\alpha)$ is obtained by applying the implicit function theorem to the equation $DL(s, \alpha) = 0$, where the function $DL(s, \alpha)$ is defined by

$$DL(s, \alpha) := \frac{d}{ds} \mathcal{L}_\alpha(s),$$

in the neighborhood of each $(s_{cusp, i}^\varepsilon, 0)$. We have

$$\partial_s DL(s_{cusp, i}^\varepsilon, 0) = \frac{d^2}{ds^2} \mathcal{L}(s_{cusp, i}^\varepsilon) \neq 0$$

and we conclude. The uniqueness of these four points, on $[0, 2\pi]$, is trivial since $DL(s, \alpha) = \frac{d}{ds} \mathcal{L}(s) + \mathcal{O}(\alpha)$.

Similarly, for $\varepsilon = \pm$, the existence of the double points $D^\varepsilon(\alpha)$ follows after applying the implicit function theorem to the equation $DP(s, s', \alpha) = 0$, where the function $DP(s, s', \alpha)$ is defined by

$$DP(s, s', \alpha) = \mathcal{L}_\alpha(s) - \mathcal{L}_\alpha(s'),$$

in the neighborhood of each $(s_1^\varepsilon, s_2^\varepsilon, 0)$. For the uniqueness, we proceed as before. \blacksquare

In the case $C > \pi/4$, we also define σ_α to be equal to \mathcal{L}_α . As a consequence, we are able to characterize $OF(\alpha, k_M\pi)$, the minimum time front at time $k_M\pi$ when $C \neq \pi/4$.

Proposition 7 For α small enough and $C \neq \pi/4$, the minimum time front at time $k_M\pi$, $OF(\alpha_k, k_M\pi)$ is equal to $\tilde{\sigma}_\alpha$, the inverse image on S^2 , by N_α , of σ_α .

Remark 11 As a consequence, we deduce that, for $C > \pi/4$ and α small enough, the optimal synthesis between $\mathcal{F}(\alpha, (k_M - 1)\pi)$ and $\mathcal{F}(\alpha, k_M\pi)$ is simply given by the extremal flow whereas, for $C < \pi/4$, there is a loss of optimality along certain extremal curves starting at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ before reaching $\mathcal{F}(\alpha, k_M\pi)$. The values of s corresponding to such curves can be deduced from the previous characterizations of $\mathcal{F}(\alpha, k_M\pi)$ and $OF(\alpha_k, k_M\pi)$.

Proof of Proposition 7. Recall that $OF(\alpha_k, k_M\pi)$ is a piecewise \mathcal{C}^1 submanifold of $\mathcal{F}(\alpha, k_M\pi)$. As in the proof of Proposition 4, the result to establish is a consequence of the fact that $\sigma_\alpha = \mathcal{L}_\alpha : [0, 2\pi] \rightarrow \mathbb{R}^2$ is homeomorphic to e^{is} , $s \in [0, 2\pi]$ and it can be achieved by means of simple topological arguments.

In the case $C < \pi/4$, σ_α is a piecewise \mathcal{C}^1 Jordan curve homeomorphic to e^{is} , $s \in [0, 2\pi]$. A simple topological argument yields the conclusion. \blacksquare

5.2 Limit of the synthesis

It remains to describe the limiting dynamics close to the south pole. In order to take the limit, as α tends to zero, in $(\Lambda)_\alpha$, one must reparameterize by the time αt . The limit is then given by the control system

$$(\Lambda) : \quad \begin{cases} \dot{z}_1 = 0, \\ \dot{z}_2 = u. \end{cases}$$

We now describe the optimal synthesis for the limit problem, i.e. for the problem of reaching in minimum time every point inside σ along (Λ) and starting from σ . Because of the symmetries of σ and because the tangent vector to σ is vertical only at $s = 0$ and $s = \pi$, there exists a unique overlap curve $(Seg)_C$, defined as the segment of the z_1 -axis between the points $(-2C, 0)$ and $(2C, 0)$. Above it, the input u takes the constant value -1 and, below that overlap curve, the constant value 1 . Integral curves are clearly vertical lines.

We next intend to prove that the optimal synthesis consisting of reaching in minimum time every point inside σ_α along $(\Lambda)_\alpha$ and starting from σ_α converges to the previous synthesis in the following sense.

Theorem 2 *Assume that $C \neq \pi/4$. As α tends to zero, the time optimal synthesis associated to $(\Lambda)_\alpha$ inside σ_α is characterized by an overlap curve $(Seg)_C^\alpha$, converging to $(Seg)_C$ in the C^0 topology, and, above $(Seg)_C^\alpha$, the control u takes the constant value -1 and below $(Seg)_C^\alpha$, it is equal to 1 . Moreover, there exist only two time optimal trajectories reaching the origin and, in the case $C < \pi/4$, these trajectories start from D_α^ε , $\varepsilon = \pm$, the double points of \mathcal{L}_α .*

Proof of Theorem 2. Fix $C \neq \pi/4$. We first notice that, for α small enough, there are not switching curves inside σ_α . Therefore, the cut-locus may only occur as images by N_α of points $M \in S^2$ such that $M = e^{\frac{t}{\alpha}X} \tilde{\sigma}(s) = e^{\frac{t}{\alpha}X} \tilde{\sigma}(s')$ for $t \in [0, \frac{2\pi}{\alpha}]$, $s \in [0, \pi]$ and $s' \in [\pi, 2\pi]$. Proceeding exactly as in the proof of Theorem 1, we apply inverse function arguments first in neighborhoods of $\sigma_\alpha(0)$ and $\sigma_\alpha(\pi)$ and second in a region enclosed by σ_α excluding such neighborhoods. It is then easy to determine the values of the input u in each connected component of the region enclosed by σ_α minus $(Seg)_C^\alpha$.

By a continuity argument, it is clear that there exist only two time optimal trajectories reaching the origin, one above $(Seg)_C^\alpha$ and one below. Finally, suppose that $C < \pi/4$. In that case, it was proved in [7] that the only extremals starting at a point $\mathcal{L}_\alpha(s)$ and reaching the origin from above the overlap curve $(Seg)_C^\alpha$ correspond to values of s verifying one of the following three possibilities as α tends to zero: (a) s tends to zero, (b) s tends to $\pi/2$, (c) $\mathcal{L}_\alpha(s)$ is a double point also associated to $s' = v(s) - s$. In view of what precedes, only possibility (c) is allowed for optimality. Theorem 2 is proved. \blacksquare

Remark 12 As a consequence of the previous argument and from the results of [7], we get that, for α small enough and $C < \pi/4$,

$$s_2^+(\alpha) = v(s_1^+(\alpha)) - s_1^+(\alpha), \quad s_2^-(\alpha) = 2\pi + v(s_1^-(\alpha) - \pi) - s_1^-(\alpha),$$

where $s_i^\varepsilon(\alpha)$, $i = 1, 2$ $\varepsilon = \pm$, were defined in (68).

6 Case $r(\alpha) = 0$

We assume here that $r(\alpha) = 0$, i.e. $\alpha_k = \frac{\pi}{2k}$ for $k \geq 1$. From Proposition 3, we know that the extremal front at time $(\lfloor \frac{\pi}{2\alpha} \rfloor - 1)\pi = \frac{\pi}{2\alpha} - \pi$, encloses the south pole, is optimal and is approximately (in the C^1 sense) a circle of radius $2r(\alpha)\alpha$ around the south pole. Moreover, at time $\lfloor \frac{\pi}{2\alpha} \rfloor \pi$, we know that the extremal front must contain the south pole and is equal, up to $\mathcal{O}(\alpha^3)$, to $(\alpha^2 \mathcal{L}, -1)^T$ given in (63) and (65) with $C = 0$. In that case, the minimum time front reduces to the south pole.

In this case it is interesting to consider the synthesis starting from the extremal front at time $(k_M - 1)\pi$ and it is natural to compare it with the synthesis of the linear pendulum studied in Section 4.2 and corresponding to $\rho = 2$. Let us first describe briefly that synthesis. Let D_2 and C_2 be the disc and the circle centered at the origin and of radius 2 respectively. The overlap curve inside D_2 coincides with the switching curves and with the trajectories, corresponding to $u = \pm 1$, connecting the points $(\pm 2, 0)$ to the origin. In particular, it means

that an optimal trajectory of the synthesis starting at any point $P \in C_2$ reaches the origin, and thus, there exists an infinite number of optimal trajectories from C_2 to the origin.

For $\alpha > 0$ and $r(\alpha) = 0$, the situation is rather different. Let us first define $\tilde{F}(\alpha, (k_M - 1)\pi)$ to be the image of $\mathcal{F}(\alpha, (k_M - 1)\pi)$ by M_α . Then, for α small enough, it was shown in [7], that the only optimal trajectories starting from $\tilde{F}(\alpha, (k_M - 1)\pi)$ and reaching the origin are those starting at P_+^α and P_-^α . Let us refer to them as γ^+ and γ^- . Therefore, in the case $r(\alpha) = 0$, the synthesis for $\alpha > 0$ is rather different than the synthesis of the limit candidate when α tends to zero. It is a clear indication that the case $r(\alpha) = 0$ is more delicate than the cases $r(\alpha)$ positive constant or $r(\alpha) = C\alpha$. However we are still able to give a partial description of the limit synthesis as the next proposition shows.

Proposition 8 *Assume that $r(\alpha) = 0$ and α is small enough. Then the switching curve $C_{k_M}^+$ (resp. $C_{k_M}^-$) is optimal for some interval $[0, s(\alpha)]$, $s(\alpha) < \pi$, and it is above (resp. below) γ^+ (resp. γ^-) as long as it is optimal. Moreover, we have*

$$\lim_{\alpha \rightarrow 0, r(\alpha)=0} s(\alpha) = \bar{s} := \arccos \sqrt{1/3}. \quad (69)$$

Proof of Proposition 8. We only provide an argument for $C_{k_M}^+$, being the other case analogous. To prove the first statement of the proposition, we reason by contradiction. If the switching curve is not optimal on any interval $[0, \tau]$, $\tau > 0$, we get the existence of an optimal trajectory starting at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ above P_α^+ and reaching the origin, which is equal to the concatenation of an integral curve of X_- and a piece of γ^+ . Therefore, an optimal integral curve of X_- , starting above γ^+ , must either switch or lose optimality before reaching γ^+ . If the second possibility occurs, we must have an overlap i.e., at that point an optimal integral curve of X_+ arrives. Close to P_α^+ , the latter would imply that the optimal integral curve of X_+ starts at $\mathcal{F}(\alpha, (k_M - 1)\pi)$ above P_α^+ . This is impossible because, from every point of $\mathcal{E}^+(\alpha, (k_M - 1)\pi)$, the value of the optimal control is -1 . Let $s(\alpha) \leq \pi$ be the first value of s for which $C_{k_M}^+$ ceases to be optimal. Define

$$H(s) := \det \left(X_+(C_{k_M}^+(s)), \frac{dC_{k_M}^+}{ds}(s), C_{k_M}^+(s) \right),$$

for $s \in [0, \pi]$. Then, $s(\alpha)$ is the smallest solution in $(0, \pi]$ of $H(s) = 0$. It is easy to see that H must take the value zero before π . We deduce that $s(\alpha) < \pi$. By taking the asymptotic expansion of the previous expression as α tends to zero, we get

$$H(s) = \frac{\pi}{4} s_s \alpha^3 (1 + 3 \cos(2s)) + \mathcal{O}(\alpha).$$

Then $s(\alpha)$ must converge to \bar{s} as α tends to zero, the smallest solution in $[0, \pi]$ of $1 + 3 \cos(2s) = 0$. ■

7 Appendix

The following version of the inverse function theorem is used in the argument of Proposition 5.

Theorem 3 *Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a \mathcal{C}^1 map and $\mathcal{K} \subset \mathbb{R}^n$ a compact set such that $\Phi|_{\mathcal{K}} : \mathcal{K} \rightarrow \Phi(\mathcal{K})$ is bijective and the differential $D\Phi(x)$ is invertible for $x \in \mathcal{K}$. Then, there exists an open neighborhood $U \supset \mathcal{K}$ such that $\Phi|_U$ is a \mathcal{C}^1 diffeomorphism.*

Let now $(\Phi_k)_{k \geq 1}$ be a sequence of \mathcal{C}^1 maps converging in the \mathcal{C}_{loc}^1 sense to Φ . Then, for every open set \tilde{U} with closure included in U , there exists \bar{k} such that, for every $k \geq \bar{k}$, $\Phi_{k|\tilde{U}}$ is a \mathcal{C}^1 diffeomorphism and, for every compact subset $\tilde{\mathcal{K}}$ of \tilde{U} , $\Phi(\tilde{\mathcal{K}}) \subset \Phi_k(\tilde{U})$ and $\lim_{k \rightarrow \infty} \Phi_k^{-1}(v) = \Phi^{-1}(v)$ uniformly with respect to $v \in \Phi(\tilde{\mathcal{K}})$.

Proof. Let us define, for $k \geq 0$, the following open neighborhoods of \mathcal{K}

$$A := \{x \in \mathbb{R}^n : \det D\Phi(x) \neq 0\} \quad A_k := \cup_{x \in \mathcal{K}} B\left(x, \frac{1}{k}\right) \cap A.$$

In view of the inverse function theorem, in order to conclude the proof of the first part, it is enough to show that for k large enough the restriction $\Phi|_{A_k}$ is one-to-one.

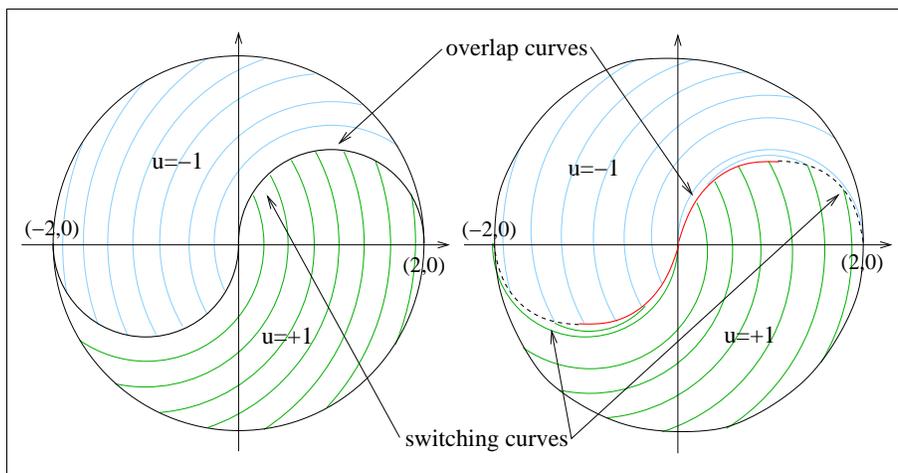


Figure 10: Comparison between the optimal synthesis for the linear pendulum and the optimal synthesis on the bottom of the sphere in the case $r(\alpha) = 0$

We argue by contradiction. Let $x_k \neq y_k \in A_k$ such that $\Phi(x_k) = \Phi(y_k) \quad \forall k$. Then, up to extractions of subsequences, we can assume that the two sequences converge to \bar{x} and \bar{y} respectively. Since $\bar{x}, \bar{y} \in \bigcap_k A_k = \mathcal{K}$ and $\Phi(\bar{x}) = \Phi(\bar{y})$, we deduce that $\bar{x} = \bar{y}$. However, since $\det D\Phi(\bar{x}) \neq 0$, we have that Φ is bijective in a neighborhood of \bar{x} , which contradicts the assumption $\Phi(x_k) = \Phi(y_k)$ for k large enough.

The proof of the second part is similar. First, fix a subset \tilde{U} of U . By the uniform convergence of $D\phi_k$ to $D\phi$ on every compact subset of U , we get $\det D\Phi_k(x) \neq 0$ for every $x \in \tilde{U}$ and k large enough. We also obtain that Φ_k is one-to-one with the same argument as above. For the remaining results to establish, they simply follow from the uniform convergence of Φ_k to Φ on every compact subset of U . ■

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