

Semidefnite Relaxation Bounds for Indefinite Homogeneous Quadratic Optimization

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Abstract

In this paper we study the relationship between the optimal value of a homogeneous quadratic optimization problem and that of its Semidefinite Programming (SDP) relaxation. We consider two quadratic optimization models: (1) $\min\{x^*Cx \mid x^*A_kx \geq 1, x \in \mathbb{F}^n, k = 0, 1, \dots, m\}$; and (2) $\max\{x^*Cx \mid x^*A_kx \leq 1, x \in \mathbb{F}^n, k = 0, 1, \dots, m\}$. If *one* of A_k 's is indefinite while others and C are positive semidefinite, we prove that the ratio between the optimal value of (1) and its SDP relaxation is upper bounded by $O(m^2)$ when \mathbb{F} is the real line \mathbb{R} , and by $O(m)$ when \mathbb{F} is the complex plane \mathbb{C} . This result is an extension of the recent work of Luo *et al.* [8]. For (2), we show that the same ratio is bounded from below by $O(1/\log m)$ for both the real and complex case, whenever all but one of A_k 's are positive semidefinite while C can be indefinite. This result improves the so-called approximate S-Lemma of Ben-Tal *et al.* [2]. We also consider (2) with multiple indefinite quadratic constraints and derive a general bound in terms of the problem data and the SDP solution. Throughout the paper, we present examples showing that all of our results are essentially tight.

Keywords: Quadratic optimization, SDP relaxation, approximation ratio, randomized solution.

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1 Introduction

We consider in this paper homogeneous quadratic optimization problems in either the minimization form

$$\begin{aligned} \min \quad & x^* C x \\ \text{s.t.} \quad & x^* A_k x \geq 1, \quad k = 0, 1, \dots, m \\ & x \in \mathbb{F}^n, \end{aligned} \tag{1.1}$$

or the maximization form

$$\begin{aligned} \max \quad & x^* C x \\ \text{s.t.} \quad & x^* A_k x \leq 1, \quad k = 0, 1, \dots, m \\ & x \in \mathbb{F}^n, \end{aligned} \tag{1.2}$$

where matrices A_k and C are $n \times n$, \mathbb{F} can be the field of real numbers \mathbb{R} or the field of complex numbers \mathbb{C} , and the superscript $*$ represents Hermitian transpose (or regular transpose in case of real numbers). Both of above quadratic optimization problems are NP-hard [8, 2], even when all the data matrices, C and A_k , $k = 1, \dots, m$, are positive semidefinite. Homogeneous quadratic optimization problems (1.1)–(1.2) arise naturally in telecommunications and robust control applications; see [8, 2] and the references therein. A popular approach to approximately solving the NP-hard quadratic programs (1.1)–(1.2) is to use the so-called *Semidefinite Programming* (SDP) relaxations as follows:

$$\begin{aligned} \min \quad & \text{Tr}(CX) \\ \text{s.t.} \quad & \text{Tr}(A_k X) \geq 1, \quad k = 0, 1, \dots, m \\ & X \in \mathbb{SF}_+^n, \end{aligned}$$

and, respectively,

$$\begin{aligned} \max \quad & \text{Tr}(CX) \\ \text{s.t.} \quad & \text{Tr}(A_k X) \leq 1, \quad k = 0, 1, \dots, m \\ & X \in \mathbb{SF}_+^n, \end{aligned}$$

where $\text{Tr}(\cdot)$ represents the trace of a matrix, \mathbb{SF}_+^n denotes the convex cone of positive semidefinite matrices in the space of all (Hermitian) symmetric matrices \mathbb{SF}^n . The above two SDPs are convex and can be solved efficiently via interior point methods. After the SDP relaxation problems are solved, we can apply a *randomization procedure* to the corresponding optimal SDP solutions to extract rank-one feasible solutions for (1.1) and (1.2) respectively. Theoretically, even though the randomized solutions obtained in this manner are not globally optimal for either (1.1) or (1.2), they can be shown to be high quality approximate solutions; see, e.g. [2, 8, 9]. Specifically, Nemirovski *et al.* [9] proved that for the maximization problem (1.2), if all A_k 's are positive semidefinite, then the ratio between the optimal value of the SDP relaxation problem and that of the original quadratic problem is bounded above by $O(\log m)$. More generally, Ben-Tal *et al.* [2] established a so-called *approximate S-Lemma* which shows that the approximation ratio for the SDP relaxation is at most $O(\log(n^2 m))$ when all but one of the matrices A_k , $k = 0, 1, \dots, m$ are positive semidefinite.

In a parallel development, Luo *et al.* [8] considered the homogeneous quadratic optimization in minimization form (1.1). It turns out that the SDP approximation ratio for the minimization version

of the problem takes a quite different form as compared with its maximization counterpart. When all the matrices A_k and C are positive semidefinite, Luo *et al.* [8] showed that the ratio between the original optimal value and the SDP relaxation optimal value is bounded above by $O(m^2)$ when $\mathbb{F} = \mathbb{R}$ and by $O(m)$ when $\mathbb{F} = \mathbb{C}$. All these bounds are shown to be tight in the worst case, although the simulation studies in [8] showed that the ratios are typically close to 1. In other words, the average performance can be much better than the stated worst-case bounds for randomly generated instances. Recently, So *et al.* [11] developed methods for finding approximate low rank solutions for linear matrix inequalities. Their results unify the approximation bounds of Nemirovski *et al.* [9] and Luo *et al.* [8] as special cases (rank being 1), when all the data matrices are positive semidefinite.

In this paper, we study the approximation ratio of the SDP relaxation for homogeneous quadratic optimization problems (1.1)–(1.2) when some of the constraint matrices $\{A_k\}$ are *indefinite*. Our results are as follows. In Section 3, we show that, for the problem in minimization form (1.1), the upper bounds for the approximation ratios of the SDP relaxation as presented in [8] ($O(m^2)$ and $O(m)$ for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ respectively) hold true even when *one* of the constraint matrices is indefinite. If there are more than one indefinite quadratic constraints, we show by an example that the approximation ratio can be infinite. Therefore, our bounds are essentially best possible. In Section 4, we consider the problem in maximization form (1.2). We improve the approximate S-Lemma of Ben-Tal *et al.* [2] by reducing their upper bound on approximation ratio from $O(\log(n^2m))$ to $O(\log m)$ when *one* quadratic inequality is indefinite. In the process of establishing this new bound, we resolve a conjecture by Ben-Tal *et al.* [2] on a possible universal lower bound for the probability that a homogeneous quadratic form of binary i.i.d. Bernoulli random variables lies below its mean. Finally, in Section 5 we present a new and unifying upper bound on the ratio of the optimal value of SDP relaxation over that of the original quadratic maximization problem (1.2) *without* any definiteness assumptions. This new general bound involves the problem data and the SDP optimal solution, which are computable in polynomial time. We also present an example showing that this bound is essentially tight.

2 Estimating Asymmetry of a Random Variable About its Mean

To facilitate the technical analysis in subsequent sections, we establish in this section a bound on the probability for a general random variable to be above (or symmetrically, below) its mean value, using only the high order moment information of the random variable. This problem is of importance on its own in statistics and probability theory. The following lemma is a generalization of Theorem 2.1 in [7].

Lemma 2.1. *Suppose that a random variable Φ satisfies $\mathbb{E}\Phi = 0$, $\text{Var}(\Phi) = 1$ and $\mathbb{E}|\Phi|^t \leq \tau$ for some $t > 2$ and $\tau > 0$. Then $\text{Prob}\{\Phi \geq 0\} > 0.25\tau^{-\frac{2}{t-2}}$ and $\text{Prob}\{\Phi \leq 0\} > 0.25\tau^{-\frac{2}{t-2}}$.*

Proof. Let $p_1 = \text{Prob}\{\Phi \geq 0\}$ and $p_2 = \text{Prob}\{\Phi \leq 0\}$. Also let $Y_1 = \max(\Phi, 0)$ and $Y_2 = -\min(\Phi, 0)$. Since $\mathbb{E}\Phi = 0$, we know $\mathbb{E}Y_1 - \mathbb{E}Y_2 = 0$. Let $s := \mathbb{E}Y_1 = \mathbb{E}Y_2$. By Hölder's inequality it follows that

$(\mathbb{E}Y_1^t)^{1/(t-1)}(\mathbb{E}Y_1)^{(t-2)/(t-1)} \geq \mathbb{E}Y_1^2$ and $(\mathbb{E}Y_2^t)^{1/(t-1)}(\mathbb{E}Y_2)^{(t-2)/(t-1)} \geq \mathbb{E}Y_2^2$. Since $\mathbb{E}Y_1^t + \mathbb{E}Y_2^t = \mathbb{E}|\Phi|^t$, we have

$$\tau \geq \mathbb{E}|\Phi|^t = \mathbb{E}Y_1^t + \mathbb{E}Y_2^t \geq \frac{(\mathbb{E}Y_1^2)^{t-1} + (\mathbb{E}Y_2^2)^{t-1}}{s^{t-2}}.$$

Let $u = \mathbb{E}Y_1^2 \in [0, 1]$. Since $\mathbb{E}Y_1^2 + \mathbb{E}Y_2^2 = \mathbb{E}\Phi^2 = \text{Var}(\Phi) = 1$, it follows that $s^{t-2} \geq \frac{u^{t-1} + (1-u)^{t-1}}{\tau}$. On the other hand, by the Cauchy-Schwartz inequality, we have

$$s^2 = (\mathbb{E}Y_1)^2 = (\mathbb{E}(\text{sign}(Y_1)Y_1))^2 \leq \mathbb{E}(\text{sign}(Y_1)^2)\mathbb{E}Y_1^2 \leq p_1 u$$

which implies that

$$\begin{aligned} p_1 &\geq u^{-1} \left[\frac{u^{t-1} + (1-u)^{t-1}}{\tau} \right]^{\frac{2}{t-2}} \\ &= \frac{(u^{t-1} + (1-u)^{t-1})^{\frac{2}{t-2}}}{u} \tau^{-\frac{2}{t-2}} \\ &\geq (u^{t-1} + (1-u)^{t-1})^{\frac{2}{t-2}} \tau^{-\frac{2}{t-2}} \\ &\geq \left(2 \left(\frac{1}{2} \right)^{t-1} \right)^{\frac{2}{t-2}} \tau^{-\frac{2}{t-2}} \\ &= 0.25 \tau^{-\frac{2}{t-2}}, \end{aligned}$$

where the third inequality follows from the convexity of the function u^{t-1} when $t > 2$. Obviously, the equality can not hold throughout. Therefore, $p_1 > 0.25 \tau^{-\frac{2}{t-2}}$. By symmetry, we also have $p_2 > 0.25 \tau^{-\frac{2}{t-2}}$. \square

In case $t = 4$, Lemma 2.1 asserts that $\text{Prob}\{\Phi \geq 0\} \geq \frac{1}{4\tau}$ and $\text{Prob}\{\Phi \leq 0\} \geq \frac{1}{4\tau}$. However, in this particular case, this specific bound can in fact be further sharpened.

Lemma 2.2. *Suppose that a random variable Φ satisfies $\mathbb{E}\Phi = 0$, $\text{Var}(\Phi) = 1$ and $\mathbb{E}\Phi^4 \leq \tau$. Then $\text{Prob}\{\Phi \geq 0\} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}$ and $\text{Prob}\{\Phi \leq 0\} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}$.*

Proof. It follows from the proof in the Lemma 2.1 that

$$p_1 \geq \frac{u^3 + (1-u)^3}{\tau u} = \left(\frac{1}{u} + 3u - 3 \right) \frac{1}{\tau} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}.$$

By symmetry, $p_2 > \frac{9}{20\tau}$ holds as well. \square

3 Homogenous Quadratic Minimization and SDP Relaxation

Consider the homogeneous quadratic optimization

$$\begin{aligned} v_{qp}^{\min} &:= \min x^* C x \\ \text{s.t. } &x^* A_k x \geq 1, \quad k = 0, 1, \dots, m \\ &x \in \mathbb{F}^n, \end{aligned} \tag{3.1}$$

where $C, A_1, A_2, \dots, A_m \in \mathbb{S}\mathbb{F}^n$ are symmetric matrices. This problem is generally NP-hard [8]. A natural semidefinite programming (SDP) relaxation to the above quadratic optimization problem is

$$\begin{aligned} v_{sdp}^{\min} &:= \min \quad \text{Tr}(CZ) \\ \text{s.t.} \quad &\text{Tr}(A_k Z) \geq 1, \quad k = 0, 1, \dots, m \\ &Z \in \mathbb{S}\mathbb{F}_+^n. \end{aligned} \quad (3.2)$$

Obviously, the SDP relaxation provides a lower bound, i.e., $v_{sdp}^{\min} \leq v_{qp}^{\min}$. In the case $C = I_n$, and A_0, A_1, \dots, A_m are all positive semidefinite, Luo *et al.* [8] proved that $v_{qp}^{\min}/v_{sdp}^{\min} \leq \frac{27(m+1)^2}{\pi}$ for $\mathbb{F} = \mathbb{R}$, and $v_{qp}^{\min}/v_{sdp}^{\min} \leq 8(m+1)$ for $\mathbb{F} = \mathbb{C}$. Moreover, when there are two or more of A_0, A_1, \dots, A_m are indefinite, there is in general no data-independent upper bound on $v_{qp}^{\min}/v_{sdp}^{\min}$, as shown by the following example [8]:

$$\begin{aligned} \min \quad &x_1^2 + x_2^2 \\ \text{s.t.} \quad &x_1^2 \geq 1 \\ &x_1^2 + Mx_1x_2 \geq 1 \\ &x_1^2 - Mx_1x_2 \geq 1 \end{aligned}$$

where $M > 0$ is a constant. In the above example, $v_{sdp}^{\min} = 1$, and the last two constraints imply $x_1^2 \geq M|x_1||x_2| + 1$ which, together with the first constraint $x_2^2 \geq 1$, yield $x_1^2 \geq M|x_1| + 1$ or, equivalently, $|x_1| \geq (M + \sqrt{M^2 + 4})/2$. Therefore, $v_{qp}^{\min} \geq 1 + \frac{1}{4}(M + \sqrt{M^2 + 4})^2$. That is, $v_{qp}^{\min}/v_{sdp}^{\min} \geq 1 + \frac{1}{4}(M + \sqrt{M^2 + 4})^2$, which can be arbitrarily large, depending on the problem data $M > 0$.

In this section, we consider the homogeneous quadratic optimization (3.1) under the assumption that $C, A_1, A_2, \dots, A_m \in \mathbb{S}\mathbb{F}_+^n$ are positive semidefinite while $A_0 \in \mathbb{S}\mathbb{F}^n$ can be indefinite. Throughout this section, we assume that (3.1) is feasible, and that there is $\mu_k \geq 0$, $k = 0, 1, \dots, m$, such that $\sum_{k=0}^m \mu_k A_k \prec 0$. This assumption guarantees that the SDP relaxation is primal feasible while its dual problem satisfies the Slater condition. Hence the strong duality holds and the primal problem (3.2) has an optimal solution that attains its infimum.

Our analysis shall treat the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ separately, leading to different bounds and flavors. For clarity, the analysis will be presented in the next two subsections.

3.1 The real case

Let us start with a useful lemma regarding a lower bound on worst asymmetric mass distributions for a χ^2 -distribution around its mean vector. In fact this result is interesting on its own right.

Lemma 3.1. *Let τ_i be any real numbers, $i = 1, \dots, n$, and let $\eta \sim N(0, I_n)$ be an n -dimensional normal distribution with zero mean and covariance matrix I_n . Then we have*

$$\text{Prob} \left\{ \sum_{i=1}^n \tau_i (\eta_i^2 - 1) \geq 0 \right\} > \frac{3}{100}, \quad \text{Prob} \left\{ \sum_{i=1}^n \tau_i (\eta_i^2 - 1) \leq 0 \right\} > \frac{3}{100}.$$

Proof. Note that $E(\eta_i^2 - 1)^2 = E(\eta_i^4 - 2\eta_i^2 + 1) = 3 - 2 + 1 = 2$. Let $\Psi = \sum_{i=1}^n \tau_i(\eta_i^2 - 1)$, and $\Phi = \frac{\Psi}{\sqrt{2 \sum_{i=1}^n \tau_i^2}}$. Then $E\Phi = 0$ and $\text{Var}(\Phi) = 1$. Since $E(\eta_i^2 - 1)^2 = 2$, and $E(\eta_i^2 - 1)^4 = 60$, direct calculation shows

$$E\Psi^4 = 48 \sum_{i=1}^n \tau_i^4 + 12 \left(\sum_{i=1}^n \tau_i^2 \right)^2 \leq 60 \left(\sum_{i=1}^n \tau_i^2 \right)^2.$$

Therefore, we have

$$E\Phi^4 = \frac{E\Psi^4}{4(\sum_{i=1}^n \tau_i^2)^2} \leq 15.$$

It follows from Lemma 2.2 that $\text{Prob}\{\Phi \geq 0\} > \frac{3}{100}$. Similarly, we have $\text{Prob}\{\Phi \leq 0\} > \frac{3}{100}$ by symmetry. \square

Using Hölder's inequality, we also have $E|\Psi|^3 \leq 60^{\frac{3}{4}} (\sum_{i=1}^n \tau_i^2)^{\frac{3}{2}}$ and $E|\Phi|^3 \leq 15^{\frac{3}{4}}$ which can be used to lower $\text{Prob}\{\Phi \geq 0\}$ (c.f. Theorem 2.1 in [7]). However, in this particular case, the bound so obtained is slightly worse than the one that we derived in Lemma 3.1.

Lemma 3.2. *Let A, Z be two real symmetric matrices with $Z \succeq 0$ and $\text{Tr}(AZ) \geq 0$. Let $\xi \in N(0, Z)$ be a normal random vector with zero mean and covariance matrix Z . Then for any $0 \leq \gamma \leq 1$ we have*

$$\text{Prob}\{\xi^T A \xi < \gamma E(\xi^T A \xi)\} < 1 - \frac{3}{100}.$$

Proof. Let $r = \text{rank}(AZ)$, and $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix such that

$$Q^T (Z^{\frac{1}{2}} A Z^{\frac{1}{2}}) Q = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0).$$

Since $\text{Tr}(AZ) \geq 0$ we have $\sum_{i=1}^r \lambda_i \geq 0$. Let $\bar{\xi} \in N(0, I_n)$ and $\xi := Z^{\frac{1}{2}} Q \bar{\xi}$. Then ξ follows a Gaussian distribution $N(0, Z)$. Moreover, we have $\xi^T A \xi = \sum_{i=1}^r \lambda_i \bar{\xi}_i^2$, where $\bar{\xi}_i$, $i = 1, \dots, r$, are independent and follow the normal distribution $N(0, 1)$. Therefore, we have $E(\xi^T A \xi) = \sum_{i=1}^r \lambda_i$ and

$$\begin{aligned} \text{Prob}\{\xi^T A \xi < \gamma E(\xi^T A \xi)\} &= \text{Prob}\left\{ \sum_{i=1}^r \lambda_i \bar{\xi}_i^2 < \gamma \sum_{i=1}^r \lambda_i \right\} \\ &= \text{Prob}\left\{ \sum_{i=1}^r \lambda_i (\bar{\xi}_i^2 - 1) < (\gamma - 1) \sum_{i=1}^r \lambda_i \right\} \\ &\leq \text{Prob}\left\{ \sum_{i=1}^r \lambda_i (\bar{\xi}_i^2 - 1) < 0 \right\} < 1 - \frac{3}{100}, \end{aligned}$$

where the first inequality follows from $\gamma \in [0, 1]$ and $\sum_{i=1}^r \lambda_i \geq 0$, and the last step is due to Lemma 3.1. \square

Now we are ready to establish the following quality bound for the SDP relaxation. The argument follows closely those of [8].

Theorem 3.3. Consider the real quadratic program (3.1) and its SDP relaxation (3.2), where $\mathbb{F} = \mathbb{R}$. Then, there holds

$$\frac{v_{qp}^{\min}}{v_{sdp}^{\min}} \leq \frac{10^6 m^2}{\pi}.$$

Proof. Let \hat{Z} be an optimal solution of the SDP relaxation (3.2) with rank r satisfying $\frac{(r+1)r}{2} \leq m$. The existence of such matrix solution is well known; cf. Pataki [10]. Moreover, this low rank matrix can be constructed in polynomial-time; cf. [6]. Clearly, $r < \sqrt{2m}$. Since \hat{Z} is feasible, $\text{Tr}(A_0 \hat{Z}) \geq 1$. For any $0 < \gamma \leq 1$ and $\mu > 0$ we have

$$\begin{aligned} & \text{Prob} \left\{ \min_{0 \leq k \leq m} \xi^T A_k \xi \geq \gamma, \xi^T C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} \\ &= \text{Prob} \left\{ \xi^T A_k \xi \geq \gamma \text{ for all } k = 0, 1, \dots, m, \text{ and } \xi^T C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} \\ &\geq \text{Prob} \left\{ \xi^T A_k \xi \geq \gamma \text{Tr}(A_k \hat{Z}) \text{ for all } k = 0, 1, \dots, m, \text{ and } \xi^T C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} \\ &= \text{Prob} \left\{ \xi^T A_k \xi \geq \gamma \mathbb{E}(\xi A_k \xi) \text{ for all } k = 0, 1, \dots, m, \text{ and } \xi^T C \xi \leq \mu \mathbb{E}(\xi^T C \xi) \right\} \\ &\geq 1 - \sum_{k=0}^m \text{Prob} \left\{ \xi^T A_k \xi < \gamma \mathbb{E}(\xi A_k \xi) \right\} - \text{Prob} \left\{ \xi^T C \xi > \mu \mathbb{E}(\xi^T C \xi) \right\}. \end{aligned}$$

Since $A_k \succeq 0$ for $k = 1, \dots, m$, it follows from Lemma 3.1 of [8] that

$$\text{Prob} \left\{ \xi^T A_k \xi < \gamma \mathbb{E}(\xi^T A_k \xi) \right\} \leq \max \left\{ \sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi-2} \right\}.$$

Although A_0 is indefinite, we can use Lemma 3.2 to obtain

$$\text{Prob} \left\{ \xi^T A_0 \xi < \gamma \mathbb{E}(\xi^T A_0 \xi) \right\} < 1 - \frac{3}{100}.$$

Also, since $C \succeq 0$, we can apply Markov inequality to obtain

$$\text{Prob} \left\{ \xi^T C \xi > \mu \mathbb{E}(\xi^T C \xi) \right\} \leq \frac{1}{\mu}.$$

Combining the above estimates yields

$$\text{Prob} \left\{ \min_{0 \leq k \leq m} \xi^T A_k \xi \geq \gamma, \xi^T C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} > \frac{3}{100} - m \max \left\{ \sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi-2} \right\} - \frac{1}{\mu}.$$

Let $\mu = 100$ and $\gamma = \frac{\pi}{10^4 m^2}$. Since $r < \sqrt{2m}$, we have $\sqrt{\gamma} \geq \frac{2(r-1)\gamma}{\pi-2}$. For these values of μ and γ , we have

$$\frac{3}{100} - m \max \left\{ \sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi-2} \right\} - \frac{1}{\mu} = \frac{3}{100} - m \frac{\sqrt{\pi}}{100m} - \frac{1}{100} > \frac{1}{500}.$$

Therefore, there exists a vector $\xi \in \mathbb{R}^n$ such that

$$\xi^T A_k \xi \geq \gamma, \quad k = 0, 1, \dots, m, \quad \text{and} \quad \xi^T C \xi \leq \mu \text{Tr}(C \hat{Z}).$$

Now let $x = \frac{1}{\sqrt{\gamma}}\xi$. Then, $x^T A_k x \geq 1$, $k = 0, 1, \dots, m$, and

$$v_{qp}^{\min} \leq x^T C x = \frac{1}{\gamma} \xi^T C \xi \leq \frac{\mu}{\gamma} \text{Tr}(C \hat{Z}) = \frac{10^6 m^2}{\pi} v_{sdp}^{\min},$$

which establishes the desired bound \square

3.2 The complex case

Recall that the density function of a complex-valued normal distribution¹ $\eta \sim N_c(0, 1)$ is

$$\frac{1}{\pi} e^{-|u|^2}, \forall u \in \mathbb{C}.$$

In polar coordinates, the density function becomes

$$\frac{\rho}{\pi} e^{-\rho^2}, \forall \rho \in [0, +\infty), \theta \in [0, 2\pi).$$

The argument θ is uniformly distributed in $[0, 2\pi)$, and the modulus ρ has the distribution

$$f(\rho) = \begin{cases} 2\rho e^{-\rho^2}, & \text{if } \rho \geq 0; \\ 0, & \text{if } \rho < 0. \end{cases}$$

Thus squared modulus $|\eta|^2$ has the exponential distribution

$$\text{Prob}\{|\eta|^2 \leq \alpha\} \leq 1 - e^{-\alpha}.$$

Lemma 3.4. *For any real numbers τ_i , and i.i.d. exponential random variables η_i with unit variance, $i = 1, \dots, n$, there holds*

$$\text{Prob}\left\{\sum_{i=1}^n \tau_i(\eta_i - 1) \geq 0\right\} > \frac{1}{20}, \quad \text{Prob}\left\{\sum_{i=1}^n \tau_i(\eta_i - 1) \leq 0\right\} > \frac{1}{20}.$$

Proof. Note that $\mathbb{E}(\eta_i - 1)^2 = 1$. Let $\Psi = \sum_{i=1}^n \tau_i(\eta_i - 1)$ and $\Phi = \frac{\Psi}{\sqrt{\sum_{i=1}^n \tau_i^2}}$. Clearly, $\mathbb{E}\Phi = 0$ and $\text{Var}(\Phi) = 1$. Since $\mathbb{E}(\eta_i - 1)^4 = 9$, direct calculation shows

$$\mathbb{E}\Psi^4 = 6 \sum_{i=1}^n \tau_i^4 + 3 \left(\sum_{i=1}^n \tau_i^2 \right)^2 \leq 9 \left(\sum_{i=1}^n \tau_i^2 \right)^2.$$

This further implies

$$\mathbb{E}\Phi^4 = \frac{\mathbb{E}\Psi^4}{(\sum_{i=1}^n \tau_i^2)^2} \leq 9.$$

Using Lemma 2.2 we have $\text{Prob}\{\Phi \geq 0\} > \frac{1}{20}$. Similarly, $\text{Prob}\{\Phi \leq 0\} > \frac{1}{20}$. \square

¹For a discussion on the complex normal distribution and the related references, see Zhang and Huang [13].

Interestingly, it is possible to find a closed formula (see e.g. [4] and [1]) for the above probability. In particular, if all the τ_i 's are distinctive, then

$$\text{Prob} \left\{ \sum_{i=1}^n \tau_i (\eta_i - 1) \geq 0 \right\} = \sum_{i=1}^n \frac{e^{-\frac{1}{\tau_i}}}{\prod_{j \neq i} \left(1 - \frac{\tau_j}{\tau_i}\right)}.$$

Therefore, we have

$$\frac{1}{20} < \sum_{i=1}^n \frac{e^{-\frac{1}{\tau_i}}}{\prod_{j \neq i} \left(1 - \frac{\tau_j}{\tau_i}\right)} < \frac{19}{20}$$

for any distinctive real values τ_i , $i = 1, \dots, n$.

In fact, we conjecture that the following tighter inequalities

$$\frac{1}{e} < \sum_{i=1}^n \frac{e^{-\frac{1}{\tau_i}}}{\prod_{j \neq i} \left(1 - \frac{\tau_j}{\tau_i}\right)} < \frac{e-1}{e}, \quad (3.3)$$

hold for any real values τ_i , $i = 1, \dots, n$. Inequality (3.3) can be shown to hold for $n = 2, 3$. It also admits a geometric interpretation. Specifically, let us consider the joint exponential distribution on \mathbb{R}_+^n with density $e^{-\sum_{i=1}^n x_i}$. Then, the mean vector of this distribution, or equivalently, the center of gravity of \mathbb{R}_+^n is $x^c := (1, 1, \dots, 1)^T$. Given any real numbers τ_i , $i = 1, \dots, n$, the set

$$\mathcal{H} = \left\{ (\eta_1, \eta_2, \dots, \eta_n)^T \left| \sum_{i=1}^n \tau_i (\eta_i - 1) = 0 \right. \right\}$$

represents a hyperplane passing through x^c . If we let \mathcal{H}_+ denote the half space in \mathbb{R}^n created by the positive side of \mathcal{H} , then inequality (3.3) can be interpreted as follows:

$$\text{Prob}(\mathbb{R}_+^n \cap \mathcal{H}_+) \geq e^{-1}, \quad \text{for any hyperplane } \mathcal{H} \text{ passing through } x^c.$$

Interestingly, the well-known theorem of Grünbaum [5] can also be viewed from this perspective: for any bounded convex body $\mathcal{C} \subset \mathbb{R}^n$, if we assign the uniform distribution to \mathcal{C} , then the mean vector of this distribution is given by the center of gravity

$$x^c = \frac{1}{\text{Volume}(\mathcal{C})} \int_{\mathcal{C}} dx;$$

as a result, if we consider any hyperplane \mathcal{H} passing through x^c and let \mathcal{H}_+ denote the positive side of the hyperplane, then Grünbaum inequality

$$\text{Volume}(\mathcal{C} \cap \mathcal{H}_+) \geq e^{-1} \text{Volume}(\mathcal{C})$$

can be written as

$$\text{Prob}(\mathcal{C} \cap \mathcal{H}_+) \geq e^{-1}, \quad \text{for any hyperplane } \mathcal{H} \text{ passing through } x^c.$$

Thus, inequality (3.3) can be viewed as an extension of Grünbaum's theorem to the exponential distribution over the unbounded convex set $\mathcal{C} = \mathbb{R}_+^n$.

Lemma 3.5. *Let A, Z be two Hermitian matrices satisfying $Z \succeq 0$ and $\text{Tr}(AZ) \geq 0$. Let $\xi \sim N_c(0, Z)$ be a complex normal random vector. Then, for any $0 \leq \gamma \leq 1$, we have*

$$\text{Prob} \{ \xi^* A \xi < \gamma \mathbb{E}(\xi^* A \xi) \} < 1 - \frac{1}{20}.$$

Proof. Let $Q \in \mathbb{C}^{n \times n}$ be an unitary matrix such that

$$Q^*(Z^{\frac{1}{2}}AZ^{\frac{1}{2}})Q = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$$

where $r = \text{rank}(AZ)$. Since $\text{Tr}(AZ) \geq 0$, it follows that $\sum_{i=1}^r \lambda_i \geq 0$. Let $\hat{\xi} \in \mathbb{C}^n$ be a random Gaussian vector drawn from the complex normal distribution $N_c(0, I_n)$. Then the random vector $\xi = Z^{\frac{1}{2}}Q\hat{\xi}$ follows the Gaussian distribution $N_c(0, Z)$. As a result, there holds

$$\begin{aligned} \text{Prob} \{ \xi^* A \xi < \gamma \mathbb{E}(\xi^* A \xi) \} &= \text{Prob} \left\{ \sum_{i=1}^r \lambda_i |\hat{\xi}_i|^2 < \gamma \sum_{i=1}^n \lambda_i \right\} \\ &= \text{Prob} \left\{ \sum_{i=1}^n \lambda_i (|\hat{\xi}_i|^2 - 1) < (\gamma - 1) \sum_{i=1}^n \lambda_i \right\} \\ &\leq \text{Prob} \left\{ \sum_{i=1}^n \lambda_i (|\hat{\xi}_i|^2 - 1) < 0 \right\}, \end{aligned}$$

where the last step follows from $\gamma \in [0, 1]$ and $\sum_{i=1}^r \lambda_i \geq 0$. Since $|\xi_i|^2$ is exponentially distributed, by Lemma 3.4, we have

$$\text{Prob} \left\{ \sum_{i=1}^n \lambda_i (|\hat{\xi}_i|^2 - 1) \geq 0 \right\} > \frac{1}{20}$$

which proves the lemma. \square

Theorem 3.6. *Consider (3.1) and (3.2), where $\mathbb{F} = \mathbb{C}$. Then*

$$\frac{v_{qp}^{\min}}{v_{sdp}^{\min}} \leq 2400m.$$

Proof. It is known that in this case, if v_{sdp}^{\min} is finite and $m \leq 3$, then $v_{qp}^{\min}/v_{sdp}^{\min} = 1$ (cf. e.g. [6] and [12]). Below we shall only consider the case where $m \geq 4$. Let \hat{Z} be a low rank optimal solution of the SDP relaxation (3.2), such that $r = \text{rank}(\hat{Z}) \leq \sqrt{m}$ (see [6], §5). The feasibility of \hat{Z} implies that $\text{Tr}(A_0 \hat{Z}) \geq 1$. Similar to Theorem 3.3, we can use the union bound to obtain the following inequality

$$\begin{aligned} &\text{Prob} \left\{ \min_{0 \leq k \leq m} \xi^* A_k \xi \geq \gamma, \xi^* C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} \\ &\geq 1 - \sum_{k=0}^m \text{Prob} \{ \xi^* A_k \xi < \gamma \mathbb{E}(\xi^* A_k \xi) \} - \text{Prob} \{ \xi^* C \xi > \mu \mathbb{E}(\xi^* C \xi) \}. \end{aligned}$$

Since $A_k \succeq 0$, $k = 1, \dots, m$, it follows from Lemma 3.4 in [8] that

$$\text{Prob} \{ \xi^* A_k \xi < \gamma E(\xi^* A_k \xi) \} \leq \max \left\{ \frac{4}{3} \gamma, 16(r-1)^2 \gamma^2 \right\}.$$

Although A_0 is indefinite, Lemma 3.5, asserts that

$$\text{Prob} \{ \xi^* A_0 \xi < \gamma E(\xi^* A_0 \xi) \} < 1 - \frac{1}{20}.$$

Therefore, combining these estimates and using Markov inequality, we have

$$\text{Prob} \left\{ \min_{0 \leq k \leq m} \xi^* A_k \xi \geq \gamma, \xi^* C \xi \leq \mu, \text{Tr}(C \hat{Z}) \right\} > \frac{1}{20} - m \max \left\{ \frac{4}{3} \gamma, 16(r-1)^2 \gamma^2 \right\} - \frac{1}{\mu}.$$

Now choose $\mu = 60$ and $\gamma = \frac{1}{40m}$. In this case, $\frac{4}{3} \gamma \geq 16(r-1)^2 \gamma^2$. We also have a strict lower bound of the above probability

$$\text{Prob} \left\{ \min_{0 \leq k \leq m} \xi^* A_k \xi \geq \gamma, \xi^* C \xi \leq \mu \text{Tr}(C \hat{Z}) \right\} > 0.$$

This implies that there exists $\xi \in \mathbb{C}^n$ such that

$$\xi^* A_k \xi \geq \gamma, \quad k = 0, 1, \dots, m; \quad \xi^* C \xi \leq \mu \text{Tr}(C \hat{Z}).$$

Now let $x := \frac{1}{\sqrt{\gamma}} \xi$. Then $x^* A_k x \geq 1$, $k = 0, 1, \dots, m$, and so

$$v_{qp}^{\min} \leq x^* C x \leq \frac{\xi^* C \xi}{\gamma} \leq \frac{\mu \text{Tr}(C \hat{Z})}{\gamma} = 2400m \cdot v_{sdp}^{\min}.$$

The theorem is proven. □

Notice that there are examples (see [8]) which show that the worst-case ratios of $v_{qp}^{\min}/v_{sdp}^{\min}$ are indeed $O(m^2)$ and $O(m)$ in the real and complex case respectively, even in the absence of indefinite constraint $x^* A_0 x \geq 1$. Thus, the bounds of Theorems 3.3 and 3.6 are essentially tight.

Finally, we may also wonder what happens if there are more than one indefinite quadratic constraint. The following example shows that in this case the SDP relaxation does not admit any *finite* quality bound.

Example 3.7.

$$\begin{aligned} \min \quad & x_4^2 \\ \text{s.t.} \quad & x_1 x_2 + x_3^2 + x_4^2 \geq 1 \\ & -x_1 x_2 + x_3^2 + x_4^2 \geq 1 \\ & \frac{1}{2} x_1^2 - x_3^2 \geq 1 \\ & \frac{1}{2} x_2^2 - x_3^2 \geq 1 \\ & x_1, x_2, x_3, x_4 \in \mathbb{R}. \end{aligned}$$

The first two constraints are equivalent to $|x_1x_2| \leq x_3^2 + x_4^2 - 1$. At the same time, the last two constraints imply $|x_1x_2| \geq 2(x_3^2 + 1)$. Combining these two inequalities yields

$$x_3^2 + x_4^2 - 1 \geq 2(x_3^2 + 1),$$

which further implies $x_4^2 \geq 3$. Therefore, we must have $v_{qp}^{\min} \geq 3$ in this case. However,

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is feasible for the corresponding SDP relaxation problem and attains an objective value of 0. Thus, it must be optimal and thus $v_{sdp}^{\min} = 0$. Hence, $v_{qp}^{\min}/v_{sdp}^{\min} = \infty$ in this case.

4 Quadratic Maximization and the Approximate S -Lemma

In this section, we consider the nonconvex homogeneous quadratic optimization in the maximization form

$$\begin{aligned} v_{qp}^{\max} &:= \max && x^*Cx \\ \text{s.t.} &&& x^*A_kx \leq 1, \quad k = 0, 1, \dots, m \\ &&& x \in \mathbb{F}^n, \end{aligned} \tag{4.1}$$

where $A_k \in \mathbb{SF}_+^n$, $k = 1, \dots, m$, are positive semidefinite, while $C, A_0 \in \mathbb{SF}^n$ may be indefinite. For convenience, from now on we shall focus on the case $\mathbb{F} = \mathbb{R}^n$. Unlike the case of minimization form, this choice does not significantly affect the quality of SDP approximation ratios, since in the complex case the bounds are of the same order of magnitude. We assume that there is $\mu_k \geq 0$, $k = 0, 1, \dots, m$, such that

$$\sum_{k=0}^m \mu_k A_k \succ 0.$$

Under this condition, the SDP relaxation satisfies the dual Slater condition. Thus the primal-dual optimal solutions exist and the primal-dual optimal objective values are attainable. Let the SDP relaxation optimal value be

$$\begin{aligned} v_{sdp}^{\max} &:= \max && \text{Tr}(CX) \\ \text{s.t.} &&& \text{Tr}(A_kX) \leq 1, \quad k = 0, 1, \dots, m \\ &&& X \succeq 0. \end{aligned} \tag{4.2}$$

Obviously $v_{qp}^{\max} \leq v_{sdp}^{\max}$.

Lemma 4.1. *Let w_{ij} ($1 \leq i < j \leq n$) be any real numbers, and ξ_i ($1 \leq i \leq n$) be random variables such that $\text{Prob}\{\xi_i = -1\} = \text{Prob}\{\xi_i = 1\} = 0.5$. Then there holds*

$$\text{Prob} \left\{ \sum_{1 \leq i < j \leq n} w_{ij} \xi_i \xi_j \leq 0 \right\} > \frac{1}{87}.$$

Proof. Let $\Psi = \sum_{1 \leq i < j \leq n} w_{ij} \xi_i \xi_j$. Then $\mathbb{E} \Psi = 0$, $\mathbb{E}(\Psi^2) = \sum_{1 \leq i < j \leq n} w_{ij}^2$ and

$$\mathbb{E}(\Psi^4) = \sum_{1 \leq i < j \leq n} w_{ij}^4 + 6 \sum_{(i,j) < (k,\ell)} w_{ij}^2 w_{k\ell}^2 + W$$

where $(i, j) < (k, \ell)$ means $i < k$ or $i = k$ and $j < \ell$, and

$$\begin{aligned} W &= 24 \sum_{1 \leq i < j < k < \ell \leq n} (w_{ij} w_{ik} w_{j\ell} w_{k\ell} + w_{ij} w_{i\ell} w_{jk} w_{k\ell} + w_{ik} w_{i\ell} w_{jk} w_{j\ell}) \\ &\leq 6 \sum_{1 \leq i < j < k < \ell \leq n} ((w_{ij}^2 + w_{k\ell}^2)(w_{ik}^2 + w_{j\ell}^2) + (w_{ij}^2 + w_{k\ell}^2)(w_{i\ell}^2 + w_{jk}^2) + (w_{ik}^2 + w_{j\ell}^2)(w_{i\ell}^2 + w_{jk}^2)) \\ &\leq 36 \left(\sum_{1 \leq i < j \leq n} w_{ij}^2 \right)^2. \end{aligned}$$

Therefore we have $\mathbb{E}(\Psi^4) \leq 39(\sum_{1 \leq i < j \leq n} w_{ij}^2)^2$, since

$$\sum_{1 \leq i < j \leq n} w_{ij}^4 + 6 \sum_{(i,j) < (k,\ell)} w_{ij}^2 w_{k\ell}^2 \leq 3 \left(\sum_{1 \leq i < j \leq n} w_{ij}^2 \right)^2.$$

Now let $\Phi = \frac{\Psi}{\sqrt{\sum_{1 \leq i < j \leq n} w_{ij}^2}}$. Then $\mathbb{E}(\Phi) = 0$, $\text{Var}(\Phi) = 1$ and $\mathbb{E}(\Phi^4) \leq 39$. By Lemma 2.2, we have

$$\text{Prob} \{ \Phi \leq 0 \} > \frac{1}{87}.$$

The desired result follows. \square

Lemma 4.1 settles in the affirmative an open question of Ben-Tal *et al.* [2, Conjecture A.5] who conjectured that

$$\text{Prob} \left\{ \sum_{1 \leq i < j \leq n} w_{ij} \xi_i \xi_j \leq 0 \right\} \geq \frac{1}{4}, \quad \forall w_{ij},$$

except that we have a smaller constant of $1/87$. The above inequality was needed to establish the so called *approximate S-Lemma* — an extension of the well-known *S-Lemma*, which is important in the context of robust optimization and is closely related to our analysis in this section. In their work [9], Ben-Tal *et al.* derived a weaker lower bound of $1/8n^2$, which goes to zero as $n \rightarrow \infty$.

We can now use Lemma 4.1 to analyze the performance of SDP relaxation for (4.2). Let $\hat{X} = UU^T$ be one optimal solution of (4.2), where $U \in \mathbb{R}^{n \times r}$ and $r = \text{rank}(\hat{X})$. Suppose $Q \in \mathbb{R}^{n \times r}$ is the orthogonal matrix such that $\hat{C} := Q^T U^T C U Q$ is diagonal. Let ξ_k , $k = 1, \dots, r$, be i.i.d. random variables taking values -1 or 1 with equal probabilities, and let

$$x(\xi) := \frac{1}{\sqrt{\max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi}} U Q \xi,$$

where $\hat{A}_k = Q^T U^T A_k U Q$. Note that the above random vector $x(\xi)$ is always well-defined, since the assumption $\sum_{k=0}^m \mu_k A_k \succ 0$ implies

$$\max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi > 0 \text{ for any } \xi \neq 0.$$

Let $\mu = \min\{m, \max_i \text{rank}(A_i \hat{X})\}$. We have the following estimate of the SDP approximation ratio.

Theorem 4.2. *There holds*

$$v_{qp}^{\max} \leq v_{sdp}^{\max} \leq 2 \log(174 m \mu) v_{qp}^{\max}.$$

Proof. Notice that $\hat{C} = Q^T U^T C U Q$ is diagonal and hence

$$x(\xi)^T C x(\xi) = \frac{1}{\max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi} \xi^T Q^T U^T C U Q \xi = \frac{1}{\max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi} \text{Tr}(CX).$$

Therefore for any $\alpha > 1$ we have

$$\begin{aligned} & \text{Prob} \left\{ x(\xi)^T C x(\xi) \geq \frac{1}{\alpha} \text{Tr}(CX) \right\} \\ &= \text{Prob} \left\{ \max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi \leq \alpha \right\} \\ &= 1 - \text{Prob} \left\{ \max_{0 \leq k \leq m} \xi^T \hat{A}_k \xi > \alpha \right\} \\ &\geq 1 - \text{Prob} \left\{ \max_{1 \leq k \leq m} \xi^T \hat{A}_k \xi > \alpha \right\} - \text{Prob} \left\{ \xi^T \hat{A}_0 \xi > \alpha \right\}. \end{aligned}$$

Since $\text{Tr}(A_0) \leq 1$ and so $\alpha - \text{Tr}(A_0) \geq 0$, it follows from Lemma 4.1 that

$$\text{Prob} \left\{ \xi^T \hat{A}_0 \xi > \alpha \right\} \leq \text{Prob} \left\{ \sum_{1 \leq i < j \leq m} (\hat{A}_0)_{ij} \xi_i \xi_j > 0 \right\} < 1 - \frac{1}{87}.$$

Since $\hat{A}_k \succeq 0$ for $k = 1, \dots, m$, and $\text{Tr}(\hat{A}_k) \leq 1$, it follows from (12) in [9] that

$$\text{Prob} \left\{ \max_{1 \leq k \leq m} \xi^T \hat{A}_k \xi > \alpha \right\} < 2m\mu e^{-\frac{1}{2}\alpha}.$$

Hence we have

$$\text{Prob} \left\{ x(\xi)^T C x(\xi) \geq \frac{1}{\alpha} \text{Tr}(CX) \right\} > \frac{1}{87} - 2m\mu e^{-\frac{1}{2}\alpha}.$$

Letting $\alpha = 2 \log(174 m \mu)$ ensures the above probability to be positive. Therefore, there exists a random vector ξ such that $\text{Tr}(CX) \leq \alpha x(\xi)^T C x(\xi)$, and the theorem is proven. \square

We point out that Theorem 4.2 is an improvement of the so-called approximate S -Lemma of Ben-Tal, Nemirovski, and Roos [2] (Lemma A.6). In particular, Ben-Tal *et al.* showed that $\alpha \leq 2 \log(16n^2 m \mu)$, in contrast to our bound $\alpha \leq 2 \log(174 m \mu)$.

Notice that in (4.1) there is only one indefinite inequality. A natural question arises: can we allow more? The following example shows that the answer is “no” if we wish to have a data-independent worst-case approximation ratio. (Data-dependent approximation ratio bounds will be discussed in Section 5 where we do allow multiple indefinite constraints.)

Example 4.3. Consider

$$\begin{aligned} \max \quad & x_1^2 + \frac{1}{M}x_2^2 \\ \text{s.t.} \quad & Mx_1x_2 + x_2^2 \leq 1 \\ & -Mx_1x_2 + x_2^2 \leq 1 \\ & M(x_1^2 - x_2^2) \leq 1, \end{aligned}$$

where $M > 0$ is an arbitrarily large positive constant. Its SDP relaxation is

$$\begin{aligned} \max \quad & X_{11} + \frac{1}{M}X_{22} \\ \text{s.t.} \quad & MX_{12} + X_{22} \leq 1, -MX_{12} + X_{22} \leq 1, M(X_{11} - X_{22}) \leq 1 \\ & \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0. \end{aligned}$$

For this quadratic program, the first two constraints imply that $|x_1x_2| \leq \frac{1-x_2^2}{M} \leq \frac{1}{M}$ and so $x_1^2 \leq \frac{1}{M^2x_2^2}$. The third inequality assures that $x_1^2 \leq \frac{1}{M} + x_2^2$. Therefore, $x_1^2 \leq \min \left\{ \frac{1}{M^2x_2^2}, \frac{1}{M} + x_2^2 \right\} \leq \frac{\sqrt{5}+1}{2M} \approx \frac{1.618}{M}$. Moreover, $x_2^2 \leq 1$, and so $v_{qp}^{\max} \leq \frac{2.618}{M}$.

The SDP relaxation satisfies both primal and dual Slater conditions, so the primal-dual optimal solutions exist. A feasible solution for the SDP relaxation (primal problem) is the 2 by 2 identity matrix, with the objective value being $1 + \frac{1}{M} > 1$. On the other hand, since $X_{22} \leq M|X_{12}| + X_{22} \leq 1$, and $X_{11} \leq X_{22} + \frac{1}{M}$, an upper bound for the SDP optimal value is $1 + \frac{2}{M}$. Therefore, for this example, the ratio $\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \geq \frac{M}{2.618} \approx 0.382M$, which can be arbitrarily large, depending on the size of M .

If there are at most two homogeneous quadratic constraints, and moreover if the SDP relaxation has a primal-dual complementary optimal solution, then the SDP optimal value will be equal to the optimal value of the quadratic model; see e.g. Ye and Zhang [12] (Corollary 2.6). In other words, if there are no more than two inequality constraints, then under the primal-dual Slater condition, we will have $v_{sdp}^{\max}/v_{qp}^{\max} = 1$. In this sense, Example 4.3 is the smallest possible in size. By removing the requirement that the SDP relaxation has a finite optimal value, then it is possible to construct an example which involves only two inequality constraints.

Example 4.4. Consider

$$\begin{aligned} \max \quad & x_1x_2 + x_1^2 \\ \text{s.t.} \quad & x_1x_2 \leq 1 \\ & x_1^2 - x_2^2 \leq 1, \end{aligned}$$

with the SDP relaxation

$$\begin{aligned} \max \quad & X_{12} + X_{11} \\ \text{s.t.} \quad & X_{12} \leq 1, X_{11} - X_{22} \leq 1, \\ & \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0. \end{aligned}$$

In terms of polar coordinates, $(x_1, x_2) \longrightarrow (r \cos \theta, r \sin \theta)$, the original quadratic problem can be turned into

$$\begin{aligned} \max \quad & r^2(\sin 2\theta + \cos 2\theta + 1)/2 \\ \text{s.t.} \quad & r^2 \sin 2\theta \leq 2 \\ & r^2 \cos 2\theta \leq 1. \end{aligned}$$

By a further change of variables $(r^2 \cos 2\theta, r^2 \sin 2\theta) \longrightarrow (y_1, y_2)$, we can reformulate the original quadratic problem as

$$\begin{aligned} \max \quad & \frac{1}{2} (y_1 + y_2 + \sqrt{y_1^2 + y_2^2}) \\ \text{s.t.} \quad & y_1 \leq 2 \\ & y_2 \leq 1. \end{aligned}$$

This optimization problem has a unique optimal solution at $(y_1^*, y_2^*) = (2, 1)$ with the optimal value being $\frac{3+\sqrt{5}}{2} \approx 2.618$. The SDP relaxation problem is clearly unbounded, as any positive multiple of the identity matrix is feasible. Therefore, $v_{sdp}^{\max}/v_{qp}^{\max} = +\infty$. This example is possible because the dual of the SDP relaxation problem is infeasible.

5 Quadratic Optimization with Multiple Indefinite Constraints

Unlike the minimization form (1.1) for which the SDP approximation ratio can be infinite when there are more than one indefinite constraints (see Example 3.7), the maximization form (1.2) can still admit a finite SDP approximation ratio in this case. In particular, consider a general homogeneous quadratic maximization problem

$$\begin{aligned} \max \quad & x^T C x \\ \text{s.t.} \quad & x^T A_k x \leq 1, k = 0, 1, \dots, m \\ & x \in \mathbb{F}^n. \end{aligned} \tag{5.1}$$

Suppose that \mathcal{I}, \mathcal{D} are two index sets, $\mathcal{I} \cup \mathcal{D} = \{0, 1, \dots, m\}$ and $\mathcal{I} \cap \mathcal{D} = \emptyset$, such that $A_k \succeq 0$ for $k \in \mathcal{D}$ and A_k indefinite for $k \in \mathcal{I}$. The SDP relaxation for (5.1) is

$$\begin{aligned} \max \quad & \text{Tr}(CX) \\ \text{s.t.} \quad & \text{Tr}(A_k X) \leq 1, k = 0, 1, \dots, m \\ & X \succeq 0. \end{aligned} \tag{5.2}$$

We begin our analysis with a technical lemma which bounds the probability of an exponential tail. Similar bounds exist in the literature, e.g. [3]. However, the lemma below serves our needs exactly; for completeness we include a proof here.

Lemma 5.1. Let $\{\lambda_i\}_{i=1}^n$ be any given real numbers and $\{\eta_i\}_{i=1}^n$ be i.i.d. random variables drawn from either the real or complex valued zero mean Gaussian distribution with unit variance. Let $\sigma = \sqrt{\sum_{i=1}^n \lambda_i^2}$ and $\delta = \max\{\max\{\lambda_i \mid 1 \leq i \leq n\}, 0\}$. Then, for any $\alpha > 0$ there holds

$$\text{Prob} \left\{ \sum_{i=1}^n \lambda_i \eta_i^2 - \sum_{i=1}^n \lambda_i \geq \alpha \sigma \right\} \leq \begin{cases} \exp \left(-\min \left\{ \alpha, \frac{\sigma}{\delta} \right\} \frac{\alpha}{8} \right), & \text{if } \eta_i \sim N(0, 1) \text{ is real Gaussian,} \\ \exp \left(-\min \left\{ \alpha, \frac{\sigma}{\delta} \right\} \frac{\alpha}{4} \right), & \text{if } \eta_i \sim N_c(0, 1) \text{ is complex Gaussian.} \end{cases}$$

Proof. We will only prove the real Gaussian case; the complex case is similar and therefore omitted. Let $\beta := \frac{1}{4} \min\{\frac{1}{\delta}, \frac{\alpha}{\sigma}\}$. Then, $2\beta\lambda_i \leq 1/2$ for all $i = 1, \dots, n$, and $\beta\sigma = \frac{1}{4} \min\{\frac{\sigma}{\delta}, \alpha\}$. Note that for any $t \leq 1/2$ the following inequality holds:

$$\frac{1}{1-t} \leq e^{t+t^2}. \quad (5.3)$$

Let $\zeta := e^{\beta \sum_{i=1}^n \lambda_i \eta_i^2}$. Since $\{\eta_i^2\}_{i=1}^n$ are standard i.i.d. χ^2 random variables, it follows that

$$\mathbb{E}(\zeta) = \prod_{i=1}^n \mathbb{E} \left(e^{\beta \lambda_i \eta_i^2} \right) = \prod_{i=1}^n \frac{1}{\sqrt{1-2\beta\lambda_i}} = \left(\prod_{i=1}^n \frac{1}{1-2\beta\lambda_i} \right)^{\frac{1}{2}} \leq \left(\prod_{i=1}^n e^{2\beta\lambda_i + 4\beta^2\lambda_i^2} \right)^{\frac{1}{2}} = e^{2\beta^2\sigma^2 + \beta \sum_{i=1}^n \lambda_i}$$

where the inequality is due to (5.3). This together with the Markov inequality implies

$$\begin{aligned} \text{Prob} \left\{ \sum_{i=1}^n \lambda_i \eta_i^2 - \sum_{i=1}^n \lambda_i \geq \alpha \sigma \right\} &= \text{Prob} \left\{ \zeta \geq e^{\beta(\alpha\sigma + \sum_{i=1}^n \lambda_i)} \right\} \\ &\leq \frac{\mathbb{E}(\zeta)}{e^{\beta(\alpha\sigma + \sum_{i=1}^n \lambda_i)}} \\ &\leq e^{2\beta^2\sigma^2 - \beta\sigma\alpha} = e^{\beta\sigma(2\beta\sigma - \alpha)} \leq e^{\beta\sigma(\frac{\alpha}{2} - \alpha)} \\ &= e^{-\min\{\alpha, \frac{\sigma}{\delta}\} \frac{\alpha}{8}}. \end{aligned}$$

The lemma is proven. \square

We are now ready to pursue the performance analysis for the real case $\mathbb{F} = \mathbb{R}$. Assume that (5.2) has an optimal solution \hat{X} . Denote the set of (real) eigenvalues of $A_k \hat{X}$ as $\lambda_1^k, \dots, \lambda_n^k$, $k = 0, 1, \dots, m$. Since $\text{Tr}(A_k \hat{X}) \leq 1$, it follows that $\sum_{i=1}^n \lambda_i^k \leq 1$. Moreover, $\|A_k \hat{X}\|_F^2 = \sum_{i=1}^n (\lambda_i^k)^2$, $k = 0, 1, \dots, m$, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Let ξ be a random vector drawn from the Gaussian distribution $N(0, \hat{X})$. For any $\alpha > 1$ and $0 \leq k \leq m$, we consider the probability of the event $\text{Prob}\{\xi^T A_k \xi > \alpha\}$. By diagonalization, we have $\text{Prob}\{\xi^T A_k \xi > \alpha\} = \text{Prob}\{\sum_{i=1}^n \lambda_i^k \eta_i^2 > \alpha\}$, where $\eta = (\eta_1, \dots, \eta_n)^T$ is a random vector following the normal distribution $N(0, I_n)$.

If we let $\sigma^k := \sqrt{\sum_{i=1}^n (\lambda_i^k)^2} = \|A_k \hat{X}\|_F$, and $\delta^k := \max\{0, \max\{\lambda_i^k \mid 1 \leq i \leq n\}\}$, then Lemma 5.1 leads to

$$\text{Prob}\{\xi^T A_k \xi > \alpha\} \leq \exp \left(-\min \left\{ \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{\sigma^k}, \frac{\sigma^k}{\delta^k} \right\} \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{8\sigma^k} \right), \quad \forall 0 \leq k \leq m. \quad (5.4)$$

Alternatively, we can bound the tail probability using Chebyshev's inequality. In particular, since $\text{Var}(\sum_{i=1}^n \lambda_i^k \eta_i^2) = 2 \sum_{i=1}^n (\lambda_i^k)^2 = 2 \|A_k \hat{X}\|_F^2$, it follows from Chebyshev's inequality

$$\begin{aligned} \text{Prob} \left\{ \sum_{i=1}^n \lambda_i^k \eta_i^2 > \alpha \right\} &= \text{Prob} \left\{ \sum_{i=1}^n \lambda_i^k \eta_i^2 - \sum_{i=1}^n \lambda_i^k > \alpha - \sum_{i=1}^n \lambda_i^k \right\} \\ &\leq \text{Prob} \left\{ \left| \sum_{i=1}^n \lambda_i^k \eta_i^2 - \sum_{i=1}^n \lambda_i^k \right| > \alpha - \sum_{i=1}^n \lambda_i^k \right\} \\ &\leq \frac{\text{Var}(\sum_{i=1}^n \lambda_i^k \eta_i^2)}{(\alpha - \sum_{i=1}^n \lambda_i^k)^2} \leq \frac{2 \|A_k \hat{X}\|_F^2}{(\alpha - 1)^2}, \quad \forall 0 \leq k \leq m, \end{aligned} \quad (5.5)$$

where we have used the fact $\alpha > 1 \geq \sum_{i=1}^n \lambda_i^k$. Applying Lemma 3.1 and using (5.5)–(5.4) gives

$$\begin{aligned} &\text{Prob} \left\{ \xi^T A_k \xi \leq \alpha, k = 0, 1, \dots, m; \xi^T C \xi \geq \text{Tr}(C \hat{X}) \right\} \\ &\geq 1 - \text{Prob} \left\{ \xi^T C \xi < \text{Tr}(C \hat{X}) \right\} - \sum_{k=0}^m \text{Prob} \left\{ \xi^T A_k \xi > \alpha \right\} \\ &\geq \frac{3}{100} - \sum_{k=0}^m \min \left\{ \exp \left(- \min \left\{ \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{\sigma^k}, \frac{\sigma^k}{\delta^k} \right\} \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{8\sigma^k} \right), \frac{2 \|A_k \hat{X}\|_F^2}{(\alpha - 1)^2} \right\}. \end{aligned}$$

Notice that $\delta^k \leq \sigma^k$ and $\sum_{i=1}^n \lambda_i^k \leq 1$ for any k . Therefore, we have, for any $\alpha > 1$,

$$\begin{aligned} &\text{Prob} \left\{ \xi^T A_k \xi \leq \alpha, k = 0, 1, \dots, m; \xi^T C \xi \geq \text{Tr}(C \hat{X}) \right\} \\ &\geq \frac{3}{100} - \sum_{i \in \mathcal{D}} \exp \left(- \min \left\{ \frac{\alpha - 1}{\sigma^k}, 1 \right\} \frac{\alpha - 1}{8\sigma^k} \right) \\ &\quad - \sum_{i \in \mathcal{I}} \min \left\{ \exp \left(- \min \left\{ \frac{\alpha - 1}{\sigma^k}, 1 \right\} \frac{\alpha - 1}{8\sigma^k} \right), \frac{2 \|A_k \hat{X}\|_F^2}{(\alpha - 1)^2} \right\}. \end{aligned}$$

Let us choose

$$\alpha = 1 + \max \left\{ 20 + 8 \log |\mathcal{D}|, \min \left\{ (20 + 8 \log |\mathcal{I}|) \max_{k \in \mathcal{I}} \|A_k \hat{X}\|_F, \sqrt{200 \sum_{k \in \mathcal{I}} \|A_k \hat{X}\|_F^2} \right\} \right\}.$$

Since $\sigma^k \leq \sum_{i=1}^n \lambda_i^k \leq 1$ for $k \in \mathcal{D}$, it follows from the choice of α that

$$\exp \left(- \min \left\{ \frac{\alpha - 1}{\sigma^k}, 1 \right\} \frac{\alpha - 1}{8\sigma^k} \right) = \exp \left(- \frac{\alpha - 1}{8\sigma^k} \right) \leq \exp \left(- \frac{\alpha - 1}{8} \right) \leq \frac{1}{100|\mathcal{D}|}, \quad \forall k \in \mathcal{D},$$

and

$$\sum_{i \in \mathcal{I}} \min \left\{ \exp \left(- \min \left\{ \frac{\alpha - 1}{\sigma^k}, 1 \right\} \frac{\alpha - 1}{8\sigma^k} \right), \frac{2 \|A_k \hat{X}\|_F^2}{(\alpha - 1)^2} \right\} \leq \frac{1}{100}.$$

This further implies that

$$\text{Prob} \left\{ \xi^T A_k \xi \leq \alpha, k = 0, 1, \dots, m; \xi^T C \xi \geq \text{Tr}(C \hat{X}) \right\} \geq \frac{1}{100}.$$

Summarizing, we obtain the following worst-case performance ratio bounds on the SDP relaxation for a real-valued homogeneous (indefinite) quadratic maximization problem. [We also state the complex case without proof.]

Theorem 5.2. *For the quadratic optimization problem (5.1) with $\mathbb{F} = \mathbb{R}$ and its SDP relaxation (5.2), suppose that an optimal solution, say \hat{X} , for (5.2) exists. Then,*

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \max \left\{ 20 + 8 \log |\mathcal{D}|, \min \left\{ (20 + 8 \log |\mathcal{I}|) \max_{k \in \mathcal{I}} \|A_k \hat{X}\|_F, \sqrt{200 \sum_{k \in \mathcal{I}} \|A_k \hat{X}\|_F^2} \right\} \right\}.$$

Similarly, for the complex case $\mathbb{F} = \mathbb{C}$, we have

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \max \left\{ 15 + 4 \log |\mathcal{D}|, \min \left\{ (15 + 4 \log |\mathcal{I}|) \max_{k \in \mathcal{I}} \|A_k \hat{X}\|_F, \sqrt{40 \sum_{k \in \mathcal{I}} \|A_k \hat{X}\|_F^2} \right\} \right\}.$$

Let us consider two special cases of Theorem 5.2. First, if $\mathcal{I} = \emptyset$, then Theorem 5.2 becomes $\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 20 + 8 \log m$ (in the real case), which recovers the approximation result of Nemirovski *et al.* [9]. The second case is $\mathcal{D} = \emptyset$, where Theorem 5.2 becomes

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \min \left\{ (20 + 8 \log(m+1)) \max_{0 \leq k \leq m} \|A_k \hat{X}\|_F, \sqrt{200 \sum_{k=0}^m \|A_k \hat{X}\|_F^2} \right\}.$$

Below is an example showing that this bound is also tight (in the order of magnitude). Specifically, consider Example 4.3 again:

$$\begin{aligned} \max \quad & x_1^2 + \frac{1}{M} x_2^2 \\ \text{s.t.} \quad & M x_1 x_2 + x_2^2 \leq 1 \\ & -M x_1 x_2 + x_2^2 \leq 1 \\ & M(x_1^2 - x_2^2) \leq 1. \end{aligned}$$

In this case we know that the SDP relaxation has an optimal solution $\hat{X} = \begin{bmatrix} 1 + \frac{1}{M} & 0 \\ 0 & 1 \end{bmatrix}$, while the approximation ratio is $v_{sdp}^{\max}/v_{qp}^{\max} = O(M)$. There are three constraints, all indefinite, $\mathcal{I} = \{1, 2, 3\}$, with

$$A_1 = \begin{bmatrix} 0 & \frac{M}{2} \\ \frac{M}{2} & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -\frac{M}{2} \\ -\frac{M}{2} & 1 \end{bmatrix}, A_3 = \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix},$$

and so one may compute that

$$A_1 \hat{X} = \begin{bmatrix} 0 & \frac{M}{2} \\ \frac{M}{2} + \frac{1}{2} & 1 \end{bmatrix}, A_2 \hat{X} = \begin{bmatrix} 0 & -\frac{M}{2} \\ -\frac{M}{2} - \frac{1}{2} & 1 \end{bmatrix}, A_3 \hat{X} = \begin{bmatrix} M+1 & 0 \\ 0 & -M \end{bmatrix}.$$

Thus, $\|A_k \hat{X}\|_F = O(M^2)$, for $k = 1, 2, 3$. Theorem 5.2 predicts that $v_{sdp}^{\max}/v_{qp}^{\max} \leq O(M)$, and this upper bound is exactly attained in this example.

6 Simulations and Discussions

This paper studies the quality bound of SDP relaxation for solving nonconvex quadratic optimization problems (3.1) and (4.1). For problem (3.1), a quality bound $O(m^2)$ was derived for $\mathbb{F} = \mathbb{R}$, and a quality bound $O(m)$ for $\mathbb{F} = \mathbb{C}$, when there is only one constraint $x^*A_0x \geq 1$ with A_0 indefinite. For problem (4.1), a quality bound $O(\log m)$ was derived when there is only one nonconvex constraint $x^*A_0x \leq 1$ with A_0 indefinite. These quality bounds are independent of the problem dimension n or data matrices, and only depend on the number of constraints.

For problem (3.1), if there are two or more constraints in the form of $x^*Ax \geq 1$ with A indefinite, then there is no general quality bound as shown by Example 3.7. For problem (4.1), if there are two or more nonconvex constraints, a quality bound is given in Theorem 5.2, albeit the bound is dependent not only on the number of constraints but also the data of the problem.

As shown in the preceding sections, these quality bounds are derived based on the worst-case analysis, and they are indeed tight, in the worst case, up to some constant. This analysis is important as a theoretical guide. The empirical tests, on the other side, serve a quite different purpose. Next, we present some numerical experiments on randomly generated instances. These numerical experiments show that the average approximation ratios are much better than the worst-case ratio, even though they appear to still follow the same growth trend (as a function of m).

More specifically, we generate various random symmetric matrices A_k in the following way: for a full rank positive semidefinite A_k , we set $A_k = \text{rand} \cdot Q^T \cdot \text{diag}(\text{abs}(\text{randn}(n, 1))) \cdot Q$, where ‘rand’, ‘randn’ are Matlab notations, and Q is an orthogonal matrix obtained by QR factorization of a random matrix $\text{randn}(n)$; for a rank-one positive semidefinite A_k , we set $A_k = \text{rand} \cdot Q^T \cdot \text{diag}([\text{abs}(\text{randn}); \text{zeros}(n - 1, 1)]) \cdot Q$; and for an indefinite A_k , we set $A_k = \text{rand} \cdot Q^T \cdot \text{diag}(\text{randn}(n, 1)) \cdot Q$ (Q defined as before). To examine the performance of SDP relaxation for randomly generated problem of form (3.1) with $\mathbb{F} = \mathbb{R}$, we simply set C to be the identity matrix: fix $n = 10$, and choose m from 5, 10, 15, ..., 100. For each m , we do the following: (a) Generate 1,000 random problems such that only one of the A_k ’s is indefinite, and all the other A_k ’s are positive definite; (b) Generate 1,000 random problems such that 10% of the A_k ’s are indefinite, and all the other A_k ’s are positive definite; (c) Generate 1,000 random problems such that only one of the A_k ’s is indefinite, and all the other A_k are rank one and positive semidefinite; (d) Generate 1,000 random problems such that 10% of the A_k ’s are indefinite, and all the other A_k are rank one and positive semidefinite. For each instance of the above randomly generated problems, we solve its SDP relaxation to obtain an optimal solution Z^* and optimal value v_{sdp}^{\min} . Then we find one approximate solution for (3.1) by the following randomization process. Generate 100 random vectors ξ^1, \dots, ξ^{100} . For each $k = 1, \dots, 100$, let $x^i = \xi^i / \sqrt{\min_{k=1}^m (\xi^i)^T A_k \xi^i}$. Then $v_{qp}^{\min} \leq \hat{v}_{qp}^{\min} := \min_{i=1}^{100} (x^i)^T C x^i$. We use the empirical quality bound $\hat{v}_{qp}^{\min} / v_{sdp}^{\min}$ to estimate the real quality bound $v_{qp}^{\min} / v_{sdp}^{\min}$, since the former is at least the latter. These empirical quality bounds are plotted in Figure 1.

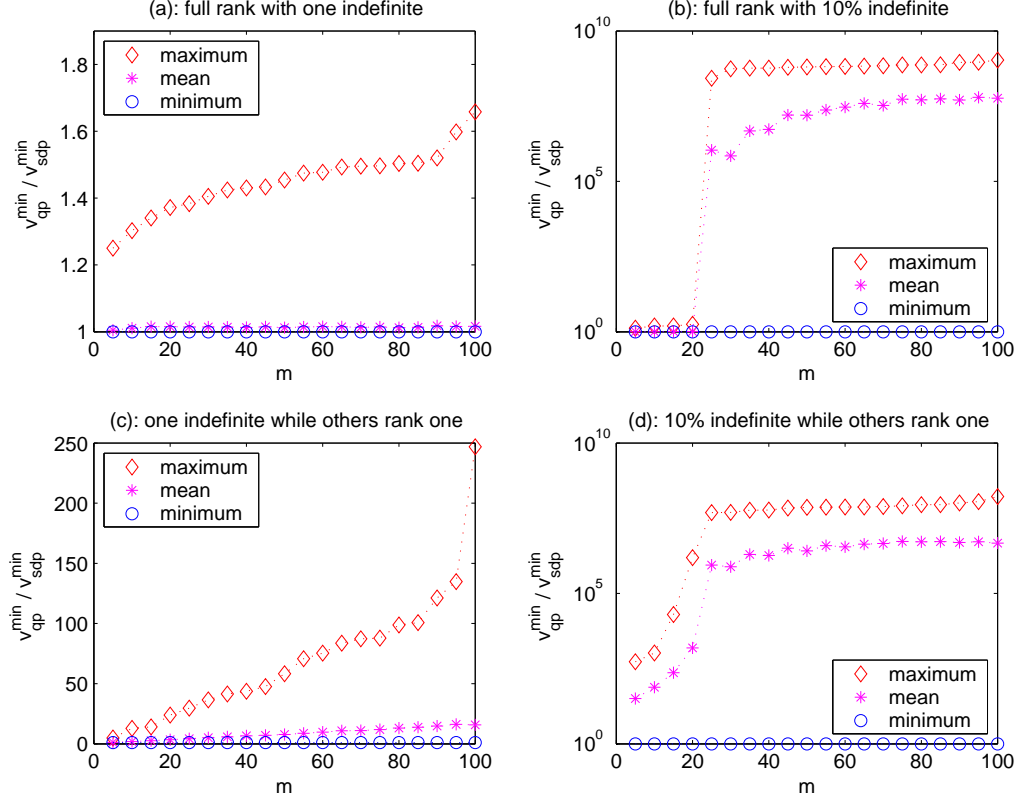


Figure 1: Empirical quality bounds for problem (3.1) with $\mathbb{F} = \mathbb{R}$

In this figure, diamonds \diamond are the maximum quality bounds in the 1,000 random problems for each m , stars $*$ are the mean quality bounds, and circles \circ are the minimum quality bounds. For cases (a) and (c), we have quality bound $\mathcal{O}(m^2)$ for the worst case, while for cases (b) and (d), there is no worst-case theoretical quality bound. As we can see, the computed empirical bounds are very small for case (a), and are moderate for case (c), and are indeed big for cases (b) and (d).

To examine the SDP relaxation performance for the maximization problem (4.1), we generate four classes of random problems in the same way as for problem (3.1), except that the matrix C in the objective is now indefinite (generated the same way as an indefinite A_k). After the SDP relaxation is solved, we apply a similar randomization procedure to find a lower bound \hat{v}_{qp}^{\max} for v_{qp}^{\max} as we did for problem (3.1). The empirical quality bound $v_{sdp}^{\max} / \hat{v}_{qp}^{\max}$ is an upper bound for the actual quality bound $v_{sdp}^{\max} / v_{qp}^{\max}$. The empirical quality bounds are plotted in Figure 2. The legends \diamond , $*$ and \circ carry the same meaning as they did in Figure 1. This figure shows that the computed empirical bounds are close to one for cases (a) and (c), and are somewhat larger for cases (b) and (d). This is consistent with the bounds in Theorem 5.2.

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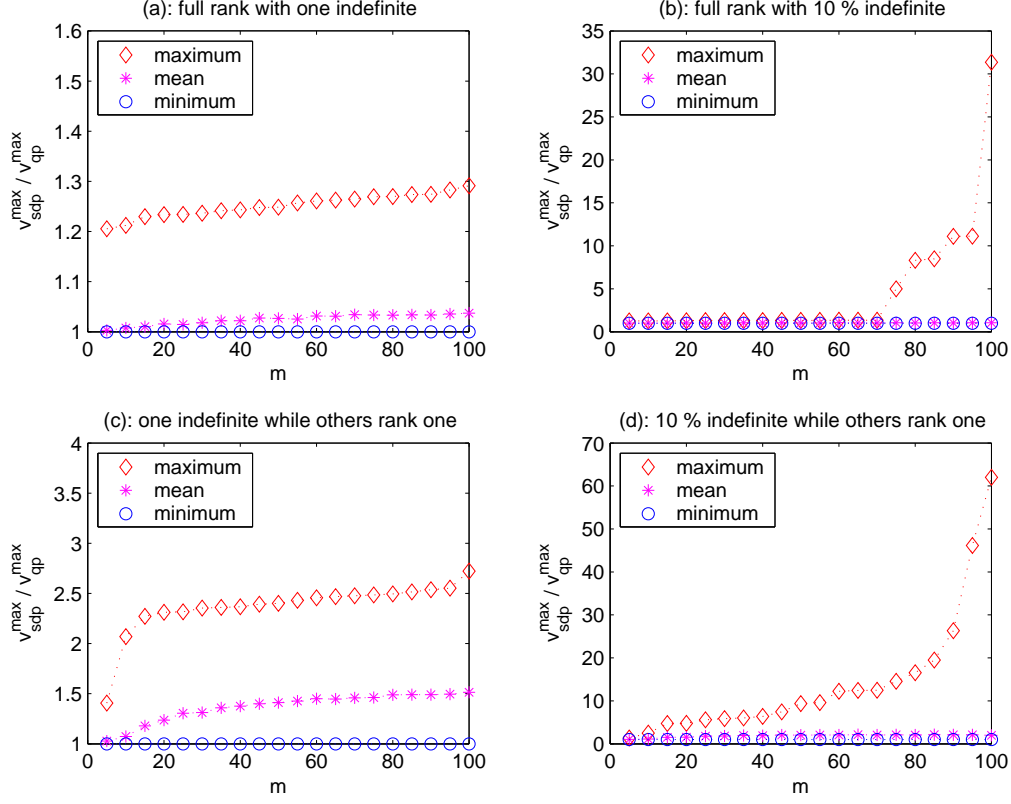


Figure 2: Empirical quality bounds for problem (4.1)

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