# On convergence of solutions of fractal Burgers equation toward rarefaction waves 

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#### Abstract

In the paper, the large time behavior of solutions of the Cauchy problem for the one dimensional fractal Burgers equation $u_{t}+\left(-\partial_{x}^{2}\right)^{\alpha / 2} u+u u_{x}=0$ with $\alpha \in(1,2)$ is studied. It is shown that if the nondecreasing initial datum approaches the constant states $u_{ \pm}\left(u_{-}<u_{+}\right)$as $x \rightarrow \pm \infty$, respectively, then the corresponding solution converges toward the rarefaction wave, i.e. the unique entropy solution of the Riemann problem for the nonviscous Burgers equation.


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## 1 Introduction

The goal of this work is to study asymptotic properties of solutions to the Cauchy problem for the nonlocal conservation law

$$
\begin{align*}
& u_{t}+\Lambda^{\alpha} u+u u_{x}=0, \quad x \in \mathbb{R}, t>0,  \tag{1.1}\\
& u(0, x)=u_{0}(x) \tag{1.2}
\end{align*}
$$

where $\Lambda^{\alpha}=\left(-\partial^{2} / \partial x^{2}\right)^{\alpha / 2}$ is the pseudodifferential operator defined via the Fourier transform

$$
\begin{equation*}
\widehat{\left(\Lambda^{\alpha} v\right)}(\xi)=|\xi|^{\alpha} \widehat{v}(\xi) \tag{1.3}
\end{equation*}
$$

Following [2], we will call (1.1) the fractal Burgers equation. Equations of this type appear in the study of growing interfaces in the presence of selfsimilar hopping surface diffusion [17]. Moreover, in their recent papers, Jourdain, Méléard, and Woyczynski [11, 12] gave probabilistic motivations to study equations with the anomalous diffusion, when Laplacian (the generator of the Wiener process) is replaced by a more general pseudodifferential operator generating the Lévy process. In particular, the authors of 12 studied problem (1.1)-(1.2), where the initial condition $u_{0}$ is assumed to be a nonconstant function with bounded variation on $\mathbb{R}$. In other words, a.e. on $\mathbb{R}$,

$$
\begin{equation*}
u_{0}(x)=c+\int_{-\infty}^{x} m(d y)=c+H * m(x) \tag{1.4}
\end{equation*}
$$

with $c \in \mathbb{R}, m$ being a finite signed measure on $\mathbb{R}$, and $H(y)$ denoting the unit step function $\mathbb{I}_{\{y \geq 0\}}$. Observe that the gradient $v(x, t)=u_{x}(x, t)$ satisfies

$$
\begin{equation*}
v_{t}+\Lambda^{\alpha} v+(v H * v)_{x}=0, \quad v(\cdot, 0)=m \tag{1.5}
\end{equation*}
$$

If m is a probability measure on $\mathbb{R}$, the equation (1.5) is a nonlinear Fokker-Planck equation. In the case of an arbitrary finite signed measure, the authors of [12] associate (1.5) with a suitable nonlinear martingale problem. Next, they study the convergence of systems of particles with jumps as the number of particles tends to $+\infty$. As a consequence, the weighted empirical cumulative distribution functions of the particles converge to the solution of the martingale problem connected to (1.5). This phenomena is called the propagation of chaos for problem (1.1)-(1.2) and we refer the reader to [12] for more details and additional references.

Motivated by the results from [12], we study problem (1.1)-(1.2) under the crucial assumption $\alpha \in(1,2)$ and with the initial condition of the form (1.4). In our main result, we assume that $u_{0}$ is a function satisfying

$$
\begin{equation*}
u_{0}-u_{-} \in L^{1}((-\infty, 0)) \quad \text { and } \quad u_{0}-u_{+} \in L^{1}((0,+\infty)) \quad \text { with } \quad u_{-}<u_{+}, \tag{1.6}
\end{equation*}
$$

where $u_{-}=c$ and $u_{+}=c+\int_{\mathbb{R}} m(d x)$.
It is well known (cf. [10, 18, 9] and Lemma [2.4, below) that the asymptotic profile as $t \rightarrow \infty$ of solutions to the viscous Burgers equation

$$
\begin{equation*}
u_{t}-u_{x x}+u u_{x}=0 \tag{1.7}
\end{equation*}
$$

(i.e. equation (1.1) with $\alpha=2$ ) supplemented with an initial datum satisfying (1.6) is given by the so-called rarefaction wave. This is the continuous function

$$
w^{R}(x, t)=W^{R}(x / t)= \begin{cases}u_{-}, & x / t \leq u_{-}  \tag{1.8}\\ x / t, & u_{-} \leq x / t \leq u_{+} \\ u_{+}, & x / t \geq u_{+}\end{cases}
$$

which is the unique entropy solution of following Riemann problem

$$
\begin{equation*}
w_{t}^{R}+w^{R} w_{x}^{R}=0 \tag{1.9}
\end{equation*}
$$

$$
w^{R}(x, 0)=w_{0}^{R}(x)= \begin{cases}u_{-}, & x<0  \tag{1.10}\\ u_{+}, & x>0\end{cases}
$$

Below, we use the solution of the Burgers equation (1.7) with the initial datum (1.10) as the smooth approximation of the rarefaction wave (1.8).

The authors of this work were inspired by the fundamental paper of Il'in and Oleinik [10] who showed the convergence toward rarefaction waves of solutions to the nonlinear equation $u_{t}-u_{x x}+f(u)_{x}=0$ under strict convexity assumption imposed on $f$. That idea was next extended in several different directions and we refer the reader e.g. to [9, 16, 18, 19, 21, 22 for an overview of know results and additional references.

In this work, we contribute to the existing theory by developing tools which allows to obtain analogous results for equations with a nonlocal and anomalous diffusion. Basic properties of solutions (namely, their existence and the regularity) of quasilinear evolution equations with $(-\Delta)^{\alpha / 2}, \alpha \in(1,2)$, (or, more generally, with the Lévy diffusion) were shown in [7, 8]. On the other hand, one may expect singularities in finite time of solutions to (1.1) with $\alpha \in(0,1)$, see [1] for more details. If $u_{0} \in L^{1}(\mathbb{R})$, Biler, Karch, and Woyczynski [3] proved that the large time asymptotics of solutions to (1.1)-(1.2) is described by the self-similar fundamental solution of equation $v_{t}+\Lambda^{\alpha} v=0$. Analogous asymptotic properties of solutions to multidimensional generalizations of problem (1.1)-(1.2) with $u_{0} \in L^{1}\left(\mathbb{R}^{N}\right)$ are studied in [4, 5].

Finally, we would like to report the recent progress in the understanding of properties of solutions of the quasi-geostrophic equation with an anomalous diffusion, cf. [6, 13] and the references therein. We are convinced that our techniques can be applied to that equation, as well.

The purpose of the present paper is to prove the convergence of solutions to the Cauchy problem for the fractal Burgers equation (1.1)-(1.2) toward rarefaction waves. We state our main result in the following theorem.

Theorem 1.1. Let $\alpha \in(1,2)$. Assume that $w^{R}=w^{R}(x, t)$ is the rarefaction wave (1.8). Denote by $u=u(x, t)$ the unique solution of problem (1.1)-(1.2) corresponding to the initial datum $u_{0}$ of the form (1.4) and satisfying (1.6). For every $p \in((3-\alpha) /(\alpha-1), \infty]$ there exists $C>0$ independent of $t$ such that

$$
\left\|u(t)-w^{R}(t)\right\|_{p} \leq C t^{-[\alpha-1-(3-\alpha) / p] / 2} \log (2+t)
$$

for all $t>0$.
Remark 1.1. Our result and its proof hold true also for $\alpha=2$ (observe that (3 $\alpha) /(\alpha-1) \rightarrow 1$ as $\alpha \rightarrow 2)$. However, we pass over this case for simplicity of the exposition and because the large time asymptotics of solutions to the Burgers equation (1.7) is wellknown, see Lemma 2.4, below.

In the next section, we gather several preliminary properties of the operator $\Lambda^{\alpha}$ and of solutions to problem (1.1)-(1.2). Theorem 1.1 is shown in Section 3. In Section 4, we discuss possible generalizations of our main result.

Notation. For $1 \leq p \leq \infty$, the $L^{p}$-norm of a Lebesgue measurable, real-valued function $v$ defined on $\mathbb{R}$ is denoted by $\|v\|_{p}$. For a finite signed measure $m$ on $\mathbb{R}$, we put $\|m\|=|m|(\mathbb{R})$, where $|m|$ is the total variation of $m$. The Fourier transform of $v$ is $\widehat{v}(\xi) \equiv(2 \pi)^{-1 / 2} \int_{\mathbb{R}} e^{-i x \xi} v(x) d x$. Given a function $v=v(x)$, we are going to use the decomposition $v=v^{-}+v^{+}$, where as usual $v^{-}=\max \{0,-v\}$ and $v^{+}=\max \{0, v\}$. The constants (always independent of of $t$ ) will be denoted by the same letter $C$, even if they may vary from line to line. Occasionally, we write, e.g., $C=C(\alpha, \ell)$ when we want to emphasize the dependence of $C$ on parameters $\alpha$ and $\ell$.

## 2 Preliminary results

We begin by recalling that the basic questions on the existence and the uniqueness of solutions of problem (1.1)-(1.2) were answered in the papers [7, 8].

Theorem 2.1. ([7, Thm. 1.1], [8, Thm. 7]) Let $\alpha \in(1,2)$ and $u_{0} \in L^{\infty}(\mathbb{R})$. There exists the unique solution $u=u(x, t)$ to problem (1.1)-(1.2) in the following sense: for all $T>0$,

$$
\begin{aligned}
& u \in C_{b}((0, T) \times \mathbb{R}) \text { and, for all } a \in(0, T), u \in C_{b}^{\infty}((a, T) \times \mathbb{R}), \\
& u \text { satisfies }(1.1) \text { on }(0, T) \times \mathbb{R}, \\
& u(t, \cdot) \rightarrow u_{0} \text { in } L^{\infty}(\mathbb{R}) \text { weak }-* \text {, as } t \rightarrow 0
\end{aligned}
$$

Moreover, the following inequality holds true

$$
\begin{equation*}
\|u(t)\|_{\infty} \leq\left\|u_{0}\right\|_{\infty} \quad \text { for all } \quad t>0 \tag{2.1}
\end{equation*}
$$

The main goal of the section is to complete this result by additional properties of $u_{x}$ if the initial conditions are of the form (1.4).

Theorem 2.2. Let $\alpha \in(1,2)$. Assume that the initial datum $u_{0}$ can be written in the form (1.4) for a constant $c \in \mathbb{R}$ and a signed finite measure $m$ on $\mathbb{R}$. Then the solution $u=u(x, t)$ of problem (1.1) -(1.2) satisfies $u_{x} \in C\left((0, T] ; L^{p}(\mathbb{R})\right)$ for each $1 \leq p \leq \infty$ and every $T>0$.

Consider $u$ and $\widetilde{u}$ two such solutions with initial conditions $u_{0}$ and $\widetilde{u}_{0}$, respectively. Suppose that $\widetilde{u}_{x}(x, t)$ is nonnegative a.e. and $u_{0}-\widetilde{u}_{0} \in L^{1}(\mathbb{R})$. Then

$$
\begin{equation*}
\|u(t)-\widetilde{u}(t)\|_{1} \leq\left\|u_{0}-\widetilde{u}_{0}\right\|_{1} \tag{2.2}
\end{equation*}
$$

for all $t>0$.
Theorem 2.3. Under the assumption of Theorem 2.2, if the measure $m$ in the initial datum (1.4) is nonnegative, we have
i) $u_{x}(x, t) \geq 0$ for all $x \in \mathbb{R}$ and $t>0$,
ii) for every $p \in[1, \infty]$ there exists $C=C(p)>0$ such that

$$
\begin{equation*}
\left\|u_{x}(t)\right\|_{p} \leq t^{-1+1 / p}\|m\|^{1 / p} \tag{2.3}
\end{equation*}
$$

In the proofs of Theorems 2.2 and 2.3 as well as in our study of the large time asymptotics, we shall require several properties of the operator $\Lambda^{\alpha}$ and of the semigroup of linear operators generated by it. First of all, note that the operator defined by (1.3) has the integral representation for every $\alpha \in(1,2)$ (cf. eg. [8, Thm. 1])

$$
\begin{equation*}
\Lambda^{\alpha} w(x)=-C(\alpha) \int_{\mathbb{R}} \frac{w(x+z)-w(x)-w_{x}(x) z}{|z|^{1+\alpha}} d z . \tag{2.4}
\end{equation*}
$$

This formula allows us to apply $\Lambda^{\alpha}$ to functions which are bounded and sufficiently smooth, however, not necessary decaying at infinity.

Lemma 2.1. Let $1<\alpha<2$. For every $p \in[1, \infty]$ there exists $C=C(p, \alpha)>0$ such that

$$
\begin{equation*}
\left\|\Lambda^{\alpha} w\right\|_{p} \leq C\left\|w_{x}\right\|_{p}^{2-\alpha}\left\|w_{x x}\right\|_{p}^{\alpha-1} \tag{2.5}
\end{equation*}
$$

all functions $w$ satisfying $w_{x}, w_{x x} \in L^{p}(\mathbb{R})$.
Proof. We can easily deduce the interpolation inequality (2.5) from (2.4). Indeed, it follows from the Taylor formula that for any fixed $R>0$ we have

$$
\begin{aligned}
\left\|\Lambda^{\alpha} w\right\|_{p} & \leq C\left\|w_{x x}\right\|_{p} \int_{|z| \leq R}|z|^{1-\alpha} d z+C\left\|w_{x}\right\|_{p} \int_{|z|>R}|z|^{-\alpha} d z \\
& \leq C\left(R^{2-\alpha}\left\|w_{x x}\right\|_{p}+R^{1-\alpha}\left\|w_{x}\right\|_{p}\right) .
\end{aligned}
$$

Choosing $R=\left\|w_{x}\right\|_{p} /\left\|w_{x x}\right\|_{p}$ we complete the proof of inequality (2.5).
Now, we prove the Nash inequality for the operator $\Lambda^{\alpha}$.
Lemma 2.2. Let $0<\alpha$. There exists a constant $C_{N}>0$ such that

$$
\begin{equation*}
\|w\|_{2}^{2(1+\alpha)} \leq C_{N}\left\|\Lambda^{\alpha / 2} w\right\|_{2}^{2}\|w\|_{1}^{2 \alpha} \tag{2.6}
\end{equation*}
$$

for all functions $w$ satisfying $w \in L^{1}(\mathbb{R})$ and $\Lambda^{\alpha / 2} w \in L^{2}(\mathbb{R})$.
Proof. For every $R>0$, we decompose the $L^{2}$-norm of the Fourier transform of $w$ as follows

$$
\begin{aligned}
\|w\|_{2}^{2} & =C \int_{\mathbb{R}}|\widehat{w}(\xi)|^{2} d \xi \\
& \leq C\|\widehat{w}\|_{\infty}^{2} \int_{|\xi| \leq R} d \xi+C R^{-\alpha} \int_{|\xi|>R}|\xi|^{\alpha}|\widehat{w}(\xi)|^{2} d \xi \\
& \leq C R\|w\|_{1}^{2}+C R^{-\alpha}\left\|\Lambda^{\alpha / 2} w\right\|_{2}^{2}
\end{aligned}
$$

For $R=\left(\left\|\Lambda^{\alpha / 2} w\right\|_{2}^{2} /\|w\|_{1}^{2}\right)^{1 /(1+\alpha)}$ we obtain (2.6).

Lemma 2.3. Let $0 \leq \alpha \leq 2$. For every $p>1$, we have

$$
\begin{equation*}
\int_{\mathbb{R}}\left(\Lambda^{\alpha} w\right)|w|^{p-2} w d x \geq \frac{4(p-1)}{p^{2}} \int_{\mathbb{R}}\left(\Lambda^{\frac{\alpha}{2}}|w|^{\frac{p}{2}}\right)^{2} d x \tag{2.7}
\end{equation*}
$$

for all $w \in L^{p}(\mathbb{R})$ such that $\Lambda^{\alpha} w \in L^{p}(\mathbb{R})$. If $\Lambda^{\alpha} w \in L^{1}(\mathbb{R})$, we obtain

$$
\begin{equation*}
\int_{\mathbb{R}}\left(\Lambda^{\alpha} w\right) \operatorname{sgn} w d x \geq 0 \tag{2.8}
\end{equation*}
$$

and, if $w, \Lambda^{\alpha} w \in L^{2}(\mathbb{R})$, it follows

$$
\begin{equation*}
\int_{\mathbb{R}}\left(\Lambda^{\alpha} w\right) w^{+} d x \geq 0 \quad \text { and } \quad \int_{\mathbb{R}}\left(\Lambda^{\alpha} w\right) w^{-} d x \geq 0 \tag{2.9}
\end{equation*}
$$

where $w^{+}=\max \{0, w\}$ and $w^{-}=\max \{0,-w\}$.
Inequality (2.7) is well-known in the theory of sub-Markovian operators and its statement and the proof is given e.g. in [15, Theorem 2.1 combined with the Beurling-Deny condition (1.7)], see also [6, 13]. Observe that if $\alpha=2$, integrating by parts we obtain (2.7) with the equality. Inequality (2.8) (called the Kato inequlity) is used in [8] to construct entropy solutions of (1.1) and it can be easily deduced from [8, Lemma 1] by an approximation argument (see also [2, Inequality (3.5)]). The proof of (2.9) can be found, for example, in [15, Proposition 1.6].

We also recall that, by Duhamel's principle, the solution to problem (1.1)-(1.2) can be written in the equivalent integral form

$$
\begin{equation*}
u(t)=S_{\alpha}(t) u_{0}-\int_{0}^{t} S_{\alpha}(t-\tau) u(\tau) u_{x}(\tau) d t a u \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{\alpha}(t) u_{0}=p_{\alpha}(t) * u_{0}(x) \tag{2.11}
\end{equation*}
$$

Here, the fundamental solution $p_{\alpha}(x, t)$ of the linear equation $\partial_{t} v+\Lambda^{\alpha} v=0$ can be computed via the Fourier transform $\widehat{p}_{\alpha}(\xi, t)=e^{-t|\xi|^{\alpha}}$. Hence, $p_{\alpha}(x, t)=t^{-1 / \alpha} P_{\alpha}\left(x t^{-1 / \alpha}\right)$, where $P_{\alpha}$ is the inverse Fourier transform of $e^{-|\xi|^{\alpha}}$. It is well known that for every $\alpha \in(0,2]$ the function $P_{\alpha}$ has the property $\int_{\mathbb{R}} P_{\alpha}(x) d x=1$ and is smooth, nonnegative, and satisfies

$$
\begin{equation*}
0 \leq P_{\alpha}(x) \leq C(1+|x|)^{-(\alpha+1)} \quad \text { and } \quad\left|\partial_{x} P_{\alpha}(x)\right| \leq C(1+|x|)^{-(\alpha+2)} \tag{2.12}
\end{equation*}
$$

for a constant $C$ and all $x \in \mathbb{R}$. Using these properties of the convolution operator $S_{\alpha}(t)$ defined by (2.11) we obtain the estimates

$$
\begin{align*}
\left\|S_{\alpha}(t) v\right\|_{p} & \leq C t^{-(1-1 / p) / \alpha)}\|v\|_{1}  \tag{2.13}\\
\left\|\left(S_{\alpha}(t) v\right)_{x}\right\|_{p} & \leq C t^{-(1-1 / p) / \alpha-1 / \alpha}\|v\|_{1} \tag{2.14}
\end{align*}
$$

for every $p \in[1, \infty]$ and all $t>0$. Moreover, we can replace $v$ in (2.13) and in (2.14) by any signed measure $m$. In that case, $\|v\|_{1}$ should be replaced by $\|m\|$.

Proof of Theorem [2.2. It follows from the integral equation (2.10) that $u_{x}$ is the solution of

$$
\begin{equation*}
u_{x}(t)=S_{\alpha}(t) m-\int_{0}^{t} \partial_{x} S_{\alpha}(t-\tau) V(\tau) u_{x}(\tau) d \tau \tag{2.15}
\end{equation*}
$$

where $V(x, t)=u(x, t)$ is treated as given, smooth, and bounded. Now the standard argument involving the Banach fixed point theorem allows us to show that the solution of the "linear" equation (2.15) has the solution in $C\left((0, T] ; L^{p}(\mathbb{R})\right)$ for each $p \in[1, \infty]$ and every $T>0$. Here, we should use the following estimate of the operator $\mathcal{T}(u)$ defined by the right-hand side of (2.15)

$$
\begin{aligned}
\|\mathcal{T}(u)(t)\|_{p} & \leq\left\|S_{\alpha}(t) m\right\|_{p}+\int_{0}^{t}\left\|\partial_{x} S_{\alpha}(t-\tau) V(\tau) u_{x}(\tau)\right\|_{p} d \tau \\
& \leq C t^{-(1-1 / p) / \alpha}\|m\|+C \sup _{\tau \in[0, T]}\|V(\tau)\|_{\infty} \int_{0}^{t}(t-\tau)^{-1 / \alpha}\left\|u_{x}(\tau)\right\|_{p} d \tau
\end{aligned}
$$

being the immediate consequence of (2.13) and (2.14). Let us skip other details of this well-known argument (cf. [20]).

Now, we prove inequality (2.2). A direct calculation shows that the function $v(x, t)=$ $u(x, t)-\widetilde{u}(x, t)$ satisfies

$$
\begin{equation*}
\left.v_{t}+\Lambda^{\alpha} v+\frac{1}{2}\left(v^{2}+2 v \widetilde{u}\right)\right)_{x}=0 . \tag{2.16}
\end{equation*}
$$

First, we multiply equation (2.16) by $\operatorname{sgn} v=v|v|^{-1}$ :

$$
\left.\frac{d}{d t} \int_{\mathbb{R}}|v| d x+\int_{\mathbb{R}}\left(\Lambda^{\alpha} v\right) \operatorname{sgn} v d x+\frac{1}{2} \int_{\mathbb{R}}\left[v^{2}+2 v \widetilde{u}\right)\right]_{x} \operatorname{sgn} v d x=0
$$

The second term is nonnegative by (2.8). To show the same property for the third term, we replace the sgn function by smooth and nondecreasing $\varphi=\varphi(x)$. In this case, we obtain

$$
\int_{\mathbb{R}}\left[v^{2}+2 v \widetilde{u}\right]_{x} \varphi(v) d x=-\int_{\mathbb{R}}\left(v^{2}+2 v \widetilde{u}\right) \varphi^{\prime}(v) v_{x} d x=-\int_{\mathbb{R}} \Psi(v)_{x} d x+\int_{\mathbb{R}} \widetilde{u}_{x} \Phi(v) d x
$$

where $\Psi(s)=\int_{0}^{s} z^{2} \varphi^{\prime}(z) d z$ and $\Phi(s)=\int_{0}^{s} 2 z \varphi^{\prime}(z) d z$. Obviously, the first term on the right hand side is equal to zero and the second one is nonnegative because $\widetilde{u}_{x} \geq 0$ and $\Phi(s) \geq 0$ for all $s \in \mathbb{R}$. Now, the standard approximation argument gives $\int_{\mathbb{R}}\left[v^{2}+\right.$ $2 v \widetilde{u}]_{x} \operatorname{sgn} v d x \geq 0$. Hence $\|v(t)\|_{1}=\|u(t)-\widetilde{u}(t)\|_{1} \leq\left\|u_{0}-\widetilde{u}_{0}\right\|_{1}=\left\|v_{0}\right\|_{1}$ for all $t>0$.

Proof of Theorem 2.3. To show part (i) of Theorem [2.3, we deal first with the smooth initial datum $u_{0}$ satisfying $u_{0, x}(x) \geq 0$ and $u_{0, x} \in L^{p}(\mathbb{R})$ for every $p \in[1, \infty]$. In this case, differentiating equation (1.1) with respect to $x$ we have

$$
\begin{equation*}
\left(u_{x}\right)_{t}+\Lambda^{\alpha} u_{x}+\left(u u_{x}\right)_{x}=0 \tag{2.17}
\end{equation*}
$$

Note the well known property

$$
\int_{\mathbb{R}} v_{t} v^{-} d x=\int_{v \leq 0} v_{t}^{-} v^{-} d x=\frac{1}{2} \frac{d}{d t} \int_{v \leq 0}\left(v^{-}\right)^{2} d x
$$

Hence, multiplying (2.17) by $u_{x}^{-}$, integrating the resulting equation over $\mathbb{R}$, and integrating by parts on the right hand side, we obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{u_{x} \leq 0}\left(u_{x}^{-}\right)^{2} d x+\int_{\mathbb{R}}\left(\Lambda^{\alpha} u_{x}\right) u_{x}^{-} d x & =-\int_{u_{x} \leq 0}\left(u u_{x}^{-}\right)_{x} u_{x}^{-} d x \\
& =-\frac{1}{2} \int_{u_{x} \leq 0}\left(u_{x}^{-}\right)^{3} d x
\end{aligned}
$$

Since $\int_{\mathbb{R}}\left(\Lambda^{\alpha} u_{x}\right) u_{x}^{-} d x \geq 0$ by (2.9) and $\int_{u_{x} \leq 0}\left(u_{x}^{-}(x, 0)\right)^{2} d x=0$ by the assumption imposed on $u_{0}$, the Gronwall inequality implies $\int_{u_{x} \leq 0}\left(u_{x}^{-}(x, t)\right)^{2} d x=0$ for all $t \geq 0$. Consequently, $u_{x}^{-}(x, t) \equiv 0$ and the proof of (i) for regular initial conditions is finished.

Now, the proof of (i) for the solution $u=u(x, t)$ corresponding to the initial datum $u_{0}$ of the form (1.4) with the nonnegative finite measure $m$ can be completed by the following approximation argument. We consider the sequence of regular initial conditions $u_{0}^{n}$ as in the first part of this proof. Moreover, we assume that $u_{0, x}^{n}$ converges weakly to $m$ and $\left\|u_{0}^{n}-u_{0}\right\|_{1} \rightarrow 0$ as $n \rightarrow \infty$. Inequality (2.2) allows us to prove that the corresponding solutions $u^{n}(\cdot, t)$ satisfy $\left\|u^{n}(\cdot, t)-u(\cdot, t)\right\|_{1} \rightarrow 0$ as $n \rightarrow \infty$ for any $t>0$. Hence, there is a subsequence $n_{k} \rightarrow \infty$ such that $u^{n_{k}}(x, t) \rightarrow u(x, t)$ a.e. Since each $u^{n}(x, t)$ is nondecreasing as function of $x$, the same conclusion holds true for $u(x, t)$.

In order to show inequality (2.3), we first observe that integrating equation (2.15) over $\mathbb{R}$ and using the equalities

$$
\int_{\mathbb{R}} S_{\alpha}(t) m d t=\int_{\mathbb{R}} m(d x) \quad \text { and } \quad \int_{\mathbb{R}} \partial_{x} S_{\alpha}(t-\tau)\left(u(\tau) u_{x}(\tau)\right) d x=0
$$

we obtain the identity $\int_{\mathbb{R}} u_{x}(x, t) d x=\int_{\mathbb{R}} m(d x)$ which for nonnegative $u_{x}$ means

$$
\begin{equation*}
\left\|u_{x}(t)\right\|_{1}=\|m\| \quad \text { for all } \quad t>0 \tag{2.18}
\end{equation*}
$$

Now, for fixed $p \in(1, \infty)$ and $u_{x} \geq 0$, we multiply (2.17) by $u_{x}^{p-1}$ and integrate the resulting equation over $\mathbb{R}$. After some manipulations involving integrations by parts on the right hand side we arrive at

$$
\begin{align*}
\frac{1}{p} \frac{d}{d t}\left\|u_{x}\right\|_{p}^{p}+\int_{\mathbb{R}} u_{x}^{p-1} \Lambda^{\alpha} u_{x} d x & =-\int_{\mathbb{R}}\left(u u_{x}\right)_{x} u_{x}^{p-1} d x \\
& =-\frac{p-1}{p} \int_{\mathbb{R}} u_{x}^{p+1} d x \tag{2.19}
\end{align*}
$$

Recall now that $\int_{\mathbb{R}} u_{x}^{p-1} \Lambda^{\alpha} u_{x} d x \geq 0$ by inequality (2.7). Moreover, it follows from the Hölder inequality combined with (2.18) that

$$
\left\|u_{x}(t)\right\|_{p}^{p^{2} /(p-1)} \leq\left\|u_{x}(t)\right\|_{p+1}^{p+1}\|m\|^{1 /(p-1)}
$$

Applying those two inequalities to (2.19) (note $u_{x} \geq 0$ ) we obtain the following differential inequality for $\left\|u_{x}(t)\right\|_{p}^{p}$

$$
\frac{d}{d t}\left\|u_{x}(t)\right\|_{p}^{p} \leq-(p-1)\|m\|^{-1 /(p-1)}\left(\left\|u_{x}(t)\right\|_{p}^{p}\right)^{p /(p-1)}
$$

Integrating it we complete the proof of (2.3) for any $p \in(1, \infty)$.
The case of $p=\infty$ is obtained immediately passing to the limit $p \rightarrow \infty$ in inequality (2.3).

We conclude this section by recalling some results on smooth approximations of rarefaction waves, namely, the solutions of the following Cauchy problem

$$
\begin{align*}
& w_{t}-w_{x x}+w w_{x}=0  \tag{2.20}\\
& w(x, 0)=w_{0}(x)= \begin{cases}u_{-}, & x<0 \\
u_{+}, & x>0\end{cases} \tag{2.21}
\end{align*}
$$

Lemma 2.4. Let $u_{-}<u_{+}$. Problem (2.20)-(2.21) has the unique, smooth, global-in-time solution $w(x, t)$ satisfying
i) $u_{-}<w(t, x)<u_{+}$and $w_{x}(t, x)>0$ for all $(x, t) \in \mathbb{R} \times(0, \infty)$;
ii) for every $p \in[1, \infty]$, there exists a constant $C=C\left(p, u_{-}, u_{+}\right)>0$ such that

$$
\left\|w_{x}(t)\right\|_{p} \leq C t^{-1+1 / p}, \quad\left\|w_{x x}(t)\right\|_{p} \leq C t^{-3 / 2+1 /(2 p)}
$$

and

$$
\left\|w(t)-w^{R}(t)\right\|_{p} \leq C t^{-(1-1 / p) / 2}
$$

for all $t>0$, where $w^{R}(x, t)$ is the rarefaction wave (1.8).
All results stated in Lemma 2.4 are deduced from the explicit formula for solutions to $(2.20)-(2.21)$ and detailed calculations can be found in [9] with some additional improvements contained in [14, Section 3].

## 3 Convergence toward rarefaction waves

For simplicity of the exposition, we divide the proof of Theorem 1.1 into a sequence of Lemmata.

Lemma 3.1. Let $\alpha \in(1,2)$. Assume that $u$ and $\widetilde{u}$ are two solutions of problem (1.1)-(1.2) with initial conditions $u_{0}$ and $\widetilde{u}_{0}$, the both of the from (1.4) with finite signed measures $m$ and $\widetilde{m}$, respectively. Suppose, moreover, that the measure $\widetilde{m}$ of $\widetilde{u}_{0}$ is nonnegative and $u_{0}-\widetilde{u}_{0} \in L^{1}(\mathbb{R})$. Then, for every $p \in[1, \infty]$ there exists a constant $C=C(p)>0$ such that

$$
\begin{equation*}
\|u(t)-\widetilde{u}(t)\|_{p} \leq C t^{-(1-1 / p) / \alpha}\left\|u_{0}-\widetilde{u}_{0}\right\|_{1} \tag{3.1}
\end{equation*}
$$

for all $t>0$.

Proof. In our reasoning, we denote $v(x, t)=u(x, t)-\tilde{u}(x, t)$ which satisfies equation (2.16). It follows from Theorem (2.2, inequality (2.2), that $\|v(t)\|_{1} \leq\left\|v_{0}\right\|_{1}$.

Now, we multiply equation (2.16) by $|v|^{p-2} v$ with $p>1$

$$
\begin{equation*}
\frac{1}{p} \frac{d}{d t} \int_{\mathbb{R}}|v|^{p} d x+\int_{\mathbb{R}}\left(\Lambda^{\alpha} v\right)\left(|v|^{p-2} v\right) d x+\frac{1}{2} \int_{\mathbb{R}}\left[v^{2}+2 v \widetilde{u}\right]_{x}|v|^{p-2} v d x=0 \tag{3.2}
\end{equation*}
$$

The third term on the left hand side of (3.2) is nonnegative by the following calculations

$$
\begin{align*}
\int_{\mathbb{R}}\left[v^{2}+2 v \widetilde{u}\right]_{x}|v|^{p-2} v d x & =\int_{\mathbb{R}} 2 v_{x}|v|^{p} d x+\int_{\mathbb{R}} 2 \widetilde{u} v_{x}|v|^{p-2} v d x+\int_{\mathbb{R}} 2 \widetilde{u}_{x}|v|^{p} d x  \tag{3.3}\\
& =2\left(1-\frac{1}{p}\right) \int_{\mathbb{R}} \widetilde{u}_{x}|v|^{p} d x \geq 0
\end{align*}
$$

because $\int_{\mathbb{R}} v_{x}|v|^{p} d x=0$ and $\widetilde{u}_{x} \geq 0$. Hence, using inequality (2.7), we obtain from (3.2)

$$
\begin{equation*}
\frac{d}{d t} \int_{\mathbb{R}}|v|^{p} d x+4\left(1-\frac{1}{p}\right) \int_{\mathbb{R}}\left(\Lambda^{\alpha / 2}|v|^{p / 2}\right)^{2} d x \leq 0 \tag{3.4}
\end{equation*}
$$

From now on, we proceed by induction. Applying the Nash inequality (2.6) combined with (2.2), we deduce from (3.4) with $p=2$ the following differential inequality

$$
\frac{d}{d t}\|v(t)\|_{2}^{2}+2 C_{N}^{-1}\left\|v_{0}\right\|_{1}^{-2 \alpha}\|v(t)\|_{2}^{2(1+\alpha)} \leq 0
$$

which, after integration, leads to

$$
\begin{equation*}
\|v(t)\|_{2} \leq C_{1}\left\|v_{0}\right\|_{1} t^{-1 /(2 \alpha)} \quad \text { with } \quad C_{1}=\left(C_{N} / 2 \alpha\right)^{1 /(2 \alpha)} \tag{3.5}
\end{equation*}
$$

This is estimate (3.1) with $p=2$.
Suppose now that

$$
\begin{equation*}
\|v(t)\|_{2^{n}} \leq C_{n} t^{-\left(1-2^{-n}\right) / \alpha}\left\|v_{0}\right\|_{1} \quad \text { for all } \quad t>0 \tag{3.6}
\end{equation*}
$$

We consider (3.4) with $p=2^{n+1}$, where the second term is estimated, first, by the Nash inequality (2.6) with $w=|v|^{2^{n}}$, next, by the inductive hypothesis (3.6). This two-step estimate leads to the differential inequality

$$
\frac{d}{d t}\|v(t)\|_{2^{n+1}}^{2^{n+1}}+4\left(1-2^{-n-1}\right) C_{N}^{-1}\left(C_{n}\left\|v_{0}\right\|_{1}\right)^{-\alpha 2^{n+1}} t^{2^{n+1}-2}\left(\|v(t)\|_{2^{n+1}}^{2^{n+1}}\right)^{1+\alpha} \leq 0
$$

Integrating it we obtain

$$
\begin{equation*}
\|v(t)\|_{2^{n+1}} \leq C_{n+1} t^{-\left(1-2^{-n-1}\right) / \alpha}\left\|v_{0}\right\|_{1} \quad \text { for all } \quad t>0 \tag{3.7}
\end{equation*}
$$

with

$$
C_{n+1}=C_{n}\left(\left(C_{N} /(2 \alpha)\right)^{1 / \alpha}\right)^{2^{-n-1}}\left(2^{n 2^{-n-1}}\right)^{1 / \alpha}
$$

This is inequality (3.1) for any $p=2^{n+1}$ with $n \in \mathbb{N}$.
We leave to the reader the proof that $\lim _{\sup }^{n \rightarrow \infty}$ $C_{n}<\infty$. Hence, passing to the limit $n \rightarrow \infty$ in (3.7) we obtain inequality (3.1) for $p=\infty$.

The Hölder inequality

$$
\|v\|_{p} \leq\|v\|_{2^{n}}^{2^{n+1} / p-1}\|v\|_{2^{n+1}}^{2-2^{n+1} / p}
$$

completes the proof for every $p \in\left(2^{n}, 2^{n+1}\right)$.
Lemma 3.2. Let $\alpha \in(1,2)$. Assume that $w=w(x, t)$ is the smooth approximation of the rarefaction wave, namely, the solution of problem (2.20)-(2.21). Then for each $t_{0}>0$ we have

$$
\int_{t_{0}}^{\infty}\left\|w_{x x}(t)\right\|_{p} d t<\infty \quad \text { for every } \quad p \in(1, \infty]
$$

and

$$
\int_{t_{0}}^{t}\left\|\Lambda^{\alpha} w(t)\right\|_{p} d t \leq C \log (2+t) \quad \text { for } \quad p=\frac{3-\alpha}{\alpha-1}
$$

all $t \geq t_{0}$ and $C>0$ independent of $t$.
Proof. It follows from the decay estimates recalled in Lemma 2.4 that

$$
\int_{t_{0}}^{\infty}\left\|w_{x x}(t)\right\|_{p} d t \leq C \int_{t_{0}}^{\infty} t^{-3 / 2+1 /(2 p)} d t<\infty \quad \text { for every } \quad p \in(1, \infty]
$$

By the interpolation inequality (2.5) and Lemma 2.4, we obtain

$$
\begin{aligned}
\left\|\Lambda^{\alpha} w(t)\right\|_{p} & \leq C(1+t)^{(-1+1 / p)(2-\alpha)}(1+t)^{(-3 / 2+1 /(2 p))(\alpha-1)} \\
& =C(1+t)^{-(1+\alpha) / 2+(3-\alpha) /(2 p)}
\end{aligned}
$$

Hence, the rate of decay on the right hand side is equal to -1 for $p=(3-\alpha) /(\alpha-1)$.
Lemma 3.3. Let $\alpha \in(1,2)$. Assume that $u=u(x, t)$ is the solution of (1.1)-(1.2) and $w=w(x, t)-$ of (2.20) $-(2.21)$. Suppose that $u_{0}-w_{0} \in L^{p}(\mathbb{R})$ for $p=(3-\alpha) /(\alpha-1)$. Then

$$
\|u(t)-w(t)\|_{p} \leq C \log (2+t)
$$

Proof. Denoting $v=u-w$, we see that this new function satisfies

$$
v_{t}+\Lambda^{\alpha} v+\frac{1}{2}\left[v^{2}+2 v w\right]_{x}=-\Lambda^{\alpha} w+w_{x x}
$$

We multiply this equation by $|v|^{p-2} v$ and we integrate over $\mathbb{R}$ to obtain

$$
\begin{align*}
& \frac{1}{p} \frac{d}{d t} \int|v|^{p} d x+\int\left(\Lambda^{\alpha} v\right)\left(|v|^{p-2} v\right)+\frac{1}{2} \int\left[v^{2}+2 v w\right]_{x}|v|^{p-2} v d x \\
= & \int\left(-\Lambda^{\alpha} w+w_{x x}\right)\left(|v|^{p-2} v\right) d x \tag{3.8}
\end{align*}
$$

It follows from Lemma 2.3 that $\int_{\mathbb{R}}\left(\Lambda^{\alpha} v\right)\left(|v|^{p-2} v\right) d x \geq 0$. The third term on the left hand side of (3.8) is nonnegative by Lemma 2.4 and the same argument as the one used in the proof of Lemma 3.1, cf. identity (3.3). Moreover, using the Hölder inequality, we have

$$
\left|\int_{\mathbb{R}}\left(-\Lambda^{\alpha} w+w_{x x}\right)\left(|v|^{p-2} v\right) d x\right| \leq\left(\left\|\Lambda^{\alpha} w\right\|_{p}+\left\|w_{x x}\right\|_{p}\right)\|v\|_{p}^{p-1} .
$$

Consequently, (3.8) implies the following differential inequality

$$
\frac{d}{d t}\|v(t)\|_{p}^{p} \leq p\left(\left\|\Lambda^{\alpha} w(t)\right\|_{p}+\left\|w_{x x}(t)\right\|_{p}\right)\|v(t)\|_{p}^{p-1}
$$

which, after integration, leads to

$$
\|v(t)\|_{p} \leq\left\|v\left(t_{0}\right)\right\|_{p}+\int_{t_{0}}^{t}\left\|\Lambda^{\alpha} w(\tau)\right\|_{p}+\left\|w_{x x}(\tau)\right\|_{p} d \tau
$$

The proof is completed by the result stated in Lemma 3.2.

Now, we are in a position to prove the main result of this paper
The proof of Theorem 1.1. First, we consider the auxiliary solution $\widetilde{u}=\widetilde{u}(x, t)$ of the fractal Burgers equation (1.1) with the step-like initial condition (1.10). In this case, the measure $\widetilde{m}=\left(u_{+}-u_{-}\right) \delta_{0}$ is nonnegative, hence by Theorem 2.2, $\widetilde{u}_{x} \geq 0$ and by Lemma 3.1 .

$$
\|u(t)-\widetilde{u}(t)\|_{p} \leq C t^{-(1-1 / p) / \alpha}\left\|u_{0}-\widetilde{u}_{0}\right\|_{1}
$$

for every $p \in[1, \infty]$ and all $t>0$.
Next, we compare $\widetilde{u}$ with the smooth approximation of the rarefaction wave that is with the solution $w=w(x, t)$ of (2.20) $-(2.21)$ (note that $\left.\widetilde{u}_{0}=w_{0}\right)$. By Theorem 2.2 and Lemma 2.4, we obtain

$$
\left\|\widetilde{u}_{x}(t)\right\|_{\infty}+\left\|w_{x}(t)\right\|_{\infty} \leq C t^{-1} .
$$

Moreover, using the following Gagliardo-Nirenberg inequality

$$
\begin{equation*}
\|v\|_{p} \leq C\left\|v_{x}\right\|_{\infty}^{a}\|v\|_{p_{0}}^{1-a}, \tag{3.9}
\end{equation*}
$$

valid for any $1<p_{0}<p \leq \infty$ and $a=\left(1 / p_{0}-1 / p\right) /\left(1+1 / p_{0}\right)$, we have

$$
\begin{aligned}
\|\widetilde{u}(t)-w(t)\|_{p} & \leq C\left(\left\|\widetilde{u}_{x}(t)\right\|_{\infty}+\left\|w_{x}(t)\right\|_{\infty}\right)^{a}\|\widetilde{u}(t)-w(t)\|_{p_{0}}^{1-a} \\
& \leq C t^{-a}\|\widetilde{u}(t)-w(t)\|_{p_{0}}^{1-a} .
\end{aligned}
$$

Choosing $p_{0}=(3-\alpha) /(\alpha-1)$ (hence $\left.a=[\alpha-1-(3-\alpha) / p] / 2\right)$, by Lemma 3.3, we conclude $\|\widetilde{u}(t)-w(t)\|_{p} \leq C t^{-a} \log (2+t)$ for every $p \in\left(p_{0}, \infty\right]$. Here, we are allowed to use Lemma 3.3 because $\widetilde{u}_{0}-w_{0} \in L^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \subset L^{p}(\mathbb{R})$ for every $p \in[1, \infty]$.

Finally, it follows from Lemma 2.4 that the large time asymptotics of $w(t)$ is described in $L^{p}(\mathbb{R})$ by the rarefaction wave $w^{R}(x, t)$.

The proof is complete because for $1<\alpha<2$ we have $(1-1 / p) / \alpha>(1-1 / p) / 2$. Moreover, since $1<p_{0}<p$, we have $(1-1 / p) / 2>\left(1 / p_{0}-1 / p\right) /\left(1+1 / p_{0}\right)$.

## 4 Additional comments and possible generalizations

Our main result is stated and shown in the simplest case of equation (1.1), however, several generalizations are possible.

First of all, the operator $\Lambda^{\alpha}$ can be replaced by the Lévy operator $\mathcal{L}$ which is a pseudodifferential operator defined by the symbol $a=a(\xi) \geq 0, \widehat{\mathcal{L} v}(\xi)=a(\xi) \widehat{v}(\xi)$. Here, the function $e^{-t a(\xi)}$ should be positive-definite, so the symbol $a(\xi)$ can be represented by the Lévy-Khintchine formula in the Fourier variables

$$
\begin{equation*}
a(\xi)=i b \xi+Q(\xi)+\int_{\mathbb{R}}\left(1-e^{-i \eta \xi}-i \eta \xi \mathbb{I}_{\{|\eta|<1\}}(\eta)\right) \Pi(d \eta) \tag{4.1}
\end{equation*}
$$

Here, $b \in \mathbb{R}$ is fixed, $Q(\xi)=q \xi^{2}$ with some $q \geq 0$, and $\Pi$ is a Borel measure such that $\Pi(\{0\})=0$ and $\int_{\mathbb{R}} \min \left(1,|\eta|^{2}\right) \Pi(d \eta)<\infty$.

Detailed analysis of conservation laws with the anomalous diffusion operator $\mathcal{L}$ is contained in papers [3, 4, 5]. Here, we would like to emphasize that the fundamental nature of the operator $\mathcal{L}$ is clear from the perspective of probability theory. It represents the most general form of generator of a stochastically continuous Markov process with independent and stationary increments. This fact was our basic motivation for the development of the theory presented above.

In order to show the convergence toward rarefaction waves of solutions of conservation laws with the Lévy operator, we need the counterparts of estimates (2.13)-(2.14) of the semigroup of linear operators $e^{-t \mathcal{L}}$ generated by $-\mathcal{L}$. They are valid e.g. under the assumption that the symbol $a$ of $\mathcal{L}$ has the form

$$
\begin{equation*}
a(\xi)=\ell|\xi|^{\alpha}+k(\xi) \tag{4.2}
\end{equation*}
$$

where $\ell>0,0<\alpha \leq 2$ and $k$ is a symbol of another Lévy operator $\mathcal{K}$ such that

$$
\begin{equation*}
\lim _{\xi \rightarrow 0} \frac{k(\xi)}{|\xi|^{\alpha}}=0 \tag{4.3}
\end{equation*}
$$

The assumptions (4.2) and (4.3) are fulfilled, for example, by multifractal diffusion operators

$$
\mathcal{L}=-a_{0} \partial_{x}^{2}+\sum_{j=1}^{k} a_{j}\left(-\partial_{x}^{2}\right)^{\alpha_{j} / 2}
$$

with $a_{0} \geq 0, a_{j}>0,1<\alpha_{j}<2$, and $\alpha=\min _{1 \leq j \leq k} \alpha_{j}$. We refer the reader to [4, 5] for the reasoning leading to the decay estimates of solution of nonlinear problem with operator $\mathcal{L}$ satisfying (4.2)-(4.3). That argument can be directly adapted to obtain counterparts of Theorem 2.2 and Lemma 3.1 with $\Lambda^{\alpha}$ replaced by $\mathcal{L}$. Note here that the $L^{p}-L^{q}$ estimates of the semigroup $e^{-t \mathcal{L}}$ are equivalent to a certain Nash inequality, see the papers [15, 4] and the references therein.

Our result also holds true, if we replace the nonlinear term $u u_{x}$ in (1.1) by $f(u)_{x}$ with a strictly convex $C^{2}$-function $f$ (as in the paper of Il'in and Oleinik [10]) satisfying
$f^{\prime \prime}(u) \geq \kappa$ for some fixed $\kappa>0$ and all $u \in \mathbb{R}$. Under this assumption, we immediately generalize Theorem 2.2 and we obtain the decay estimate (2.3).

In order to show the counterpart of Lemma 3.1, we should use the assumption $f^{\prime \prime}(u) \geq$ $\kappa$ and to replace equalities (3.3) by the following (recall that $v=u-\widetilde{u}$ )

$$
\int_{\mathbb{R}}[f(u)-f(\widetilde{u})]_{x}|v|^{p-2} v d x \geq \kappa\left(1-\frac{1}{p}\right) \int_{\mathbb{R}} \widetilde{u}_{x}|v|^{p} d x \geq 0
$$

This argument, however, is known and used systematically e.g. in [22, inequality (3.5)] (see, also [10, 18, 19, 21]), hence we skip other details.

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## References

[1] N. Alibaud, J. Droniou, and J. Vovelle, Occurrence and non-appearance of shocks in fractal Burgers equations, (2006) preprint.
[2] P. Biler, T. Funaki, and W. Woyczynski, Fractal Burgers equations, J. Differential Equations 148 (1998), 9-46.
[3] P. Biler, G. Karch, and W. Woyczynski, Asymptotics for multifractal conservation laws, Studia Math. 135 (1999), 231-252.
[4] P. Biler, G. Karch, W. A. Woyczyńsky, Asymptotics for conservation laws involving Lévy diffusion generators, Studia Math. 148 (2001), 171-192.
[5] P. Biler, G. Karch, W. A. Woyczyńsky, Critical nonliearity exponent and self-similar asymptotics for Lévy conservation laws, Ann. Inst. Henri Poincaré, Analyse nonlinéaire, 18 (2001), 613-637
[6] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004), 511-528.
[7] J. Droniou, T. Gallou et, J. Vovelle, Global solution and smoothing effect for a nonlocal regularization of a hyperbolic equation, J. Evol. Equ. 3 (2002), 499 - 521.
[8] J. Droniou and C. Imbert, Fractal first order partial differential equations, Arch. Rat. Mech. Anal. 182 (2006), 299-331.
[9] Y. Hattori and K. Nishihara, A note on the stability of the rarefaction wave of the Burgers equation, Japan J. Indust. Appl. Math. 8 (1991), 85-96.
[10] A. M. Il'in and O. A. Oleinik, Asymptotic behavior of solutions of the Cauchy problem for some quasi-linear equations for large values of the time, Mat. Sb. (N.S.) 51 (93) (1960), 191-216.
[11] B. Jourdain, S. Méléard and W. Woyczynski, A probabilistic approach for nonlinear equations involving the fractional Laplacian and singular operator, Potential Analysis 23 (2005), 55-81.
[12] B. Jourdain, S. Méléard and W. Woyczynski, Probabilistic approximation and inviscid limits for one-dimensional fractional conservation laws, Bernoulli 11 (2005), 689-714.
[13] N. Ju, The maximum principle and the global attractor for the dissipative 2D QuasiGeostrophic equations, Comm. Math. Phys., 255 (2005), 161-181.
[14] S. Kawashima and Y. Tanaka, Stability of rarefaction waves for a model system of a radiating gas, Kyushu J. Math. 58 (2004), 211-250.
[15] V. A. Liskevich, Yu. A. Semenov, Some problems on Markov semigroups, Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras, 163-217, Math. Top., 11, Akademie Verlag, Berlin, 1996.
[16] T.-P. Liu, A. Matsumura, and K. Nishihara, Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves, SIAM J. Math. Anal., 29 (1998), 293-308.
[17] J.A. Mann and W.A. Woyczynski, Growing fractal interfaces in the presence of selfsimilar hopping surface diffusion, Phys. A 291 (2001), 159-183.
[18] A. Matsumura and K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, Japan J. Appl. Math. 3 (1986), 1-13.
[19] A. Matsumura and K. Nishihara, Global stability of the rarefaction wave of a onedimensional model system for compressible viscous gas, Comm. Math. Phys. 144 (1992), 325-335.
[20] C. Miao and B. Yuan and B. Zhang, Well-posedness of the Cauchy problem for fractional power dissipative equations, arXiv:math.AP/0607456 v2 (2006), 1-30, to appear in Nonlinear Analysis.
[21] T. Nakamura, Asymptotic decay toward the rarefaction waves of solutions for viscous conservation laws in a one-dimensional half space, SIAM J. Math. Anal. 34 (2003), 1308-1317.
[22] M. Nishikawa and K. Nishihara, Asymptotics toward the planar rarefaction wave for viscous conservation law in two space dimensions, Trans. Amer. Math. Soc. 352 (1999), 1203-1215.

