

# A PROXIMAL-PROJECTION METHOD FOR FINDING ZEROS OF SET-VALUED OPERATORS

DAN BUTNARIU\* AND GÁBOR KASSAY\*\*

ABSTRACT. In this paper we study the convergence of an iterative algorithm for finding zeros with constraints for not necessarily monotone set-valued operators in a reflexive Banach space. This algorithm, which we call the proximal-projection method is, essentially, a fixed point procedure and our convergence results are based on new generalizations of Lemma Opial. We show how the proximal-projection method can be applied for solving ill-posed variational inequalities and convex optimization problems with data given or computable by approximations only. The convergence properties of the proximal-projection method we establish also allow us to prove that the proximal point method (with Bregman distances), whose convergence was known to happen for maximal monotone operators, still converges when the operator involved in it is monotone with sequentially weakly closed graph.

## 1. INTRODUCTION

In that follows  $X$  denotes a real reflexive Banach space with norm  $\|\cdot\|$  and  $X^*$  denotes the (topological) dual of  $X$  with the dual norm  $\|\cdot\|_*$ . Let  $f : X \rightarrow (-\infty, +\infty]$  be a proper, lower semicontinuous convex function with domain  $\text{dom } f$ . Then, the Fenchel conjugate  $f^* : X^* \rightarrow (-\infty, +\infty]$  is also a proper lower semicontinuous convex function and  $f^{**} := (f^*)^* = f$ . We assume that  $f$  is a Legendre function in the sense given to this term in [19, Definition 5.2], that is,  $f$  is essentially smooth and essentially strictly convex. Then, according to [19, Theorem 5.4], the function  $f^*$  is a Legendre function too. Moreover, by [19, Theorem 5.6 and Theorem 5.10], both functions  $f$  and  $f^*$  have domains with nonempty interior, are (Gâteaux) differentiable on the interiors of their respective domains,

$$(1.1) \quad \text{ran } \nabla f = \text{dom } \nabla f^* = \text{int dom } f^* = \text{dom } \partial f^*,$$

---

*Date:* February 4, 2007.

*2000 Mathematics Subject Classification.* Primary 90C25, 47J25, 47J20; Secondary 90C30, 90C48, 47N10.

*Key words and phrases.* Bregman distance, Bregman projection,  $D_f$ -antiresolvent,  $D_f$ -nonexpansivity pole,  $D_f$ -coercive operator,  $D_f$ -firm operator,  $D_f$ -nonexpansive operator,  $D_f$ -resolvent, firmly nonexpansive operator, Legendre function, maximal monotone operator, monotone operator, nonexpansive operator, proximal mapping, proximal point method, proximal projection method, relative projection, sequentially consistent function, projected subgradient method, Tikhonov-Browder regularization, strongly monotone operator, uniformly convex function, variational inequality.

\*Dan Butnariu's work on this paper was partially done during his visit within The Graduate Center of the City University of New York and he gratefully acknowledges Professor Gabor Herman's support and the informative discussions and activities in which he was invited to take part.

\*\*Gabor Kassay gratefully acknowledges the support of Cesarea Benjamin de Rothschild Foundation during his May 2006 visit within the Department of Mathematics of the University of Haifa, Israel, when part of this research was completed.

$$(1.2) \quad \text{ran } \nabla f^* = \text{dom } \nabla f = \text{int dom } f = \text{dom } \partial f,$$

and

$$(1.3) \quad \nabla f = (\nabla f^*)^{-1}.$$

With the function  $f$  we associate the function  $W_f : X^* \times X \rightarrow (-\infty, +\infty]$  defined by

$$(1.4) \quad W_f(\xi, x) = f(x) - \langle \xi, x \rangle + f^*(\xi).$$

By the Young-Fenchel inequality the function  $W_f$  is nonnegative and  $\text{dom } W_f = (\text{dom } f^*) \times (\text{dom } f)$ . It is known (see [1], [2], [18], [36] and see also Section 2 below) that, for any nonempty closed convex set  $E$  contained in  $X$  such that  $E \cap \text{int dom } f \neq \emptyset$ , the function  $\text{Proj}_E^f : \text{int dom } f^* \rightarrow X$  given by

$$(1.5) \quad \text{Proj}_E^f \xi = \arg \min \{W_f(\xi, x) : x \in E\},$$

is well defined and its range is contained in  $E \cap \text{int dom } f$ . In fact, this function is a particular proximal projection in the sense given to this term in [20] which was termed in [36] projection onto  $E$  relative to  $f$  because in the particular case when  $X$  is a Hilbert space and  $f(x) = \frac{1}{2} \|x\|^2$  the vector  $\text{Proj}_E^f \xi$  coincides with the usual (metric) projection of  $\xi$  onto  $E$ .

In this paper we are interested in the following problem:

**Problem 1.1:** *Given an operator  $A : X \rightarrow 2^{X^*}$  and a nonempty closed subset  $C$  of  $X$  such that*

$$(1.6) \quad \emptyset \neq C \cap \text{dom } A \subseteq \text{int dom } f,$$

*find  $x \in C$  such that  $0^* \in Ax$ , where  $0^*$  denotes the null vector in  $X^*$ .*

Our purpose is to discover sufficient conditions for the following iterative procedure, which we call *the proximal-projection method*,

$$(1.7) \quad \begin{aligned} x^0 &\in C_0 \cap \text{dom } A \cap \text{int dom } f \quad \text{and} \\ x^{k+1} &\in \text{Proj}_{C_{k+1} \cap \text{dom } A}^f (\nabla f(x^k) - Ax^k), \quad \forall k \in \mathbb{N}, \end{aligned}$$

to generate approximations of solutions to the Problem 1.1 when  $\{C_k\}_{k \in \mathbb{N}}$  is a sequence of closed convex subsets of  $X$  approximating weakly (see Definition 4.2 below) the set  $C$  under the following conditions of compatibility of  $f$  and  $C_k$  with the data of Problem 1.1:

**Assumption 1.1:** *For each  $k \in \mathbb{N}$ , the set  $C_k \cap \text{dom } A$  is convex and closed,*

$$(1.8) \quad C \subseteq C_k \quad \text{and} \quad (\nabla f - A)(C_k) \subseteq \text{int dom } f^*.$$

In this paper (1.6) and Assumption 1.1 are standing assumptions, even if not explicitly mentioned, whenever we refer to the Problem 1.1 or to the proximal-projection method. In view of (1.6) and (1.8) the sets  $C_k \cap \text{dom } A \cap \text{int dom } f$  are necessarily nonempty and, consequently,  $(\nabla f - A)(C_k)$  is nonempty too. This fact, Assumption 1.1 and Lemma 2.1 below which ensures that

$$\text{dom Proj}_{C_k \cap \text{dom } A}^f = \text{int dom } f^* \quad \text{and} \quad \text{ran Proj}_{C_k \cap \text{dom } A}^f \subseteq C_k \cap \text{dom } A \cap \text{int dom } f,$$

taken together, guarantee that the procedure of generating sequences in (1.7) is well defined.

The proximal-projection method described above is a natural generalization of the method of finding zeros of linear operators due to Landweber [49], of Shor's [70]

(see also [71]) and Ermoliev's [44] "gradient descent" methods for finding unconstrained minima of convex functions and of the "projected-subgradient" method for finding constrained minima of convex functions studied by Polyak [62]. These methods inspired the construction of a plethora of algorithms for finding zeros of various operators as well as for other purposes. Among them are the algorithms presented in [1], [3], [4], [6], [7], [8], [9], [10], [11], [12], [13], [14], [17], [20], [29], [35], [36], [69] which, in turn, inspired this research. The main formal differences between the proximal-projection method (1.7) and its already classical counterparts developed in the 50-ies and 60-ies consist of the use of the proximal projections instead of metric projections and of projecting not on the set  $C$  involved in Problem 1.1, but on some convex approximations  $C_k$  of it. The use of proximal projections instead of metric projections is mostly due to the fact that metric projections in Banach spaces which are not Hilbertian do not have many of those properties (like single valuedness and nonexpansivity) which make them so useful in a Hilbert space setting for establishing convergence of algorithms based on them. As far as we know, the idea of using proximal projections instead of metric projections in projected-subgradient type algorithms goes back to Alber's works [1], [2].

As we make clear in Section 3.3, there is an intimate connection between the proximal-projection method and the well-known proximal point method with Bregman distances – see (3.23). The proximal point method with Bregman distances considered in this paper is itself a generalization of the classical proximal point algorithm developed since the ninety-fifties by Krasnoselskii [48], Moreau [55], [56], [57], Yosida [73], Martinet [53], [54] and Rockafellar [66], [67] among others (see [50] for a survey of the literature concerning the classical proximal point method). It emerged from the works of Erlander [43], Eriksson [42], Eggermont [41] and Eckstein [40] who studied various instances of the algorithm in  $\mathbb{R}^n$ . Its convergence analysis in Banach spaces which are not necessarily Hilbertian was initiated in [30], [31] and [47] (see [32] and [30] for related references on this topic). Lemma 3.5 shows that, in our setting, the proximal point method with Bregman distances is a particular instance of the proximal-projection method.

Computing projections, metric or proximal, onto a closed convex set with complicated geometry is, in itself, a challenging problem. It requires (see (1.5)) solving convex nonlinear programming problems with convex constraints. Specific techniques for finding proximal projections are presented in [5], [21] and [33]. It is obvious from these works that it is much easier to find proximal projections onto sets with simple geometry like, for instance, hyperplanes, half spaces or finite intersections of such sets. These facts naturally led to the question of whether it is possible to replace in the process of computing iterates of metric or proximal projections algorithms the constraint set  $C$  by some approximations  $C_k$  of it whose geometry is simple enough to allow relatively easy calculation of the required metric or proximal projections at each iterative step  $k$ . That this approach is sound is quite clear from the works of Mosco [58] and its subsequent developments due to Attouch [15], Aubin and Frankowska [16], Dontchev and Zolezzi [39], and from the studies of Liskovets [51], [52]. Its main difficulty in the case of the proximal-projection method is that the approximations  $C_k$  one uses should converge to  $C$  in a manner that ensures stable convergence of the algorithm to solutions of the problem. For some variants of the proximal-projection method, types of convergence of the sets  $C_k$  to  $C$  which are sufficiently good for this purpose are presented in [4], [7], [9],

[12], [13]. They mostly are relaxed forms of Hausdorff metric convergence. It was shown in [8] that, in some circumstances, fast Mosco convergence of the sets  $C_k$  to  $C$  (see [8, Definition 2.1]), a form of convergence significantly less demanding than Hausdorff convergence, is sufficient to make the proximal-projection method (1.7) applied to variational inequalities converge. As we show below, these convergence requirements in the case of the proximal-projection method (1.7) can be further weakened. In fact, in our convergence theorems for the proximal-projection method we only require weak Mosco convergence of the sets  $C_k$  to  $C$  (see Definition 4.3) and this is significantly demanding than Hausdorff metric or fast Mosco convergence.

The purpose of this work is to find general conditions which guarantee that the proximal-projection method converges weakly or strongly to solutions of the Problem 1.1. Observe that there is no apparent connection between the data of Problem 1.1 and the function  $f$  involved in the definition of the proximal-projection method. Our main question is how the function  $f$  should be chosen in order to ensure (weak or strong) convergence of the proximal-projection method to solutions of Problem 1.1 without excessively conditioning the problem data. The function  $f$  is a parameter of the proximal-projection method whose appropriate choice, as we show below, can make the procedure converge to solutions of Problem 1.1 even if the problem data are quite "bad" in the sense that they do not have some, usually difficult to verify in practice properties like maximal monotonicity, strict monotonicity, various forms of nonexpansivity, continuity or closedness properties of some kind or another. Theorem 4.1 and Theorem 5.1 are our responses to the question posed above.

Theorem 4.1 shows that for guaranteeing that the proximal-projection method produces weak approximations of solutions for Problem 1.1 it is sufficient to choose a function  $f$  which, besides the conditions (1.6) and (1.8) which are meant to make the procedure consistent with the problem data, should satisfy some requirements which, most of them, are common features of the powers of the norm  $\|\cdot\|^p$  with  $p > 1$  in uniformly convex and smooth Banach spaces. The only somehow outstanding condition which we require for  $f$  is that it should be such that the operator  $A$  involved in the problem be  $D_f$ -coercive on  $C$  or, if the set  $C$  is approximated by sets  $C_k$ , then  $D_f$ -coercivity of  $A$  should happen on the union of those sets.  $D_f$ -coercivity, a notion introduced in this paper (see Definition 3.2), is a generalization of the notion of firm nonexpansivity for operators in a Hilbert space (see (4.34)). Although in Hilbert spaces provided with the function  $f = \frac{1}{2} \|\cdot\|^2$  this notion coincides with the notion of firm nonexpansivity and, also, with the notion of  $D_f$ -firmness introduced in [20] (see Definition 3.3 below), outside this particular setting the notions of  $D_f$ -coercivity and  $D_f$ -firmness complement each other (cf. Section 3.2). In Section 4.3 we present several corollaries of Theorem 4.1 and examples which clearly show that fitting a function  $f$  to the specific data of Problem 1.1 may come naturally in many situations. If  $X$  is a Hilbert space and  $A$  is a firmly nonexpansive operator, then the natural choice is  $f = \frac{1}{2} \|\cdot\|^2$  (see Corollary 4.1). In this case the convergence of the proximal-projection method happens to be strong if the set of solutions of the problem has nonempty interior. If in Problem 1.1 we have  $C = X$ , then the problem is equivalent to that of finding a zero for the operator  $\nabla f - \nabla f \circ A_f$ , where  $A_f$  is the  $D_f$ -resolvent of  $A$  (a notion introduced in [20] - see also (3.19)) and application of the proximal-projection method with  $C_k = X$  to

$\nabla f - \nabla f \circ A_f$  is no more and no less than the proximal point method with Bregman distances mentioned above. The operator  $\nabla f - \nabla f \circ A_f$  is  $D_f$ -coercive whenever the operator  $A$  is monotone (cf. Lemma 3.4). This leads us to the application of Theorem 4.1 to the operator  $\nabla f - \nabla f \circ A_f$  which is Corollary 4.2. It shows that the proximal point algorithm with Bregman distances converges subsequentially weakly (and when  $A$  has a single zero, sequentially weakly) for a large class of functions  $f$ , whenever  $A$  is monotone and provided that its graph is sequentially weakly closed (as happens, for instance, when  $\text{Graph } A$  is convex and closed in  $X \times X^*$ ). It seems to us that this is the first time when weak convergence of the proximal point algorithm with Bregman distances is proved without requiring maximal monotonicity of  $A$ . The proximal-projection method is also a tool for solving monotone variational inequalities via their Tikhonov-Browder regularization. This is shown by Corollary 4.3, another consequence of Theorem 4.1. Corollary 4.3 also asks for the monotone operator  $B : X \rightarrow 2^{X^*}$  involved in the variational inequality to be such that  $\nabla f - \nabla f \circ \text{Proj}_C^f \circ [(1 - \alpha)\nabla f - B]$  is  $D_f$ -coercive (or, equivalently, such that  $\text{Proj}_C^f \circ [(1 - \alpha)\nabla f - B]$  to be  $D_f$ -firm). This happens in many situations of practical interest. Several such situations are described in the Examples 4.1 and 4.2.

A careful analysis of the proof of Theorem 4.1 reveals the fact that the proximal-projection method (1.7) is a procedure of approximating fixed points for the operator  $\text{Proj}_C^f \circ (\nabla f - A)$  by iterating the operator. A customary tool of proving convergence of such algorithms in Hilbert spaces is the already classical Opial Lemma [59, Lemma 2]. Unfortunately, this result can not be extrapolated into a nonhilbertian setting in its original form. Our Theorem 4.1 is based on Proposition 4.1, a generalization of the Opial Lemma which works in reflexive Banach spaces and which is of interest by itself. If the Banach space  $X$  has finite dimension, Proposition 4.1 can be substantially improved – see Proposition 5.1. Thus, in spaces with finite dimension we can also improve Theorem 4.1 by dropping some of the requirements made on the problem data. This is, in fact, our Theorem 5.1 which guarantees convergence of the proximal-projection method with less demanding conditions than closedness of the graph of  $A$ . Accordingly, in finite dimensional spaces the conclusions of Corollaries 4.2 and 4.3 can be reached at lesser cost for the operators involved in them as shown by Corollaries 5.1 and 5.2, respectively.

This paper continues and develops a series of concepts, methods and techniques initiated in [1], [18], [20], [32], [36] and [47]. In Sections 2 and 3 we present in a unified approach the notions, notations and preliminary results on which our convergence analysis of the proximal-projection method is based. It should be noted that, in Section 2, some of the notions and results are presented in a more general setting than strictly needed in the subsequent parts of the material. This is done so because we hope to use the framework created in the current paper as a base for a forthcoming study of methods of solving nonclassical variational inequalities which are only tangentially approached here.

## 2. PROXIMAL MAPPINGS, RELATIVE PROJECTIONS AND VARIATIONAL INEQUALITIES

In this section we present the notions, notations and results concerning proximal projections and variational inequalities which are essential for the convergence analysis of the proximal-projection method done in the sequels.

**2.1. Proximal mappings and relative projections.** All over this paper we denote by  $\mathcal{F}_f$  the set of proper, lower semicontinuous, convex functions  $\varphi : X \rightarrow (-\infty, +\infty]$  which satisfy the conditions that

$$(2.1) \quad \text{dom } \varphi \cap \text{int dom } f \neq \emptyset,$$

and

$$(2.2) \quad \varphi_f := \inf \{ \varphi(x) : x \in \text{dom } \varphi \cap \text{dom } f \} > -\infty.$$

With every  $\varphi \in \mathcal{F}_f$  we associate the function  $\text{Env}_\varphi^f : X^* \rightarrow (-\infty, +\infty]$  given by

$$(2.3) \quad \text{Env}_\varphi^f(\xi) = \inf \{ \varphi(x) + W_f(\xi, x) : x \in X \}.$$

This is a natural generalization of the notion of *Moreau envelope function* (see [68, Definition 1.22]). By (2.1) and (2.2) it results that the function  $\text{Env}_\varphi^f$  is proper and  $\text{dom } \text{Env}_\varphi^f = \text{dom } f^*$ . Another generalization of the *Moreau envelope function* is the function  $\text{env}_\varphi^f := \text{Env}_\varphi^f \circ \nabla f$  introduced and studied in [22]. Using Fenchel's duality theorem, it is easy to deduce that if  $\varphi \in \mathcal{F}_f$ , then

$$\text{Env}_\varphi^f(\xi) = f^*(\xi) - (\varphi + f)^*(\xi) = f^*(\xi) - (\varphi^* \square f^*)(\xi).$$

where  $\varphi^* \square f^*$  denotes the infimal convolution of  $\varphi^*$  and  $f^*$ .

The next result shows a way of generalizing the notion of Moreau proximal mapping (in the sense given to this term in [68]) whose study was initiated in [55], [56], [57] and further developed in [66], [67]. As we will make clear below, the generalization we propose here slightly differs from the notion of  $D_f$ -proximal mapping introduced and studied in [20]. In fact, most of the next lemma can be also deduced from [20, Propositions 3.22 and 3.23] due to the equality (2.7) established below. We prefer to present it here with a direct proof for sake of completeness.

**Lemma 2.1:** *Suppose that  $\varphi \in \mathcal{F}_f$ . For any  $\xi \in \text{int dom } f^*$  there exists a unique global minimizer, denoted  $\text{Prox}_\varphi^f(\xi)$ , of the function  $\varphi(\cdot) + W_f(\xi, \cdot)$ . The vector  $\text{Prox}_\varphi^f(\xi)$  is contained in  $\text{dom } \partial\varphi \cap \text{int dom } f$  and we have*

$$(2.4) \quad \text{Prox}_\varphi^f(\xi) = [\partial(\varphi + f)]^{-1}(\xi) = (\partial\varphi + \nabla f)^{-1}(\xi).$$

**Proof.** Let  $\xi \in \text{int dom } f^*$ . Then  $\text{Env}_\varphi^f(\xi)$  is finite and the function  $f - \langle \xi, \cdot \rangle$  is coercive (see [64, Theorem 7A] or [19, Fact 3.1]), that is, its sublevel sets

$$\text{lev}_{\leq}^f(\alpha) := \{x \in X : f(x) \leq \alpha\},$$

are bounded for all  $\alpha \in \mathbb{R}$ . Consequently, the function  $W_f(\xi, \cdot)$  is coercive too. Let  $\{x^k\}_{k \in \mathbb{N}}$  be a sequence contained in  $\text{dom } \varphi \cap \text{dom } f$  and such that

$$\lim_{k \rightarrow \infty} [\varphi(x^k) + W_f(\xi, x^k)] = \text{Env}_\varphi^f(\xi).$$

The sequence  $\{\varphi(x^k) + W_f(\xi, x^k)\}_{k \in \mathbb{N}}$  being convergent is also bounded. So, for some real number  $M > 0$  we have

$$W_f(\xi, x^k) \leq M - \varphi(x^k) \leq M - \varphi_f,$$

showing that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  is contained in the sublevel set  $\text{lev}_{\leq}^\psi(M - \varphi_f)$  of the function  $\psi := W_f(\xi, \cdot)$ . By the coercivity of  $W_f(\xi, \cdot)$  it follows that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  is bounded. Since the space  $X$  is reflexive, it results that  $\{x^k\}_{k \in \mathbb{N}}$  has a weakly convergent subsequence  $\{x^{i_k}\}_{k \in \mathbb{N}}$ . Let  $\bar{x} = \text{w-lim}_{k \rightarrow \infty} x^{i_k}$ . The functions  $f$  and  $\varphi$  are sequentially weakly lower semicontinuous because they

are lower semicontinuous and convex. Hence,  $\varphi(\cdot) + W_f(\xi, \cdot)$  is also sequentially weakly lower semicontinuous and, thus, we have

$$\varphi(\bar{x}) + W_f(\xi, \bar{x}) \leq \liminf_{k \rightarrow \infty} [\varphi(x^{i_k}) + W_f(\xi, x^{i_k})] = \text{Env}_\varphi^f(\xi) < +\infty.$$

This implies that  $\bar{x} \in \text{dom } \varphi \cap \text{dom } f$  and that  $\bar{x}$  is a minimizer of  $\varphi(\cdot) + W_f(\xi, \cdot)$ .

Suppose that  $y$  is any minimizer of  $\varphi(\cdot) + W_f(\xi, \cdot)$ . Then  $y$  is also a minimizer of  $\varphi + f - \xi$ . Therefore, we have that  $0 \in \partial(\varphi + f - \xi)(y)$ , that is,  $\xi \in \partial(\varphi + f)(y)$ . The function  $f$  is continuous on  $\text{int dom } f$  (as being convex and lower semicontinuous). This and (2.1) imply (see [65]) that  $\partial(\varphi + f)(y) = \partial\varphi(y) + \nabla f(y)$ . Hence,

$$(2.5) \quad \xi \in \partial\varphi(y) + \nabla f(y).$$

Since  $\text{dom } \nabla f = \text{int dom } f$  (see (1.2)), this implies that

$$y \in \text{dom } \partial\varphi \cap \text{dom } \nabla f = \text{dom } \partial\varphi \cap \text{int dom } f.$$

Hence, all minimizers of  $\varphi(\cdot) + W_f(\xi, \cdot)$  are contained in  $\text{dom } \partial\varphi \cap \text{int dom } f \subseteq \text{dom } \varphi \cap \text{int dom } f$ . The Legendre function  $f$  is strictly convex on the convex subsets of  $\text{dom } \partial f$  and, in particular, on the convex set  $\text{dom } \varphi \cap \text{int dom } f = \text{dom } \varphi \cap \text{dom } \partial f$ . Thus,  $\varphi(\cdot) + W_f(\xi, \cdot)$  is strictly convex on this set too. Consequently, there is at most one minimizer of  $\varphi(\cdot) + W_f(\xi, \cdot)$  on the convex set  $\text{dom } \varphi \cap \text{int dom } f$  and this proves that the minimizer  $\bar{x}$  whose existence was established above is unique. Formula (2.4) follows from (2.5) when  $y = \bar{x}$ .  $\square$

Lemma 2.1 ensures well definedness of the function

$$(2.6) \quad \text{Prox}_\varphi^f : \text{int dom } f^* \rightarrow \text{dom } \partial\varphi \cap \text{int dom } f, \quad \xi \rightarrow \text{Prox}_\varphi^f(\xi)$$

when  $\varphi \in \mathcal{F}_f$ . We call this function *the proximal mapping relative to  $f$  associated to  $\varphi$* . Well definedness of the proximal mappings relative to the Legendre function  $f$  can also be deduced from [20, Theorem 3.18] where well definedness of the resolvent

$$(2.7) \quad \text{prox}_\varphi^f := \text{Prox}_\varphi^f \circ \nabla f$$

was established. In more particular circumstances for  $f$  and  $\varphi$ , well definedness of  $\text{Prox}_\varphi^f$  was proved in [1], [31], and in [36].

Let  $E$  be a closed convex subset of  $X$  satisfying

$$(2.8) \quad E \cap \text{int dom } f \neq \emptyset.$$

Then the indicator function of the set  $E$ , that is, the function  $\iota_E : X \rightarrow (-\infty, +\infty]$  defined by  $\iota_E(x) = 0$  if  $x \in E$ , and  $\iota_E(x) = +\infty$ , otherwise, is contained in  $\mathcal{F}_f$ . The operator  $\text{Prox}_{\iota_E}^f$  is called *projection onto  $E$  relative to  $f$*  (cf. [36]) and is denoted  $\text{Proj}_E^f$  in that follows. According to Lemma 2.1, we have

$$\text{Proj}_E^f = (N_E + \nabla f)^{-1}$$

where  $N_E$  denotes the normal cone operator associated to the set  $E$ . The operator

$$(2.9) \quad \text{proj}_E^f := \text{prox}_{\iota_E}^f = \text{Proj}_E^f \circ \nabla f$$

is exactly the *Bregman projection onto  $E$  relative to  $f$*  whose importance in convex optimization was first emphasized in [23]. To see that it is sufficient to recall that the *Bregman distance*  $D_f : X \times \text{int dom } f \rightarrow (-\infty, +\infty]$  is the function defined by

$$(2.10) \quad D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle = W_f(\nabla f(x), y),$$

with  $\text{dom } D_f = (\text{dom } f) \times (\text{int dom } f)$ .

**2.2. Variational inequalities.** There is an intimate connection between proximal mappings and variational inequalities. It is based on the following result which extends the variational characterization of  $\text{Proj}_E^f$  given in [36] to a variational characterization of  $\text{Prox}_\varphi^f$ .

**Lemma 2.2.** *Suppose that  $\varphi \in \mathcal{F}_f$  and  $\xi \in \text{int dom } f^*$ . If  $\hat{x} \in \text{dom } \partial\varphi \cap \text{int dom } f$  then the following conditions are equivalent:*

- (a)  $\hat{x} = \text{Prox}_\varphi^f(\xi)$ ;
- (b)  $\hat{x}$  is a solution of the variational inequality

$$(2.11) \quad \langle \xi - \nabla f(x), y - x \rangle \leq \varphi(y) - \varphi(x), \quad \forall y \in \text{dom } \varphi \cap \text{dom } f.$$

- (c)  $\hat{x}$  is a solution of the variational inequality

$$W_f(\xi, x) + W_f(\nabla f(x), y) - W_f(\xi, y) \leq \varphi(y) - \varphi(x), \quad \forall y \in \text{dom } \varphi \cap \text{dom } f.$$

**Proof:**  $\boxed{(a) \Rightarrow (b)}$  Suppose that  $\hat{x} = \text{Prox}_\varphi^f(\xi)$ . Take  $y \in \text{dom } \varphi \cap \text{dom } f$  and  $t \in (0, 1)$ . Then  $(1-t)\hat{x} + ty \in \text{dom } \varphi \cap \text{dom } f$  and we have

$$\varphi(\hat{x}) + W_f(\xi, \hat{x}) \leq \varphi((1-t)\hat{x} + ty) + W_f(\xi, (1-t)\hat{x} + ty),$$

that is

$$(2.12) \quad \varphi((1-t)\hat{x} + ty) - \varphi(\hat{x}) \geq W_f(\xi, \hat{x}) - W_f(\xi, (1-t)\hat{x} + ty).$$

Since  $\hat{x} \in \text{int dom } f$ , there exists  $t_0 \in (0, 1)$  such that for any  $t \in (0, t_0)$  we have that  $(1-t)\hat{x} + ty \in \text{int dom } f$ . Consequently, for any  $t \in (0, t_0)$  the function  $W_f(\xi, \cdot)$  is differentiable at  $(1-t)\hat{x} + ty$ . Clearly, we also have

$$\nabla W_f(\xi, \cdot)(u) = \nabla f(u) - \xi, \quad \forall u \in \text{int dom } f.$$

Therefore, by the convexity of  $W_f(\xi, \cdot)$  and (2.12), we deduce

$$\begin{aligned} t^{-1} [\varphi((1-t)\hat{x} + ty) - \varphi(\hat{x})] &\geq \langle \nabla W_f(\xi, \cdot)((1-t)\hat{x} + ty), \hat{x} - y \rangle \\ &= \langle \nabla f((1-t)\hat{x} + ty) - \xi, \hat{x} - y \rangle \end{aligned}$$

for any  $t \in (0, t_0)$ . The function  $\nabla f(\cdot)$  is norm to weak continuous (see, for instance, [61, Proposition 2.8]). Hence, letting  $t \rightarrow 0^+$  in the last inequality we get

$$\varphi^\circ(\hat{x}; y - \hat{x}) \geq \langle \nabla f(\hat{x}) - \xi, \hat{x} - y \rangle,$$

where  $\varphi^\circ$  stands for the right-hand side derivative of  $\varphi$ , that is,

$$\varphi^\circ(x; d) = \lim_{s \rightarrow 0^+} \frac{\varphi(x + sd) - \varphi(x)}{s}.$$

Taking into account that

$$\varphi(y) - \varphi(\hat{x}) \geq \varphi^\circ(\hat{x}; y - \hat{x})$$

we obtain (2.11).

$\boxed{(b) \Rightarrow (a)}$  Suppose that

$$\langle \xi - \nabla f(\hat{x}), y - \hat{x} \rangle \leq \varphi(y) - \varphi(\hat{x}), \quad \forall y \in \text{dom } \varphi \cap \text{dom } f.$$

Observe that

$$(2.13) \quad \nabla W_f(\xi, \cdot) = \nabla f(\cdot) - \xi.$$

Then, by the convexity of  $W_f(\xi, \cdot)$  and (2.13), for any  $y \in \text{dom } \varphi \cap \text{dom } f$  we have that

$$\begin{aligned} W_f(\xi, y) - W_f(\xi, \hat{x}) &\geq \langle \nabla W_f(\xi, \cdot)(\hat{x}), y - \hat{x} \rangle \\ &= \langle \nabla f(\hat{x}) - \xi, y - \hat{x} \rangle \\ &\geq \varphi(\hat{x}) - \varphi(y). \end{aligned}$$

This shows that  $\hat{x} = \text{Prox}_\varphi^f(\xi)$ . The equivalence  $\boxed{(b) \Leftrightarrow (c)}$  results immediately by observing that

$$W_f(\xi, x) + W_f(\nabla f(x), y) - W_f(\xi, y) = \langle \xi - \nabla f(x), y - x \rangle,$$

whenever  $y \in \text{dom } f$ ,  $\xi \in \text{int dom } f^*$  and  $x \in \text{int dom } f$ .  $\square$

A consequence of Lemma 2.2 is the following generalization of the variational characterization of the Bregman projections originally given in [5].

**Corollary 2.1.** *Let  $x \in \text{int dom } f$  and let  $E$  be a nonempty, closed and convex set such that  $E \cap \text{int dom } f \neq \emptyset$ . If  $\hat{x} \in E$ , then the following conditions are equivalent:*

- (i) *The vector  $\hat{x}$  is the Bregman projection of  $x$  onto  $E$  with respect to  $f$ ;*
- (ii) *The vector  $\hat{x}$  is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall y \in E;$$

- (iii) *The vector  $\hat{x}$  is the unique solution of the variational inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in K.$$

Now we are in position to establish the connection between the proximal mappings and a class of variational inequalities. It extends similar results known to hold in less general settings (see, for instance, [1] and [45, Proposition 1.5.8]). The variational inequality we consider here is

(2.14) Find  $x \in \text{int dom } f$  such that

$$\exists \xi \in Bx : [\langle \xi, y - x \rangle \geq \varphi(x) - \varphi(y), \quad \forall y \in \text{dom } f],$$

where  $\varphi \in \mathcal{F}_f$  and  $B : X \rightarrow 2^{X^*}$  is an operator which satisfies the condition

(2.15)  $\emptyset \neq \text{dom } B \cap \text{dom } \varphi \cap \text{int dom } f$  and  $\text{ran } (\nabla f - B) \subseteq \text{int dom } f^*$ .

Condition (2.15) guarantees that the operator

$$\text{Prox}_\varphi^f(\nabla f - B) := \text{Prox}_\varphi^f \circ (\nabla f - B)$$

is well defined. Therefore, the following statement makes sense.

**Lemma 2.3.** *Let  $\varphi \in \mathcal{F}_f$  and  $\hat{x} \in \text{dom } \partial \varphi \cap \text{int dom } f$ . Suppose that  $B : X \rightarrow 2^{X^*}$  is an operator which satisfies (2.15). Then  $\hat{x}$  is a solution of the variational inequality (2.14) if and only if it is a fixed point of the operator  $\text{Prox}_\varphi^f(\nabla f - B)$ .*

**Proof:** Note that  $\hat{x}$  is a solution of (2.14) if and only if there exists  $\xi \in B\hat{x}$  such that

$$(2.16) \quad \langle (\nabla f(\hat{x}) - \xi) - \nabla f(\hat{x}), y - \hat{x} \rangle \leq \varphi(y) - \varphi(\hat{x}), \quad \forall y \in \text{dom } \varphi \cap \text{dom } f.$$

According to Lemma 2.2, this is equivalent to  $\hat{x} = \text{Prox}_\varphi^f(\nabla f(\hat{x}) - \xi)$  for some  $\xi \in B\hat{x}$  which, in turn, is equivalent to  $\hat{x} \in \text{Prox}_\varphi^f(\nabla f(\hat{x}) - B\hat{x})$ , i.e., to the condition that  $\hat{x}$  is a fixed point of  $\text{Prox}_\varphi^f(\nabla f - B)$ .  $\square$

Let  $B : X \rightarrow 2^{X^*}$  be an operator and suppose that the closed convex subset  $C$  of  $X$  satisfies

$$(2.17) \quad \emptyset \neq \text{dom } B \cap C \cap \text{int dom } f \text{ and } \text{ran } (\nabla f - B) \subseteq \text{int dom } f^*.$$

Note that if  $\varphi := \iota_C$ , then the variational inequality (2.14) is exactly a *classical variational inequality*

$$(2.18) \quad \text{Find } x \in C \cap \text{int dom } f \text{ such that}$$

$$\exists \xi \in Bx : [\langle \xi, y - x \rangle \geq 0, \forall y \in C \cap \text{dom } f].$$

Applying Lemma 2.3 and (1.3) in this case, we re-discover the following known result (cf. [2]):

**Lemma 2.4.** *Suppose that the condition (2.17) is satisfied and that  $\hat{x} \in C \cap \text{int dom } f$ . Then the following statements are equivalent:*

- (a) *The vector  $\hat{x}$  is a solution of the classical variational inequality (2.18);*
- (b) *The vector  $\hat{x}$  is a fixed point of the operator  $\text{Proj}_C^f(\nabla f - B)$ ;*
- (c) *The vector  $\hat{x}$  is a zero for the operator  $V[B; C; f] : X \rightarrow 2^{X^*}$  given by*

$$(2.19) \quad V[B; C; f] := \nabla f - \nabla f \circ \text{Proj}_C^f(\nabla f - B).$$

Lemma 2.3 and Lemma 2.4 provided the initial motivation for this research. Observe that Lemma 2.4 reduces the problem of finding a solution for the classical variational inequality (2.18) to the problem of finding a fixed point for the operator  $\text{Proj}_C^f(\nabla f - B)$ . It is well known that, in many instances, by iterating an operator starting from initial points located in its definition domain, one produces sequences which converge to fixed points of the operator. This suggests that for solving the classical variational inequality (2.18) we would have to produce sequences defined by the iterative rule

$$(2.20) \quad x^{k+1} \in \text{Proj}_C^f(\nabla f(x^k) - Bx^k)$$

in hope that such sequences will converge to fixed points of  $\text{Proj}_C^f(\nabla f - B)$ . Note that any fixed point of the operator  $\text{Proj}_C^f(\nabla f - B)$  is a zero for the operator  $V[B; C; f]$  given by (2.19) and conversely. According to (2.9), we have that

$$(2.21) \quad \begin{aligned} \text{Proj}_C^f(\nabla f - V[B; C; f]) &= \text{Proj}_C^f \circ \nabla f \circ \text{Proj}_C^f \circ (\nabla f - B) \\ &= \text{proj}_C^f \circ \text{proj}_C^f \circ (\nabla f)^{-1} \circ (\nabla f - B) \\ &= \text{proj}_C^f \circ \nabla f^* \circ (\nabla f - B) \\ &= \text{Proj}_C^f(\nabla f - B). \end{aligned}$$

Thus, we are naturally led to the question of whether, and in which conditions, the sequences generated according to the rule  $x^{k+1} \in \text{Proj}_C^f(\nabla f(x^k) - V[B; C; f]x^k)$ , which are the same (see (2.21)) as the sequences generated according to rule (2.20), converge to zeros of the operator  $V[B; C; f]$ . This is, in fact, a particular instance of the more general question of whether, and in which conditions, the proximal-projection method (1.7) approximates zeros of a given operator  $V[B; C; f]$ , provided that such zeros exist. In the sequels we present answers to this question. It is interesting to observe that by focusing in our convergence analysis on conditions concerning the operator  $V[B; C; f]$  instead of  $B$  we do not mean that computing

values of  $\text{Proj}_C^f(\nabla f - V[B; C; f])$  is easier than computing values of the same operator via the formula  $\text{Proj}_C^f(\nabla f - B)$ . However, from a theoretical point of view, the operator  $V[B; C; f]$  associated to  $B$  via formula (2.19) may happen to be better conditioned than  $B$  for a convergence analysis of the proximal-projection method. This aspect can be clearly seen after a careful dissection of the considerations which lead to our main convergence results presented in this paper. It should be taken into account that the operator  $B$  may have not zeros in  $C$  even if the operator  $V[B; C; f]$ , associated to  $B$  by (2.19), has. For example, take  $X = \mathbb{R}$ ,  $f(x) = \frac{1}{2}x^2$ ,  $C = [1, 2]$  and  $Bx = x$  for all  $x \in X$ . Then  $V[B; C; f]x = x - 1$  vanishes at  $x = 1$ , but  $B$  does not have any zero in  $C$ .

### 3. $D_f$ -NONEXPANSIVITY POLES, $D_f$ -COERCIVITY AND $D_f$ -FIRMNESS OF OPERATORS IN BANACH SPACES

In this section we introduce the notion of  $D_f$ -coercivity for operators from  $X$  to  $2^{X^*}$ . We clarify how this notion is related with the notions of  $D_f$ -nonexpansivity pole introduced in [32] and of  $D_f$ -firm operator introduced in [20]. Using these relations we show that the proximal point method with Bregman distances can be seen as a particular instance of the proximal-projection method applied to a  $D_f$ -coercive operator.

**3.1.  $D_f$ -nonexpansivity poles.** In that follows, to the function  $f$  described in Section 1 and to any operator  $A : X \rightarrow 2^{X^*}$  we associate the operator  $A^f : X \rightarrow 2^X$  given by

$$(3.1) \quad A^f := \nabla f^* \circ (\nabla f - A).$$

We call this operator the  $D_f$ -antiresolvent of  $A$ . Observe that

$$(3.2) \quad \text{dom } A^f \subseteq \text{dom } A \cap \text{int dom } f \quad \text{and} \quad \text{ran } A^f \subseteq \text{int dom } f,$$

and that, if  $x \in \text{int dom } f$ , then  $0^* \in Ax$  if and only if  $x \in \text{Fix } A^f$ , where  $\text{Fix } A^f$  denotes the set of fixed points of  $A^f$ . Therefore, the (possibly empty) set of solutions of the Problem 1.1 situated in  $\text{int dom } f$ , denoted  $\mathcal{S}_f(A, C)$ , is exactly

$$(3.3) \quad \mathcal{S}_f(A, C) = C \cap \text{Fix } A^f.$$

We are going to prove that, for operators  $A$  which are  $D_f$ -coercive, the set  $\mathcal{S}_f(A, C)$  is exactly the set of  $D_f$ -nonexpansivity poles of  $A^f$  over the set  $C$  and this fact will be later used in our convergence analysis of the proximal-projection method. To this end, recall the following notion.

**Definition 3.1** (cf. [32]) Let  $T : X \rightarrow 2^X$  be an operator and let  $Y$  be a subset of  $X$  such that

$$(3.4) \quad \emptyset \neq T(Y \cap \text{int dom } f) \subseteq \text{int dom } f.$$

The vector  $z \in X$  is called a  $D_f$ -nonexpansivity pole of  $T$  over  $Y$  if the following conditions are satisfied:

$$(3.5) \quad z \in Y \cap \text{int dom } f,$$

$$(3.6) \quad (x \in Y \cap \text{int dom } f \quad \text{and} \quad u \in Tx) \Rightarrow \langle \nabla f(x) - \nabla f(u), z - u \rangle \leq 0.$$

We denote by  $\text{Nexp}_Y^f T$  the set of  $D_f$ -nonexpansivity poles of  $T$  over  $Y$ .

Operators having  $D_f$ -nonexpansivity poles were termed totally nonexpansive operators in [32]. Operators  $T$  such that  $\text{ran } T \subseteq \text{dom } T = \text{int dom } f$  and having  $\text{Nexp}_{\text{int dom } f}^f T \supseteq \text{Fix } T$  were called  $\mathcal{B}$ -class operators in [19] and [20].  $\mathcal{B}$ -class operators necessarily have  $\text{Nexp}_{\text{int dom } f}^f T = \text{Fix } T$  (cf. [20, Proposition 3.3]). However, not every operator having  $D_f$ -nonexpansivity poles over some subset  $Y$  of  $X$  is  $\mathcal{B}$ -class. For example, the operator  $Tx = \{x^2\}$  when  $X = \mathbb{R}$ ,  $f = \frac{1}{2}|\cdot|^2$  and  $Y = \text{int dom } f = \mathbb{R}$  has  $z \in \text{Nexp}_{\text{int dom } f}^f T$  if and only if

$$x(1-x)(z-x^2) \leq 0, \quad \forall x \in \mathbb{R},$$

and this last inequality can not hold because, irrespective of  $z$ , we have  $\lim_{x \rightarrow \infty} x(1-x)(z-x^2) = \infty$ . Hence,  $\text{Nexp}_{\text{int dom } f}^f T = \emptyset$ . In spite of that,  $\text{Fix } T = \{0, 1\}$  and, if  $Y = [0, 1]$ , then (3.6) holds for  $z = 0$  only, i.e.,  $\text{Nexp}_{[0,1]}^f T = \{0\} \neq \text{Fix } T$ .

The following lemma summarizes several properties of operators having nonexpansivity poles which are used in this work.

**Lemma 3.1.** *Let the operator  $T : X \rightarrow 2^X$  and the set  $Y \subseteq X$  be such that condition (3.4) holds. Then the following statements are true:*

(a) *The (possibly empty) set  $\text{Nexp}_Y^f T$  is convex and closed when  $Y \subseteq \text{int dom } f$  is convex and closed;*

(b) *A vector  $z \in Y \cap \text{int dom } f$  is a  $D_f$ -nonexpansivity pole of  $T$  over  $Y$  if and only if*

$$(3.7) \quad (x \in Y \cap \text{int dom } f \quad \text{and} \quad u \in Tx) \Rightarrow D_f(z, u) + D_f(u, x) \leq D_f(z, x).$$

(c)  *$\text{Nexp}_Y^f T \subseteq \text{Fix } T$  and  $T$  is single-valued at any  $D_f$ -nonexpansivity pole.*

**Proof.** Statement (a) results from the fact that the function  $z \rightarrow \langle \nabla f(x) - \nabla f(u), z - u \rangle$  is linear and continuous of  $z$ . Statement (b) follows from (3.6) and (2.10). Now, by taking in (3.7)  $x = z \in \text{Nexp}_Y^f T$  one gets

$$(3.8) \quad D_f(z, u) = D_f(u, z) = 0, \quad \forall u \in Tz.$$

By (3.4), if  $u \in Tz$ , then  $u \in \text{int dom } f$ . The function  $f$  being essentially strictly convex is strictly convex on  $\text{int dom } f$ . Hence, the equalities in (3.8) can not hold unless  $u = z$  (cf. [32, Proposition 1.1.4]). In other words, we have the following implication

$$(3.9) \quad z \in \text{Nexp}_Y^f T \Rightarrow Tz = \{z\}.$$

It shows that  $\text{Nexp}_Y^f T \subseteq \text{Fix } T$  and that  $T$  is single-valued at  $D_f$ -nonexpansivity poles.  $\square$

**3.2.  $D_f$ -coercivity and  $D_f$ -firmness.** The notion of  $D_f$ -coercive operator, introduced in this section, and the notion of  $D_f$ -firm operator, originally introduced in [20, Definition 3.4], are generalizations of the notion of firmly nonexpansive operator in a Hilbert space. Recall (cf. [46, pp. 41-42]) that if  $X$  is a Hilbert space (which we always identify with its dual  $X^*$ ), then an operator  $A : X \rightarrow X$  is firmly nonexpansive on a subset  $Y$  of  $X$  if and only if

$$(3.10) \quad \langle Ax - Ay, x - y \rangle \geq \|Ax - Ay\|^2, \quad \forall x, y \in Y.$$

It can be easily seen from the definitions given below that if  $X$  is a Hilbert space and if  $f = \frac{1}{2}\|\cdot\|^2$ , then the operator  $A$  is  $D_f$ -coercive on its domain if and only if it is firmly nonexpansive on its domain and this happens if and only if  $A$  is  $D_f$ -firm.

Returning to the general context in which  $X$  and  $f$  are as described in Section 1, we introduce the following notion.

**Definition 3.2.** Let  $Y$  be a subset of the space  $X$ . The operator  $A : X \rightarrow 2^{X^*}$  is called  $D_f$ -coercive on the set  $Y$  if

$$(3.11) \quad Y \cap (\text{dom } A) \cap (\text{int dom } f) \neq \emptyset$$

and

$$(3.12) \quad \left. \begin{array}{l} x, y \in Y \cap \text{int dom } f \\ \xi \in Ax \text{ and } \eta \in Ay \end{array} \right\} \Rightarrow \langle \xi - \eta, \nabla f^*(\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta) \rangle \geq 0.$$

Operators satisfying a somewhat less restrictive condition than (3.12) were studied in [36, Section 5] under the name of inverse-monotone operators relative to  $f$ . In general, an operator  $A$  (even in a Hilbert space provided that  $f$  is not the function  $f = \frac{1}{2} \|\cdot\|^2$ ) does not have to satisfy (3.10) in order to be  $D_f$ -coercive on  $Y$ . For instance, if the function  $f$  has a minimizer in  $\text{int dom } f$  (i.e., if the equation  $\nabla f(x) = 0$  has a solution), and if  $\alpha \in (0, 1)$ , then the operator  $A = \alpha \nabla f$  is  $D_f$ -coercive on  $Y = \text{dom } A = \text{int dom } f$  without necessarily satisfying (3.10) on  $Y = \text{int dom } f$ . Indeed, in this case, if  $x \in \text{int dom } f$  then  $\{\nabla f(x), 0^*\} \subset \text{ran } \nabla f = \text{int dom } f^*$  and, due to the convexity of  $\text{int dom } f^*$ , we have that

$$(1 - \alpha)\nabla f(x) = (1 - \alpha)\nabla f(x) + \alpha 0^* \in \text{int dom } f^* = \text{dom } \nabla f^*,$$

and, consequently, the operator

$$A^f x = \nabla f^*((1 - \alpha)\nabla f(x))$$

has  $\text{dom } A^f = \text{int dom } f$ . If  $x, y \in \text{int dom } f$ , and if  $\beta = 1 - \alpha$ , then, by the monotonicity of  $\nabla f^*$ , we deduce that

$$\begin{aligned} \langle Ax - Ay, \nabla f^*(\nabla f(x) - Ax) - \nabla f^*(\nabla f(y) - Ay) \rangle = \\ \alpha\beta^{-1} \langle \beta\nabla f(x) - \beta\nabla f(y), \nabla f^*(\beta\nabla f(x)) - \nabla f^*(\beta\nabla f(y)) \rangle \geq 0, \end{aligned}$$

i.e., the operator  $A = \alpha \nabla f$  is  $D_f$ -coercive on  $\text{int dom } f$ . However, the operator  $A = \alpha \nabla f$  does not have to be firmly nonexpansive on its domain even if  $X$  is a Hilbert space. For example, take in the considerations above  $X = L^2[0, 1]$ ,  $f = \frac{1}{3} \|\cdot\|^3$  and  $\alpha \in (1/4, 1)$ . Then  $\nabla f(x) = \|x\|x$  and an easy verification shows that (3.10) does not hold for any  $x, y \in X$ . For instance, (3.10) is violated when  $x = 2\alpha^{-1/2}$  and  $y = 0$ .

We are going to show that there are strong connections between the  $D_f$ -coercivity of the operator  $A$  and the  $D_f$ -firmness of its  $D_f$ -antiresolvent  $A^f$ . For this purpose we recall the following:

**Definition 3.3.** (cf. [20, Definition 3.4]) An operator  $T : X \rightarrow 2^X$  is called  $D_f$ -firm if it satisfies the conditions

$$(3.13) \quad \emptyset \neq \text{dom } T \cup \text{ran } T \subseteq \text{int dom } f$$

and

$$(3.14) \quad u \in Tx \text{ and } v \in Ty \Rightarrow \langle \nabla f(u) - \nabla f(v), u - v \rangle \leq \langle \nabla f(x) - \nabla f(y), u - v \rangle.$$

We start with the following result which summarizes some basic properties of  $D_f$ -coercive operators.

**Lemma 3.2.** Let  $A : X \rightarrow 2^{X^*}$  be an operator and let  $Y$  be a subset of  $X$  which satisfies (3.11). The following statements are true:

(a) The operator  $A$  is  $D_f$ -coercive on  $Y$  if and only if it satisfies the following condition for any  $x, y \in Y \cap \text{int dom } f$  :

$$\left. \begin{array}{l} u \in A^f x \\ v \in A^f y \end{array} \right\} \Rightarrow D_f(u, v) + D_f(v, u) + D_f(u, x) + D_f(v, y) \leq D_f(v, x) + D_f(u, y);$$

(b) If  $A$  is  $D_f$ -coercive on  $Y$ , then  $A$  is  $D_f$ -coercive on any subset of  $Y$  which intersects  $\text{dom } A \cap \text{int dom } f$ ;

(c) The operator  $A$  is  $D_f$ -coercive on its domain if and only if its  $D_f$ -antiresolvent  $A^f$  is  $D_f$ -firm;

(d) If  $A$  is  $D_f$ -coercive on its domain, then  $A^f$  is single valued and all its fixed points are  $D_f$ -nonexpansivity poles on  $\text{int dom } f$ .

**Proof.** Statements (a), (b) and (c) result from (3.1), (3.2), (3.13), (3.14) and (2.10). Single valuedness of  $A^f$  in statement (d) is a consequence of (c) and of [20, Proposition 3.5(iii)]. Letting  $y = z \in \text{Fix } A^f$  in the inequality of (a), and taking into account the single valuedness of  $A^f$ , one obtains that  $z$  satisfies (3.7) for  $T = A^f$ .  $\square$

Whenever the operator  $A$  involved in Problem 1.1 is  $D_f$ -coercive on the domain  $C$  of the problem, we have

$$(3.15) \quad \mathcal{S}_f(A, C) = C \cap \text{Fix } A^f = \text{Nexp}_C^f A^f.$$

This immediately follows from the next result.

**Lemma 3.3.** *The following statements are true:*

(a) *If  $z \in \text{Nexp}_C^f A^f$ , then  $Az = \{0^*\}$ ;*

(b) *If the operator  $A$  is  $D_f$ -coercive on  $C$  and  $z \in \text{int dom } f$  is a solution of Problem 1.1, then  $z \in \text{Nexp}_C^f A^f$ .*

**Proof.** The statement (a) results from (3.1) and Lemma 3.1(c). In order to prove (b), note that for any  $x \in C \cap \text{int dom } f$  and  $y \in A^f x$  we have

$$D_f(z, x) - D_f(z, y) = D_f(y, x) - \langle \nabla f(x) - \nabla f(y), z - y \rangle,$$

and  $y = \nabla f^*(\nabla f(x) - \xi)$  for some  $\xi \in Ax$ . Thus,  $\nabla f(y) = \nabla f(x) - \xi$  and

$$(3.16) \quad D_f(z, x) - D_f(z, y) = D_f(y, x) + \langle 0^* - \xi, z - y \rangle.$$

If  $z$  is a solution of Problem 1.1, then  $z \in C$ ,  $0^* \in Az$  and  $z = \nabla f^*(\nabla f(z) - 0^*)$ . By consequence,

$$(3.17) \quad \langle 0^* - \xi, z - y \rangle = \langle 0^* - \xi, \nabla f^*(\nabla f(z) - 0^*) - \nabla f^*(\nabla f(x) - \xi) \rangle.$$

Since  $A$  is  $D_f$ -coercive on  $C$ , it results that the right-hand side of (3.17) is non-negative (see (3.12)) for any  $x \in C \cap \text{int dom } f$ . Hence, by (3.16) and (3.17), the inequality in (3.7) results and it shows that  $z \in \text{Nexp}_C^f A^f$ .  $\square$

The class of operators which are  $D_f$ -coercive contains some meaningful operators. Among them are all operators  $A[T] : X \rightarrow 2^{X^*}$  given by

$$(3.18) \quad A[T] = \nabla f - \nabla f \circ T,$$

where  $T : X \rightarrow 2^X$  is a  $D_f$ -firm operator. In particular,  $A[T]$  is  $D_f$ -coercive when  $T = B_f$ , where  $B_f : X \rightarrow 2^X$  is the  $D_f$ -resolvent (cf. [20]) of a monotone operator  $B : X \rightarrow 2^{X^*}$ , i.e.,

$$(3.19) \quad B_f := (\nabla f + B)^{-1} \circ \nabla f.$$

According to [20, Proposition 3.8], the operator  $T = B_f$  satisfies the condition (3.13). These facts are summarized in the following lemma.

**Lemma 3.4.** *Let  $T : X \rightarrow 2^X$  be an operator which satisfies (3.13). Then the following statements are true:*

- (a)  $\text{dom } A[T] = \text{dom } T$  and its antiresolvent is  $A[T]^f = T$ ;
- (b) The operator  $T$  is  $D_f$ -firm if and only if the operator  $A[T] : X \rightarrow 2^{X^*}$  defined by (3.18) is  $D_f$ -coercive on its domain.
- (c) If  $B : X \rightarrow 2^{X^*}$  is a monotone operator with  $\text{dom } B \cap \text{int dom } f \neq \emptyset$ , then  $B_f$  is  $D_f$ -firm, single valued on its domain,

$$(3.20) \quad A[B_f] = \nabla f - \nabla f \circ (\nabla f + B)^{-1} \circ \nabla f,$$

and  $A[B_f]$  is  $D_f$ -coercive on its domain.

**Proof.** Statement (a) results from (3.1) and (3.18). To prove (b) observe that for any  $x, y \in \text{dom } T$ , for any  $\xi \in A[T]x$  and for any  $\eta \in A[T]y$ , we have

$$(3.21) \quad \xi = \nabla f(x) - \nabla f(u) \quad \text{and} \quad \eta = \nabla f(y) - \nabla f(v)$$

for some  $u \in Tx$  and for some  $v \in Ty$ . Therefore,

$$(3.22) \quad \begin{aligned} & \langle \xi - \eta, \nabla f^*(\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta) \rangle \\ &= \langle \nabla f(x) - \nabla f(y), u - v \rangle - \langle \nabla f(u) - \nabla f(v), u - v \rangle. \end{aligned}$$

If  $T$  is  $D_f$ -firm, then the right hand side of (3.22) is nonnegative and this implies that  $A[T]$  is  $D_f$ -coercive on  $\text{dom } T$ . Conversely, suppose that  $A[T]$  is  $D_f$ -coercive on  $\text{dom } T$ . If  $u \in Tx$  and  $v \in Ty$ , then the vectors  $\xi$  and  $\eta$  given by (3.21) satisfy (3.22) and the left hand side of this equality is nonnegative. Hence,  $T$  is  $D_f$ -firm. This proves (b). In order to prove (c) recall that, since  $B$  is monotone, its resolvent,  $B_f$ , is necessarily  $D_f$ -firm and single valued on its domain (cf. [20, Proposition 3.8]) and, as noted above, it satisfies (3.13). Thus, according to (b), the operator  $A[B_f]$  is  $D_f$ -coercive on its domain.  $\square$

**3.3. Connection between the proximal-projection method and the proximal point method.** Lemma 3.4 helps establishing a connection between the proximal-projection method and the *proximal point method (with Bregman distances)*. The proximal point method we are referring to in this paper is the iterative procedure which, in our setting, can be described by

$$(3.23) \quad x^0 \in \text{dom } B_f \quad \text{and} \quad x^{k+1} = B_f(x^k), \quad \forall k \in \mathbb{N},$$

where  $B : X \rightarrow 2^{X^*}$  is a monotone operator with  $\text{dom } B \cap \text{int dom } f \neq \emptyset$ . Its well definedness is guaranteed when

$$(3.24) \quad \emptyset \neq \text{ran } B_f \subseteq \text{dom } B_f.$$

For ensuring that the inclusion in this condition holds it is sufficient to make sure that  $\text{dom } B_f = X$ . This implicitly happens when one considers the classical proximal point method (see [66]) where  $X$  is a Hilbert space,  $f = \frac{1}{2} \|\cdot\|^2$  and  $B$  is presumed to be maximal monotone. Alternative conditions which imply that  $\text{dom } B_f = X$  when  $B$  is maximal monotone are presented in [30] in a more general setting. In particular, those conditions hold if  $X$  is a uniformly convex and uniformly smooth Banach space,  $B$  is maximal monotone and  $f = \frac{1}{2} \|\cdot\|^2$ .

Maximal monotonicity of  $B$  is a commonly used condition for ensuring well definedness of the proximal point method because, if  $B$  is maximal monotone and

$f = \frac{1}{2} \|\cdot\|^2$ , then  $\nabla f + B$  and  $\nabla f^*$  are surjective and, thus,  $\nabla f^* \circ (\nabla f + B)$  is surjective too, that is,  $\text{dom } B_f = \text{ran } [\nabla f^* \circ (\nabla f + B)] = X$ . However, well definedness of the proximal point method can be sometimes ensured for operators  $B$  which are monotone without being maximal monotone. In such cases, it is interesting to know whether the proximal point method preserves the convergence properties which make it so useful in applications requiring finding zeros of maximal monotone operators. Here is an example of a monotone operator which is not maximal and for which (3.24) holds (in spite of the fact that  $\nabla f + B$  is not surjective). Take  $X = \mathbb{R}$ ,  $f = \frac{1}{2} |\cdot|^2$  and let  $B : X \rightarrow 2^{X^*}$  be given by  $Bx = \{0\}$  if  $x \leq 0$ , and  $Bx = \emptyset$  if  $x > 0$ . The operator  $B$  is monotone, but it is not maximal monotone since the operator defined by  $B'x = \{0\}$  for all  $x \in X$  is a proper monotone extension of  $B$ . Obviously, in this case  $\nabla f$  is the identity,  $(\nabla f + B)x = \{x\}$  if  $x \leq 0$ ,  $(\nabla f + B)x = \emptyset$  if  $x > 0$  and, therefore,  $\nabla f + B$  is not surjective. However,  $B_f = (\nabla f + B)^{-1} = \nabla f + B$  and, hence,  $\text{ran } B_f = \text{dom } B_f = (-\infty, 0]$  showing that (3.24) is satisfied, that is, the proximal point algorithm is well defined.

The next result establishes the connection between the proximal-projection method and the proximal point method. It requires that  $\text{dom } B_f$  should be convex and closed. This necessarily happens if  $\nabla f + B$  is surjective and  $\text{dom } f = X$ . However,  $\text{dom } B_f$  may happen to be convex and closed even if  $\nabla f + B$  is not surjective as one can see from the example above. Other instances in which  $\text{dom } B_f$  is convex and closed are described in the remarks preceding Corollary 4.2 as well as in the body of that corollary.

**Lemma 3.5.** *Let  $B : X \rightarrow 2^{X^*}$  be a monotone operator such that  $\text{dom } B_f$  is convex and closed and suppose that (3.24) is satisfied. Then the proximal point method (3.23) is exactly the proximal-projection method applied to the  $D_f$ -coercive operator  $A[B_f]$  with  $C_k = X$  for all  $k \in \mathbb{N}$ .*

**Proof.** Observe that, by (3.20), we have

$$\begin{aligned} \text{Proj}_{\text{dom } B_f}^f (\nabla f - A[B_f]) &= \text{Proj}_{\text{dom } B_f}^f \left[ \nabla f \circ (\nabla f + B)^{-1} \circ \nabla f \right] \\ &= \text{proj}_{\text{dom } B_f}^f \left[ (\nabla f + B)^{-1} \circ \nabla f \right] = B_f, \end{aligned}$$

where the last equality results holds because (3.24) is satisfied. This shows that the proximal point method and the proximal-projection method are overlapping when one takes  $A = A[B_f]$  and  $C_k = X$  for all  $k \in \mathbb{N}$  in (1.7).  $\square$

#### 4. CONVERGENCE ANALYSIS OF THE PROXIMAL-PROJECTION METHOD

In this section we present a convergence theorem for the proximal-projection method in reflexive Banach spaces. Our convergence analysis is based on a generalization of Lemma 5.7 in [36] which, in turn, is a generalization of a result known as Opial's Lemma [59, Lemma 2]. All over this section we assume that the function  $f$  and the Banach space  $X$  are as described in Section 1.

**4.1. A generalization of Opial's Lemma.** Opial's Lemma says that if  $X$  is a Hilbert space and if  $T : Y \rightarrow X$  is a nonexpansive mapping on the nonempty closed convex subset  $Y$  of  $X$ , then for any sequence  $\{z^k\}_{k \in \mathbb{N}} \subseteq Y$  which is weakly convergent and has  $\lim_{k \rightarrow \infty} \|Tz^k - z^k\| = 0$ , the vector  $z = w - \lim_{k \rightarrow \infty} z^k$  is necessarily a fixed point of  $T$ . In [36, Lemma 5.7] a similar result was shown to hold in Banach spaces which are not necessarily Hilbert spaces. Namely, it was proved

that the conclusion of Opial's Lemma still holds for operators  $T : X \rightarrow X$  which are nonexpansive relative to  $f$  (in the sense given to this term in [37]), i.e., such that

$$(4.1) \quad D_f(Tx, Ty) \leq D_f(x, y), \quad \forall x, y \in Y,$$

provided that  $\text{dom } f = \text{dom } \nabla f = X$  and that  $f$  is not only Legendre, but it is also totally convex (see [32]) and bounded on bounded sets, while  $T$  satisfies  $\lim_{k \rightarrow \infty} D_f(Tz^k, z^k) = 0$ .

Our current generalization of Opial's Lemma concerns set-valued operators  $T$  satisfying a somehow less stringent nonexpansivity condition than (4.1) with respect to a function  $f$  subjected to weaker requirements than those involved in [36, Lemma 5.7]. In the sequels we use the following notion which generalizes that of nonexpansive operator relative to  $f$ .

**Definition 4.1.** The operator  $T : X \rightarrow 2^X$  is said to be  $D_f$ -nonexpansive if it satisfies (3.13) and for any  $x \in \text{dom } T$  there exists  $u \in Tx$  such that

$$(4.2) \quad (\forall y \in \text{dom } T) : [v \in Ty \Rightarrow D_f(v, u) \leq D_f(y, x)].$$

In that follows (see Theorem 4.1 below) we will be interested in operators whose antiresolvents are simultaneously  $D_f$ -firm and  $D_f$ -nonexpansive. It should be noted that the notions of  $D_f$ -nonexpansivity and  $D_f$ -firmness are not equivalent, although some operators may have both properties. If  $X$  is a Hilbert space provided with  $f = \frac{1}{2} \|\cdot\|^2$ , then it is obvious that any  $D_f$ -firm operator (i.e., any firmly nonexpansive operator) is  $D_f$ -nonexpansive (i.e., nonexpansive). Even in this context, the converse implication does not generally hold. Take, for example, the case where the Hilbert space is  $X = \mathbb{R}$  and  $Ax = 2x$ . Its antiresolvent is  $A^f x = -x$  and it is  $D_f$ -nonexpansive without being  $D_f$ -firm. However, operators whose antiresolvents are simultaneously  $D_f$ -nonexpansive and  $D_f$ -firm are not specific to the setting of Hilbert spaces provided with  $f = \frac{1}{2} \|\cdot\|^2$ . For instance, if  $X = \mathbb{R}$  and  $f(x) = \frac{1}{4}x^4$ , then the antiresolvent  $A^f$  of the operator  $A = \alpha \nabla f$  with  $\alpha \in (0, 1)$  is  $D_f$ -nonexpansive and  $D_f$ -firm at the same time. It is  $D_f$ -firm because, as shown in Section 3.2,  $A$  is  $D_f$ -coercive and Lemma 3.2(c) applies.  $A^f$  is also  $D_f$ -nonexpansive because

$$D_f(A^f y, A^f x) = (1 - \alpha)^4 D_f(y, x) \leq D_f(y, x), \quad \forall x, y \in X.$$

One still may hope that (as happens in the particular situation noted above when  $X$  is a Hilbert space provided with  $f = \frac{1}{2} \|\cdot\|^2$ ) a  $D_f$ -firm operator is always  $D_f$ -nonexpansive. The following example shows that this is not the case.

**Example 4.1.** A  $D_f$ -firm operator which is not  $D_f$ -nonexpansive. Let  $X = \mathbb{R}$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the Legendre function given by  $f(x) = |x|^{3/2}$ . Take the continuous, single-valued operator  $T : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$T(x) = \begin{cases} \frac{1}{4}, & \text{if } x < \frac{1}{16}, \\ \sqrt{x}, & \text{if } x \geq \frac{1}{16}. \end{cases}$$

We first show that  $T$  is  $D_f$ -firm, that is, we verify that for every  $x, y \in \mathbb{R}$  one has

$$(4.3) \quad [f'(T(x)) - f'(T(y))][T(x) - T(y)] \leq [f'(x) - f'(y)][T(x) - T(y)].$$

For symmetry reasons we can assume that  $x > y$ . The case  $x, y < 1/16$  being trivial, we shall consider the following two cases.

*Case 1.*  $x, y \geq 1/16$ . In this case (4.3) reduces to  $[\sqrt[4]{x} - \sqrt[4]{y}][\sqrt{x} - \sqrt{y}] \leq (\sqrt{x} - \sqrt{y})^2$ , which is equivalent with the obviously true inequality  $1 \leq \sqrt[4]{x} + \sqrt[4]{y}$ .

*Case 2.*  $x \geq 1/16$  and  $y < 1/16$ . In this case we distinguish two subcases:

(i)  $y \geq 0$ . In this situation (4.3) can be re-written as  $[\sqrt[4]{x} - \sqrt{1/4}][\sqrt{x} - 1/4] \leq [\sqrt{x} - \sqrt{y}][\sqrt{x} - 1/4]$ , which, in turn, is equivalent with  $\sqrt[4]{x} + \sqrt{y} \leq \sqrt{x} + 1/2$ . This last inequality holds since for  $x \geq 1/16$  and  $y < 1/16$  we obviously have  $\sqrt[4]{x} + \sqrt{y} \leq \sqrt[4]{x} + 1/4$  and it is easy to verify that

$$(4.4) \quad \sqrt[4]{x} + 1/4 \leq \sqrt{x} + 1/2, \quad \forall x \in [0, \infty).$$

(ii)  $y < 0$ . In this case, the inequality (4.3) is equivalent to  $[\sqrt[4]{x} - \sqrt{1/4}][\sqrt{x} - 1/4] \leq [\sqrt{x} + \sqrt{-y}][\sqrt{x} - 1/4]$ , that is, to  $\sqrt[4]{x} - 1/2 \leq \sqrt{x} + \sqrt{-y}$ . This last inequality is true because  $\sqrt{x} + \sqrt{-y} \geq \sqrt{x}$  and, by (4.4), we also have  $\sqrt[4]{x} \leq \sqrt{x} + 1/4 < \sqrt{x} + 1/2$  for every  $x > 0$ .

These show that the operator  $T$  is  $D_f$ -firm. Now we verify that  $T$  is not  $D_f$ -nonexpansive, that is, that there exists two real numbers  $x$  and  $y$  such that

$$(4.5) \quad f(T(y)) - f(T(x)) - f'(T(x))(T(y) - T(x)) > f(y) - f(x) - f'(x)(y - x)$$

Taking  $x = 1/8$  and  $y = 1/16$ , a simple computation shows that the left hand side of (4.5) equals  $2^{13/2}[2^{3/4} + 2^{1/2} - 3] \simeq 8.6895$ , while the right hand side equals  $2^{15/4}[2^{1/2} - 1] \simeq 5.5730$ .  $\square$

Before proceeding with the presentation of our generalization of Opial's Lemma several observations concerning its hypothesis are in order.

**Remark 4.1.** (a) The next result is a proper generalization of Lemma 5.7 in [36] which, in turn, is a proper generalization of Opial's Lemma. To see this, observe that when the operator  $T : X \rightarrow X$  is  $D_f$ -nonexpansive, it satisfies (4.7) below.

(b) The hypothesis of the next result implicitly require that the space  $X$  should be reflexive. The fact is that a function  $f$  which is lower semicontinuous with  $\text{int dom } f \neq \emptyset$  and uniformly convex on bounded subsets of  $\text{int dom } f$  exists on a Banach space  $X$  only if  $X$  is reflexive (cf. [34, Corollary 4.3] in conjunction with [36, Theorem 2.10(ii)]).

With these facts in mind we now proceed with the presentation of the generalization of Opial's Lemma.

**Proposition 4.1.** *Suppose that the function  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ . Let  $T : X \rightarrow 2^X$  be an operator satisfying (3.13) and suppose that  $\nabla f$  is bounded on bounded subsets of  $\text{dom } T \cup \text{ran } T$ . If  $\{z^k\}_{k \in \mathbb{N}} \subseteq \text{dom } T$  is a sequence which converges weakly to a vector  $z \in \text{dom } T$  and if, for some sequence  $\{u^k\}_{k \in \mathbb{N}}$  satisfying*

$$(4.6) \quad (\forall k \in \mathbb{N} : u^k \in Tz^k) \quad \text{and} \quad \lim_{k \rightarrow \infty} D_f(u^k, z^k) = 0,$$

there exists  $u \in T(z)$  such that

$$(4.7) \quad \liminf_{k \rightarrow \infty} D_f(u^k, u) \leq \liminf_{k \rightarrow \infty} D_f(z^k, z),$$

then the vector  $z$  is a fixed point of  $T$ .

**Proof.** For any  $x \in \text{int dom } f$  one has

$$D_f(z^k, x) - D_f(z^k, z) = D_f(z, x) + \langle \nabla f(x) - \nabla f(z), z - z^k \rangle.$$

This implies that

$$(4.8) \quad \liminf_{k \rightarrow \infty} D_f(z^k, x) \geq D_f(z, x) + \liminf_{k \rightarrow \infty} D_f(z^k, z),$$

because, since  $\{z^k\}_{k \in \mathbb{N}}$  converges weakly to  $z$ , we have

$$\lim_{k \rightarrow \infty} \langle \nabla f(x) - \nabla f(z), z - z^k \rangle = 0.$$

Since the function  $f$  is Legendre, it is strictly convex on  $\text{int dom } f$ . This implies that  $D_f(z, x) > 0$  whenever  $x \neq z$  (cf. [32, Proposition 1.1.4]) and, consequently, by (4.8) we obtain

$$(4.9) \quad x \neq z \Rightarrow \liminf_{k \rightarrow \infty} D_f(z^k, x) > \liminf_{k \rightarrow \infty} D_f(z^k, z).$$

We claim that

$$(4.10) \quad \liminf_{k \rightarrow \infty} D_f(u^k, u) = \liminf_{k \rightarrow \infty} D_f(z^k, u).$$

To prove this claim, observe that

$$(4.11) \quad D_f(u^k, u) = D_f(z^k, u) + [f(u^k) - f(z^k)] - \langle \nabla f(u), u^k - z^k \rangle.$$

The sequence  $\{z^k\}_{k \in \mathbb{N}}$  is bounded as being weakly convergent. Since the function  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ , it is also sequentially consistent (cf. [36, Theorem 2.10]). Therefore, by (4.6), we deduce that

$$(4.12) \quad \lim_{k \rightarrow \infty} \|u^k - z^k\| = 0.$$

Hence,  $\{u^k\}_{k \in \mathbb{N}}$  is bounded and

$$(4.13) \quad \lim_{k \rightarrow \infty} \langle \nabla f(u), u^k - z^k \rangle = 0.$$

The convexity of  $f$  on  $\text{int dom } f$  implies

$$(4.14) \quad \langle \nabla f(u^k), u^k - z^k \rangle \geq f(u^k) - f(z^k) \geq \langle \nabla f(z^k), u^k - z^k \rangle, \quad \forall k \in \mathbb{N}.$$

By hypothesis,  $\nabla f$  is bounded on bounded subsets of  $\text{dom } T \cup \text{ran } T$ . Therefore, the sequences  $\{\nabla f(z^k)\}_{k \in \mathbb{N}}$  and  $\{\nabla f(u^k)\}_{k \in \mathbb{N}}$  are bounded. Thus, by (4.12) and (4.14), we deduce that

$$\lim_{k \rightarrow \infty} [f(u^k) - f(z^k)] = 0.$$

This, combined with (4.11) and (4.13), implies (4.10) and the claim above is proved.

Suppose by contradiction that  $z \notin Tz$ . Then the vector  $u \in Tz$  whose existence is guaranteed by hypothesis has  $u \neq z$  and then, by (4.9), we deduce

$$(4.15) \quad \liminf_{k \rightarrow \infty} D_f(z^k, u) > \liminf_{k \rightarrow \infty} D_f(z^k, z).$$

On the other hand, by (4.7) and (4.10), we have that

$$(4.16) \quad \liminf_{k \rightarrow \infty} D_f(z^k, z) \geq \liminf_{k \rightarrow \infty} D_f(u^k, u) = \liminf_{k \rightarrow \infty} D_f(z^k, u),$$

which contradicts (4.15). This completes the proof.  $\square$

**4.2. A convergence theorem for the proximal-projection algorithm.** At this stage we are in position to consider the question of convergence of the procedure (1.7) towards solutions of Problem 1.1. For this purpose, we recall the following:

**Definition 4.2.** (Cf. [58]) (a) The *weak upper limit of the sequence*  $\{E_k\}_{k \in \mathbb{N}}$  of subsets of  $X$  is the set denoted  $w\text{-}\overline{\lim}_{k \rightarrow \infty} E_k$  and consisting of all  $x \in X$  such that there exists a subsequence  $\{E_{i_k}\}_{k \in \mathbb{N}}$  of  $\{E_k\}_{k \in \mathbb{N}}$  and a sequence  $\{x^k\}_{k \in \mathbb{N}}$  in  $X$  which converges weakly to  $x$  and has the property that  $x^k \in E_{i_k}$  for each  $k \in \mathbb{N}$ .

(b) The operator  $A : X \rightarrow 2^{X^*}$  is *sequentially weakly-strongly closed* if its graph is sequentially closed in  $X \times X^*$  provided with the *(weak)  $\times$  (strong)*-topology, that is,

$$(4.17) \quad \left. \begin{array}{l} \forall k \in \mathbb{N} : \xi^k \in Av^k \\ v^k \rightharpoonup v \text{ and } \xi^k \rightarrow \xi \end{array} \right\} \Rightarrow \xi \in Av.$$

Before proceedings towards the main result of this paper the following observations may be of use.

**Remark 4.2.** (a) The subset  $E$  of  $X$  may happen not to be convex even if there exists a sequence of closed convex sets  $\{E_k\}_{k \in \mathbb{N}}$  contained in  $X$  such that  $w\text{-}\overline{\lim}_{k \rightarrow \infty} E_k = E$ . Indeed, take  $X = \mathbb{R}$  and

$$E_k := \begin{cases} \left[ -1 - \frac{1}{k+1}, -1 + \frac{1}{k+1} \right] & \text{if } k \text{ is even,} \\ \left[ 1 - \frac{1}{k+1}, 1 + \frac{1}{k+1} \right] & \text{if } k \text{ is odd.} \end{cases}$$

It is easy to verify that the weak upper limit of this sequence of closed convex sets is the nonconvex set  $\{-1, +1\}$ . This fact explains why, in the next theorem, convexity of  $C$  can not be derived from the convexity of the sets  $C_k$ .

(b) Any nonempty closed convex subset  $E$  of  $X$  is the weak upper limit of a sequence of half spaces (corresponding to support hyperplanes) containing it.

(c) Among the operators which are sequentially weakly-strongly closed are all the maximal monotone operators (see, for instance, [60]).

(d) An essential part of condition (b) of the theorem below is the requirement that the gradient  $\nabla f$  of the Legendre function  $f$  should be bounded on bounded subsets of  $\text{int dom } f$ . If the Legendre function  $f$  has the property that  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ , then  $\text{dom } f = X$ . Indeed, since  $f$  is essentially smooth, it follows that  $\text{int dom } f \neq \emptyset$  and for any sequence  $\{x^k\}_{k \in \mathbb{N}}$  contained in  $\text{int dom } f$  and converging to a point of the boundary of  $\text{int dom } f$  has the property that  $\lim_{k \rightarrow \infty} \|\nabla f(x^k)\|_* = \infty$  (cf. [18, Theorem 5.6]). Now, suppose that  $\{x^k\}_{k \in \mathbb{N}}$  is a convergent sequence contained in  $\text{int dom } f$  and denote by  $x$  its limit. Then the sequence  $\{\nabla f(x^k)\}_{k \in \mathbb{N}}$  is bounded because the sequence  $\{x^k\}_{k \in \mathbb{N}}$  is bounded and  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ . We claim that  $x \in \text{int dom } f$ . Assume by contradiction that  $x \notin \text{int dom } f$ . Then  $x$  belongs to the boundary of  $\text{int dom } f$ . Hence,  $\lim_{k \rightarrow \infty} \|\nabla f(x^k)\|_* = \infty$  and this contradicts the boundedness of  $\{\nabla f(x^k)\}_{k \in \mathbb{N}}$ . Since  $\text{int dom } f$  contains the limit of any convergent sequence of vectors contained in it, it follows that  $\text{int dom } f$  is, simultaneously, a closed and open set. The space  $X$ , being a Banach space, it is necessarily arcways connected and, thus, a connected space (cf. [38, Theorem 10.3.2]). Consequently,  $X$  is the

only nonempty subset of  $X$  which is open and closed at the same time (cf. [38, Theorem 10.1.8]), that is,  $\text{int dom } f = X$ .

The following theorem establishes the basic convergence properties of the proximal-projection method. It should be observed that, in view of Remark 4.2(d), the hypothesis of point (b) of the theorem implicitly requires that  $\text{dom } f = X$ . At point (ii) of the theorem sequential weak-weak continuity of  $\nabla f$  is mentioned as a sufficient condition for weak convergence (as opposed to subsequential convergence) of the proximal-projection method. This condition is obviously satisfied whenever the space  $X$  has finite dimension. It is also satisfied if  $X$  is a Hilbert space and  $f = \frac{1}{2} \|\cdot\|^2$  as well as in some nonhilbertian Banach spaces like  $\ell^p$  provided with  $f = \frac{1}{p} \|\cdot\|^p$  for any  $p \in (1, +\infty)$  – see [28, Proposition 8.2].

**Theorem 4.1.** *Suppose that the function  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ ,  $\nabla f^*$  is bounded on bounded subsets of  $\nabla f(\text{dom } A)$  and that, in addition to (1.6) and Assumption 1.1, the subsets  $C_k$  of  $X$  satisfy*

$$(4.18) \quad C = \text{w-}\overline{\lim}_{k \rightarrow \infty} C_k.$$

*If Problem 1.1 has at least one solution, if the operator  $A$  is  $D_f$ -coercive on the set  $Q := \bigcup_{k \in \mathbb{N}} C_k$ , and if at least one of the following two conditions is satisfied:*

(a)  *$\nabla f$  is uniformly continuous on bounded subsets of  $\text{int dom } f$  and  $A$  is sequentially weakly-strongly closed;*

(b)  *$\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ ,  $A^f$  is  $D_f$ -nonexpansive and  $C \subseteq \text{dom } A$ ;*

*then any sequence  $\{x^k\}_{k \in \mathbb{N}}$  generated by the proximal-projection method (1.7) has the following properties:*

(i) *It is bounded, has weak accumulation points and any such point is a solution of Problem 1.1;*

(ii) *If Problem 1.1 has unique solution or if  $\nabla f$  is weakly-weakly sequentially continuous, then the sequence  $\{x^k\}_{k \in \mathbb{N}}$  converges weakly and its weak limit is solution to Problem 1.1;*

(iii) *If the Banach space  $X$  has finite dimension, then  $\{x^k\}_{k \in \mathbb{N}}$  converges in norm to a solution of Problem 1.1.*

**Proof.** Let  $z \in C$  be a solution of Problem 1.1. Then, clearly  $z \in \text{dom } A$  and, by (1.6), we deduce that  $z \in \text{int dom } f$ . For each  $k \in \mathbb{N}$ , let  $\zeta^k \in Ax^k$  be such that

$$(4.19) \quad x^{k+1} = \text{Proj}_{C_{k+1} \cap \text{dom } A}^f(\nabla f(x^k) - \zeta^k).$$

Denote by

$$(4.20) \quad u^k := \nabla f^*(\nabla f(x^k) - \zeta^k).$$

Observe that

$$(4.21) \quad u^k \in A^f x^k \quad \text{and} \quad x^{k+1} = \text{proj}_{C_{k+1} \cap \text{dom } A}^f u^k, \quad \forall k \in \mathbb{N}.$$

By hypothesis, the operator  $A$  is  $D_f$ -coercive on the set  $Q$  and  $z$  is a solution of Problem 1.1. Note that, since  $0^* \in Az$  and  $z \in C \subseteq Q$ , Lemma 3.3 applies with  $C$  replaced by  $Q$ . It implies that  $z \in \text{Nexp}_Q^f A^f$ . By Lemma 2.1 we have that

$$(4.22) \quad x^k \in C_k \cap \text{dom } A \cap \text{int dom } f \subseteq Q \cap \text{dom } A \cap \text{int dom } f, \quad \forall k \in \mathbb{N}.$$

First we prove the following:

*Claim 1: The sequence  $\{x^k\}_{k \in \mathbb{N}}$  is bounded.*

In order to show this notice that, by applying Lemma 3.1(b) to  $z \in \text{Nexp}_Q^f A^f$  we deduce

$$(4.23) \quad D_f(z, u^k) + D_f(u^k, x^k) \leq D_f(z, x^k), \quad \forall k \in \mathbb{N}.$$

This implies

$$(4.24) \quad D_f(z, u^k) \leq D_f(z, x^k), \quad \forall k \in \mathbb{N}.$$

By Assumption 1.1, we have that  $z \in C \subseteq C_k$ , for all  $k \in \mathbb{N}$ . Thus, taking into account (2.10), (4.21) and Lemma 2.2 applied to  $\varphi = \iota_{C_{k+1} \cap \text{dom } A}$ , we obtain that

$$(4.25) \quad D_f(z, x^{k+1}) + D_f(x^{k+1}, u^k) \leq D_f(z, u^k), \quad \forall k \in \mathbb{N}.$$

Combining (4.25) with (4.24) yields

$$(4.26) \quad D_f(z, x^{k+1}) \leq D_f(z, x^k), \quad \forall k \in \mathbb{N},$$

showing that the nonnegative sequence  $\{D_f(z, x^k)\}_{k \in \mathbb{N}}$  is nonincreasing and, therefore, bounded. Let  $\beta$  be an upper bound of  $\{D_f(z, x^k)\}_{k \in \mathbb{N}}$ . According to (1.4), (2.10) and (4.26), we deduce that

$$f^*(\nabla f(x^k)) - \langle \nabla f(x^k), z \rangle + f(z) = W_f(\nabla f(x^k), z) = D_f(z, x^k) \leq \beta, \quad \forall k \in \mathbb{N}.$$

This implies that the sequence  $\{\nabla f(x^k)\}_{k \in \mathbb{N}}$  is contained in the sublevel set  $\text{lev}_{\leq}^{\psi}(\beta - f(z))$  of the function  $\psi := f^* - \langle \cdot, z \rangle$ . Since  $z \in \text{int dom } f = \text{int dom } (f^*)^*$ , and since the function  $f^*$  is proper and lower semicontinuous, application of the Moreau-Rockafellar Theorem ([64, Theorem 7(A)] or [19, Fact 3.1]) shows that  $f^* - \langle \cdot, z \rangle$  is coercive. Consequently, all sublevel sets of  $\psi$  are bounded. Hence, the sequence  $\{\nabla f(x^k)\}_{k \in \mathbb{N}}$  is bounded. By hypothesis,  $\nabla f^*$  is bounded on bounded subsets of  $\nabla f(\text{dom } A)$  and, according to (1.7),  $\{x^k\}_{k \in \mathbb{N}}$  is contained in  $\nabla f(\text{dom } A)$ . Hence, the sequence  $x^k = \nabla f^*(\nabla f(x^k))$ ,  $k \in \mathbb{N}$ , is bounded. This proves Claim 1.

Now we are going to prove the following:

*Claim 2: The sequence  $\{x^k\}_{k \in \mathbb{N}}$  has weak accumulation points and any such point is a solution of Problem 1.1.*

The space  $X$  being reflexive, there exists a weakly convergent subsequence  $\{x^{i_k}\}_{k \in \mathbb{N}}$  of  $\{x^k\}_{k \in \mathbb{N}}$ . Let  $\bar{x} = \text{w-lim}_{k \rightarrow \infty} x^{i_k}$ . In order to show that  $\bar{x} \in C$ , denote  $y^k = x^{i_k}$  for each  $k \in \mathbb{N}$ . According to (4.22), we have that  $y^k \in C_{i_k}$ , for every  $k \in \mathbb{N}$ . By hypothesis  $\text{w-lim}_{k \rightarrow \infty} C_k = C$  and this implies that  $\bar{x} \in C$  (see Definition 4.2). It remains to prove that  $0^* \in A\bar{x}$ . To this end, observe that, according to (2.10), (4.25) and (4.24) we have

$$(4.27) \quad 0 \leq D_f(z, u^k) - D_f(z, x^{k+1}) \leq D_f(z, x^k) - D_f(z, x^{k+1}), \quad \forall k \in \mathbb{N}.$$

As noted above, the sequence  $\{D_f(z, x^k)\}_{k \in \mathbb{N}}$  is nonincreasing and nonnegative and, therefore, it converges. By (4.27), this implies that the sequence  $\{D_f(z, u^k)\}_{k \in \mathbb{N}}$  converges and has the same limit as  $\{D_f(z, x^k)\}_{k \in \mathbb{N}}$ . By (4.23) we also have that

$$(4.28) \quad D_f(u^k, x^k) \leq D_f(z, x^k) - D_f(z, u^k), \quad \forall k \in \mathbb{N},$$

and, thus,

$$(4.29) \quad \lim_{k \rightarrow \infty} D_f(u^k, x^k) = 0.$$

Since  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ , it results that it is sequentially consistent too (cf. [36, Theorem 2.10]). Therefore, the equality (4.29) implies that

$$(4.30) \quad \lim_{k \rightarrow \infty} \|u^k - x^k\| = 0.$$

Now, we distinguish two possible situations. First, suppose that condition (a) is satisfied. Note that, according to (4.20), we have

$$\zeta^k = \nabla f(x^k) - \nabla f(u^k), \quad \forall k \in \mathbb{N}.$$

Since  $\nabla f$  is uniformly continuous on bounded subsets of its domain, we deduce by (4.30) that  $\lim_{k \rightarrow \infty} \zeta^k = 0^*$ . The operator  $A$  being sequentially weakly-strongly closed, this and the fact that  $\{x^{i_k}\}_{k \in \mathbb{N}}$  converges weakly to  $\bar{x}$  imply that  $0^* \in A\bar{x}$ . Hence,  $\bar{x}$  is a solution of Problem 1.1 when condition (a) is satisfied.

Alternatively, suppose that condition (b) is satisfied. Recall that, in this case, we necessarily have  $\text{dom } f = X$  (cf. Remark 4.2(d)). By hypothesis (b), we have that  $C \subseteq \text{dom } A$ . By Assumption 1.1, we have that

$$(\nabla f - A)(C) \subseteq (\nabla f - A)(C_0) \subseteq \text{int dom } f^*,$$

and, as shown above,  $\bar{x} \in C$ . Hence,

$$\emptyset \neq (\nabla f - A)(\bar{x}) \subseteq \text{int dom } f^*$$

which implies that  $\bar{x} \in \text{dom } A^f$ . From (4.2) written with  $x^{i_k}$  instead of  $y$  and  $u^{i_k}$  instead of  $v$ , we obtain that there exists  $\bar{u} \in A^f \bar{x}$ , such that

$$(4.31) \quad D_f(u^{i_k}, \bar{u}) \leq D_f(x^{i_k}, \bar{x}), \quad \forall k \in \mathbb{N}.$$

This implies

$$(4.32) \quad \liminf_{k \rightarrow \infty} D_f(u^{i_k}, \bar{u}) \leq \liminf_{k \rightarrow \infty} D_f(x^{i_k}, \bar{x}).$$

Proposition 4.1, (4.29) and (4.32) imply that  $\bar{x}$  is a fixed point of  $A^f$ , that is,  $0^* \in A\bar{x}$ . Hence,  $\bar{x}$  is a solution of Problem 1.1. This completes the proof of (i).

The fact that if the Problem 1.1 has unique solution then the sequence  $\{x^k\}_{k \in \mathbb{N}}$  converges weakly to that solution is an immediate consequence of (i). Assume that the function  $f$  has sequentially weakly-weakly continuous gradient. We show next that, in this case, the sequence  $\{x^k\}_{k \in \mathbb{N}}$  can not have more than one weak accumulation point. Suppose by contradiction that this is not the case and that  $x'$  and  $x''$  are two different weak accumulation points of  $\{x^k\}_{k \in \mathbb{N}}$ . Let  $\{x^{i_k}\}_{k \in \mathbb{N}}$  and  $\{x^{j_k}\}_{k \in \mathbb{N}}$  be subsequences of  $\{x^k\}_{k \in \mathbb{N}}$  converging weakly to  $x'$  and  $x''$ , respectively. By (i) combined with Lemma 3.3(b) it results that  $\{x', x''\} \subseteq \mathcal{S}_f(A, C) = \text{Nexp}_C^f(A)$ . Hence, the inequality (4.24) still holds for any  $z \in \{x', x''\}$ . It implies that the sequences  $\{D_f(x', x^k)\}_{k \in \mathbb{N}}$  and  $\{D_f(x'', x^k)\}_{k \in \mathbb{N}}$  are convergent. Let  $a$  and  $b$  be their respective limits. For any  $k \in \mathbb{N}$  we have

$$D_f(x', x^k) - D_f(x'', x^k) = D_f(x', x'') + \langle \nabla f(x'') - \nabla f(x^k), x' - x'' \rangle$$

because of (2.10). Replacing in this equation  $x^k$  by  $x^{j_k}$  and letting  $k \rightarrow \infty$  we deduce that

$$(4.33) \quad a - b = D_f(x', x''),$$

because  $\nabla f$  is sequentially weakly-weakly continuous. A similar reasoning with  $x'$  and  $x''$  interchanged shows that

$$b - a = D_f(x'', x').$$

Adding this equality with (4.33) we obtain that  $D_f(x', x'') = 0$ . This can not happen unless  $x' = x''$  because the function  $f$  is strictly convex on  $C \cap \text{int dom } f$  as being Legendre. Thus, we reached a contradiction and this completes the proof of (ii). It is clear that (iii) follows from (i) and (ii) since the gradient of any convex function in a finite dimensional space is continuous on the interior of its domain (see, for instance, [61, Proposition 2.8]). This completes the proof of the theorem.  $\square$

**4.3. Consequences of Theorem 4.1.** If  $X$  is a Hilbert space provided with the function  $f = \frac{1}{2} \|\cdot\|^2$ , then Theorem 4.1 has a somewhat simpler form and even strong convergence of the sequence generated by the proximal-projection method can be sometimes ensured. Note that in this case the operator  $A : X \rightarrow 2^X$  is  $D_f$ -coercive if and only if it is *firmly nonexpansive* in the sense that

$$(4.34) \quad \left. \begin{array}{l} x, y \in X \\ \xi \in Ax, \zeta \in Ay \end{array} \right\} \Rightarrow \langle \xi - \zeta, x - y \rangle \geq \|\xi - \zeta\|^2.$$

Clearly, if  $A$  has this property, then the operator  $A^f$  (which is exactly  $I - A$ ) is nonexpansive and, thus,  $D_f$ -nonexpansive. Since in this situation  $\text{Proj}_{C_k}^f$  is exactly the metric projection operator  $\text{Proj}_{C_k}$ , we obtain the following result:

**Corollary 4.1.** *Let  $X$  be a Hilbert space. Suppose that  $A : X \rightarrow 2^X$  is a firmly nonexpansive operator (i.e., it satisfies (4.34)). If  $C$  is a nonempty, closed and convex subset of  $X$ , if  $\{C_k\}_{k \in \mathbb{N}}$  is a sequence of subsets of  $X$  satisfying (4.18) and such that  $C_k \cap \text{dom } A$  is convex and closed and contains  $C$  for each  $k \in \mathbb{N}$ , and if Problem 1.1 has at least one solution, then the sequence  $\{x^k\}_{k \in \mathbb{N}}$  generated according to the rule*

$$(4.35) \quad x^0 \in C_0 \cap \text{dom } A \quad \text{and} \quad x^{k+1} \in \text{Proj}_{C_{k+1} \cap \text{dom } A}(x^k - Ax^k),$$

*converges weakly to a solution of Problem 1.1. If  $\text{int } \mathcal{S}_f(A, C) \neq \emptyset$ , then  $\{x^k\}_{k \in \mathbb{N}}$  converges strongly.*

**Proof.** As noted above, the operator  $A$  satisfying (4.34) is  $D_f$ -coercive and  $D_f$ -nonexpansive. Applying Theorem 4.1(b) with  $f = \frac{1}{2} \|\cdot\|^2$  which has  $\nabla f = I$  (and, hence, has  $\nabla f$  sequentially weakly-weakly continuous) and taking into account that  $A^f = I - A$ , one deduces that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  converges weakly to a vector in  $\mathcal{S}_f(A, C)$ . The set  $\mathcal{S}_f(A, C) = \text{Nexp}_C^f A^f$  is convex and closed (cf. Lemma 3.1(a)). In the current circumstances, the inequality (4.26) still holds for all  $z \in \mathcal{S}_f(A, C)$ . It is equivalent to the condition

$$\|z - x^{k+1}\| \leq \|z - x^k\|, \quad \forall z \in \mathcal{S}_f(A, C), \quad \forall k \in \mathbb{N}.$$

Therefore, one can apply Theorem 4.5.10 in [25] and this result implies that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  converges strongly when  $\text{int } \mathcal{S}_f(A, C) \neq \emptyset$ .  $\square$

It was pointed out in Subsection 3.3 that there is a strong connection between the proximal-projection method (1.7) and the proximal point method (3.23) – see Lemma 3.5. As far as we know, convergence of the proximal point method in reflexive Banach spaces was established for maximal monotone operators only. We use the connection between the proximal point method and the proximal-projection

method in order to obtain convergence of the proximal point method for operators which are monotone with sequentially weakly-weakly closed graphs (but are not necessarily maximal monotone). Clearly, in spaces with finite dimension any monotone operator with closed graph and, in general, monotone operators with closed convex graphs have this property. The other requirement of the next corollary that  $\text{dom } B_f$  should be convex is necessarily satisfied if  $\nabla f^*$  and  $\nabla f + B$  are surjective because, in this case,  $\text{dom } B_f = \text{ran } \nabla f^* \circ (\nabla f + B) = X$ . This condition is sufficient without being necessary as the example preceding Lemma 3.5 shows. It can be easily verified that this also happens whenever  $\text{Graph } B$  is convex and  $\nabla f$  is linear. Since the corollary is based on Theorem 4.1(a), the remarks preceding Theorem 4.2 concerning the implications of the hypothesis on the domains of  $f$  and  $f^*$  still apply here.

**Corollary 4.2.** *Suppose that the following conditions are satisfied:*

- (a)  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ ;
- (b)  $\nabla f$  is uniformly continuous on bounded subsets of  $\text{int dom } f$  as well as sequentially weakly to weak continuous;
- (c)  $\nabla f^*$  is bounded on bounded subsets of  $\text{int dom } f^*$ .

If  $B : X \rightarrow 2^{X^*}$  is a monotone operator with sequentially weakly-closed graph in  $X \times X^*$ , satisfying (3.24) and such that  $\text{dom } B_f$  is convex, if  $B$  has at least one zero in  $\text{int dom } f$ , and if either

- (d)  $\text{dom } B_f$  is closed in  $X$ ,

or

- (e)  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ ,

then the sequences generated by the proximal point method (3.23) converge weakly to zeros of the operator  $B$ .

**Proof.** We start by observing that, due to the boundedness on bounded subsets of its domain of  $\nabla f^*$ , we have that  $\text{dom } f^* = X^*$  - see Remark 4.2(d). We first prove that the conclusion holds when  $\text{dom } B_f$  is closed in  $X$ . Subsequently we will show that, if  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ , then  $\text{dom } B_f$  is necessarily closed in  $X$  and, thus, the conclusion is true in this case too.

So, assume that  $\text{dom } B_f$  is closed in  $X$ . By Lemma 3.5, the proximal point method is identical to the proximal-projection method applied to the operator  $A[B_f]$  given by (3.20). Therefore, for proving the corollary in this case, it is sufficient to show that  $A[B_f]$  satisfies the requirements of Theorem 4.1(a). In order to do that it is sufficient to ensure that the operator  $A[B_f]$  is  $D_f$ -coercive on its domain and has sequentially weakly-strongly closed graph. Observe that, according to Lemma 3.4, the operator  $A[B_f]$  is  $D_f$ -coercive on its domain. It remains to show that  $A[B_f]$  is sequentially weakly-strongly closed. Let  $\{y^k\}_{k \in \mathbb{N}}$  be a weakly convergent sequence contained in  $\text{dom } A[B_f]$  and denote  $\xi^k = A[B_f]y^k$ . Suppose that  $\{\xi^k\}_{k \in \mathbb{N}}$  converges strongly in  $X^*$  to some vector  $\xi$ . Let  $y = \text{w-lim}_{k \rightarrow \infty} y^k$ . By hypothesis (b), the sequence  $\{\nabla f(y^k)\}_{k \in \mathbb{N}}$  converges weakly in  $X^*$  to  $\nabla f(y)$ . Thus, the sequence

$$(4.36) \quad \nabla f(y^k) - \xi^k = \left[ \nabla f \circ (\nabla f + B)^{-1} \circ \nabla f \right] (y^k)$$

converges weakly in  $X^*$  to  $\nabla f(y) - \xi$ . Denote  $u^k := \nabla f^* \left( \nabla f(y^k) - \xi^k \right)$  and observe that, by (4.36), we have that

$$(4.37) \quad u^k = \left[ (\nabla f + B)^{-1} \circ \nabla f \right] (y^k) = B_f y^k, \quad \forall k \in \mathbb{N}.$$

According to hypothesis (c), the sequence  $\{u^k\}_{k \in \mathbb{N}}$  is bounded because the sequence  $\left\{ \nabla f(y^k) - \xi^k \right\}_{k \in \mathbb{N}}$  is bounded (as shown above this sequence is weakly convergent). Let  $\{u^{i_k}\}_{k \in \mathbb{N}}$  be a weakly convergent subsequence of  $\{u^k\}_{k \in \mathbb{N}}$  and let  $u$  be the weak limit of this subsequence. By (4.37) we deduce that  $\nabla f(y^{i_k}) \in (\nabla f + B)u^{i_k}$  for all  $k \in \mathbb{N}$ , and thus we obtain

$$(4.38) \quad \nabla f(y^{i_k}) - \nabla f(u^{i_k}) \in Bu^{i_k}, \quad \forall k \in \mathbb{N}.$$

By hypothesis (b), we have that

$$(4.39) \quad \text{w-} \lim_{k \rightarrow \infty} \left[ \nabla f(y^{i_k}) - \nabla f(u^{i_k}) \right] = \nabla f(y) - \nabla f(u).$$

Since Graph  $B$  is sequentially weakly-weakly closed and  $\{u^{i_k}\}_{k \in \mathbb{N}}$  converges weakly to  $u$ , the relations (4.38) and (4.39) imply that  $\nabla f(y) - \nabla f(u) \in Bu$ , i.e.,  $\nabla f(y) \in \nabla f(u) + Bu$ . Consequently, we have that

$$(4.40) \quad u = (\nabla f + B)^{-1} (\nabla f(y)) = B_f y,$$

because the operator  $B_f$  is single valued (cf. Lemma 3.4). On the other hand, by (4.36), (4.37), (4.39) and (4.40), we have that

$$\xi = \nabla f(y) - \text{w-} \lim_{k \rightarrow \infty} \nabla f(u^{i_k}) = \nabla f(y) - \nabla f(u) = \nabla f(y) - (\nabla f \circ B_f) y,$$

showing that  $(y, \xi) \in \text{Graph } A[B_f]$ . Hence,  $A[B_f]$  is sequentially weakly-strongly closed and the proof, in this case, is complete.

Now, assume that  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ . Then, by Remark 4.2(d),  $\text{dom } f = X$ . We are going to show that, in this case, the set  $\text{dom } B_f$  is closed. As shown above, if  $\text{dom } B_f$  is closed, then the conclusion holds. In order to prove that  $\text{dom } B_f$  is closed, let  $\{z^k\}_{k \in \mathbb{N}}$  be a sequence contained in  $\text{dom } B_f$  and converging in  $X$  to some vector  $\bar{z}$ . Denote  $w^k = B_f z^k$ . We claim that the sequence  $\{w^k\}_{k \in \mathbb{N}}$  is bounded. To show that, note that, since  $A[B_f]$  is  $D_f$ -coercive (cf. Lemma 3.4(c)), all solutions of Problem 1.1 are in the set  $\text{Nexp}_{\text{dom } A[B_f]}^f (A[B_f])^f$  (cf. Lemma 3.3). Taking into account that, by Lemma 3.4(a, b),  $B_f = (A[B_f])^f$  and  $\text{dom } B_f = \text{dom } A[B_f]$ , we deduce that all solutions of Problem 1.1 are contained in  $\text{Nexp}_{\text{dom } B_f}^f B_f$ . Let  $z$  be such a solution. Then

$$D_f(z, w^k) + D_f(w^k, z^k) \leq D_f(z, z^k), \quad \forall k \in \mathbb{N}.$$

According to the definition of the modulus of total convexity of  $f$  on the bounded set  $\{z^k\}_{k \in \mathbb{N}}$ , denoted  $\nu_f(\{z^k\}_{k \in \mathbb{N}}, \cdot)$ , – see [36] – we obtain

$$(4.41) \quad 0 \leq \nu_f \left( \{z^k\}_{k \in \mathbb{N}}; \|w^k - z^k\| \right) \leq D_f(w^k, z^k) \leq D_f(z, z^k), \quad \forall k \in \mathbb{N},$$

Since  $\nabla f$  is bounded on bounded subsets of  $X$ , the function  $f$  is also bounded on bounded subsets of  $X$ . Consequently, taking into account (2.10), we deduce that the sequence  $\{D_f(z, z^k)\}_{k \in \mathbb{N}}$  is bounded. Let  $M$  be an upper bound of this

sequence. Suppose by contradiction that the sequence  $\{w^k\}_{k \in \mathbb{N}}$  contains a subsequence  $\{w^{j_k}\}_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} \|w^{j_k}\| = \infty$ . Then there exists a positive integer  $k_0$  such that for all integers  $k \geq k_0$  we have  $\|w^{j_k} - z^{j_k}\| \geq 1$ . By [36, Proposition 2.1(ii)] and (4.41), we deduce that for any  $k \geq k_0$  we have

$$(4.42) \quad \|w^{j_k} - z^{j_k}\| \nu_f \left( \{z^k\}_{k \in \mathbb{N}}; 1 \right) \leq \nu_f \left( \{z^k\}_{k \in \mathbb{N}}; \|w^{j_k} - z^{j_k}\| \right) \leq M, \quad \forall k \in \mathbb{N}.$$

The function  $f$  is, by hypothesis, uniformly convex on bounded subsets of  $\text{int dom } f$  and, consequently, it is also totally convex on bounded subsets of  $\text{int dom } f$  – cf. [36, Theorem 2.10]. Therefore,  $\nu_f \left( \{z^k\}_{k \in \mathbb{N}}; 1 \right) > 0$ . Taking this into account together with the fact that  $\{z^{j_k}\}_{k \in \mathbb{N}}$  is bounded (as being convergent) and letting  $k \rightarrow \infty$  in (4.42), we reach a contradiction. Hence, the sequence  $\{w^k\}_{k \in \mathbb{N}}$  is bounded. Let  $\{w^{s_k}\}_{k \in \mathbb{N}}$  be a weakly convergent subsequence of  $\{w^k\}_{k \in \mathbb{N}}$  and let  $\bar{w}$  be the weak limit of this subsequence. According to (3.19), we have that

$$(4.43) \quad \nabla f(z^k) - \nabla f(w^k) \in Bw^k, \quad \forall k \in \mathbb{N}.$$

Since  $\nabla f$  is sequentially weakly-weakly continuous, and since  $B$  has sequentially weakly-weakly closed graph, the relation (4.43), written with  $s_k$  instead of  $k$ , implies that  $\nabla f(\bar{z}) - \nabla f(\bar{w}) \in B\bar{w}$ . This shows that  $\bar{w} = B_f \bar{z}$ , that is,  $\bar{z} \in \text{dom } B_f$ . Hence,  $\text{dom } B_f$  is closed and the proof of the corollary is complete.  $\square$

Another result which follows from Theorem 4.1 concerns a method of regularizing and solving classical variational inequalities in the form (2.18). Since the problem of solving (2.18) may be ill-posed (in the sense that it may not have solutions or it may have multiple solutions) and, therefore, many algorithms for approximating solutions may not converge, or may converge only subsequentially, to solutions of the problem, one "regularizes" the original problem by solving an auxiliary problem which has unique solution and whose solution is in the vicinity of the solution set of the original problem, provided that the later is not empty. A regularization technique, which originates in the works of Tikhonov [72] and Browder [26], [27], consists of replacing the original variational inequality (2.18) by the regularized variational inequality

$$(4.44) \quad \text{Find } x \in C \cap \text{int dom } f \text{ such that}$$

$$\exists \xi \in Bx : [\langle \xi + \alpha \nabla f(x), y - x \rangle \geq 0, \quad \forall y \in C \cap \text{dom } f],$$

for some real number  $\alpha > 0$ . If  $B$  is a monotone operator, then  $B + \alpha \nabla f$  is strictly monotone and, therefore, the variational inequality (4.44) can not have more than one solution. Moreover, in many practically interesting situations, the variational inequality (4.44) has solution even if the original variational inequality (2.18) has not and, if  $\alpha$  is sufficiently small, then the solution of (4.44) is close to the solution set of the unperturbed variational inequality (2.18) whenever the later has solutions. This is, for instance, the case (cf. [7, Theorem 3.2]) when the Banach space  $X$  is simultaneously uniformly convex and uniformly smooth and endowed with the Legendre function  $f := \frac{1}{p} \|\cdot\|^p$  for some  $p > 1$  and  $B$  is maximal monotone. Theorem 4.1 allows us to prove the next corollary which extends the applicability of this regularization technique to reflexive Banach spaces which are not necessarily uniformly convex and uniformly smooth and to produce a weakly convergent algorithm for solving (4.44) in this, more general setting, even if  $B$  is

not maximal monotone. This is of interest because closedness of the solution of (4.44) with small  $\alpha > 0$  to the (presumed nonempty) solution set of the original variational inequality (2.18) can be guaranteed even if  $B$  is not maximal monotone (cf. [9, Theorem 2.1]).

**Corollary 4.3.** *Let  $B : X \rightarrow 2^{X^*}$  be a monotone operator and let  $C$  be a nonempty, convex and closed subset of  $\text{dom } B \cap \text{int dom } f$ . Suppose that  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$ ,  $\nabla f^*$  is bounded on bounded subsets of  $\text{int dom } f^*$  and that, for some real number  $\alpha > 0$ , we have that*

$$(4.45) \quad \emptyset \neq ((1 - \alpha)\nabla f - B)(C) \subseteq \text{int dom } f^*,$$

and the operator  $\text{Proj}_C^f[(1 - \alpha)\nabla f - B]$  is  $D_f$ -firm. If one of the following conditions is satisfied:

(a)  $X$  has finite dimension,  $\text{dom } f = X$ ,  $\nabla f$  is uniformly continuous on bounded subsets of  $X$ , and  $B$  has closed graph and is bounded on bounded subsets of its domain;

(b)  $\nabla f$  is bounded on bounded subsets of  $\text{int dom } f$ , the operator  $\text{Proj}_C^f[(1 - \alpha)\nabla f - B]$  is  $D_f$ -nonexpansive;

then the iterative procedure defined by

$$(4.46) \quad x^0 \in C \text{ and } x^{k+1} \in \text{Proj}_C^f[(1 - \alpha)\nabla f(x^k) - Bx^k], \quad \forall k \in \mathbb{N},$$

is well defined and converges weakly to the necessarily unique solution of the variational inequality (4.44), provided that such a solution exists.

**Proof.** Well definedness of the procedure results from (4.45). The variational inequality (4.44) can not have more than one solution since the operator  $B' := B + \alpha\nabla f$  is strictly monotone. According to Lemma 2.4, finding a solution of (4.44) is equivalent to finding a zero for the operator  $V := V[B'; C; f]$  defined by (2.19). Note that

$$(4.47) \quad V = \nabla f - \nabla f \circ \text{Proj}_C^f((1 - \alpha)\nabla f - B),$$

and

$$(4.48) \quad V^f = \text{Proj}_C^f((1 - \alpha)\nabla f - B)$$

and that, by (2.21) applied to  $B'$  instead of  $B$ , we have

$$(4.49) \quad \text{Proj}_C^f(\nabla f - V) = \text{Proj}_C^f((1 - \alpha)\nabla f - B).$$

Therefore, we can equivalently re-write the procedure (4.46) as

$$(4.50) \quad x^0 \in C \quad \text{and} \quad x^{k+1} \in \text{Proj}_C^f(\nabla f(x^k) - Vx^k), \quad \forall k \in \mathbb{N}.$$

This is exactly (1.7) applied to  $V$  instead of  $A$  with the sequence of sets  $C_k = C$  for all  $k \in \mathbb{N}$ . Now, suppose that condition (a) of our corollary is satisfied. In this case, if we show that the graph of  $V$  is closed in  $X \times X$  and that  $V$  is  $D_f$ -coercive, then Theorem 4.1(a) applies and leads to the conclusion of the corollary. Also, Theorem 4.1(b) implies that, if condition (b) of the corollary holds, then the procedure (4.50) is weakly convergent to the unique solution of (4.44), provided that  $V$  is  $D_f$ -coercive.  $D_f$ -coercivity of  $V$  results in both cases from Lemma 3.2 combined with (4.48) and with our hypothesis that  $V^f = \text{Proj}_C^f[(1 - \alpha)\nabla f - B]$  is  $D_f$ -firm. So, it remains to prove that, under assumption (a) of the corollary, the

graph of  $V$  is closed. To this end, let  $\{y^k\}_{k \in \mathbb{N}}$  be a sequence in  $\text{dom } V$  and assume that this sequence converges to  $y \in X$ . Let  $\{\xi^k\}_{k \in \mathbb{N}}$  be the sequence

$$\xi^k = \nabla f(y^k) - \nabla f \circ \text{Proj}_C^f((1 - \alpha)\nabla f(y^k) - \zeta^k),$$

where  $\zeta^k \in B y^k$  for all  $k \in \mathbb{N}$ . Suppose that  $\lim_{k \rightarrow \infty} \xi^k = \xi$ . Then, by Lemma 2.1, we have

$$\nabla f(y^k) - \xi^k = \left[ \nabla f \circ (\nabla f + N_C)^{-1} \right] ((1 - \alpha)\nabla f(y^k) - \zeta^k).$$

This implies that

$$(4.51) \quad (1 - \alpha)\nabla f(y^k) - \zeta^k \in (\nabla f + N_C) \left[ \nabla f^* \left( \nabla f(y^k) - \xi^k \right) \right], \quad \forall k \in \mathbb{N}.$$

Since  $B$  is bounded on the bounded set  $\{y^k\}_{k \in \mathbb{N}}$ , it follows that the sequence  $\{\zeta^k\}_{k \in \mathbb{N}}$  is bounded. Let  $\{\zeta^{i_k}\}_{k \in \mathbb{N}}$  be a convergent subsequence of  $\{\zeta^k\}_{k \in \mathbb{N}}$  and let  $\zeta$  be its limit. Since  $\nabla f$  and  $\nabla f^*$  are continuous on their respective domains and the normality operator  $N_C$  is maximal monotone (and, hence, has closed graph), (4.51) implies

$$(1 - \alpha)\nabla f(y) - \zeta \in (\nabla f + N_C) [\nabla f^* (\nabla f(y) - \xi)].$$

Therefore, we have

$$\nabla f^* (\nabla f(y) - \xi) = \text{Proj}_C^f [(1 - \alpha)\nabla f(y) - \zeta]$$

showing that

$$\xi = \nabla f(y) - \nabla f \circ \text{Proj}_C^f [(1 - \alpha)\nabla f(y) - \zeta] \in V y.$$

This completes the proof.  $\square$

The requirement made in Corollary 4.3 that the operator  $V^f = \text{Proj}_C^f [(1 - \alpha)\nabla f - B]$  should be  $D_f$ -firm may seem unusual. Here are several examples which show that this condition is quite often satisfied without excessively costly demands on the operator  $B$  or the function  $f$ . The next example shows that Corollary 4.3(b) is applicable to solve variational inequalities in the form (4.44) when  $X$  is any Hilbert space,  $f = \frac{1}{2} \|\cdot\|^2$ ,  $C = \text{dom } B = X$  and  $B$  is a monotone operator which is either contractive with some constant  $\gamma > 0$  or strongly monotone with some constant  $\delta > 0$  and even in more general conditions (when (4.52) holds for some  $\alpha \in (0, \frac{1}{2})$  and  $\beta > 0$ ). Obviously, in this setting, (2.18) is exactly the problem of finding a zero of  $B$ . To follow the considerations in the examples below one should first note that by replacing in (2.18) the operator  $B$  by  $\beta B$ , where  $\beta$  is a positive constant, one obtains a variational inequality which is equivalent to the original one.

**Example 4.2.** *Let  $X$  be a Hilbert space and let  $f = \frac{1}{2} \|\cdot\|^2$ . Suppose that  $C = \text{dom } B = X$ . If  $B$  is contractive with some constant  $\gamma > 0$  or strongly monotone with some constant  $\delta > 0$ , then the operator  $V^f = \text{Proj}_C^f [(1 - \alpha)\nabla f - \beta B]$  is  $D_f$ -firm and  $D_f$ -nonexpansive whenever  $\alpha \in (0, \frac{1}{2})$  with  $\beta = \sqrt{\alpha(1 - \alpha)\gamma^{-1}}$  in the contractive case and  $\beta = (1 - 2\alpha)\delta$  in the strongly monotone case.*

In order to prove this observe that, in the current setting,  $\nabla f = \text{Proj}_C^f = I$ . Also, the  $D_f$ -firmness condition (3.14) for  $T = V^f$  is equivalent to (4.34) and this is exactly

$$\langle [(1 - \alpha)x - \beta\xi] - [(1 - \alpha)y - \beta\eta], x - y \rangle$$

$$\geq \|[(1-\alpha)x - \beta\xi] - [(1-\alpha)y - \beta\eta]\|^2,$$

which can be equivalently re-written as

$$(4.52) \quad \alpha(1-\alpha)\|x-y\|^2 + (1-2\alpha)\beta\langle\xi-\eta, x-y\rangle \geq \beta^2\|\xi-\eta\|^2,$$

for all pairs  $(x, \xi), (y, \eta) \in \text{Graph } B$ . Moreover, if  $V^f$  is  $D_f$ -firm, then it is also  $D_f$ -nonexpansive. Due to the monotonicity of  $B$  the inequality (4.52) holds whenever  $\alpha \in (0, \frac{1}{2})$  and

$$(4.53) \quad \alpha(1-\alpha)\|x-y\| \geq \beta\|\xi-\eta\|, \quad \forall(x, \xi) \in \text{Graph } B, \forall(y, \eta) \in \text{Graph } B.$$

If  $B$  is contractive with constant  $\gamma$ , then we have

$$\gamma\|x-y\| \geq \|\xi-\eta\|, \quad \forall(x, \xi) \in \text{Graph } B, \forall(y, \eta) \in \text{Graph } B$$

and multiplying this inequality by  $\beta = \sqrt{\alpha(1-\alpha)}\gamma^{-1}$  we deduce that (4.53) holds. Similarly, note that (4.52) is satisfied when

$$(4.54) \quad (1-2\alpha)\langle\xi-\eta, x-y\rangle \geq \beta\|\xi-\eta\|^2, \quad \forall(x, \xi) \in \text{Graph } B, \forall(y, \eta) \in \text{Graph } B.$$

If  $B$  is strongly monotone with constant  $\delta$ , then

$$\langle\xi-\eta, x-y\rangle \geq \delta\|\xi-\eta\|^2, \quad \forall(x, \xi) \in \text{Graph } B, \forall(y, \eta) \in \text{Graph } B.$$

Multiplying this inequality by  $(1-2\alpha)$  we deduce that (4.54) holds in this case.  $\square$

In situations in which  $X$  is not a Hilbert space, or  $X$  is a Hilbert space but the monotone operator  $B$  does not satisfy (4.52) for some  $\alpha \in (0, \frac{1}{2})$  and  $\beta > 0$ , the question of how to choose the "regularization function"  $f$  and the "regularization parameter"  $\alpha$  in the perturbed variational inequality (4.44) in order to force  $D_f$ -firmness and/or  $D_f$ -nonexpansivity on  $V^f = \text{Proj}_C^f((1-\alpha)\nabla f - B)$  is relevant. Here are examples of situations when  $V^f$  is  $D_f$ -firm even if  $X$  is not a Hilbert space.

**Example 4.3.** *If the monotone operator  $B : X \rightarrow 2^{X^*}$  and the nonempty closed convex subset  $C$  of  $\text{dom } B \cap \text{int dom } f$  have the property (4.45), then  $V^f = \text{Proj}_C^f((1-\alpha)\nabla f - B)$  is  $D_f$ -firm in any of the following situations:*

- (a)  $C = X$ ,  $\alpha \in (0, 1)$  and the operator  $\alpha\nabla f + B$  is  $D_f$ -coercive on its domain;
- (b)  $X$  is a Hilbert space,  $C = X$ ,  $f = \frac{1}{2}\|\cdot\|^2$ ,  $\alpha \in (0, \frac{1}{2}]$  and  $B$  is contractive with constant  $\alpha(1-\alpha)$ .

In order to show this observe that in case (a), according to Lemma 2.1, we have  $V^f = (\alpha\nabla f + B)^f$  and, by Lemma 3.2(c), the conclusion follows. In case (b) an easy calculation shows that the operator  $\alpha\nabla f + B$  is  $D_f$ -coercive and, therefore, the conclusion of case (a) applies.  $\square$

Example 4.3(a), in conjunction with Corollary 4.3, leads to an algorithm for solving the variational inequality (4.44) whenever is possible to find a Legendre function  $f$  and a number  $\alpha \in (0, 1)$  such that  $\alpha\nabla f + B$  is  $D_f$ -coercive on its domain. Example 4.3(b) covers the class of expansive operators in Hilbert spaces for which, as far as we know, few methods of finding zeros are available.

**Example 4.4.** *If the monotone operator  $B : X \rightarrow 2^{X^*}$  and the nonempty closed convex subset  $C$  of  $\text{dom } B \cap \text{int dom } f$  have the property (4.45) and if*

$$(4.55) \quad \langle[\alpha\nabla f(x) + \xi] - [\alpha\nabla f(y) + \eta], \text{Proj}_C^f[(1-\alpha)\nabla f(x) - \xi] - \text{Proj}_C^f[(1-\alpha)\nabla f(y) - \eta]\rangle \geq 0,$$

for any  $(x, \xi)$  and  $(y, \eta)$  in  $\text{Graph } B$ , then the operator  $V^f = \text{Proj}_C^f[(1 - \alpha)\nabla f - B]$  is  $D_f$ -firm. In particular, this happens in any of the following situations:

- (a)  $B = (\frac{1}{2} - \alpha)\nabla f$  and  $\alpha \in (0, \frac{1}{2})$ ;
- (b) The operator  $P : X \times X^* \rightarrow 2^{X^* \times X}$ , defined by

$$(4.56) \quad P(z, \zeta) = (0^*, \text{Proj}_C^f((1 - \alpha)\nabla f(z) - \zeta)),$$

for some  $\alpha \in (0, 1)$  is monotone when  $X \times X^*$  is provided with the norm  $\|(z, \zeta)\| = (\|z\|^2 + \|\zeta\|_*^2)^{1/2}$  and with the duality pairing  $\langle (z, \zeta), (\zeta', z') \rangle = \langle \zeta', z \rangle + \langle \zeta, z' \rangle$  (and, therefore, its dual is isometric with  $X^* \times X$ ).

Let  $x, y \in \text{dom } V^f$  and let  $\xi \in Bx$  and  $\eta \in By$ . Denote

$$x' = \text{Proj}_C^f((1 - \alpha)\nabla f(x) - \xi) \text{ and } y' = \text{Proj}_C^f((1 - \alpha)\nabla f(y) - \eta).$$

By Lemma 2.1 we have

$$(1 - \alpha)\nabla f(x) - \xi - \nabla f(x') \in N_C(x') \text{ and } (1 - \alpha)\nabla f(y) - \eta - \nabla f(y') \in N_C(y'),$$

which imply

$$\langle (1 - \alpha)\nabla f(x) - \xi, y' - x' \rangle \leq \langle \nabla f(x'), y' - x' \rangle$$

and, respectively,

$$\langle (1 - \alpha)\nabla f(y) - \eta, x' - y' \rangle \leq \langle \nabla f(y'), x' - y' \rangle.$$

Summing up the last two inequalities we obtain that

$$(4.57) \quad \begin{aligned} & \langle [(1 - \alpha)\nabla f(x) - \xi] - [(1 - \alpha)\nabla f(y) - \eta], x' - y' \rangle \\ & \geq \langle \nabla f(x') - \nabla f(y'), x' - y' \rangle. \end{aligned}$$

The  $D_f$ -firmness condition (3.14) for the operator  $T = V^f$  is exactly

$$\langle \nabla f(x) - \nabla f(y), x' - y' \rangle \geq \langle \nabla f(x') - \nabla f(y'), x' - y' \rangle.$$

According to (4.57), this is satisfied when

$$\langle \nabla f(x) - \nabla f(y), x' - y' \rangle \geq \langle [(1 - \alpha)\nabla f(x) - \xi] - [(1 - \alpha)\nabla f(y) - \eta], x' - y' \rangle$$

and this last inequality is equivalent to (4.55). Hence, if (4.55) holds, then the operator  $V^f$  is  $D_f$ -firm. In case (a) we have that  $\alpha\nabla f + B = (1 - \alpha)\nabla f - B$  and using the monotonicity of  $\text{Proj}_C^f$  (cf. [36, Theorem 4.6]) one deduces that (4.55) holds. Suppose that we are in case (d) and the operator  $P$ , given by (4.56), is monotone. Then observe that

$$\begin{aligned} & \langle [\alpha\nabla f(x) + \xi] - [\alpha\nabla f(y) + \eta], \\ & \text{Proj}_C^f[(1 - \alpha)\nabla f(x) - \xi] - \text{Proj}_C^f[(1 - \alpha)\nabla f(y) - \eta] \rangle = \\ & \langle (x, \alpha\nabla f(x) + \xi) - (y, \alpha\nabla f(y) + \eta), P(x, \xi) - P(y, \eta) \rangle, \end{aligned}$$

and that the last expression is nonnegative due to the monotonicity of  $P$  (see [36, Proposition 4.7]). Hence, (4.55) holds in this case too.  $\square$

Note that problem (4.44) in which  $B = \beta\nabla f$  for some  $\beta > 0$  is equivalent to the problem of finding the minimizer of  $f$  over  $C$ . The facts observed in Example 4.4(a), in conjunction with Corollary 4.3, leads to a proximal-projection method of finding that minimizer, provided that  $f$  satisfies the other requirements there. Obviously, the effectiveness of that method, as well as of the other methods discussed in this work, depends on the possibility of computing proximal projections onto  $C$ . Algorithms for computing proximal projections are presented in [5], [21] and

[36]. Compared with already classical projection methods (see [63]) in which the iterations are, usually, of the form  $x^{k+1} = \text{Proj}_C(x^k - \lambda_k \nabla f(x^k))$  with a converging to zero positive step size  $\lambda_k$ , the proximal-projection method presents the advantage of not requiring arbitrarily small step sizes which, in practical applications, may force the procedure to become stationary long before the iterates are close to the minimizer of  $f$  (due to the computer identification of  $\lambda_k \nabla f(x^k)$  with the null vector).

In general, it would be nice to have a Legendre function  $f$  for which the condition in Example 4.4(b) is satisfied. Whether such a function exists in a Banach space  $X$  is an *open question*. This question is relevant because such a function  $f$ , if any, would be a "universal regularizer" for variational inequalities in the form (2.18) in the sense that it would be such that the regularized variational inequality (4.44) will be solvable by the proximal-projection method, no matter how the monotone operator  $B$  is, provided that (4.44) has solution.

## 5. CONVERGENCE OF THE PROXIMAL-PROJECTION METHOD IN SPACES OF FINITE DIMENSION

Theorem 4.1 and its corollaries ensure weak and, sometimes, strong convergence of the proximal-projection method to solutions of the Problem 1.1 under conditions which, besides the  $D_f$ -coercivity of the operator  $A$ , require sequential weak-strong closedness of the Graph  $A$  or, alternatively,  $D_f$ -nonexpansivity of  $A^f$ . In this section we show that, when the space  $X$  has finite dimension, some of these requirements can be dropped or weakened. This is possible due to the validity in spaces of finite dimension of another generalization of Opial's Lemma which we present below.

**5.1. Another variant of a generalized Opial's Lemma.** The following result applies to operators  $T : X \rightarrow 2^X$  which are not necessarily  $D_f$ -nonexpansive, but satisfy condition (5.2) below which is more general than  $D_f$ -firmness (compare condition (5.2) with Definition 3.3). It is interesting to observe that, if  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$  and if  $T$  has closed graph, then the conclusion of the next result holds even if the hypothesis that  $u$  satisfies (5.2) is removed. This happens because the equality in (5.1) implies that the sequences  $\{z^k\}_{k \in \mathbb{N}}$  and  $\{u^k\}_{k \in \mathbb{N}}$  converge to the same limit  $z$  and, then, closedness of the graph of  $T$  guarantees that  $z \in Tz$ .

**Proposition 5.1.** *Suppose that the space  $X$  has finite dimension,  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$  and  $T : X \rightarrow 2^X$  is an operator satisfying condition (3.13). Let  $\{z^k\}_{k \in \mathbb{N}}$  be a sequence in  $\text{dom } T$  converging to an element  $z \in \text{dom } T$ . If for some sequence  $\{u^k\}_{k \in \mathbb{N}}$  satisfying*

$$(5.1) \quad (\forall k \in \mathbb{N} : u^k \in Tz^k) \quad \text{and} \quad \lim_{k \rightarrow \infty} D_f(u^k, z^k) = 0,$$

*there exists  $u \in Tz$  such that*

$$(5.2) \quad \liminf_{k \rightarrow \infty} \langle \nabla f(u^k) - \nabla f(u), u^k - u \rangle \leq 0,$$

*then the vector  $z$  is a fixed point of  $T$ .*

**Proof.** Since the function  $f$  is convex and differentiable on  $\text{int dom } f$ , the gradient  $\nabla f$  is continuous on  $\text{int dom } f$ . This fact and the strict convexity of  $f$  on

int dom  $f$  imply that

$$(5.3) \quad \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(x), z^k - x \rangle = \langle \nabla f(z) - \nabla f(x), z - x \rangle > 0.$$

whenever  $x \in (\text{int dom } f) \setminus \{z\}$ . The function  $f$  being uniformly convex on bounded subsets of int dom  $f$ , it is also sequentially consistent (cf. [36, Theorem 2.10]). Therefore, the equality in (5.1) implies that

$$(5.4) \quad \lim_{k \rightarrow \infty} \|z^k - u^k\| = 0.$$

Consequently, the sequences  $\{z^k\}_{k \in \mathbb{N}}$  and  $\{u^k\}_{k \in \mathbb{N}}$  converge to the same limit  $z$ . By condition (3.13), the boundedness of  $\{u^k\}_{k \in \mathbb{N}}$  and the continuity of  $\nabla f$  we deduce that

$$(5.5) \quad \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(z), z^k - z \rangle = 0 = \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(z), u^k - u \rangle.$$

Note that

$$(5.6) \quad \begin{aligned} \langle \nabla f(u^k) - \nabla f(u), u^k - u \rangle &= \langle \nabla f(u^k) - \nabla f(u), u^k - z^k \rangle \\ &\quad + \langle \nabla f(u^k) - \nabla f(u), z^k - u \rangle \\ &= \langle \nabla f(u^k) - \nabla f(u), u^k - z^k \rangle \\ &\quad + \langle \nabla f(z^k) - \nabla f(u), z^k - u \rangle \\ &\quad + \langle \nabla f(u^k) - \nabla f(z^k), z^k - u \rangle. \end{aligned}$$

Since the sequence  $\{\nabla f(u^k)\}_{k \in \mathbb{N}}$  is bounded, it follows from (5.4) that the first term of the last sum in (5.6) converges to zero. The second term of the same sum is nonnegative because of the monotonicity of  $\nabla f$ . Taking the limit as  $k \rightarrow \infty$  on both sides of (5.6), we obtain that

$$(5.7) \quad \lim_{k \rightarrow \infty} \langle \nabla f(u^k) - \nabla f(u), u^k - u \rangle \geq \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(u), z^k - u \rangle.$$

In order to conclude the proof, suppose by contradiction that  $z \notin T(z)$ . Then,  $u \neq z$  and, therefore, by (5.5), (5.2), (5.7) and (5.3), respectively, we obtain

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(z), z^k - z \rangle = \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(z), u^k - u \rangle \\ &\geq \lim_{k \rightarrow \infty} \langle \nabla f(u^k) - \nabla f(u), u^k - u \rangle \geq \lim_{k \rightarrow \infty} \langle \nabla f(z^k) - \nabla f(u), z^k - u \rangle > 0, \end{aligned}$$

which is a contradiction.  $\square$

**5.2. A convergence theorem for the proximal-projection method in spaces of finite dimension.** The following theorem shows that, in finite dimensional spaces, convergence of the proximal-projection method to solutions of Problem 1.1 can be ensured with lesser requirements on the operator  $A$  in addition to the  $D_f$ -coercivity than those involved in Theorem 4.1 and its corollaries.

**Theorem 5.1.** *Suppose that the space  $X$  has finite dimension,  $f$  is uniformly convex on bounded subsets of int dom  $f$ ,  $\nabla f^*$  is bounded on bounded subsets of  $\nabla f(\text{dom } A)$  and that (1.6), Assumption 1.1 hold and*

$$(5.8) \quad C \cap \text{dom } A = \overline{\text{w-lim}_{k \rightarrow \infty} (C_k \cap \text{dom } A)}.$$

*If the Problem 1.1 has at least one solution, if the operator  $A : X \rightarrow 2^{X^*}$  is  $D_f$ -coercive on  $Q = \bigcup_{k=0}^{\infty} C_k$  and if  $C \cap \text{dom } A$  is closed, then the sequences generated*

by the proximal-projection method (1.7) are well defined and converge to solutions of the Problem 1.1.

**Proof.** Well definedness of the sequences generated by (1.7) follows from (1.6) and Assumption 1.1. Suppose that, for each  $k \in \mathbb{N}$ ,  $\zeta^k$  and  $u^k$  are as in (4.19) and (4.20), respectively. Then, clearly, condition (4.21) holds too. The operator  $A$  being  $D_f$ -coercive on its domain, the operator  $A^f$  is  $D_f$ -firm (cf. Lemma 3.2). This means that

$$(5.9) \quad \langle \nabla f(u^k) - \nabla f(u), u^k - u \rangle \leq \langle \nabla f(x^k) - \nabla f(x), u^k - u \rangle,$$

for any pair  $(x, u) \in \text{Graph } A^f$  and for any  $k \in \mathbb{N}$ . Now, repeating without change the arguments in the proof of Theorem 4.1 one can see that Claim 1 proven there still holds in our setting and implies that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  is bounded. Let  $\{x^{i_k}\}_{k \in \mathbb{N}}$  be a convergent subsequence of  $\{x^k\}_{k \in \mathbb{N}}$  and let  $\bar{x}$  be its limit. An argument identical to that made in the proof of Theorem 4.1 (Claim 2) for the same purpose shows that  $\bar{x} \in C$  and (4.30) holds. According to (5.8), since  $x^{i_k} \in C_{i_k} \cap \text{dom } A$ , it results that  $\bar{x} \in C \cap \text{dom } A$ . Hence,  $A\bar{x} \neq \emptyset$ . Writing (5.9) for  $i_k$  instead of  $k$  and  $\bar{x}$  instead of  $x$ , and for any  $u \in A\bar{x}$ , letting in the resulting inequality  $k \rightarrow \infty$  and taking into account that  $\nabla f$  is continuous on  $\text{int dom } f$ , we deduce that

$$(5.10) \quad \liminf_{k \rightarrow \infty} \langle \nabla f(u^{i_k}) - \nabla f(u), u^{i_k} - u \rangle \leq 0.$$

This shows that the sequence  $\{u^{i_k}\}_{k \in \mathbb{N}}$  satisfies (5.2) for  $T = A^f$ ,  $z = \bar{x}$  and any  $u \in A^f \bar{x}$ . Also, by (4.30) and (5.10), the sequences  $\{x^{i_k}\}_{k \in \mathbb{N}}$  and  $\{u^{i_k}\}_{k \in \mathbb{N}}$  satisfy (5.1). Hence, Proposition 5.1 applies to  $T = A^f$ ,  $z = \bar{x}$  and  $u \in A^f \bar{x}$ . By consequence, we have that  $\bar{x}$  is a fixed point of  $A^f$  and, hence, a zero of  $A$ . It remains to prove that  $\bar{x}$  is the only accumulation point of the sequence  $\{x^k\}_{k \in \mathbb{N}}$ . The proof in this respect reproduces without modifications the arguments made in the proof of Theorem 4.1 in order to show that the sequence  $\{x^k\}_{k \in \mathbb{N}}$  has a single weak accumulation point when  $\nabla f$  is sequentially weakly-weakly continuous.  $\square$

**5.3. Consequences of Theorem 5.1.** Using Theorem 5.1 instead of Theorem 4.1 we can prove again Corollary 4.2 and Corollary 4.3 in a finite dimensional setting with different and less demanding conditions on  $A$ . Here is the new version of Corollary 4.2.

**Corollary 5.1.** *Suppose that the space  $X$  has finite dimension,  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$  and  $\nabla f^*$  is bounded on bounded subsets of  $\text{int dom } f^*$ . If  $B : X \rightarrow 2^{X^*}$  is a monotone operator satisfying (3.24) and having at least one zero, and if any of the following conditions holds*

- (a)  $\text{ran } (\nabla f + B)$  is closed in  $X^*$  and  $\nabla f^*(\text{ran } (\nabla f + B))$  is convex;
- (b)  $\text{ran } (\nabla f + B) = X^*$ ;

*then the sequences generated by the proximal point method (3.23) converge to zeros of the operator  $B$ .*

**Proof.** Recall that in this setting  $\text{dom } f^* = X^*$  (cf. Remark 4.2(d)). Observe that, according to Lemma 3.4, we have that

$$\text{dom } A[B_f] = \text{dom } B_f = \nabla f^*(\text{ran } (\nabla f + B)).$$

Therefore, if condition (a) holds, the set  $\text{dom } A[B_f] = \text{dom } B_f$  is closed and convex. Since condition (b) implies (a), this remains true when (b) holds. Again by Lemma

3.4, the operator  $A[B_f]$  is  $D_f$ -coercive on its domain. Consequently, the operator  $A := A[B_f]$  satisfies the requirements of Theorem 5.1 when  $C = C_k = X$  for all  $k \in \mathbb{N}$ . Applying Theorem 5.1 to  $A[B_f]$  and taking into account Lemma 3.5 the conclusion follows.  $\square$

Now we give another variant of Corollary 4.3(a) in which the condition that  $A$  should have closed graph is replaced by less demanding requirements.

**Corollary 5.2.** *Let  $B : X \rightarrow 2^{X^*}$  be a monotone operator. Suppose that the space  $X$  has finite dimension,  $f$  is uniformly convex on bounded subsets of  $\text{int dom } f$  and  $\nabla f^*$  is bounded on bounded subsets of  $\text{int dom } f^*$ . If  $C$  is a closed convex subset of  $\text{dom } B \cap \text{int dom } f$  such that, for some real number  $\alpha > 0$ ,*

$$(5.11) \quad \emptyset \neq ((1 - \alpha)\nabla f - B)(C) \subseteq \text{int dom } f^*,$$

*and the operator  $\text{Proj}_C^f \circ [(1 - \alpha)\nabla f - B]$  is  $D_f$ -firm, then the iterative procedure given by (4.46) is well defined and converges to the necessarily unique solution of the variational inequality (4.44), provided that such a solution exists.*

**Proof.** Since (4.48) and (4.49) still hold, the operator  $V$  given by (4.47) is  $D_f$ -coercive on its domain (cf. Lemma 3.4). By (5.11) and by the fact that  $C \subseteq \text{dom } B \cap \text{int dom } f$ , it results that  $C \subseteq \text{dom } V$ . Hence, Theorem 5.1 applies to the operator  $A = V$  and the sets  $C_k = C$  and the conclusion follows.  $\square$

## REFERENCES

- [1] Alber, Ya.I, Generalized projection operators in Banach spaces: Properties and applications, in: "Functional Differential Equations" edited by M.E. Draklin and E. Litsyn, Vol. 1, The research Institute of Judea and Samaria, Kedumim-Ariel, 1993.
- [2] Alber Ya.I.: Metric and generalized projection operators in Banach spaces: properties and applications, in "Theory and Applications of Nonlinear Operators of Accretive and Monotone Type", pp. 15–50, edited by A. G. Kartsatos, Lecture Notes in Pure and Appl. Math., 178, *Marcel Dekker*, New York, 1996.
- [3] Alber, Ya.I.: Generalized projections, decompositions, and the Pythagorean-type theorem in Banach spaces, *Appl. Math. Lett.* **11** (1998), 115–121.
- [4] Alber, Ya.I.: Stability of the proximal projection algorithm for nonsmooth convex optimization problems with perturbed constraint sets, *J. Nonlinear Convex Anal.* **4** (2003), 1–14.
- [5] Alber, Ya.I. and Butnariu, D.: Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces. *J. Optim. Theory Appl.* **92** (1997), 33–61.
- [6] Alber, Ya.I., Butnariu, D. and Kassay, G.: On the convergence and stability of a regularization method for maximal monotone inclusions and its applications to convex optimization, in: F. Giannessi and G. Maugeri (eds.) "Variational Inequalities and Applications", pp. 89–132, *Springer Verlag*, New York 2005.
- [7] Alber, Ya.I., Butnariu, D. and Ryazantseva, I.: Regularization methods for ill-posed inclusions and variational inequalities with domain perturbations, *J. Convex Nonlin. Anal.* **2** (2001), 53–79.
- [8] Alber, Ya.I., Butnariu, D. and Ryazantseva, I.: Regularization of monotone variational inequalities with Mosco approximations of the constraint sets, *Set-Valued Anal.* **13** (2005), 265–290.
- [9] Alber, Ya.I., Butnariu, D. and Ryazantseva, I.: Regularization and resolution of monotone variational inequalities with operators given by hypomonotone approximations, *J. Nonlinear Convex Anal.* **6** (2005), 23–53.
- [10] Alber, Ya.I. and Guerre-Delabriere, S.: On the projection method for fixed point problems, *Analysis (Munich)* **21** (2001), 17–39.
- [11] Alber, Ya.I., Iusem, A.N. and Solodov, M.V.: Minimization of nonsmooth convex functionals in Banach spaces, *J. Convex Anal.* **4** (1997), 235–255.

- [12] Alber, Y.I., Kartsatos, A.G. and Litsyn E.: Iterative solutions of unstable variational inequalities on approximately given sets, *Abstr. Appl. Anal.* **1** (1996), 45-54.
- [13] Alber, Ya.I. and Nashed, Z.M.: Iterative-projection regularization of ill-posed variational inequalities, *Analysis (Munich)* **24** (2004), 19–39.
- [14] Alber, Ya.I. and Reich, S.: An iterative method for solving a class of nonlinear operator equations in Banach spaces, *Panamer. Math. J.* **4** (1994), 39-54.
- [15] Attouch, H.: Variational Convergence for Functions and Operators, *Pitman Advanced Publishing Program*, Boston 1984.
- [16] Aubin, J.P. and Frankowska, H.: Set-Valued Analysis, *Birkhäuser*, Boston, 1990.
- [17] Bakushinskii, A.B. and Poljak, B.T.: On the solution of variational inequalities, *Soviet Math. Dokl.* **15** (1974), 1705-1710.
- [18] Bauschke, H.H. and Borwein, J.M.: Legendre functions and the method of random Bregman projections. *J. Convex Anal.* **4** (1997) 27-67.
- [19] Bauschke, H.H.; Borwein, J.M. and Combettes, P.L.: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. *Commun. Contemp. Math.* **3** (2001), 615–647.
- [20] Bauschke, H.H., Borwein, J.M. and Combettes, P.L.: Bregman monotone optimization algorithms. *SIAM J. Control Optim.* **42** (2003) 596-636.
- [21] Bauschke, H.H. and Combettes, P.L.: Construction of best Bregman approximations in reflexive Banach spaces, *Proc. Amer. Math. Soc.* **131** (2003), 3757-3766.
- [22] Bauschke, H.H., Combettes, P.L. and Noll, D.: Joint minimization with alternating Bregman proximity operators, preprint, 2005.
- [23] Bregman, L.M.: The relaxation method for finding common points of convex sets and its application to the solution of problems in convex programming, *USSR Computational Mathematics and Mathematical Physics* **7** (1967), 200-217.
- [24] Bonnans, J.F. and Shapiro, A.: Perturbation Analysis of Optimization Problems, *Springer Verlag*, New York, 2000.
- [25] Borwein, J.M. and Zhu, Q.J.: Techniques of Variational Analysis, *Springer*, New York 2005.
- [26] Browder, F.E.: Multivalued monotone nonlinear mappings and duality mappings in Banach spaces, *Trans. Amer. Math. Soc.* **118** (1965), 338-351.
- [27] Browder, F.E.: Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A.* **56** (1966), 1080–1086.
- [28] Browder, F.E., Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces, "Proceedings of Symposia in Pure Mathematics", Vol. XVIII, Part 2, *American Mathematical Society*, Providence, Rhode Island, 1976.
- [29] Bruck, R.E.: An iterative solution for a variational inequality for certain monotone operators in Hilbert space, *Bull. Amer. Math. Soc.* **81** (1975), 890-892. [Corrigendum: *Bull. Amer. Math. Soc.* **81** (1976), p.353.]
- [30] Burachik, R.S. and Scheimberg, S.: A proximal point method for the variational inequality problem in Banach spaces, *SIAM J. Control Optim.* **39** (2000), 1633–1649.
- [31] Butnariu, D. and Iusem, A.N.: On a proximal point method for convex optimization in Banach spaces. *Numer. Funct. Anal. Optim.* **18** (1997), 723–744.
- [32] Butnariu, D. and Iusem, A.N.: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, *Kluwer Academic Publishers*, Dordrecht, 2000.
- [33] Butnariu, D., Iusem, A.N. and Resmerita, E.: Total convexity for powers of the norm in uniformly convex Banach spaces, *J. Convex Anal.* **7** (2000), 319-334.
- [34] Butnariu, D., Iusem, A.N. and Zălinescu, C.: On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Anal.* **10** (2003), 1, 35–61.
- [35] Butnariu, D. and Resmerita, E.: Averaged subgradient methods for constrained convex optimization and Nash equilibria computation *Optimization* **51** (2002), 863–888.
- [36] Butnariu, D. and Resmerita, E.: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.*, 2006, Art. ID 84919, 39 pages.
- [37] Butnariu, D., Reich, S. and Zaslavski, A.: Weak convergence for orbits of nonlinear operators in reflexive Banach spaces, *Numer. Funct. Anal. Optim.* **24** (2003), 489-508.
- [38] Csaszar, A.: General Topology, *Akademiai Kiado*, Budapest, 1978.

- [39] Dontchev, A.L. and Zolezzi, T.: Well-Posed Optimization Problems, *Springer Verlag*, Berlin, 1993.
- [40] Eckstein, J.: Nonlinear proximal point algorithms using Bregman functions, with application to convex programming, *Math. Operation Research* **18** (1993), 202-226.
- [41] Eggermont, P.P.B.: Multiplicative iterative algorithms for convex programming, *Linear Algebra and Its Applications* **130** (1990), 25-42.
- [42] Eriksson, J.: An interval primal-dual algorithm for linear programming, *Technical Report 85-10, Department of Mathematics, Linköping University*, Sweden, 1985.
- [43] Erlander, S.: Entropy in linear programs, *Math. Programming* **21** (1981), 137-151.
- [44] Ermoliev, Yu.M.: Methods for solving nonlinear extremal problems, *Kibernetika (Kiev)* **1** (1966), 1-17 (Russian).
- [45] Facchinei, F. and Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. 1, *Springer Verlag*, New York, 2003.
- [46] Goebel, K. and Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Marcel Dekker*, New York and Basel, 1984.
- [47] Kassay, G.: The proximal point algorithm for reflexive Banach spaces, *Studia Math.* **30** (1985), 9-17.
- [48] Krasnoselskii, M.A.: Two observations about the method of successive approximations (Russian), *Uspekhi Matematicheskikh Nauk* **10** (1955), 123-127.
- [49] Landweber, L.: An iterative formula for Fredholm integral equations of the first kind, *Amer. J. Math.* **73** (1951), 615-624.
- [50] Lemaire, B.: The proximal algorithm, in: J.P. Penot (Ed.), "International Series of Numerical Mathematics", No. 87, pp. 83-97, *Birkhäuser*, Basel, 1989.
- [51] Liskovets, O. A.: External approximations for the regularization of monotone variational inequalities, *Soviet Math. Dokl.* **36** (1988), 220-224.
- [52] Liskovets, O. A.: Regularized variational inequalities with pseudo-monotone operators on approximately given sets (Russian). *Differential Equations* **11** (1989), 1970-1977.
- [53] Martinet, B.: Régularization d'inéquations variationnelles par approximations successive, *Revue Française de Informatique et Recherche Opérationnelle* **2** (1970), 154-159.
- [54] Martinet, B.: Algorithmes pour la résolution des problèmes d'optimisation et minimax, Thèse d'état, *Université de Grenoble*, Grenoble, France, 1972.
- [55] Moreau, J.-J.: Fonctions convexes duales et points proximaux dans un espace hilbertien. *C. R. Acad. Sc. Paris* **255** (1962), 2897-2899.
- [56] Moreau, J.-J.: Propriétés des applications 'prox', *C. R. Acad. Sc. Paris* **256** (1963), 1069-1071.
- [57] Moreau, J.-J.: Proximité et dualité dans un espace hilbertien, *Bull. Soc. Math. France* **93** (1965), 273-299.
- [58] Mosco, U.: Convergence of convex sets and of solutions of variational inequalities, *Advances in Math.* **3** (1969), 510-585.
- [59] Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591-597.
- [60] Pascali, D. and Sburlan, S.: Nonlinear Mapping of Monotone Type, *Martinus Nijhoff*, Dordrecht, 1978.
- [61] Phelps, R.R.: Convex Functions, Monotone Operators, and Differentiability - 2nd Edition. *Springer Verlag*, Berlin, 1993.
- [62] Polyak, B.T.: A general method for solving extremum problems, *Dokl. Akad. Nauk SSSR* **174** (1967), 593-597.
- [63] Polyak, B.T.: Introduction to Optimization, *Optimization Software, Inc., Publications Division*, New York, 1987.
- [64] Rockafellar, R.T.: Level sets and continuity of conjugate convex functions. *Trans. Amer. Math. Soc.* **123** (1966), 46-63.
- [65] Rockafellar, R.T.: On the maximality of sums of nonlinear operators. *Trans. Amer. Math. Soc.* **49** (1970), 75-88.
- [66] Rockafellar, R.T.: Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.* **14** (1976), 877-898.
- [67] Rockafellar, R.T.: Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Math. Oper. Res.* **1** (1976), 97-116.
- [68] Rockafellar, R.T. and Wets, R.J.-B.: Variational Analysis. *Springer Verlag*, Berlin, 1998.

- [69] Ruszczyński, A.: A merit function approach of the subgradient method, *preprint*, 2006.
- [70] Shor, N.Z.: Application of the method of gradient descent to the solution of the network transportation problem, in: "Materials of the Scientific Seminar on Theoretical and Applied Questions of Cybernetics and Operation Research", *Ukrainian Academy of Sciences*, Kiev, 1962, pp. 1-17 (Russian).
- [71] Shor, N.Z.: Minimization Methods for Non-Differentiable Functions, *Springer Verlag*, 1985.
- [72] Tikhonov, A.N.: Regularization of incorrectly posed problems, *Soviet Math. Dokl.* **4** (1963), 1035-1038.
- [73] Yosida, K.: Lectures of Differential and Integral Equations, *Interscience*, London, 1960.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, 31905 HAIFA, ISRAEL

*E-mail address:* `dbutnaru@math.haifa.ac.il`

FACULTY OF MATHEMATICS, BABES-BOLYAI UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA

*E-mail address:* `kassay@math.ubbcluj.ro`