# Mean-variance Hedging Under Partial Information

M. Mania  $^{(1),2)}$ , R. Tevzadze  $^{(1),3)}$  and T. Toronjadze  $^{(1),2)}$ 

<sup>1)</sup> Georgian American University, Business School, 3, Alleyway II, Chavchavadze Ave. 17, A, Tbilisi, Georgia, E-mail: toronj333@yahoo.com

<sup>2)</sup> A. Razmadze Mathematical Institute, 1, M. Aleksidze St., Tbilisi, Georgia, E-mail: mania@rmi.acnet.ge

> <sup>3)</sup>Institute of Cybernetics, 5, S. Euli St., Tbilisi, Georgia, E-mail:tevza@cybernet.ge

#### Abstract

We consider the mean-variance hedging problem under partial Information. The underlying asset price process follows a continuous semimartingale and strategies have to be constructed when only part of the information in the market is available. We show that the initial mean variance hedging problem is equivalent to a new mean variance hedging problem with an additional correction term, which is formulated in terms of observable processes. We prove that the value process of the reduced problem is a square trinomial with coefficients satisfying a triangle system of backward stochastic differential equations and the filtered wealth process of the optimal hedging strategy is characterized as a solution of a linear forward equation.

2000 Mathematics Subject Classification: 90A09, 60H30, 90C39.

Key words and phrases: Backward stochastic differential equation, semimartingale market model, incomplete markets, mean-variance hedging, partial information.

## 1 Introduction

In the problem of derivative pricing and hedging it is usually assumed that the hedging strategies have to be constructed using all market information. However, in reality investors acting in a market have limited access to the information flow. E.g., an investor may observe just stock prices, but stock appreciation rates depend on some unobservable factors; one may think that stock prices can only be observed at discrete time instants or with some delay, or an investor would like to price and hedge a contingent claim whose payoff depends on an unobservable asset and he observes the prices of an asset correlated with the underlying asset. Besides, investors may not be able to use all available information even if they have access to the full market flow. In all such cases investors are forced to make decisions based only on a part of the market information.

We study a mean-variance hedging problem under partial information when the asset price process is a continuous semimartingale and the flow of observable events not necessarily contain all information on prices of the underlying asset.

We assume that the dynamics of the price process of the asset traded on the market is described by a continuous semimartingale  $S = (S_t, t \in [0, T])$  defined on a filtered probability space  $(\Omega, \mathcal{F}, F = (F_t, t \in [0, T]), P)$ , satisfying the usual conditions, where  $\mathcal{F} = F_T$  and  $T < \infty$  is the fixed time horizon. Suppose that the interest rate is equal to zero and the asset price process satisfies the structure condition, i.e., the process S admits the decomposition

$$S_t = S_0 + M_t + \int_0^t \lambda_u d\langle M \rangle_u, \quad \langle \lambda \cdot M \rangle_T < \infty \quad a.s., \tag{1.1}$$

where M is a continuous F-local martingale and  $\lambda$  is a F-predictable process.

Let us introduce an additional filtration smaller than F

$$G_t \subseteq F_t$$
, for every  $t \in [0, T]$ .

The filtration G represents the information that the hedger has at his disposal, i.e., hedging strategies have to be constructed using only information available in G.

Let H be a P-square integrable  $F_T$ -measurable random variable, representing the payoff of a contingent claim at time T.

We consider the mean-variance hedging problem

to minimize 
$$E[(X_T^{x,\pi} - H)^2]$$
 over all  $\pi \in \Pi(G)$ , (1.2)

where  $\Pi(G)$  is a class of *G*-predictable *S*-integrable processes. Here  $X_t^{x,\pi} = x + \int_0^t \pi_u dS_u$  is the wealth process starting from initial capital *x*, determined by the self-financing trading strategy  $\pi \in \Pi(G)$ .

In the case G = F of complete information the mean-variance hedging problem was introduced by Föllmer and Sondermann [8] in the case when S is a martingale and then developed by several authors for price process admitting a trend (see, e.g., [6], [12], [26], [27], [25], [10], [11]).

Asset pricing with partial information under various setups has been considered. The mean-variance hedging problem under partial information was first studied by Di Masi, Platen and Runggaldier (1995) when the stock price process is a martingale and the prices are observed only at discrete time moments. For a general filtrations and when the asset price process is a martingale this problem was solved by Schweizer (1994) in terms of G-predictable projections. Pham (2001) considered the mean-variance hedging problem for a general semimartingale model, assuming that the observable filtration contains the augmented filtration  $F^S$  generated by the asset price process S

$$F_t^S \subseteq G_t$$
, for every  $t \in [0, T]$ . (1.3)

In this paper, using the variance-optimal martingale measure with respect to the filtration G and suitable Kunita-Watanabe decomposition, the theory developed by Gourieroux, Laurent and Pham (1998) and Rheinländer and Schweizer (1997) to the case of partial information was extended.

If  $F_t^S \subseteq G_t$ , the price process is a *G*-semimartingale, the sharp bracket  $\langle M \rangle$  is *G*-adapted and the canonical decomposition of *S* with respect to the filtration *G* is of the form

$$S_t = S_0 + \int_0^t E(\lambda_u | G_u) d\langle M \rangle_s + \tilde{M}_t, \qquad (1.4)$$

where  $\tilde{M}$  is a *G*-local martingale.

In this case the problem (1.2) is equivalent to the problem

to minimize 
$$E[(X_T^{x,\pi} - E(H|G_T))^2]$$
 over all  $\pi \in \Pi(G)$  (1.5)

which is formulated in G-adapted terms, taking in mind the G-decomposition (1.4) of S. Therefore the problem (1.5) can be solved as in the case of full information using the dynamic programming method directly to (1.5), although one needs to determine  $E(H|G_T)$  and the G-decomposition terms of S.

If G is not containing  $F^S$ , then S is not a G-semimartingale and the problem is more involved, although we solve it under following additional assumptions:

- A)  $\langle M \rangle$  and  $\lambda$  are *G*-predictable,
- B) any G- martingale is a F-local martingale,
- C) the filtration F is continuous, i.e., all F- local martingales are continuous,
- D) there exists a martingale measure for S that satisfies the Reverse Hölder condition.

We shall use the notation  $\widehat{Y}_t$  for the process  $E(Y_t|G_t)$ - the *G*-optional projection of *Y*. Condition A) implies that

$$\widehat{S}_t = E(S_t | G_t) = S_0 + \int_0^t \lambda_u d\langle M \rangle_u + \widehat{M}_t.$$

Let

$$H_t = E(H|F_t) = EH + \int_0^t h_u dM_u + L_t$$
(1.6)

and

$$H_{t} = EH + \int_{0}^{t} h_{u}^{G} dM_{u} + L_{t}^{G}$$
(1.7)

be the Galtchouk-Kunita-Watanabe (GKW) decompositions of  $H_t = E(H|F_t)$  with respect to local martingales M and  $\widehat{M}$ , where  $h, h^G$  are F-predictable process and  $L, L^G$ are local martingales strongly orthogonal to M and  $\widehat{M}$  respectively.

We show (Theorem 3.1) that the initial mean variance hedging problem (1.2) is equivalent to the problem to minimize the expression

$$E\left[(x+\int_{0}^{T}\pi_{u}d\widehat{S}_{u}-\widehat{H}_{T})^{2}+\int_{0}^{T}(\pi_{u}^{2}(1-\rho_{u}^{2})+2\pi_{u}\widetilde{h}_{u})d\langle M\rangle_{u}\right],$$
(1.8)

over all  $\pi \in \Pi(G)$ , where

$$\widetilde{h}_t = \widehat{h_t^G} \rho_t^2 - \widehat{h}_t$$
 and  $\rho_t^2 = \frac{d\langle M \rangle_t}{d\langle M \rangle_t}.$ 

Thus, the problem (1.8), equivalent to (1.2), is formulated in terms of *G*-adapted processes. One can say that (1.8) is the mean variance hedging problem under complete information with additional correction term and can be solved as in the case of complete information.

Let us introduce the value process of the problem (1.8)

$$V^{H}(t,x) = \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E \left[ (x + \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - \widehat{H}_{T})^{2} + \int_{t}^{T} [\pi_{u}^{2} (1 - \rho_{u}^{2}) + 2\pi_{u} \widetilde{h}_{u}] d\langle M \rangle_{u} | G_{t} \right].$$
(1.9)

We show in section 4 that the value function of the problem (1.8) admits a representation

$$V^{H}(t,x) = V_{t}(0) - 2V_{t}(1)x + V_{t}(2)x^{2}, \qquad (1.10)$$

where the coefficients  $V_t(0)$ ,  $V_t(1)$  and  $V_t(2)$  satisfy a triangle system of backward stochastic differential equations (BSDE). Besides, the filtered wealth process of the optimal hedging strategy is characterized as a solution of the linear forward equation

$$\widehat{X}_{t}^{*} = x - \int_{0}^{t} \frac{\rho_{u}^{2} \varphi_{u}(2) + \lambda_{u} V_{u}(2)}{1 - \rho_{u}^{2} + \rho_{u}^{2} V_{u}(2)} \widehat{X}_{u}^{*} d\widehat{S}_{u} + \int_{0}^{t} \frac{\rho_{u}^{2} \varphi_{u}(1) + \lambda_{u} V_{u}(1) + \tilde{h}_{u}}{1 - \rho_{u}^{2} + \rho_{u}^{2} V_{u}(2)} d\widehat{S}_{u}.$$
(1.11)

In the case of complete information (G = F) we have  $\rho = 0$ ,  $\tilde{h} = 0$  and (1.11) gives equations for the optimal wealth process and for the coefficients of value function from [20].

In section 5 we consider a diffusion market model which consists of two assets S and Y, where  $S_t$  is a state of a process being controlled and  $Y_t$  is the observation process. Suppose that  $S_t$  and  $Y_t$  are governed by

$$dS_t = \mu_t dt + \sigma_t dw_t^0,$$
  
$$dY_t = a_t dt + b_t dw_t,$$

where  $w^0$  and w are Brownian motions with correlation  $\rho$  and the coefficients  $\mu, \sigma, a$ and b are  $F^Y$ -adapted. So, in this case  $F_t = F_t^{S,Y}$  and the flow of observable events is  $G_t = F_t^Y$ . We give in the case of markovian coefficients solution of the problem (1.2) in terms of parabolic differential equations (PDE) and an explicit solution when coefficients are constants and the contingent claim is of the form  $H = \mathcal{H}(S_T, Y_T)$ .

## 2 Main definitions and auxiliary facts

Denote by  $\mathcal{M}^{e}(F)$  the set of equivalent martingale measures for S, i.e., set of probability measures Q equivalent to P such that S is a F-local martingale under Q.

Let

$$\mathcal{M}_2^e(F) = \{ Q \in \mathcal{M}^e(F) : EZ_T^2(Q) < \infty \},\$$

where  $Z_t(Q)$  is the density process (with respect to the filtration F) of Q relative to P.

**Remark 2.1.** Since S is continuous, the existence of an equivalent martingale measure and the Girsanov theorem imply that the structure condition (1.1) is satisfied.

Note that the density process  $Z_t(Q)$  of any element Q of  $\mathcal{M}^e(F)$  is expressed as an exponential martingale of the form

$$\mathcal{E}_t(-\lambda \cdot M + N),$$

where N is a F-local martingale strongly orthogonal to M and  $\mathcal{E}_t(X)$  is the Doleans-Dade exponential of X.

If the local martingale  $Z_t^{min} = \mathcal{E}_t(-\lambda \cdot M)$  is a true martingale,  $dQ^{min}/dP = Z_T^{min}dP$  defines an equivalent probability measure called the minimal martingale measure for S.

Recall that a measure Q satisfies the Reverse Hölder inequality  $R_2(P)$  if there exists a constant C such that

$$E\left(\frac{Z_T^2(Q)}{Z_\tau^2(Q)}|F_\tau\right) \le C, \quad P-a.s$$

for every F-stopping time  $\tau$ .

**Remark 2.2.** If there exists a measure  $Q \in \mathcal{M}^e(F)$  that satisfies the Reverse Hölder inequality  $R_2(P)$ , then according to Kazamaki [15] the martingale  $M^Q = -\lambda \cdot M + N$ belongs to the class BMO and hence  $-\lambda \cdot M$  also belongs to BMO, i.e.,

$$E\left(\int_{\tau}^{T} \lambda_{u}^{2} d\langle M \rangle_{u} | F_{\tau}\right) \leq const$$

$$(2.1)$$

for every stopping time  $\tau$ . Therefore, it follows from Kazamaki [15] that  $\mathcal{E}_t(-\lambda \cdot M)$  is a true martingale. So, condition D) implies that the minimal martingale measure exists (but  $Z^{min}$  is not necessarily square integrable).

For all unexplained notations concerning the martingale theory used below we refer the reader to [5],[19],[14].

Let  $\Pi(F)$  be the space of all *F*-predictable *S*-integrable processes  $\pi$  such that the stochastic integral

$$(\pi \cdot S)_t = \int_0^t \pi_u dS_u, t \in [0, T],$$

is in the  $\mathcal{S}^2$  space of semimartingales , i.e.,

$$E\Big(\int_0^T \pi_s^2 d\langle M \rangle_s\Big) + E\Big(\int_0^T |\pi_s \lambda_s| d\langle M \rangle_s\Big)^2 < \infty.$$

Denote by  $\Pi(G)$  the subspace of  $\Pi(F)$  of G-predictable strategies.

**Remark 2.3.** Since  $\lambda \cdot M \in BMO$  (see Remark 2.2), it follows from the proof of Theorem 2.5 of Kazamaki [15]

$$E\Big(\int_0^T |\pi_u \lambda_u| d\langle M \rangle_u\Big)^2 = E\langle |\pi| \cdot M, |\lambda| \cdot M \rangle_T^2$$
$$\leq 2||\lambda \cdot M||_{\text{BMO}} E \int_0^T \pi^2 d\langle M \rangle_u < \infty.$$

Therefore, under condition D) the strategy  $\pi$  belongs to the class  $\Pi(G)$  if and only if  $E \int_0^T \pi_s^2 d\langle M \rangle_s < \infty$ .

Define  $J_T^2(F)$  and  $J_T^2(G)$  as spaces of terminal values of stochastic integrals, i.e.,

$$J_T^2(F) = \{ (\pi \cdot S)_T : \pi \in \Pi(F) \}.$$
  
$$J_T^2(G) = \{ (\pi \cdot S)_T : \pi \in \Pi(G) \}.$$

For convenience we give some assertions from [4], which establishes necessary and sufficient conditions for the closedness of the space  $J_T^2(F)$  in  $L^2$ .

**Proposition 2.1.** Let S be a continuous semimartingale. Then the following assertions are equivalent:

(1) There is a martingale measure  $Q \in \mathcal{M}^{e}(F)$  and  $J^{2}_{T}(F)$  is closed in  $L^{2}$ .

(2) There is a martingale measure  $Q \in \mathcal{M}^{e}(F)$  that satisfies the Reverse Hölder condition  $R_{2}(P)$ .

(3) There is a constant C such that for all  $\pi \in \Pi(F)$  we have

$$||\sup_{t \le T} (\pi \cdot S)_t||_{L^2(P)} \le C||(\pi \cdot S)_T||_{L^2(P)}.$$

(4) There is a constant c such that for every stopping time  $\tau$ , every  $A \in \mathcal{F}_{\tau}$  and for every  $\pi \in \Pi(F)$  with  $\pi = \pi I_{[\tau,T]}$  we have

$$||I_A - (\pi \cdot S)_T||_{L^2(P)} \ge cP(A)^{1/2}.$$

Note that assertion (4) implies that for every stopping time  $\tau$  and for every  $\pi \in \Pi(G)$  we have

$$E\left((1+\int_{\tau}^{T}\pi_{u}dS_{u})^{2}/F_{\tau}\right) \geq c.$$

$$(2.2)$$

Let us make some remarks on conditions B) and C).

**Remark 2.4.** Conditions B), C) imply that the filtration G is also continuous. By condition B any G-local martingale is F-local martingale, which are continuous by condition C). Recall that the continuity of a filtration means that all local martingales with respect to this filtration are continuous.

**Remark 2.5.** Condition B) is satisfied if and only if the  $\sigma$ -algebras  $F_t$  and  $G_T$  are conditionally independent given  $G_t$  for all  $t \in [0, T]$  (see Theorem 9.29 from Jacod 1978).

Now we recall some known assertions from the filtering theory. The following proposition can be proved similarly to [19]. **Proposition 2.2.** If conditions A, B) and C) are satisfied, then for any F-local martingale M and any G-local martingale  $m^G$ 

$$\widehat{M}_t = E(M_t|G_t) = \int_0^t E\left(\frac{d\langle M, m^G \rangle_u}{d\langle m^G \rangle_u}|G_u\right) dm_u^G + L_t^G,$$
(2.3)

where  $L^G$  is a local martingale orthogonal to  $m^G$ .

It follows from this proposition that for any G-predictable, M-integrable process  $\pi$  and any G-martingale  $m^G$ 

$$\begin{split} \widehat{\langle (\pi \cdot M), m^G \rangle} &= \int_0^t \pi_u E \Big( \frac{d \langle M, m^G \rangle_u}{d \langle m^G \rangle_u} | G_u \Big) d \langle m^G \rangle_u = \\ &= \int_0^t \pi_u d \langle \widehat{M}, m^G \rangle_u = \langle \pi \cdot \widehat{M}, m^G \rangle_t. \end{split}$$

Hence, for any G-predictable, M-integrable process  $\pi$ 

$$\widehat{(\pi \cdot M)}_t = E\Big(\int_0^t \pi_s dM_s | G_t) = \int_0^t \pi_s d\widehat{M}_s.$$
(2.4)

Since  $\pi, \lambda$  and  $\langle M \rangle$  are G-predictable, from (2.4) we have

$$\widehat{(\pi \cdot S)_t} = E\Big(\int_0^t \pi_u dS_u | G_t\Big) = \int_0^t \pi_u d\widehat{S}_u, \tag{2.5}$$

where

$$\widehat{S}_t = S_0 + \int_0^t \lambda_u d\langle M \rangle_u + \widehat{M}_t.$$

## 3 Separation principle. The optimality principle

Let us introduce the value function of the problem (1.2) defined as

$$U^{H}(t,x) = \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E\left((x + \int_{t}^{T} \pi_{u} dS_{u} - H)^{2} | G_{t}\right).$$
(3.1)

By GKW decomposition

$$H_t = E(H|F_t) = EH + \int_0^t h_u dM_u + L_t$$
(3.2)

for a *F*-predictable, *M*-integrable process *h* and a local martingale *L* strongly orthogonal to *M*. We shall use also the GKW decompositions of  $H_t = E(H|F_t)$  with respect to the local martingale  $\widehat{M}$ 

$$H_t = EH + \int_0^t h_u^G d\widehat{M}_u + L_t^G$$
(3.3)

where  $h^G$  is a *F*-predictable process and  $L^G$  is a *F*- local martingale strongly orthogonal to  $\widehat{M}$ .

It follows from Proposition 2.2 ( applied for  $m^G = \widehat{M}$ ) and Lemma A.1 that

$$\langle E(H|G_{.}), \widehat{M} \rangle_{t} = \int_{0}^{t} E(h_{u}^{G}|G_{u}) d\langle \widehat{M} \rangle_{u} = \int_{0}^{t} \widehat{h_{u}^{G}} \rho_{u}^{2} d\langle M \rangle_{u}.$$
(3.4)

We shall use the notation

$$\widetilde{h}_t = \widehat{h_t^G} \rho_t^2 - \widehat{h}_t.$$
(3.5)

Note that  $\tilde{h}$  belongs to the class  $\Pi(G)$  by Lemma A.2.

Let us introduce now a new optimization problem, equivalent to the initial mean variance hedging problem (1.2), to minimize the expression

$$E\left[(x+\int_{0}^{T}\pi_{u}d\widehat{S}_{u}-\widehat{H}_{T})^{2}+\int_{0}^{T}(\pi_{u}^{2}(1-\rho_{u}^{2})+2\pi_{u}\tilde{h}_{u})d\langle M\rangle_{u}\right],$$
(3.6)

over all  $\pi \in \Pi(G)$ . Recall that  $\widehat{S}_t = E(S_t|G_t) = S_0 + \int_0^t \lambda_u d\langle M \rangle_u + \widehat{M}_t$ .

**Theorem 3.1.** Let conditions A), B) and C) be satisfied. Then the initial meanvariance hedging problem (1.2) is equivalent to the problem (3.6). In particular, for any  $\pi \in \Pi(G)$  and  $t \in [0, T]$ 

$$E\left[(x + \int_{t}^{T} \pi_{u} dS_{u} - H)^{2} | G_{t}\right] = E\left[(H - \hat{H}_{T})^{2} | G_{t}\right]$$

$$+ E\left[(x + \int_{t}^{T} \pi_{u} d\hat{S}_{u} - \hat{H}_{T})^{2} + \int_{t}^{T} (\pi_{u}^{2}(1 - \rho_{u}^{2}) + 2\pi_{u}\tilde{h}_{u})d\langle M \rangle_{u} | G_{t}\right].$$
(3.7)

*Proof.* We have

$$E[(x + \int_{t}^{T} \pi_{u} dS_{u} - H)^{2} | G_{t}] = E[(x + \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - H + \int_{t}^{T} \pi_{u} d(M_{u} - \widehat{M}_{u}))^{2} | G_{t}]$$
  
$$= E[(x + \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - H)^{2} | G_{t}] + 2E[(x + \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - H)(\int_{t}^{T} \pi_{u} d(M_{u} - \widehat{M}_{u})) | G_{t}]$$
  
$$+ E[(\int_{t}^{T} \pi_{u} d(M_{u} - \widehat{M}_{u}))^{2} | G_{t}] = I_{1} + 2I_{2} + I_{3}.$$
(3.8)

It is evedent that

$$I_1 = E\left[(x + \int_t^T \pi_u d\widehat{S}_u - \widehat{H}_T)^2 | G_t\right] + E\left[(H - \widehat{H}_T)^2 | G_t\right].$$
(3.9)

Since  $\pi, \lambda$  and  $\langle M \rangle$  are  $G_T$ -measurable and the  $\sigma$ -algebras  $F_t$  and  $G_T$  are conditionally independent given  $G_t$  (see Remark 2.5), it follows from equation (2.4) that

$$E\left[\int_{t}^{T} \pi_{u}\lambda_{u}d\langle M\rangle_{u}\int_{t}^{T} \pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\right] = E\left[\int_{t}^{T} \pi_{u}\lambda_{u}d\langle M\rangle_{u}\int_{0}^{T} \pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\right]$$

$$-E\left[\int_{t}^{T}\pi_{u}\lambda_{u}d\langle M\rangle_{u}\int_{0}^{t}\pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\right] = E\left[\int_{t}^{T}\pi_{u}\lambda_{u}d\langle M\rangle_{u}E(\int_{0}^{T}\pi_{u}d(M_{u}-\widehat{M}_{u})|G_{T})|G_{t}\right]$$
$$-E\left[\int_{t}^{T}\pi_{u}\lambda_{u}d\langle M\rangle_{u}|G_{t}\right]E\left[\int_{0}^{t}\pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\right] = 0 \qquad (3.10)$$

On the other hand using decomposition (3.2), equality (3.4), properties of square characteristics of martingales and the projection theorem we obtain

$$E\left[H\int_{t}^{T}\pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\right] = E\left[H\int_{t}^{T}\pi_{u}dM_{u}|G_{t}\right] - E\left[\widehat{H}_{T}\int_{t}^{T}\pi_{u}d\widehat{M}_{u}|G_{t}\right]$$
$$= E\left[\int_{t}^{T}\pi_{u}d\langle M, E(H|F_{\cdot})\rangle_{u}|G_{t}\right] - E\left[\int_{t}^{T}\pi_{u}d\langle \widehat{H}, \widehat{M}\rangle_{u}|G_{t}\right]$$
$$= E\left[\int_{t}^{T}\pi_{u}h_{u}d\langle M\rangle_{u}|G_{t}\right] - E\left[\int_{t}^{T}\pi_{u}\widehat{h}_{u}\widehat{\rho}_{u}^{2}d\langle M\rangle_{u}|G_{t}\right] =$$
$$E\left[\int_{t}^{T}\pi_{u}(\widehat{h}_{u}-\widehat{h}_{u}^{G}\rho_{u}^{2})d\langle M\rangle_{u}|G_{t}\right] = -E\left[\int_{t}^{T}\pi_{u}\widetilde{h}_{u}d\langle M\rangle_{u}|G_{t}\right].$$
(3.11)

Finally, it is easy to verify that

$$2E\Big[\int_{t}^{T} \pi_{u}\widehat{M}_{u}\int_{t}^{T} \pi_{u}d(M_{u}-\widehat{M}_{u})|G_{t}\Big] + E\Big[\Big(\int_{t}^{T} \pi_{u}d(M_{u}-\widehat{M}_{u})\Big)^{2}|G_{t}\Big] = \\E\Big[\Big(\int_{t}^{T} \pi_{u}^{2}d\langle M\rangle_{u} - \int_{t}^{T} \pi_{u}^{2}d\langle \widehat{M}\rangle_{u})|G_{t}\Big] = \\= E\Big[\int_{t}^{T} \pi_{u}^{2}(1-\rho_{u}^{2})d\langle M\rangle_{u}|G_{t}\Big].$$
(3.12)

Therefore equations (3.8), (3.9),(3.10), (3.11), and (3.12) imply the validity of equality (3.7).

Thus, it follows from Theorem 3.1 that the optimization problems (1.2) and (3.6) are equivalent. Therefore it is sufficient to solve the problem (3.6), which is formulated in terms of *G*-adapted processes. One can say that (3.6) is a mean variance hedging problem under complete information with correction term and can be solved as in the case of complete information.

Let us introduce the value process of the problem (3.6)

$$V^{H}(t,x) = \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E\left[ (x + \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - \widehat{H}_{T})^{2} + \int_{t}^{T} [\pi_{u}^{2}(1 - \rho_{u}^{2}) + 2\pi_{u}\tilde{h}_{u}]d\langle M \rangle_{u}|G_{t}].$$
(3.13)

It follows from Theorem 3.1 that

$$U^{H}(t,x) = V^{H}(t,x) + E[(H - \hat{H}_{T})^{2}|G_{t}].$$
(3.14)

The optimality principle takes in this case the following form

**Proposition 3.1.** (Optimality principle). Let conditions A, B) and C) be satisfied. Then

a) For all  $x \in R$ ,  $\pi \in \Pi(G)$  and  $s \in [0,T]$  the process

$$V^H(t, x + \int_s^t \pi_u d\widehat{S}_u) + \int_s^t [\pi_u^2(1 - \rho_u^2) + 2\pi_u \widetilde{h}_u)] d\langle M \rangle_u$$

is a submartingale on [s, T], admitting an RCLL modification.

b)  $\pi^*$  is optimal if and only if the process

$$V^{H}(t, x + \int_{s}^{t} \pi_{u}^{*} d\widehat{S}_{u}) + \int_{s}^{t} [(\pi_{u}^{*})^{2} (1 - \rho_{u}^{2}) + 2\pi_{u}^{*} \tilde{h}_{u}] d\langle M \rangle_{u}$$

is a martingale.

This assertion can be proved in a standard manner (see, e.g., [7], [16]). The proof more adapted to this case one can see in [20].

Let

$$V(t,x) = \underset{\pi \in \Pi(G)}{\operatorname{essinf}} E\left[ (x + \int_t^T \pi_u d\widehat{S}_u)^2 + \int_t^T \pi_u^2 (1 - \rho_u^2) d\langle M \rangle_u | G_t \right].$$

and

$$V_t(2) = \underset{\pi \in \Pi(G)}{\text{ess inf}} E\left[ (1 + \int_t^T \pi_u d\widehat{S}_u)^2 + \int_t^T \pi_u^2 (1 - \rho_u^2) d\langle M \rangle_u | G_t \right].$$

It is evident that V(t, x) (resp.  $V_2(t)$ ) is the value process of the optimization problem (3.6) in the case H = 0 (resp. H = 0 and x = 1), i.e.,

$$V(t, x) = V^{0}(t, x)$$
 and  $V_{2}(t) = V^{0}(t, 1).$ 

Since  $\Pi(G)$  is a cone, we have that

$$V(t,x) = x^{2} \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E\left[(1 + \int_{t}^{T} \frac{\pi_{u}}{x} d\widehat{S}_{u})^{2} + \int_{t}^{T} \left(\frac{\pi_{u}}{x}\right)^{2} (1 - \rho_{u}^{2}) d\langle M \rangle_{u} | G_{t}\right] = x^{2} V_{2}(t).$$
(3.15)

Therefore from Proposition 3.1 and equality (3.15) we have the following

Corollary 3.1. a) The process

$$V_2(t)(1+\int_s^t \pi_u d\widehat{S}_u)^2 + \int_s^t (\pi_u)^2 (1-\rho_u^2) d\langle M \rangle_u,$$

 $t \geq s$ ) is a submartingale for all  $\pi \in \Pi(G)$  and  $s \in [0,T]$ . b)  $\pi^*$  is optimal iff

$$V_2(t)(1+\int_s^t \pi_u^* d\widehat{S}_u)^2 + \int_s^t (\pi_u^*)^2 (1-\rho_u^2) d\langle M \rangle_u,$$

 $t \geq s$ , is a martingale.

Note that in the case H = 0 from Theorem 3.1 we have

$$E\left[(1+\int_{t}^{T}\pi_{u}dS_{u})^{2}|G_{t}\right) =$$

$$E\left[(1+\int_{t}^{T}\pi_{u}d\widehat{S}_{u})^{2}+\int_{t}^{T}\pi_{u}^{2}(1-\rho_{u}^{2})d\langle M\rangle_{u}|G_{t}\right]$$
(3.16)

and, hence

$$V_2(t) = U^0(t, 1). (3.17)$$

**Lemma 3.1.** Let conditions A) – D) be satisfied. Then there is a constant  $1 \ge c > 0$  such that  $V_t(2) \ge c$  for all  $t \in [0,T]$  a.s. and

$$1 - \rho_t^2 + \rho_t^2 V_t(2) \ge c \qquad \mu^{\langle M \rangle} a.e.$$
 (3.18)

*Proof.* Let

$$V_t^F(2) = \underset{\pi \in \Pi(F)}{\text{essinf}} E \left[ (1 + \int_t^T \pi_u dS_u)^2 | F_t \right].$$

It follows from assertion 4) of Proposition 2.1 that there is a constant c > 0 such that  $V_t^F(2) \ge c$  for all  $t \in [0, T]$  a.s.. Note that  $c \le 1$  since  $V^F \le 1$ . Then by (3.17)

$$V_t(2) = U^0(t, 1) = \underset{\pi \in \Pi(G)}{\text{ess inf }} E\left[(1 + \int_t^T \pi_u dS_u)^2 | G_t\right] = V_t(2) = \underset{\pi \in \Pi(G)}{\text{ess inf }} E\left[E((1 + \int_t^T \pi_u dS_u)^2 | F_t) | G_t\right] \ge U_t^F(2) \ge c.$$

Therefore, since  $\rho_t^2 \leq 1$  by Lemma A.1,

$$1 - \rho_t^2 + \rho_t^2 V_t(2) \ge 1 - \rho_t^2 + \rho_t^2 c \ge \inf_{r \in [0,1]} (1 - r + rc) = c.$$

## 4 BSDEs for the value process

Let us consider the semimartingale backward equation

$$Y_{t} = Y_{0} + \int_{0}^{t} f(u, Y_{u}, \psi_{u}) d\langle m \rangle_{u} + \int_{0}^{t} \psi_{u} dm_{u} + L_{t}$$
(4.1)

with the boundary condition

$$Y_T = \eta, \tag{4.2}$$

where  $\eta$  is an integrable  $G_T$ -measurable random variable,  $f : \Omega \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$  is  $\mathcal{P} \times \mathcal{B}(\mathbb{R}^2)$  measurable and m is a local martingale. A solution of (4.1)-(4.2) is a triple

 $(Y, \psi, L)$ , where Y is a special semimartingale,  $\psi$  is a predictable *m*-integrable process and L a local martingale strongly orthogonal to *m*. Sometimes we call Y alone the solution of (4.1)-(4.2), keeping in mind that  $\psi \cdot m + L$  is the martingale part of Y.

Backward stochastic differential equations have been introduced in [1] for the linear case as the equations for the adjoint process in the stochastic maximum principle. The semimartingale backward equation, as a stochastic version of the Bellman equation in an optimal control problem, was first derived in [2]. The BSDEs with more general nonlinear generators was introduced in [22] for the case of Brownian filtration, where an existence and uniqueness of a solution of BSDEs with generators satisfying the global Lifschitz condition was established. These results were generalized for generators with quadratic growth in [17], [18] for BSDEs driven by a Brownian motion and in [21], [29] for BSDEs driven by martingales. But conditions imposed in these papers are too restrictive for our needs. We prove here existence and uniqueness of a solution by directly showing that the unique solution of the BSDE we consider is the value of the problem.

In this section we characterize optimal strategies in terms of solutions of suitable Semimartingale Backward Equations.

**Theorem 4.1.** Let H be a square integrable  $F_T$ -measurable random variable and let conditions A, B, C) and D) be satisfied. Then the value function of the problem (3.6) admits a representation

$$V^{H}(t,x) = V_{t}(0) - 2V_{t}(1)x + V_{t}(2)x^{2}, \qquad (4.3)$$

where the processes  $V_t(0), V_t(1)$  and  $V_t(2)$  satisfy the following system of backward equations

$$Y_{t}(2) = Y_{0}(2) + \int_{0}^{t} \frac{\left(\psi_{s}(2)\rho_{s}^{2} + \lambda_{s}Y_{s}(2)\right)^{2}}{1 - \rho_{s}^{2} + \rho_{s}^{2}Y_{s}(2)} d\langle M \rangle_{s} + \int_{0}^{t} \psi_{s}(2)d\widehat{M}_{s} + L_{t}(2) \quad Y_{T}(2) = 1,$$

$$f_{0}^{t} \left(\psi_{s}(2)\rho_{s}^{2} + \lambda_{s}Y_{s}(2)\right)\left(\psi_{s}(1)\rho_{s}^{2} + \lambda_{s}Y_{s}(1) - \tilde{h}_{s}\right)$$

$$(4.4)$$

$$Y_{t}(1) = Y_{0}(1) + \int_{0}^{t} \frac{\left(\psi_{s}(2)\rho_{s}^{2} + \lambda_{s}Y_{s}(2)\right)\left(\psi_{s}(1)\rho_{s}^{2} + \lambda_{s}Y_{s}(1) - h_{s}\right)}{1 - \rho_{s}^{2} + \rho_{s}^{2}Y_{s}(2)} d\langle M \rangle_{s} + \int_{0}^{t} \psi_{s}(1)d\widehat{M}_{s} + L_{t}(1), \quad Y_{T}(1) = E(H|G_{T}),$$

$$(4.5)$$

$$Y_t(0) = Y_0(0) + \int_0^t \frac{\left(\psi_s(1)\rho_s^2 + \lambda_s Y_s(1) - \tilde{h}_s\right)^2}{1 - \rho_s^2 + \rho_s^2 Y_s(2)} d\langle M \rangle_s + \int_0^t \psi_s(0) d\widehat{M}_s + L_t(0), \ Y_T(0) = E^2(H|G_T),$$
(4.6)

where L(2), L(1) and L(0) are G-local martingales orthogonal to  $\widehat{M}$ .

Besides the optimal filtered wealth process  $\widehat{X}_t^{x,\pi^*} = x + \int_0^t \pi_u^* d\widehat{S}_u$  is a solution of the linear equation

$$\widehat{X}_{t}^{*} = x - \int_{0}^{t} \frac{\rho_{u}^{2}\varphi_{u}(2) + \lambda_{u}V_{u}(2)}{1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u}(2)} \widehat{X}_{u}^{*}d\widehat{S}_{u} +$$

$$+ \int_{0}^{t} \frac{\varphi_{u}(1)\rho_{u}^{2} + \lambda_{u}V_{u}(1) - \tilde{h}_{u}}{1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u}(2)} d\widehat{S}_{u}.$$
(4.7)

*Proof.* Similarly to the case of complete information one can show that the optimal strategy exists and that  $V^{H}(t, x)$  is a square trinomial of the form (4.3) (see, e.g., [20]). More precisely the space of stochastic integrals

$$J_T^2(G) = \{(\pi \cdot S)_T : \pi \in \Pi(G)\}$$

is closed by Proposition 2.1 and condition A). Hence there exists optimal strategy  $\pi^*(t,x) \in \Pi(G)$  and  $U^H(t,x) = E[|H-x-\int_t^T \pi_u^*(t,x)dS_u|^2|\mathcal{F}_t]$ . Since  $\int_t^T \pi_u^*(t,x)dS_u$  coincides with the orthogonal projection of  $H-x \in L^2$  on the closed subspace of stochastic integrals, then the optimal strategy is linear with respect to x, i.e.,  $\pi_u^*(t,x) = \pi_u^0(t) + x\pi_u^1(t)$ . This implies that the value function  $U^H(t,x)$  is a square trinomial. It follows from the equality (3.14)that  $V^H(t,x)$  is also a square trinomial and it admits the representation (4.3).

Let us show that  $V_t(0), V_t(1)$  and  $V_t(2)$  satisfy the system (4.4)-(4.6). It is evident that

$$V_{t}(0) = V^{H}(t,0) = \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E \left[ \left( \int_{t}^{T} \pi_{u} d\widehat{S}_{u} - \widehat{H}_{T} \right)^{2} + \int_{t}^{T} [\pi_{u}^{2} (1 - \rho_{u}^{2}) + 2\pi_{u} \tilde{h}_{u}] d\langle M \rangle_{u} | G_{t} \right]$$

$$(4.8)$$

and

$$V_{t}(2) = V^{0}(t, 1) = \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} E\left[(1 + \int_{t}^{T} \pi_{u} d\widehat{S}_{u})^{2} + \int_{t}^{T} \pi_{u}^{2}(1 - \rho_{u}^{2}) d\langle M \rangle_{u} | G_{t}\right].$$

$$(4.9)$$

Therefore, it follows from the optimality principle (taking  $\pi = 0$ ) that  $V_t(0)$  and  $V_t(2)$  are RCLL *G*-submaringales and

$$V_t(2) \le E(V_2(T)|G_t) \le 1,$$
  
 $V_0(t) \le E(E^2(H|G_T)|G_t) \le E(H^2|G_t).$ 

Since

$$V_t(1) = \frac{1}{2}(V_t(0) + V_t(2) - V^H(t, 1)), \qquad (4.10)$$

the process  $V_t(1)$  is also a special semimartingale and since  $V_t(0) - 2V_t(1)x + V_t(2)x^2 = V^H(t,x) \ge 0$  for all  $x \in R$ , we have that  $V_t^2(1) \le V_t(0)V_t(2)$ , hence

$$V_t^2(1) \le E(H^2|G_t).$$

Expressions (4.8), (4.9) and (3.13) imply that  $V_T(0) = E^2(H|G_T)$ ,  $V_T(2) = 1$  and  $V^H(T,x) = (x - E(H|G_T))^2$ . Therefore from (4.10) we have  $V_T(1) = E(H|G_T)$  and V(0), V(1), V(2) satisfy the boundary conditions.

Thus, the coefficients  $V_t(i)$ , i = 0, 1, 2 are special semimartingales and they admit the decomposition

$$V_t(i) = V_0(i) + A_t(i) + \int_0^t \varphi_s(i) d\widehat{M}_s + m_t(i), \quad i = 0, 1, 2,$$
(4.11)

where m(0), m(1), m(2) are G-local martingales strongly orthogonal to  $\widehat{M}$ .

There exists an increasing continuous G-predictable process K such that

$$\langle M \rangle_t = \int_0^t \nu_u dK_u, \quad A_t(i) = \int_0^t a_u(i) dK_u, \quad i = 0, 1, 2,$$

where  $\nu$  and a(i), i = 0, 1, 2, are *G*-predictable processes. Let  $\widehat{X}_{s,t}^{x,\pi} \equiv x + \int_s^t \pi_u d\widehat{S}_u$  and

$$Y_{s,t}^{x,\pi} \equiv V^{H}(t, \widehat{X}_{s,t}^{x,\pi}) + \int_{s}^{t} [\pi_{u}^{2}(1-\rho_{u}^{2}) + 2\pi_{u}\widetilde{h}_{u})]d\langle M \rangle_{u}.$$

Then using (4.3), (4.11) and the Itô formula for any  $t \ge s$  we have

$$(\widehat{X}_{s,t}^{x,\pi})^2 = x + \int_s^t [2\pi_u \lambda_u \widehat{X}_{s,u}^{x,\pi} + \pi_u^2 \rho_u^2] d\langle M \rangle_u + 2\int_s^t \pi_u \widehat{X}_{s,u}^{x,\pi} d\widehat{M}_u$$

$$(4.12)$$

and

$$Y_{s,t}^{x,\pi} - V^{H}(s,x) = \int_{s}^{t} [(\widehat{X}_{s,u}^{x,\pi})^{2} a_{u}(2) - 2\widehat{X}_{s,u}^{x,\pi} a_{u}(1) + a_{u}(0)] dK_{u} + \int_{s}^{t} [\pi_{u}^{2}(1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u-}(2)) + 2\pi_{u}\widehat{X}_{s,u}^{x,\pi}(\lambda_{u}V_{u-}(2) + \varphi_{u}(2)\rho_{u}^{2}) - 2\pi_{u}(V_{u-}(1)\lambda_{u} + \varphi_{u}(1)\rho_{u}^{2} - \tilde{h}_{u})]\nu_{u}dK_{u} + m_{t} - m_{s},$$

$$(4.13)$$

where m is a local martingale.

Let

$$G(\pi, x) = G(\omega, t, \pi, x) = \pi^2 (1 - \rho_u^2 + \rho_u^2 V_{u-}(2)) + 2\pi x (\lambda_u V_{u-}(2) + \varphi_u(2)\rho_u^2) - 2\pi (V_{u-}(1)\lambda_u + \varphi_u(1)\rho_u^2 - \tilde{h}_u)$$

It follows from the optimality principle that for each  $\pi \in \Pi(G)$  the process

$$\int_{s}^{t} [(\widehat{X}_{s,u}^{x,\pi})^{2} a_{u}(2) - 2\widehat{X}_{s,u}^{x,\pi} a_{u}(1)) + a_{u}(0)] dK_{u} + \int_{s}^{t} G(\pi_{u}, \widehat{X}_{s,u}^{x,\pi}) \nu_{u} dK_{u}$$

$$(4.14)$$

is increasing for any s on  $s \leq t \leq T$  and for the optimal strategy  $\pi^*$  we have the equality

$$\int_{s}^{t} [(\widehat{X}_{s,u}^{x,\pi^{*}})^{2} a_{u}(2) - 2\widehat{X}_{s,u}^{x,\pi^{*}} a_{u}(1) + a_{u}(0)] dK_{u} = -\int_{s}^{t} G(\pi_{u}^{*}, \widehat{X}_{s,u}^{x,\pi^{*}}) \nu_{u} dK_{u}.$$
(4.15)

Since  $\nu_u dK_u = d\langle M \rangle_u$  is continuous, without loss of generality one can assume that the process K is continuous (see [20] for details). Therefore, taking in (4.14)  $\tau_s(\varepsilon) = \inf\{t \ge s : K_t - K_s \ge \varepsilon\}$  instead of t we have that for any  $\varepsilon > 0$  and  $s \ge 0$ 

$$\frac{1}{\varepsilon} \int_{s}^{\tau_{s}(\varepsilon)} [(\widehat{X}_{s,u}^{x,\pi})^{2} a_{u}(2) - 2\widehat{X}_{s,u}^{x,\pi} a_{u}(1) + a_{u}(0)] dK_{u} \ge -\frac{1}{\varepsilon} \int_{s}^{\tau_{s}(\varepsilon)} G(\pi_{u}, \widehat{X}_{s,u}^{x,\pi}) \nu(u) dK_{u}.$$

$$(4.16)$$

Passing to the limit in (4.16) as  $\varepsilon \to 0$ , from Proposition B of [20] we obtain that

$$x^{2}a_{u}(2) - 2xa_{u}(1) + a_{u}(0) \ge -G(\pi_{u}, x)\nu_{u} \qquad \mu^{K} - a.e$$

for all  $\pi \in \Pi(G)$ . Similarly from (4.15) we have that  $\mu^{K}$ -a.e.

$$x^{2}a_{u}(2) - 2xa_{u}(1) + a_{u}(0) = -G(\pi_{u}, x)\nu_{u}$$

and hence

$$x^{2}a_{u}(2) - 2xa_{u}(1) + a_{u}(0) = -\nu_{u} \underset{\pi \in \Pi(G)}{\operatorname{ess inf}} G(\pi_{u}, x).$$
(4.17)

The infinum in (4.17) is attained for the strategy

$$\hat{\pi}_t = \frac{V_t(1)\lambda_t + \varphi_t(1)\rho_t^2 - \tilde{h}_t - x(V_t(2)\lambda_t + \varphi_t(2)\rho_t^2)}{1 - \rho_t^2 + \rho_t^2 V_t(2)}.$$
(4.18)

From here we can conclude that

$$\underset{\pi \in \Pi(G)}{\operatorname{ess\,inf}} G(\pi_t, x) \ge G(\hat{\pi}_t, x) =$$

$$= -\frac{(V_t(1)\lambda_t + \varphi_t(1)\rho_t^2 - \tilde{h}_t - x(V_t(2)\lambda_t + \varphi_t(2)\rho_t^2))^2}{1 - \rho_t^2 + \rho_t^2 V_t(2)}.$$
(4.19)

Let  $\pi_t^n = I_{[0,\tau_n[}(t)\hat{\pi}_t, \text{ where } \tau_n = \inf\{t : |V_t(1)| \ge n\}.$ 

It follows from Lemma A.2, Lemma 3.1 and Lemma A.3 that  $\pi^n \in \Pi(G)$  for every  $n \ge 1$  and hence

$$\operatorname{ess\,inf}_{\pi\in\Pi(G)} G(\pi_t, x) \le G(\pi_t^n, x)$$

for all  $n \ge 1$ . Therefore

$$\operatorname{ess\,inf}_{\pi\in\Pi(G)} G(\pi_t, x) \le \lim_{n\to\infty} G(\pi_t^n, x) = G(\hat{\pi}_t, x).$$
(4.20)

Thus (4.17), (4.19) and (4.20) imply that

$$x^{2}a_{t}(2) - 2xa_{t}(1) + a_{t}(0) =$$

$$= \nu_{t} \frac{(V_{t}(1)\lambda_{t} + \varphi_{t}(1)\rho_{t}^{2} - \tilde{h}_{t} - x(V_{t}(2)\lambda_{t} + \varphi_{t}(2)\rho_{t}^{2})^{2}}{1 - \rho_{t}^{2} + \rho_{t}^{2}V_{t}(2)}, \quad \mu^{K} \quad a.e. \quad (4.21)$$

and equalizing the coefficients of square trinomials in (4.21) (and integrating with respect to dK) we obtain that

$$A_t(2) = \int_0^t \frac{\left(\varphi_s(2)\rho_s^2 + \lambda_s V_s(2)\right)^2}{1 - \rho_s^2 + \rho_s^2 V_s(2)} d\langle M \rangle_s, \qquad (4.22)$$

$$A_t(1) = \int_0^t \frac{\left(\varphi_s(2)\rho_s^2 + \lambda_s V_s(2)\right) \left(\varphi_s(1)\rho_s^2 + \lambda_s V_s(1) - \tilde{h}_s\right)}{1 - \rho_s^2 + \rho_s^2 V_s(2)} d\langle M \rangle_s,$$
(4.23)

$$A_t(0) = \int_0^t \frac{\left(\varphi_s(1)\rho_s^2 + \lambda_s V_s(1) - \tilde{h}_s\right)^2}{1 - \rho_s^2 + \rho_s^2 V_s(2)} d\langle M \rangle_s, \qquad (4.24)$$

which, together with (4.11), implies that the triples  $(V(i), \varphi(i), m(i))$ , i = 0, 1, 2, satisfy the system (4.4)-(4.6).

Note that A(0) and A(2) are integrable increasing processes and relations (4.22) and (4.24) imply that the strategy  $\hat{\pi}$  defined by (4.18) belongs to the class  $\Pi(G)$ .

Let us show now that if the strategy  $\pi^* \in \Pi(G)$  is optimal then the corresponding filtered wealth process  $\widehat{X}_t^{\pi^*} = x + \int_0^t \pi_u^* d\widehat{S}_u$  is a solution of equation (4.7). By the optimality principle the process

$$Y_t^{\pi^*} = V^H(t, \widehat{X}_t^{\pi^*}) + \int_0^t [(\pi_u^*)^2 (1 - \rho_u^2) + 2\pi_u^* \widetilde{h}_u] d\langle M \rangle_u$$

is a martingale. Using the Itô formula we have

$$Y_t^{\pi^*} = \int_0^t (\widehat{X}_u^{\pi^*})^2 dA_u(2) - 2 \int_0^t \widehat{X}_u^{\pi^*} dA_u(1) + A_t(0) + \int_0^t G(\pi_u^*, \widehat{X}_u^{\pi^*}) d\langle M \rangle_u + N_t,$$
(4.25)

where N is a martingale. Therefore applying equalities (4.22), (4.23) and (4.24) we obtain that +

$$Y_t^{\pi^*} = \int_0^t \left(\pi_u^* - \frac{V_u(1)\lambda_u + \varphi_u(1)\rho_u^2 - h_u}{1 - \rho_u^2 + \rho_u^2 V_u(2)} + \widehat{X}_u^{\pi^*} \frac{V_u(2)\lambda_u + \varphi_u(2)\rho_u^2}{1 - \rho_u^2 + \rho_u^2 V_u(2)}\right)^2 (1 - \rho_u^2 + \rho_u^2 V_u(2)) d\langle M \rangle_u + N_t,$$
(4.26)

which implies that  $\mu^{\langle M \rangle}$ -a.e.

$$\pi_u^* = \frac{V_u(1)\lambda_u + \varphi_u(1)\rho_u^2 - \tilde{h}_u}{1 - \rho_u^2 + \rho_u^2 V_u(2)} - \hat{X}_u^{\pi^*} \frac{(V_u(2)\lambda_u + \varphi_u(2)\rho_u^2)}{1 - \rho_u^2 + \rho_u^2 V_u(2)}$$

Integrating both parts of this equality with respect to  $d\hat{S}$  (and adding then x to the both parts) we obtain that  $\hat{X}^{\pi^*}$  satisfies equation (4.7).

The uniqueness of the system (4.4)-(4.6) we shall prove under following condition  $D^*$ ), stronger than condition D).

Assume that  $D^*$ )

$$\int_0^T \frac{\lambda_u^2}{\rho_u^2} d\langle M \rangle_u \le C.$$

Since  $\rho^2 \leq 1$  (Lemma A.1), it follows from  $D^*$ ) that the mean-variance tradeoff of S is bounded, i.e.,

$$\int_0^T \lambda_u^2 d\langle M \rangle_u \le C,$$

which implies that the minimal martingale measure for S exists and satisfies the Reverse-Hölder condition  $R_2(P)$ . So, condition  $D^*$ ) implies condition D). Besides it follows from the condition  $D^*$ ) that the minimal martingale measure  $\hat{Q}^{min}$  for  $\hat{S}$ 

$$d\widehat{Q}^{min} = \mathcal{E}_T(-\frac{\lambda}{\rho^2} \cdot \widehat{M})$$

also exists and satisfies Reverse-Hölder condition.

Recall that the process Z belongs to the class D if the family of random variables  $Z_{\tau}I_{(\tau \leq T)}$  for all stopping times  $\tau$  is uniformly integrable.

**Theorem 4.2.** Let conditions A), B), C) and  $D^*$ ) be satisfied. If a triple (Y(0), Y(1), Y(2)), where  $Y(0) \in D, Y^2(1) \in D$  and  $c \leq Y(2) \leq C$  for some constants 0 < c < C, is a solution of the system (4.4)-(4.6), then such solution is unique and coincides with the triple (V(0), V(1), V(2)).

*Proof.* Let Y(2) be a bounded strictly positive solution of (4.4) and let

$$\int_0^t \psi_u(2) d\widehat{M}_u + L_t(2)$$

be the martingale part of Y(2).

Since Y(2) solves (4.4), it follows from the Itô formula that for any  $\pi \in \Pi(G)$  the process

$$Y_t^{\pi} = Y_t(2)(1 + \int_s^t \pi_u d\widehat{S}_u)^2 + \int_s^t \pi_u^2 (1 - \rho_u^2) d\langle M \rangle_u, \qquad (4.27)$$

 $t \geq s$ , is a local submartingale.

Since  $\pi \in \Pi(G)$ , from Lemma A.1 and the Doob inequality we have

$$E \sup_{t \le T} (1 + \int_0^t \pi_u d\widehat{S})^2 \le$$
$$\le const \left( 1 + E \int_0^T \pi_u^2 \rho_u^2 d\langle M \rangle_u + E \left( \int_0^T |\pi_u \lambda_u| d\langle M \rangle_u \right)^2 < \infty$$
(4.28)

Therefore, taking in mind that Y(2) is bounded and  $\pi \in \Pi(G)$  we obtain that

$$E\Big(\sup_{s\le u\le T}Y_u^{\pi}\Big)^2<\infty$$

which implies that  $Y^{\pi} \in D$ . Thus  $Y^{\pi}$  is a submartingale (as a local submartingale from the class D) and by the boundary condition  $Y_T(2) = 1$  we obtain

$$Y_s(2) \le E\left((1+\int_s^T \pi_u d\widehat{S}_u)^2 + \int_s^T \pi_u^2 (1-\rho_u^2) d\langle M \rangle_u | G_s\right)$$

for all  $\pi \in \Pi(G)$  and hence

$$Y_t(2) \le V_t(2).$$
 (4.29)

Let

$$\tilde{\pi}_t = -\frac{\lambda_t Y_t(2) + \psi_t(2)\rho_t^2}{1 - \rho_t^2 + \rho_t^2 Y_t(2)} \mathcal{E}_t \left( -\frac{\lambda Y(2) + \psi(2)\rho^2}{1 - \rho^2 + \rho^2 Y(2)} \cdot \widehat{S} \right).$$

Since  $1 + \int_0^t \tilde{\pi}_u d\hat{S}_u = \mathcal{E}_t(-\frac{\lambda Y(2) + \psi(2)\rho^2}{1 - \rho^2 + \rho^2 Y(2)} \cdot \hat{S})$ , it follows from (4.4) and the Itô formula that the process  $Y^{\tilde{\pi}}$  defined by (4.27) is a positive local martingale and hence a supermartingale. Therefore

$$Y_{s}(2) \geq E\left((1+\int_{s}^{T} \tilde{\pi}_{u} d\widehat{S}_{u})^{2} + \int_{s}^{T} \tilde{\pi}_{u}^{2}(1-\rho_{u}^{2}) d\langle M \rangle_{u} | G_{s}\right).$$
(4.30)

Let us show that  $\tilde{\pi}$  belongs to the class  $\Pi(G)$ .

From (4.30) and (4.29) we have for every  $s \in [0, T]$ 

$$E\left((1+\int_{s}^{T}\tilde{\pi}_{u}d\widehat{S}_{u})^{2}+\int_{s}^{T}\tilde{\pi}_{u}^{2}(1-\rho_{u}^{2})d\langle M\rangle_{u}|G_{s}\right)\leq Y_{s}(2)\leq V_{s}(2)\leq 1$$
(4.31)

and hence

$$E\left(1+\int_0^T \tilde{\pi}_u d\widehat{S}_u\right)^2 \le 1,\tag{4.32}$$

$$E \int_0^T \tilde{\pi}_u^2 (1 - \rho_u^2) d\langle M \rangle_u \le 1.$$

$$(4.33)$$

By D<sup>\*</sup>) the minimal martingale measure  $\hat{Q}^{min}$  for  $\hat{S}$  satisfies the Reverse-Hölder condition and hence all conditions of Proposition 2.1 are satisfied. Therefore the norm

$$E\Big(\int_0^T \tilde{\pi}_s^2 \rho_s^2 d\langle M \rangle_s\Big) + E\Big(\int_0^T |\tilde{\pi}_s \lambda_s| d\langle M \rangle_s\Big)^2$$

is estimated by  $E(1 + \int_0^T \tilde{\pi}_u d\hat{S}_u)^2$  and hence

$$E\int_0^T \tilde{\pi}_u^2 \rho_u^2 d\langle M \rangle_u < \infty, \quad E\Big(\int_0^T |\tilde{\pi}_s \lambda_s| d\langle M \rangle_s\Big)^2 < \infty.$$

It follows from (4.33) and the latter inequality that  $\tilde{\pi} \in \Pi(G)$  and from (4.30) we obtain that

$$Y_t(2) \ge V_t(2),$$

which together with (4.29) gives the equality  $Y_t(2) = V_t(2)$ .

Thus V(2) is a unique bounded strictly positive solution of equation (4.4). Besides

$$\int_{0}^{t} \psi(2)_{u} d\widehat{M}_{u} = \int_{0}^{t} \varphi(2)_{u} d\widehat{M}_{u}, \quad L_{t}(2) = m_{t}(2)$$
(4.34)

for all t, P-a.s.

Let Y(1) be a solution of equation (4.5) such that  $Y^2(1) \in D$ . By the Itô formula the process

$$R_{t} = Y_{t}(1)\mathcal{E}_{t}\left(-\frac{\varphi(2)\rho^{2} + \lambda V(2)}{1 - \rho^{2} + \rho^{2}V(2)} \cdot \widehat{S}\right) + \int_{0}^{t} \mathcal{E}_{u}\left(-\frac{\varphi(2)\rho^{2} + \lambda V(2)}{1 - \rho^{2} + \rho^{2}V(2)} \cdot \widehat{S}\right) \frac{\tilde{h}_{u}}{1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u}(2)} d\langle M \rangle_{u}$$
(4.35)

is a local martingale. Let us show that  $R_t$  is a martingale.

As it was already shown the strategy

$$\tilde{\pi} = \frac{\psi_u(2)\rho_u^2 + \lambda_u Y_u(2)}{1 - \rho^2 + \rho^2 Y_u(2)} \mathcal{E}_t(-\frac{\psi(2)\rho^2 + \lambda Y(2)}{1 - \rho^2 + \rho^2 Y(2)} \cdot \widehat{S})$$

belongs to the class  $\Pi(G)$ .

Therefore, (see (4.28))

$$E \sup_{t \le T} \mathcal{E}_t^2 \left(-\frac{\psi(2)\rho^2 + \lambda Y(2)}{1 - \rho^2 + \rho^2 Y(2)} \cdot \widehat{S}\right) = E \sup_{t \le T} (1 + \int_0^t \tilde{\pi}_u d\widehat{S})^2 < \infty$$
(4.36)

and hence

$$Y_t(1)\mathcal{E}_t(-\frac{\varphi(2)\rho^2 + \lambda V(2)}{1 - \rho^2 + \rho^2 V(2)} \cdot \widehat{S}) \in D.$$

On the other hand equation (4.36), Lemma A.2 and Lemma 3.1 imply that

$$\begin{split} E \sup_{t \leq T} \int_{0}^{t} \mathcal{E}_{u}(-\frac{\varphi(2)\rho^{2} + \lambda V(2)}{1 - \rho^{2} + \rho^{2}V(2)} \cdot \widehat{S}) \frac{\widetilde{h}_{u}}{1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u}(2)} d\langle M \rangle_{u} \leq \\ & \leq \frac{1}{c} E \int_{0}^{T} \mathcal{E}_{u}(-\frac{\psi(2)\rho^{2} + \lambda Y(2)}{1 - \rho^{2} + \rho^{2}Y(2)} \cdot \widehat{S}) |\widetilde{h}_{u}| d\langle M \rangle_{u} \\ & \leq \frac{1}{c} E^{1/2} \sup_{t \leq T} \mathcal{E}_{t}^{2}(-\frac{\psi(2)\rho^{2} + \lambda Y(2)}{1 - \rho^{2} + \rho^{2}Y(2)} \cdot \widehat{S}) E^{1/2} \int_{0}^{T} \widetilde{h}_{u}^{2} d\langle M \rangle_{u} < \infty. \end{split}$$

Therefore, the process  $R_t$  belongs to the class D and hence it is a true martingale. Using the martingale property and the boundary condition we obtain that

$$Y_{t}(1) = E\left(\widehat{H}_{T}\mathcal{E}_{tT}\left(-\frac{\varphi(2)\rho^{2} + \lambda V(2)}{1 - \rho^{2} + \rho^{2}V(2)} \cdot \widehat{S}\right) + \int_{t}^{T} \mathcal{E}_{tu}\left(-\frac{\varphi(2)\rho^{2} + \lambda V(2)}{1 - \rho^{2} + \rho^{2}V(2)} \cdot \widehat{S}\right) \frac{\widetilde{h}_{u}}{1 - \rho_{u}^{2} + \rho_{u}^{2}V_{u}(2)} d\langle M \rangle_{u} | G_{t}\right).$$
(4.37)

Thus, any solution of (4.5) is expressed explicitly in terms of  $(V(2), \varphi(2))$  in the form (4.37). Hence the solution of (4.5) is unique and it coincides with  $V_t(1)$ .

It is evident that the solution of (4.6) is also unique.

**Corollary 4.1.** In addition to conditions A)-C) assume that  $\rho$  is a constant and the mean-variance tradeoff  $\langle \lambda \cdot M \rangle_T$  is deterministic. Then the solution of (4.4) is the triple  $(Y(2), \psi(2), L(2))$ , with  $\psi(2) = 0, L(2) = 0$  and

$$Y_t(2) = V_t(2) = \nu(\rho, 1 - \rho^2 + \langle \lambda \cdot M \rangle_T - \langle \lambda \cdot M \rangle_t), \qquad (4.38)$$

where  $\nu(\rho, \alpha)$  is the root of the equation

$$\frac{1-\rho^2}{x} - \rho^2 \ln x = \alpha.$$
 (4.39)

Besides

$$Y_{t}(1) = E\left(H\mathcal{E}_{tT}\left(-\frac{\lambda V(2)}{1-\rho^{2}+\rho^{2}V(2)}\cdot\widehat{S}\right) + \int_{t}^{T} \mathcal{E}_{tu}\left(-\frac{\lambda V(2)}{1-\rho^{2}+\rho^{2}V(2)}\cdot\widehat{S}\right)\frac{\tilde{h}_{u}}{1-\rho^{2}+\rho^{2}V_{u}(2)}d\langle M\rangle_{u}|G_{t}\right).$$
(4.40)

uniquely solves equation (4.5) and the optimal filtered wealth process satisfies the linear equation

$$\widehat{X}_{t}^{*} = x - \int_{0}^{t} \frac{\lambda_{u} V_{u}(2)}{1 - \rho^{2} + \rho^{2} V_{u}(2)} \widehat{X}_{u}^{*} d\widehat{S}_{u} 
+ \int_{0}^{t} \frac{\varphi_{u}(1)\rho^{2} + \lambda_{u} V_{u}(1) - \tilde{h}_{u}}{1 - \rho^{2} + \rho^{2} V_{u}(2)} d\widehat{S}_{u}.$$
(4.41)

*Proof.* The function  $f(x) = \frac{1-\rho^2}{x} - \rho^2 \ln x$  is differentiable, strictly decreasing on  $]0, \infty[$  and takes all values from  $] - \infty, +\infty[$ . So equation (4.39) admits a unique solution for all  $\alpha$ . Besides the inverse function  $\alpha(x)$  is differentiable. Therefore  $Y_t(2)$  is a process of finite variation and it is adapted since  $\langle \lambda \cdot M \rangle_T$  is deterministic.

By definition of  $Y_t(2)$  we have that for all  $t \in [0, T]$ 

$$\frac{1-\rho^2}{Y_t(2)} - \rho^2 \ln Y_t(2) = 1 - \rho^2 + \langle \lambda \cdot M \rangle_T - \langle \lambda \cdot M \rangle_t.$$

It is evident that for  $\alpha = 1 - \rho^2$  the solution of (4.39) is equal to 1 and it follows from (4.38) that Y(2) satisfies the boundary conditions  $Y_T(2) = 1$ . Therefore

$$\frac{1-\rho^2}{Y_t(2)} - \rho^2 \ln Y_t(2) - (1-\rho^2)$$
$$= -(1-\rho^2) \int_t^T d\frac{1}{Y_u(2)} + \rho^2 \int_t^T d\ln Y_u(2)$$
$$= \int_t^T \left(\frac{1-\rho^2}{Y_u^2(2)} + \frac{\rho^2}{Y_u(2)}\right) dY_u(2)$$

and

$$\int_{t}^{T} \frac{1 - \rho^{2} + \rho^{2} Y_{u}(2)}{Y_{u}^{2}(2)} dY_{u}(2) = \langle \lambda \cdot M \rangle_{T} - \langle \lambda \cdot M \rangle_{t}$$

for all  $t \in [0, T]$ . Hence

$$\int_0^t \frac{1-\rho^2+\rho^2 Y_u(2)}{Y_u^2(2)} dY_u(2) = \langle \lambda \cdot M \rangle_t$$

and integrating both parts of this equality with respect to  $Y(2)/(1 - \rho^2 + \rho^2 Y(2))$  we obtain that Y(2) satisfies equation

$$Y_t(2) = Y_0(2) + \int_0^t \frac{Y_u^2(2)\lambda_u^2}{1 - \rho^2 + \rho^2 Y_u(2)} d\langle M \rangle_u, \qquad (4.42)$$

which implies that the triple  $(Y(2), \psi(2) = 0, L(2) = 0)$  satisfies equation (4.4) and Y(2) = V(2) by Theorem 4.2. Equations (4.40) and (4.41) follow from (4.37) and (4.7) respectively, taking  $\varphi(2) = 0$ .

**Remark 4.1.** In the case of complete information,  $M = \widehat{M}$  and  $\rho = 1$ . Therefore equation (4.42) is linear and  $Y(2) = e^{\langle \lambda \cdot M \rangle_t - \langle \lambda \cdot M \rangle_T}$ .

**Remark 4.2.** Finally let us make a comment on condition B). It would be desirable to replace condition B) by requiring that any *G*-martingale is a *F*-semimartingale, but up to now we can't do this, although one can weaken this condition imposing that any *G*-martingale is a  $\sigma(F^S \vee G)$ - martingale, where  $\sigma(F_t^S \vee G_t)$  is the minimal  $\sigma$ -algebra containing  $F^S$  and  $G_t$ , which is satisfied if  $F_t^S \subseteq G_t$ .

### 5 Diffusion market model

Let us consider the financial market model

$$d\tilde{S}_t = \tilde{S}_t \mu_t(Y) dt + \tilde{S}_t \sigma_t(Y) dw_t^0,$$
  
$$dY_t = a_t(Y) dt + b_t(Y) dw_t,$$

subjected to initial conditions, where only the second component Y is observed. Here  $w^0$  and w are corelated Brownian motions with  $Edw_t^0 dw_t = \rho dt, \rho \in (-1, 1)$ .

Let us write

$$w_t = \rho w_t^0 + \sqrt{1 - \rho^2} w_t^1,$$

where  $w^0$  and  $w^1$  are independent Brownian motions. It is evident that  $w^{\perp} = -\sqrt{1-\rho^2}w^0 + \rho w^1$  is a Brownian motion independent of w and one can express Brownian motions  $w^0, w^1$  in terms of w and  $w^{\perp}$  as

$$w_t^0 = \rho w_t - \sqrt{1 - \rho^2} w_t^{\perp}, \quad w_t^1 = \sqrt{1 - \rho^2} w_t + \rho w_t^{\perp}.$$
(5.1)

We assume that  $b^2 > 0$ ,  $\sigma^2 > 0$  and coefficients  $\mu, \sigma, a$  and b are such that  $F_t^{S,Y} = F_t^{w^0,w}$ ,  $F_t^Y = F_t^w$ . So the stochastic basis will be  $(\Omega, \mathcal{F}, F_t, P)$ , where  $F_t$  is the natural filtration of  $(w^0, w)$  and the flow of observable events is  $G_t = F_t^w$ .

Also denote  $dS_t = \mu_t dt + \sigma_t dw_t^0$ , so that  $d\tilde{S}_t = \tilde{S}_t dS_t$  and S is the return of the stock.

Let  $\tilde{\pi}_t$  be the number shares of the stock at time t. Then  $\pi_t = \tilde{\pi}_t \tilde{S}_t$  represents an amount of money invested in the stock at the time  $t \in [0, T]$ . We consider the mean variance hedging problem

to minimize 
$$E[(x + \int_0^T \tilde{\pi}_t d\tilde{S}_t - H)^2]$$
 over all  $\tilde{\pi}$  for which  $\tilde{\pi}\tilde{S} \in \Pi(G)$ , (5.2)

which is equivalent to study the mean variance hedging problem

to minimize 
$$E[(x + \int_0^T \pi_t dS_t - H)^2]$$
 over all  $\pi \in \Pi(G)$ .

**Remark 5.1.** Since S is not G-adapted,  $\tilde{\pi}_t$  and  $\tilde{\pi}_t \tilde{S}_t$  can not be simultaneously G-predictable and the problem

to minimize 
$$E[(x + \int_0^T \tilde{\pi}_t d\tilde{S}_t - H)^2]$$
 over all  $\tilde{\pi} \in \Pi(G)$ , (5.3)

is not equivalent to the problem (5.2) and it needs separate consideration.

Comparing with (1.1) we get that in this case

$$M_t = \int_0^t \sigma_s dw_s^0, \quad \langle M \rangle_t = \int_0^t \sigma_s^2 ds, \quad \lambda_t = \frac{\mu_t}{\sigma_t^2}.$$

It is evident that w is a Brownian motion also with respect to the filtration  $F^{w^0,w^1}$  and condition B) is satisfied. Therefore by Proposition 2.2

$$\widehat{M}_t = \rho \int_0^t \sigma_s dw_s.$$

By the integral representation theorem the GKW decompositions (3.2), (3.3) take following forms

$$c_H = EH, \quad H_t = c_H + \int_0^t h_s \sigma_s dw_s^0 + \int_0^t h_s^1 dw_s^1,$$
 (5.4)

$$H_{t} = c_{H} + \rho \int_{0}^{t} h_{s}^{G} \sigma_{s} dw_{s} + \int_{0}^{t} h_{s}^{\perp} dw_{s}^{\perp}.$$
 (5.5)

Putting expressions (5.1) for  $w^0, w^1$  in (5.4) and equalizing integrands of (5.4) and (5.5) we obtain that

$$h_t = \rho^2 h_t^G - \sqrt{1 - \rho^2} \frac{h_t^\perp}{\sigma_t}$$

and hence

$$\widehat{h}_t = \rho^2 \widehat{h_t^G} - \sqrt{1 - \rho^2} \, \frac{\widehat{h}_t^\perp}{\sigma_t}.$$

Therefore by definition of  $\tilde{h}$ 

$$\widetilde{h}_t = \rho^2 \widehat{h_t^G} - \widehat{h}_t = \sqrt{1 - \rho^2} \, \frac{\widehat{h}_t^\perp}{\sigma_t} \tag{5.6}$$

Using notations

$$Z_s(0) = \rho \sigma_s \varphi_s(0), \ Z_s(1) = \rho \sigma_s \varphi_s(1), \ Z_s(2) = \rho \sigma_s \varphi_s(2), \ \theta_s = \frac{\mu_s}{\sigma_s}$$

we obtain the following corollary of Theorem 4.1

**Corollary 5.1.** Let H be a square integrable  $F_T$ -measurable random variable. Then the processes  $V_t(0), V_t(1)$  and  $V_t(2)$  from (4.3) satisfy the following system of backward equations

$$V_{t}(2) = V_{0}(2) + \int_{0}^{t} \frac{\left(\rho Z_{s}(2) + \theta_{s} V_{s}(2)\right)^{2}}{1 - \rho^{2} + \rho^{2} V_{s}(2)} ds + \int_{0}^{t} Z_{s}(2) dw_{s} \quad V_{T}(2) = 1, \quad (5.7)$$

$$V_{t}(1) = V_{0}(1) + \int_{0}^{t} \frac{\left(\rho Z_{s}(2) + \theta_{s} V_{s}(2)\right) \left(\rho Z_{s}(1) + \theta_{s} V_{s}(1) - \sqrt{1 - \rho^{2}} \, \hat{h}_{s}^{\perp}\right)}{1 - \rho^{2} + \rho^{2} V_{s}(2)} ds$$

$$+ \int_{0}^{t} Z_{s}(1) dw_{s}, \quad V_{T}(1) = E(H|G_{T}), \quad (5.8)$$

$$V_{t}(0) = V_{0}(0) + \int_{0}^{t} \frac{\left(\rho Z_{s}(1) + \theta_{s} V_{s}(1) - \sqrt{1 - \rho^{2}} \, \hat{h}_{s}^{\perp}\right)^{2}}{1 - \rho^{2} + \rho^{2} V_{s}(2)} ds$$

$$+ \int_{0}^{t} Z_{s}(0) dw_{s}, \quad V_{T}(0) = E^{2}(H|G_{T}), \quad (5.9)$$

Besides the optimal wealth process  $\widehat{X}^*$  satisfies the linear equation

$$\widehat{X}_{t}^{*} = x - \int_{0}^{t} \frac{\rho Z_{s}(2) + \theta_{s} V_{s}(2)}{1 - \rho^{2} + \rho^{2} V_{s}(2)} \widehat{X}_{s}^{*}(\theta_{s} ds + \rho dw_{s}) + \int_{0}^{t} \frac{\rho Z_{s}(1) + \theta_{s} V_{s}(1) - \sqrt{1 - \rho^{2}} \, \widehat{h}_{s}^{\perp}}{1 - \rho^{2} + \rho^{2} V_{s}(2)} (\theta_{s} ds + \rho dw_{s}).$$
(5.10)

**Example**. Suppose that  $\theta_t$  and  $\sigma_t$  are deterministic. Then the solution of (5.7) is the pair  $(V_t(2), \varphi_t(2))$ , where  $\varphi(2) = 0$  and V(2) satisfies the ordinary differential equation

$$\frac{dV_t(2)}{dt} = \frac{\theta_t^2 V_t^2(2)}{1 - \rho^2 + \rho^2 V_t(2)}, \quad V_T(2) = 1.$$
(5.11)

Solving this equation we obtain that

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$$V_t(2) = \nu(\rho, 1 - \rho^2 + \int_t^T \theta_s^2 ds) \equiv \nu_t^{\theta, \rho},$$
 (5.12)

where  $\nu(\rho, \alpha)$  is the solution of (4.39). From (5.11) follows that

$$(\ln\nu_t^{\theta,\rho})' = \frac{\theta_t^2 \nu_t^{\theta,\rho}}{1 - \rho^2 + \rho^2 \nu_t^{\theta,\rho}} \text{ and } \ln\frac{\nu_s^{\theta,\rho}}{\nu_t^{\theta,\rho}} = \int_t^s \frac{\theta_r^2 \nu_r^{\theta,\rho} dr}{1 - \rho^2 + \rho^2 \nu_r^{\theta,\rho}}.$$
 (5.13)

If we solve the linear BSDE (5.8) and use (5.13) we obtain

$$V_t(1) = E\left[\widehat{H}_T(w)\mathcal{E}_{tT}\left(-\int_0^{\cdot} \frac{\theta_r \nu_r^{\theta,\rho}}{1-\rho^2+\rho^2 \nu_r^{\theta,\rho}}(\theta_r dr + \rho dw_r)\right)|G_t\right]$$
$$\int_t^T \frac{\theta_s \nu_s^{\theta,\rho} \sigma_s}{1-\rho^2+\rho^2 \nu_s^{\theta,\rho}} E\left[\widetilde{h}_s(w)\mathcal{E}_{ts}\left(-\int_0^{\cdot} \frac{\theta_r \nu_r^{\theta,\rho}}{1-\rho^2+\rho^2 \nu_r^{\theta,\rho}}(\theta_r dr + \rho dw_r)\right)|G_t\right] ds$$
$$= \nu_t^{\theta,\rho} E\left[\widehat{H}_T(w)\mathcal{E}_{tT}\left(-\int_0^{\cdot} \frac{\theta_r \nu_r^{\theta,\rho}}{1-\rho^2+\rho^2 \nu_r^{\theta,\rho}}\rho dw_r\right)|G_t\right]$$
$$+ \nu_t^{\theta,\rho} \int_t^T \frac{\mu_s}{1-\rho^2+\rho^2 \nu_s^{\theta,\rho}} E\left[\widetilde{h}_s(w)\mathcal{E}_{ts}\left(-\int_0^{\cdot} \frac{\theta_r \nu_r^{\theta,\rho}}{1-\rho^2+\rho^2 \nu_r^{\theta,\rho}}\rho dw_r\right)|G_t\right] ds$$

Using the Girsanov theorem we finally get

$$V_t(1) = \nu_t^{\theta,\rho} E\left[\hat{H}_T\left(\int_t^{\cdot} \rho \frac{\theta_r \nu_r^{\theta,\rho}}{1 - \rho^2 + \rho^2 \nu_r^{\theta,\rho}} dr + w\right) |G_t\right] + \nu_t^{\theta,\rho} \int_t^T \frac{\mu_s}{1 - \rho^2 + \rho^2 \nu_s^{\theta,\rho}} E\left[\tilde{h}_s\left(\rho \int_t^{\cdot} \frac{\theta_r \nu_r^{\theta,\rho}}{1 - \rho^2 + \rho^2 \nu_r^{\theta,\rho}} dr + w\right) |G_t\right] ds.$$
(5.14)

Suppose now that  $H = H(w_T^0, w_T)$ , Y = w and  $\frac{\mu_t}{\sigma_t} = \theta(t, w_t)$  for some continuous function  $\theta(t, x)$  and a differentiable function H(x, y). Then using the elementary ideas of Malliavin's calculus we get

$$\rho h_t^G \sigma_t \equiv h^{\mathcal{G}}(t, w_t^0, w_t) = E[\rho \partial_x H(w_T^0, w_T) + \partial_y H(w_T^0, w_T) | \mathcal{F}_t],$$
$$h_t^{\perp} = h^{\perp}(t, w_t^0, w_t) = -\sqrt{1 - \rho^2} E[\partial_x H(w_T^0, w_T) | \mathcal{F}_t],$$

where  $\partial_x H(x, y), \partial_y H(x, y)$  denote the partial derivatives of  $\mathcal{H}$ . It is evident that  $E[f(t, w_t^0, w_t) | \mathcal{F}_t^w] = E[f(t, \rho w_t - \sqrt{1 - \rho^2} w_t^{\perp}, w_t) | w_t]$  for  $f = h^w, h^{\perp}, H$ . Thus we obtain the exact expression for  $\widehat{H}_T(y), \widehat{h}^{\mathcal{G}}(t, y)$  and  $\widehat{h}^{\perp}(t, y)$ 

$$\widehat{H}_{T}(y) = EH(\rho y - \sqrt{1 - \rho^{2}} w_{T}^{\perp}, y) \equiv E[H(w_{T}^{0}, y)|w_{T} = y], \qquad (5.15)$$

$$\widehat{h}^{\mathcal{G}}(t, y) = Eh^{\mathcal{G}}(t, \rho y - \sqrt{1 - \rho^{2}} w_{t}^{\perp}, y),$$

$$\widehat{h}^{\perp}(t, y) = Eh^{\perp}(t, \rho y - \sqrt{1 - \rho^{2}} w_{t}^{\perp}, y) = -\sqrt{1 - \rho^{2}} E[H_{x}(w_{T}^{0}, w_{T})|w_{t} = y]$$

$$\equiv -\sqrt{1 - \rho^{2}} E\partial_{x}H(\rho(y + w_{T} - w_{t}) - \sqrt{1 - \rho^{2}} w_{T}^{\perp}, \rho(y + w_{T} - w_{t})).$$

**Remark 5.2.** For deterministic  $\theta_t$  the equalities

$$E\left[\widehat{H}_{T}\left(\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}dr+w_{T}\right)|w_{t}=y\right]$$
$$=E\left[H\left(\rho\left(\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)-\sqrt{1-\rho^{2}}w_{T}^{\perp},\right.\right.\right.$$
$$\left.\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)|w_{t}=y\right]$$
$$\equiv EH\left(\rho\left(\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}-w_{t}+y\right)-\sqrt{1-\rho^{2}}w_{T}^{\perp},\right.\right.$$
$$\left.\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}-w_{t}+y\right)$$

and

$$E\left[\widehat{h}_{s}^{\perp}\left(\rho\int_{t}^{s}\frac{\theta_{r}\nu_{r}^{\theta,\rho}}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}dr+w_{s}\right)|w_{t}=y\right]$$

$$=-\sqrt{1-\rho^{2}}E\left[E\left[\partial_{x}H\left(\rho\left(\rho\int_{t}^{s}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)-\sqrt{1-\rho^{2}}w_{T}^{\perp},\right.\right.\right.\right.\right.\right.$$

$$\left.\rho\int_{t}^{s}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)|w_{s}\right]|w_{t}=y\right]$$

$$=-\sqrt{1-\rho^{2}}E\left[\partial_{x}H\left(\rho\left(\rho\int_{t}^{s}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)-\sqrt{1-\rho^{2}}w_{T}^{\perp},\right.\right.\right.\right.$$

$$\left.\rho\int_{t}^{s}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}\right)|w_{t}=y\right]$$

$$\equiv-\sqrt{1-\rho^{2}}E\partial_{x}H\left(\rho\left(\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}-w_{t}+y\right)-\sqrt{1-\rho^{2}}w_{T}^{\perp},\right.$$

$$\left.\rho\int_{t}^{T}\frac{\theta_{r}\nu_{r}^{\theta,\rho}dr}{1-\rho^{2}+\rho^{2}\nu_{r}^{\theta,\rho}}+w_{T}-w_{t}+y\right)$$

are valid.

Using the well known connection between BSDEs and PDEs we can prove the following **Proposition 5.1.** *The system of nonlinear PDEs* 

$$\partial_t v(2) + \frac{1}{2} \partial_y^2 v(2) = \frac{(\theta(t, y)v(2) + \rho \partial_y v(2))^2}{1 - \rho^2 + \rho^2 v(2)}, \quad v(2, T, y) = 1,$$
(5.16)

$$\partial_t v(1) + \frac{1}{2} \partial_y^2 v(1) \tag{5.17}$$

$$=\frac{((\theta(t,y)v(2) + \rho\partial_y v(2))((\theta(t,y)v(1) + \rho\partial_y v(1)) + (1 - \rho^2)E[\partial_x H(w_T^0, w_T)|w_t = y])}{1 - \rho^2 + \rho^2 v(2)},$$
  
$$v(1,T,y) = E[H(w_T^0, y)|w_T = y]$$

admits sufficiently smooth solution and the solution of (4.4), (4.5) can be represented as  $V_t(1) = v(1, t, w_t), Z_t(1) = \partial_y v(1, t, w_t), V_t(2) = v(2, t, w_t), Z_t(2) = \partial_y v(2, t, w_t).$ Besides the optimal strategy is of the form

$$\pi^*(t, X, y) = -\frac{\theta(t, y)v(2, t, y) + \rho\partial_y v(2, t, y)}{1 - \rho^2 + \rho^2 v(2, t, y)} X$$

$$+ \frac{(\theta(t, y)v(1, t, y) + \rho\partial_y v(1, t, y) + (1 - \rho^2)E[\partial_x H(w_T^0, w_T)|w_t = y])}{1 - \rho^2 + \rho^2 v(2, t, y)},$$
(5.18)

where X and y are states at time t of the wealth and of an observable process.

**Example(continued)**. We suppose in addition that  $\theta, \sigma$  are constants and H = $\mathcal{H}(S_T, Y_T) \equiv \mathcal{H}(\mu T + \sigma w_T^0, Y_T)$ . Then using the equality  $\frac{1}{\theta} \ln \frac{\nu_s^{\theta, \rho}}{\nu_t^{\theta, \rho}} = \int_t^s \frac{\theta \nu_r^{\theta, \rho} dr}{1 - \rho^2 + \rho^2 \nu_r^{\theta, \rho}}$  the formula (5.14) can be simplified. It is easy to see that

$$\lim_{\theta \to 0} \frac{1}{\theta} \ln \nu_t^{\theta, \rho} = 0.$$

Thus we can set that expression  $\frac{1}{\theta} \ln \nu_t^{\theta,\rho}$  is zero as  $\theta = 0$ . For simplicity we also assume that a = 0, b = 1. For v(1) using (5.14) and Remark 5.2 we get

$$v(1,t,y) = \nu_t^{\theta,\rho} E\left[\widehat{H}\left(-\frac{\rho}{\theta}\ln\nu_t^{\theta,\rho} + w_T - w_t + y\right)\right] + \nu_t^{\theta,\rho} \int_t^T \frac{\mu}{1 - \rho^2 + \rho^2 \nu_s^{\theta,\rho}} E\left[\widetilde{h}_s\left(\frac{\rho}{\theta}\ln\frac{\nu_s^{\theta,\rho}}{\nu_t^{\theta,\rho}} + w_s - w_t + y\right)\right] ds,$$

or equivalently

$$v(1,t,y) = \nu_t^{\theta,\rho}$$

$$\times E\mathcal{H}\left(\mu T + \sigma\rho(-\frac{\rho}{\theta}\ln\nu_t^{\theta,\rho} + w_T - w_t + y) - \sigma\sqrt{1-\rho^2}w_T^{\perp}, \rho(-\frac{\rho}{\theta}\ln\nu_t^{\theta,\rho} + w_T - w_t + y)\right)$$

$$+ (1-\rho^2)\mu\nu_t^{\theta,\rho}\int_t^T \frac{1}{1-\rho^2+\rho^2\nu_s^{\theta,\rho}}$$

$$\times E\partial_x\mathcal{H}\left(\mu T + \sigma\rho(\frac{\rho}{\theta}\ln\frac{\nu_s^{\theta,\rho}}{\nu_t^{\theta,\rho}} + w_T - w_t + y) - \sigma\sqrt{1-\rho^2}w_T^{\perp}, \rho(\frac{\rho}{\theta}\ln\frac{\nu_s^{\theta,\rho}}{\nu_t^{\theta,\rho}} + w_T - w_t + y)\right) ds$$

taking in mind (5.15). This formula together with (5.12) and (5.18) gives an explicit solution of the problem (5.2) for the case of constant coefficients.

#### Appendix Α

For convenience we give the proofs of the following assertions used in the paper.

**Lemma A.1.** Let conditions A)–C) be satisfied and  $\widehat{M}_t = E(M_t|G_t)$ . Then  $\langle \widehat{M} \rangle$  is absolutely continuous w.r.t  $\langle M \rangle$  and  $\mu^{\langle M \rangle}$  a.e.

$$\rho_t^2 = \frac{d\langle \widehat{M} \rangle_t}{d\langle M \rangle_t} \le 1$$

**Proof.** By (2.4) for any bounded *G*-predictable process *h* 

$$E \int_{0}^{t} h_{s}^{2} d\langle \widehat{M} \rangle_{s} = E \left( \int_{0}^{t} h_{s} d\widehat{M}_{s} \right)^{2} = E \left( E \left( \int_{0}^{t} h_{s} dM_{s} | G_{t} \right) \right)^{2} \leq \\ \leq E \left( \int_{0}^{t} h_{s} dM_{s} \right)^{2} \leq E \int_{0}^{t} h_{s}^{2} d\langle M \rangle_{s}$$
(A.1)

which implies that  $\langle \widehat{M} \rangle$  is absolutely continuous w.r.t  $\langle M \rangle$ , i.e.,

$$\langle \widehat{M} \rangle_t = \int_0^t \rho_s^2 d\langle M \rangle_s$$

for a G-predictable process  $\rho$ . Moreover (A.1) implies that the process  $\langle M \rangle - \langle \widehat{M} \rangle$  is increasing and hence  $\rho^2 \leq 1$  $\mu^{\langle M\rangle}$  a.e.

**Lemma A.2.** Let  $H \in L^2(P, F_T)$  and let conditions A) - C) be satisfied. Then

$$E \int_0^T \tilde{h}_u^2 d\langle M \rangle_u < \infty. \tag{A.2}$$

*Proof.* It is evident that

$$E\int_0^T (h_u^G)^2 d\langle \widehat{M} \rangle_u < \infty, \quad E\int_0^T h_u^2 d\langle M \rangle_u < \infty.$$

Therefore, by definition of  $\tilde{h}$  and Lemma A.1,

$$\begin{split} E \int_0^T \tilde{h}_u^2 d\langle M \rangle_u \\ &\leq 2E \int_0^T E^2(h_u | G_u) d\langle M \rangle_u + 2E \int_0^T E^2(h_u^G | G_u) \rho_u^4 d\langle M \rangle_u \\ &\leq 2E \int_0^T h_u^2 d\langle M \rangle_u + 2E \int_0^T (h_u^G)^2 \rho_u^2 d\langle \widehat{M} \rangle_u < \infty. \end{split}$$

Thus  $\tilde{h} \in \Pi(G)$  by remark 2.3.

**Lemma A.3.** a) Let  $Y = (Y_t, t \in [0, T])$  be a bounded positive submartingale with the canonical decomposition

$$Y_t = Y_0 + B_t + m_t$$

where B is a predictable increasing process and m is a martingale. Then  $m \in BMO$ .

b) In particular the martingale part of V(2) belongs to BMO. If H is bounded, then martingale parts of V(0) and V(1) also belong to the class BMO, i.e., for i = 0, 1, 2,

$$E(\int_{\tau}^{T} \varphi_{u}^{2}(i)\rho_{u}^{2}d\langle M\rangle_{u}|G_{\tau}) + E(\langle m(i)\rangle_{T} - \langle m(i)\rangle_{\tau}|G_{\tau}) \le C$$
(A.3)

for every stopping time  $\tau$ .

*Proof.* Applying the Itô formula for  $Y_T^2 - Y_\tau^2$  we have

$$\langle m \rangle_T - \langle m \rangle_\tau + 2 \int_\tau^T Y_u dB_u + 2 \int_\tau^T Y_u dm_u = Y_T^2 - Y_\tau^2 \le Const.$$
 (A.4)

Since Y is positive and B is an increasing process, taking conditional expectations in (A.4) we obtain

$$E(\langle m \rangle_T - \langle m \rangle_\tau | F_\tau) \le Const.$$

for any stopping time  $\tau$ , hence  $m \in BMO$ .

(A.3) follows from assertion a) applied for positive submartingales V(0), V(2) and V(0) + V(2) - 2V(1). For the case i = 1 one should take into account also the inequality

$$\langle m(1) \rangle_t \le const(\langle m(0) + m(2) - 2m(1) \rangle_t + \langle m(0) \rangle_t + \langle m(2) \rangle_t).$$

Acknowledgments. We would like to thank N. Lazrieva for useful remarks and comments.

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