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COMPUTING THE CONICAL FUNCTION $P^{\mu}_{-1/2+i\tau}(x)$*

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Abstract. A stable computational scheme for the conical function $P^{\mu}_{-1/2+i\tau}(x)$ for $x > -1$, real τ, and $\mu \leq 0$ or $\mu \in \mathbb{N}$ is presented. The scheme combines uniform asymptotic expansions for large $|\mu|$ with the application of the three-term recurrence relation on the μ index in the direction of decreasing $|\mu|$ when $x > 0$. When $x < 0$, the conditioning of recursion is the opposite, and conical functions can be computed in the direction of increasing $|\mu|$.

Key words. Legendre functions, conical functions, hypergeometric functions, modified Bessel functions, three-term recurrence relations, difference equations, stability of recurrence relations, numerical evaluation of special functions, asymptotic analysis

AMS subject classifications. 33C05, 33C10, 39A11, 41A60, 65D20

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1. Introduction.

Conical functions are useful functions appearing in wide number of applications in applied physics [9, 15, 16], astrophysics and cosmology [2, 13, 17], fluid mechanics [8], or geophysics [20, 19], among others. Also, they are the kernel of the Mehler–Fock transform (see, for instance, [11, p. 221] and [22, Chapter 25]).

It appears that the only existing algorithm for conical functions is given by Kölbig [10], who computed the functions $P^{0}_{-1/2+i\tau}(x)$ and $P^{1}_{-1/2+i\tau}(x)$, $x > -1$ (see also [15]). In applications, however, in general, one encounters values $P^{\mu}_{-1/2+i\tau}(x)$, where $\mu = m$ are integer numbers. We are providing a computational method for these parameter values, which uses uniform asymptotics for large $|m|$ combined with recurrence relations and numerical quadrature.

2. Basic definitions.

The associated Legendre function $P^{\mu}_{\nu}(x)$ for real $x > -1$ ($\nu = -1/2 + i\tau$ for the case of conical functions) can be written in terms of the Gauss hypergeometric function as follows:

$$P^{\mu}_{\nu}(x) = \frac{1}{\Gamma(1-\mu)} \left| \frac{1+x}{1-x} \right|^{\mu/2} {}_2F_1\left(\begin{matrix} -\nu,\ \nu+1 \\ 1-\mu \end{matrix} ;\ \tfrac{1}{2} - \tfrac{1}{2}x \right). \tag{2.1}$$

We adopt this definition for $x > -1$; this is the definition used in applications (both for $-1 < x < 1$ and $x > 1$). The absolute value in the previous formula is showing that, in fact, we are dealing with two different functions in the complex plane (though trivially related). The functions defined in such a way satisfy the associated Legendre differential equation

$$\left(1-x^2\right) y''(x) - 2xy'(x) + \left(\nu(\nu+1) - \mu^2/\left(1-x^2\right)\right) y(x) = 0\,. \tag{2.2}$$

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We can use this definition for real $x > -1$ except when $\mu \in \mathbb{Z}^+$, because for these values of μ, the hypergeometric function in (2.1) is not defined. However, the reciprocal gamma function makes the right-hand side of (2.1) regular when $\mu \in \mathbb{Z}^+$. For $x \in (-1, 1)$, this also follows from the relation

$$P_\nu^\mu(x) = \frac{\Gamma(\mu-\nu)\Gamma(\mu+\nu+1)}{\pi}\left[\sin(\pi\mu)\, P_\nu^{-\mu}(-x) - \sin(\pi\nu)\, P_\nu^{-\mu}(x)\right], \tag{2.3}$$

and we see that, when $m \in \mathbb{Z}$,

$$P_\nu^m(x) = -\frac{\Gamma(m-\nu)\Gamma(m+\nu+1)}{\pi}\sin(\pi\nu)\, P_\nu^{-m}(x). \tag{2.4}$$

This relation also holds for $x > 1$ [1, equation 8.2.5].

From these representations of the Legendre function, it is clear that when $\nu = -1/2 + i\tau$, $\tau \in \mathbb{R}$, the function $P_\nu^\mu(x)$ ($\mu \in \mathbb{R}$) remains real valued, as also does the defining differential equation and the rest of relations. For example, (2.4) reads

$$P^m_{-\frac{1}{2}+i\tau}(x) = \cosh(\pi\tau)\frac{|\Gamma(m+1/2+i\tau)|^2}{\pi}P^{-m}_{-\frac{1}{2}+i\tau}(x). \tag{2.5}$$

We will use Greek letters (μ) for denoting real values of the order and Latin letters (m) for integer orders, expect otherwise specified. Usually, we will employ the notation $P^\mu_{-\frac{1}{2}+i\tau}(x)$ for positive orders ($\mu > 0$), and we will write $P^{-\mu}_{-\frac{1}{2}+i\tau}(x)$, with $\mu > 0$ for the case of negative orders, particularly in the derivation of the asymptotic expansions of sections 4 and 6; in some cases, however, it is convenient to use $P^\mu_{-\frac{1}{2}+i\tau}(x)$ for denoting conical functions with positive or negative real orders.

3. Recurrence relation and continued fraction.

Three-term recurrence relations

$$y_{n+1} + b_n y_n + a_n y_{n-1} = 0 \tag{3.1}$$

are useful methods of computation when two starting values are available for starting the recursive process. Usually, the direction of application of the recursion cannot be chosen arbitrarily, and the conditioning of the computation of a given solution fixes the direction.

A solution f_n of a three-term recurrence relation is said to be recessive or minimal as $n \to +\infty$ if

$$\lim_{n\to+\infty}\frac{f_n}{g_n} = 0 \tag{3.2}$$

for any other solution g_n independent of f_n; g_n is said to be a dominant solution. For such a solution f_n, the computation from forward application of the recurrence (increasing n) is bad conditioned (at least for large enough n) because any small error introduces a component of a dominant solution which grows faster than the minimal solution. For a minimal solution, backward recursion (decreasing n) should be considered instead. Contrary, for a dominant solution, forward recursion (increasing n should be considered).

As we discuss next, the recursion for conical functions admits the minimal solution, which indeed fixes the direction of stable recursion. First, we study the case $x \in (-1, 1)$ and later the case $x > 1$ (the case $x = 1$ is trivial, and no minimal solution exists). The main tool is Perron's theorem [7], that we give in the form of Theorem 2.5 of [7].

THEOREM 1 (Perron's theorem). *Let $y_{n+1}+b_ny_n+a_ny_{n-1}=0$ such that $a_n \sim an^{\alpha}$ and $b_n \sim bn^{\beta}$ as $n \to +\infty$ for some real values α and β. Let $\phi(\lambda)=\lambda^2+b_n\lambda+a_n$ be the characteristic polynomial, with roots $\lambda_1(n)$ and $\lambda_2(n)$ such that*

$$\left|\lim_{n\to+\infty}\frac{\lambda_1(n)}{\lambda_2(n)}\right| \neq 1\,. \tag{3.3}$$

Then there exists two independent solutions of the recurrence f_n and g_n such that

$$\lim_{n\to+\infty}\frac{1}{\lambda_1(n)}\frac{f_{n+1}}{f_n}=1\,,\quad \lim_{n\to+\infty}\frac{1}{\lambda_2(n)}\frac{g_{n+1}}{g_n}=1, \tag{3.4}$$

and the minimal solution is the one corresponding to the smallest $|\lambda_i|$ as $n\to+\infty$.

Next, we study the existence of a minimal solution of the recurrence over μ for conical functions $P^{\mu}_{-1/2+i\tau}(x)$, both as $\mu\to+\infty$ and as $\mu\to-\infty$. We will start by applying Perron's theorem as $\mu\to-\infty$, first in the interval $(-1,1)$ and later in $(1,+\infty)$. Later, we will outline the analogous results for the case $\mu\to+\infty$. The starting points are the recurrence relations, which can be written

$$P^{\mu+1}_{-\frac12+i\tau}(x)+\frac{2\mu x}{\sqrt{1-x^2}}P^{\mu}_{-\frac12+i\tau}(x)-\left(\left(\mu-\tfrac12\right)^2+\tau^2\right)P^{\mu-1}_{-\frac12+i\tau}(x)=0 \tag{3.5}$$

for $x\in(-1,1)$ and

$$P^{\mu+1}_{-\frac12+i\tau}(x)+\frac{2\mu x}{\sqrt{x^2-1}}P^{\mu}_{-\frac12+i\tau}(x)+\left(\left(\mu-\tfrac12\right)^2+\tau^2\right)P^{\mu-1}_{-\frac12+i\tau}(x)=0 \tag{3.6}$$

for $x>1$.

3.1. Conditioning for $P^{\mu}_{-1/2+i\tau}(x)$ as $\mu\to-\infty, x\in(-1,1)$. We start by studying the conditioning of recursion for $P^{\mu}_{-1/2+i\tau}(x)$ as $\mu\to-\infty$, or, what is the same, the conditioning for $y_\mu(x)=P^{-\mu}_{-1/2+\imath\tau}(x)$ as $\mu\to+\infty$.

For these functions $y_\mu(x)$, one obtains from (3.5) the relation

$$y_{\mu+1}(x)+b_\mu(x)y_\mu(x)+a_\mu(x)y_{\mu-1}(x)=0\,, \tag{3.7}$$

with coefficients such that

$$\begin{aligned} &b_\mu(x)=b\mu^{-1}\left(1+O\left(\mu^{-1}\right)\right),\quad a_\mu(x)=a\mu^{-2}\left(1+O\left(\mu^{-1}\right)\right),\\ &b=2x/\sqrt{1-x^2},\quad a=-1,\end{aligned} \tag{3.8}$$

as $\mu\to+\infty$.

Now we can apply Perron's theorem. We get

$$\lambda_1(\mu)=t_1\mu^{-1}\left(1+O\left(\mu^{-1}\right)\right),\,\lambda_2(\mu)=t_2\mu^{-1}\left(1+O\left(\mu^{-1}\right)\right), \tag{3.9}$$

as $\mu\to+\infty$, where

$$t_1=\sqrt{\frac{1-x}{1+x}},\quad t_2=-\sqrt{\frac{1+x}{1-x}}. \tag{3.10}$$

The minimal solution is the one corresponding with the smallest $\lambda_i(\mu)$ as $\mu \to +\infty$ and therefore, with the smallest $|t_i|$ (which is t_1 for $x > 0$ and t_2 for $x < 0$).

Next, we prove that $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$ is minimal as $\mu \to +\infty$ when $0 < x < 1$ and dominant when $-1 < x < 0$, because this solution corresponds to $\lambda_1(\mu)$.

Indeed, from (2.1) we have

$$\frac{y_{\mu+1}}{y_\mu} \sim \frac{1}{\mu}\sqrt{\frac{1-x}{1+x}}\,\frac{{}_2F_1\left(\begin{matrix}-\nu,\ \nu+1\\ \mu+2\end{matrix};\ \tfrac{1}{2}-\tfrac{1}{2}x\right)}{{}_2F_1\left(\begin{matrix}-\nu,\ \nu+1\\ \mu+1\end{matrix};\ \tfrac{1}{2}-\tfrac{1}{2}x\right)}. \tag{3.11}$$

But, taking into account the asymptotic behavior of the Gauss functions as $\mu \to +\infty$ (this corresponds to the behavior for the so-called $(0\,0\,+)$ recurrence [6]), namely,

$$\lim_{\mu\to+\infty} \frac{{}_2F_1\left(\begin{matrix}a,\ b\\ c+\mu+1\end{matrix};\ z\right)}{{}_2F_1\left(\begin{matrix}a,\ b\\ c+\mu\end{matrix};\ z\right)} = 1\,,\ z \notin (1,+\infty), \tag{3.12}$$

we have, when $x \in (-1,1)$, that

$$\frac{y_{\mu+1}(x)}{y_\mu(x)} \sim \frac{1}{\mu}\sqrt{\frac{1-x}{1+x}} \sim \lambda_1(n). \tag{3.13}$$

This shows that $y_\mu(x)$ is minimal as $\mu \to +\infty$ in $(0,1)$ and dominant in $(-1,0)$. Therefore, backward recursion for the computation of $y_\mu(x)$ from large positive values of μ is well conditioned for $x \in (0,1)$, but only forward recursion (increasing μ) can be used for $x \in (-1,0)$. On the other hand, it is easy to check that the recurrence does not have minimal or dominant solutions when $x = 0$.

In terms of the recursion over μ for $P^{\mu}_{-1/2+i\tau}(x)$, this analysis reveals that, for $x > 0$, the computation of the conical function $P^{\mu}_{-1/2+i\tau}(x)$ for negative μ in the direction of increasing $|\mu|$ is bad conditioned; contrary, the direction of decreasing $|\mu|$ is well conditioned. For $x \in (-1,0)$, the situation is the contrary. For $x = 0$, no minimal solutions exist, and any direction is possible.

3.2. Conditioning for $P^{\mu}_{-1/2+i\tau}(x)$ as $\mu \to -\infty, x \in (1,+\infty)$. For $x > 1$, the situation is very similar to the case $0 < x < 1$.

Proceeding as before, now from (3.6), we see that $y_\mu(x) = P^{-\mu}_{-1/2+i\tau}(x)$ satisfies a three-term recurrence (3.7) with coefficients (3.8), but now with

$$b = -2x/\sqrt{x^2-1}, \quad a = 1. \tag{3.14}$$

The application of Perron's theorem shows that the recurrence for $y_\mu(x)$ admits a minimal solution as $\mu \to +\infty$. In this case, we can write the characteristic roots as in (3.9) with

$$t_1 = \sqrt{\frac{x-1}{x+1}}, \quad t_2 = \sqrt{\frac{x+1}{x-1}}. \tag{3.15}$$

The minimal solution corresponds to t_1, and, using again (3.12), we conclude that $y_\mu(x)$ is minimal when $x > 1$.

Therefore, the conditioning of the recursion for $P^{\mu}_{-1/2+i\tau}$ for $x > 1$ as $\mu \to -\infty$ is the same as for the recursion when $x \in (0,1)$.

3.3. Conditioning for $P^{\mu}_{-1/2+i\tau}(x)$ as $\mu \to +\infty, x \in (-1,+\infty)$. The conditioning of recursion for conical functions $P^{\mu}_{-1/2+i\tau}(x)$ as $\mu \to +\infty$ can be obtained from (2.1), and the behavior of the Gauss hypergeometric function

$$w_n(z) = {}_2F_1\left(\begin{matrix} a,\ b \\ c+n \end{matrix}; z\right) \tag{3.16}$$

as $n \to -\infty$. In [6], it was shown that the recursion process as $n \to -\infty$ for Gauss functions (3.16) is well conditioned because this set is dominant in this direction of recursion. Therefore, we can also expect that $P^{\mu}_{-1/2+i\tau}(x)$ is dominant as $\mu \to +\infty$, at least when (2.1) is well defined ($\mu \notin Z^+$).

Indeed, using the asymptotic behavior inferred from the dominance of the function $w_n(z)$ as $n \to -\infty$ [6],

$$\begin{aligned} &\lim_{n\to-\infty} \frac{w_{n+1}(z)}{w_n(z)} = 1 \qquad &&\text{if } \Re z < 1/2, \\ &\lim_{n\to-\infty} \frac{w_{n+1}(z)}{w_n(z)} = 1 - \frac{1}{z} \qquad &&\text{if } \Re z > 1/2 \end{aligned} \tag{3.17}$$

and considering (2.1), it is a straightforward matter to verify that, as $\mu \to +\infty$,

$$\begin{aligned} &\frac{P^{\mu+1}_{-1/2+i\tau}(x)}{P^{\mu}_{-1/2+i\tau}(x)} \sim -\mu\sqrt{\frac{|1+x|}{|1-x|}} \qquad &&\text{if } x > 0, \\ &\frac{P^{\mu+1}_{-1/2+i\tau}(x)}{P^{\mu}_{-1/2+i\tau}(x)} \sim \mu\sqrt{\frac{|1-x|}{|1+x|}} \qquad &&\text{if } x < 0 \end{aligned} \tag{3.18}$$

and that this is the behavior of the largest root $\lambda_2(\mu)$ (Theorem 1) of the recurrences (3.5) and (3.6). Therefore, $P^{\mu}_{-1/2+i\tau}$ is dominant as $\mu \to +\infty$, at least for $\mu \notin \mathbb{Z}^+$, when (2.1) is well defined.

For $\mu = m \in \mathbb{N}$, using (2.5) and the results of the previous analysis, we have

$$\frac{P^{m+1}_{-1/2+i\tau}(x)}{P^{m}_{-1/2+i\tau}(x)} \sim m^2 \frac{P^{-m-1}_{-1/2+i\tau}(x)}{P^{m}_{-1/2+i\tau}(x)} \sim m\sqrt{\frac{|x-1|}{|x+1|}}, \tag{3.19}$$

which shows that $P^{m}_{-1/2+i\tau}(x)$, $m \in \mathbb{Z}$ is minimal as $m \to +\infty$ when $x > 0$ and dominant when $x < 0$.

In summary, $P^{\mu}_{-1/2+i\tau}(x)$ is dominant as $\mu \to +\infty$ both when $x > 0$ and $x < 0$, except when $\mu = m \in \mathbb{Z}^+$ and $x > 0$, in which case it is minimal. For $x = 0$, no minimal and dominant solutions exist.

3.4. Continued fraction. In the case when $P^{\mu}_{-1/2+i\tau}(x)$ is minimal, one needs two starting values with large $|\mu|$ for beginning the computation with the recurrence relation. Both values could be computed by means of the asymptotic expansions to be presented later, but in some cases, particularly near $x = 1$, it may be more efficient to compute the second value using the continued fraction associated to the three-term recurrence relation, which according to Pincherle's theorem [7, Chapter 4], is convergent.

As shown previously, if $x > 0$, then $P^{\mu}_{-1/2+i\tau}(x)$ is minimal as $\mu \to -\infty$ and as $\mu = m \to +\infty$ with integer values of m. Let us consider the latter possibility for explaining the use of the associated continued fraction.

We take the ratio $H_m = P^{m}_{-1/2+i\tau}(x)/P^{m-1}_{-1/2+i\tau}(x)$, which satisfies (see recurrence relation (3.5))

$$H_m = \frac{(m-1/2)^2+\tau^2}{\dfrac{2mx}{\sqrt{1-x^2}} + H_{m+1}}, \tag{3.20}$$

and the continued fraction follows from further iterations of this ratio, we see that we can compute $P^{m-1}_{-1/2+i\tau}(x)$ once $P^{m}_{-1/2+i\tau}(x)$ is known and H_m is computed numerically (see [7, Chapter 6] for methods of computation).

The error in the computation of the continued fraction when N approximants are considered is approximately given by (see [7, Chapter 4])

$$\epsilon_k \simeq \left| \frac{g_M/f_M}{g_{M+k}/f_{M+k}} \right| , \tag{3.21}$$

where g_l and f_l are a dominant and the minimal solution of the three-term recurrence relation, respectively.

Using Perron's estimates for the ratios of functions, a rough estimate for the error is given by

$$\epsilon_N \sim \left(\frac{1-x}{1+x} \right)^N . \tag{3.22}$$

From (3.22), the number of approximants N which are needed in the computation of the continued fraction in order to obtain an accuracy ϵ can be estimated as

$$N \sim \log(\epsilon_N)/\log \left| \frac{1-x}{1+x} \right| . \tag{3.23}$$

A test for this expression is shown in Figure 1.

Similarly, as before, when $x > 1$, an analogous representation for H_m is obtained from (3.6), and the number of approximants N which are needed in the computation of the continued fraction in order to obtain an accuracy ϵ is again given by (3.23). Figure 2 shows a comparison of this estimation with the actual values of N needed.

Notice that the continued fractions (both for $x > 1$ and $0 < x < 1$) converge very fast near $x = 1$ and that therefore, around this value the continued fraction is an interesting method of computation.

As can be seen in Figure 2, for large τ, the error estimations do not work so well, not surprisingly because we are assuming that m is large with respect to the rest of the parameters. Better estimations could be obtained from the uniform asymptotic expansions to be introduced in the next sections. In any case, the main conclusion is the same: that around $x = 1$, this is an effective method and that it can be used for computing one of the two starting values for starting recursions.

4. Uniform asymptotic expansion for $x \in [-1, 1]$. In this section, we describe a powerful asymptotic expansion for $P^{-\mu}_{-1/2+i\tau}(x)$ valid for large positive values

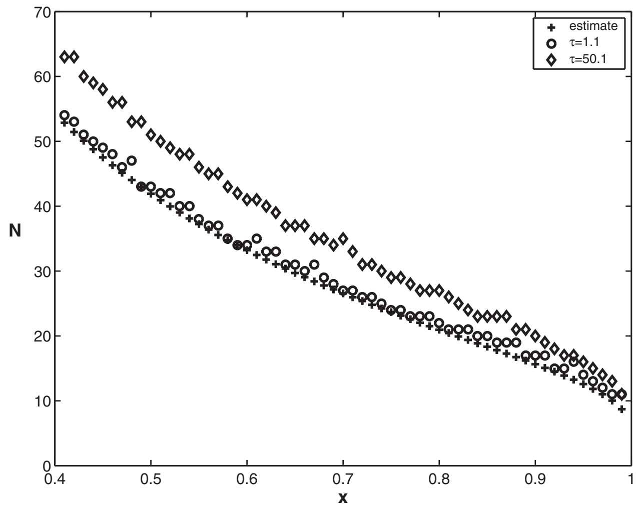

FIG. 1. *Test for* (3.23). *Comparison between the estimate for the number of approximants (crosses) of the continued fraction representation for* (3.20) *and the actual number of approximants (circles and diamonds), as a function of x (starting from $x = 0.4$) and for $m = 50$ and $\tau = 1.1, 50.1$ (circles and diamonds, respectively). In this figure, $\epsilon_N = 10^{-20}$.*

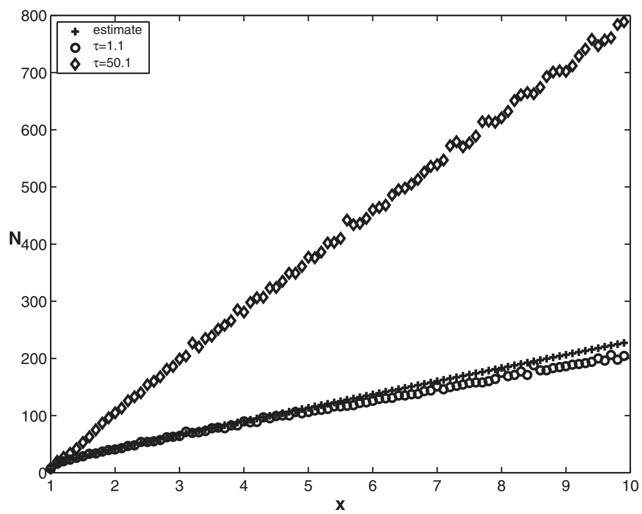

FIG. 2. *Test for* (3.23) *for $x > 1$. Comparison between the estimate for the number of approximants (crosses) of the continued fraction representation for* (3.20) *and the actual number of approximants (circles and diamonds), as a function of x and for $m = 50$ and $\tau = 1.1$ (circles) and $\tau = 50.1$ (diamonds). In this figure, $\epsilon_n = 10^{-20}$.*

of μ, which is uniformly valid with respect to $\tau \geq 0$ and $x \in [-1, 1]$. In section 8, Appendix A, we give further details of this expansion, which reads

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(x) \sim \sqrt{\frac{p}{x\mu}}\, \frac{\Gamma\left(\frac{1}{2}+\mu\right)\, \left(1-x^2\right)^{\mu/2} e^{-\mu\phi(t_0)}}{\Gamma\left(\mu+\frac{1}{2}-i\tau\right)\Gamma\left(\mu+\frac{1}{2}+i\tau\right)} \sum_{k=0}^{\infty} \frac{u_k(\beta, p)}{\mu^k}. \tag{4.1}$$

The quantities β, p, and $\phi(t_0)$ are given by

$$\beta = \frac{\tau}{\mu}, \quad p = \frac{x}{\sqrt{1+\beta^2(1-x^2)}}, \tag{4.2}$$

and

$$\phi(t_0) = \ln \frac{x(p+1)}{p(\beta^2+1)} + \beta \arccos \frac{x(1-p\beta^2)}{p(1+\beta^2)}. \tag{4.3}$$

The first few coefficients of the expansion in (4.1) are

$$\begin{aligned} &u_0(\beta,p) = 1, \quad u_1(\beta,p) = -\frac{-\beta^2+5\beta^2p^3-3\beta^2p+3p}{24(\beta^2+1)}, \\ &u_2(\beta,p) = \frac{1}{1152(\beta^2+1)^2}\big[385\beta^4p^6 + 462\beta^2(1-\beta^2)p^4 - 10\beta^4p^3 \\ &\quad + (81\beta^4 - 522\beta^2+81)p^2 + 6\beta^2(\beta^2-1)p + \beta^4 + 72\beta^2 - 72\big]. \end{aligned} \tag{4.4}$$

As it is given, the asymptotic relation in (4.1) has no meaning at the points $x = \pm 1$ because of the singularities of the conical function (and terms at the right-hand side of (4.1)) for these values of x. By using suitable scaling, we can give representations that also hold at the endpoints (see section 8.1 for details).

For $x \sim -1$, the quantity $\phi(t_0)$ becomes singular and should be combined with powers of $(1+x)$. This gives for $-1 \le x \le 0$ the expansion

$$\begin{aligned} \left(\frac{1+x}{1-x}\right)^{\mu/2} P^{-\mu}_{-\frac12+i\tau}(x) \quad \sim \quad & \sqrt{\frac{p}{x\mu}}\, \frac{\Gamma\left(\frac12+\mu\right) e^{-\tau \arccos \frac{x\left(1-p\beta^2\right)}{p(1+\beta^2)}}}{\Gamma\left(\mu+\frac12-i\tau\right)\Gamma\left(\mu+\frac12+i\tau\right)} \\ & \times \left(\frac{x(1-p)}{p(1-x)}\right)^{\mu} \sum_{k=0}^{\infty} \frac{u_k(\beta,p)}{\mu^k}. \end{aligned} \tag{4.5}$$

For $x \sim 1$, the quantity $\phi(t_0)$ remains regular, and we can write the following for $0 \le x \le 1$:

$$\begin{aligned} \left(\frac{1+x}{1-x}\right)^{\mu/2} P^{-\mu}_{-\frac12+i\tau}(x) \quad \sim \quad & \sqrt{\frac{p}{x\mu}}\, \frac{\Gamma\left(\frac12+\mu\right)(1+x)^{\mu} e^{-\mu\phi(t_0)}}{\Gamma\left(\mu+\frac12-i\tau\right)\Gamma\left(\mu+\frac12+i\tau\right)} \\ & \times \sum_{k=0}^{\infty} \frac{u_k(\beta,p)}{\mu^k}. \end{aligned} \tag{4.6}$$

When μ is an integer ($\mu = m$), the product of the complex gamma functions in (4.1), (4.5), and (4.6) can be written as

$$\Gamma\left(m+\tfrac12-i\tau\right)\Gamma\left(m+\tfrac12+i\tau\right) = \frac{\pi}{\cosh(\pi\tau)} \prod_{n=1}^{m}\left(\left(m-n+\tfrac12\right)^2+\tau^2\right). \tag{4.7}$$

4.1. Numerical tests of the expansion for $x \in [-1,1]$. In order to test the range of validity of the expansion in (4.1), we first compare it with the values obtained from (2.1) by using Maple, computing the function with as many digits as

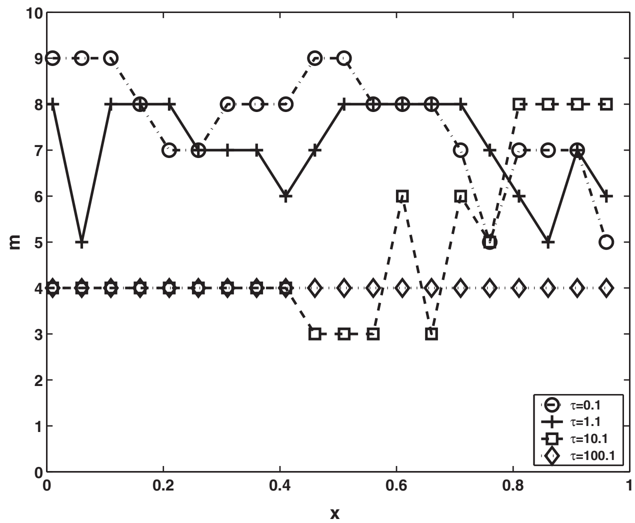

FIG. 3. *Minimum values of m for which the use of* (4.1) *allows one to get a precision better than 10^{-8} in the computation of $P^{-m}_{-1/2+i\tau}(x)$.*

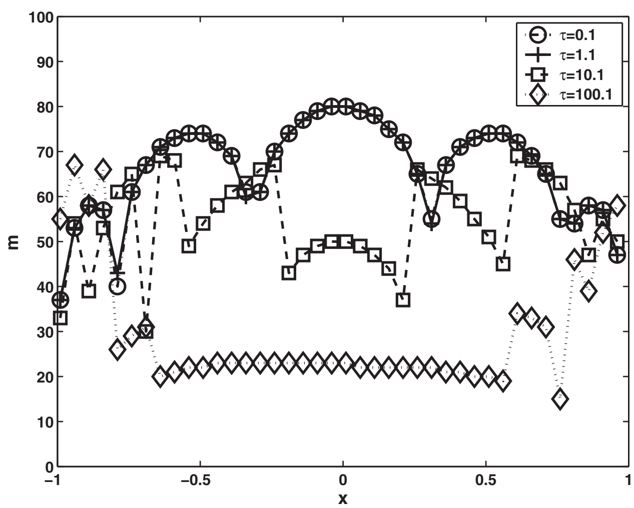

FIG. 4. *Minimum values of m for which the use of* (4.1) *allows one to get a precision better than 10^{-16} in the computation of $P^{-m}_{-1/2+i\tau}(x)$.*

needed (more digits as τ becomes larger). We consider the computation of $P^{-m}_{-1/2+i\tau}(x)$ for $m \in \mathbb{N}$, but a similar analysis can be carried out for $P^{-\mu}_{-1/2+i\tau}(x)$, with $\mu \in \mathbb{R}^+$.

Figure 3 shows, as a function of x, the minimum value of m for which the use of (4.1) allows one to get a relative accuracy of 10^{-8} (simple precision) in the computation of $P^{-m}_{-1/2+i\tau}(x)$; for larger m, the accuracy is higher. Figure 4 shows the corresponding results for double precision (10^{-16}). We have used expansion (4.1) with terms $0 \le k \le 7$ in these and the following figures.

A second test, independent of Maple computations, has been considered. It consists of testing that the values computed from asymptotics are consistent with three-

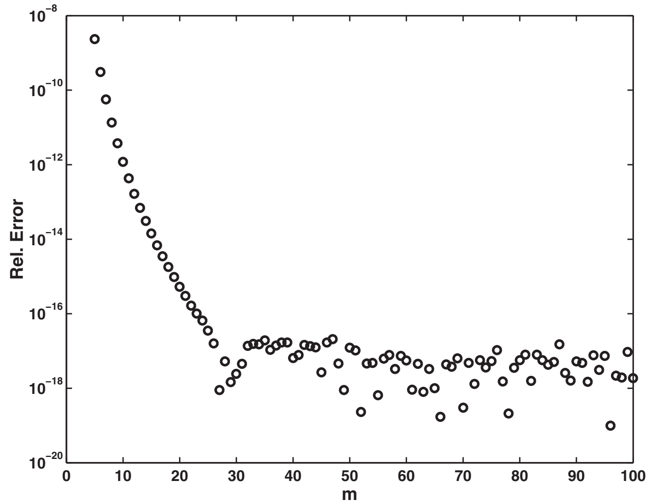

FIG. 5. *Relative errors in the computation of* $P^{-m+1}_{-1/2+i\tau}(x)$ *obtained by comparing* (4.1) *and* (3.5). *In this figure,* $\tau = 100.1$ *and* $x = 0.4$.

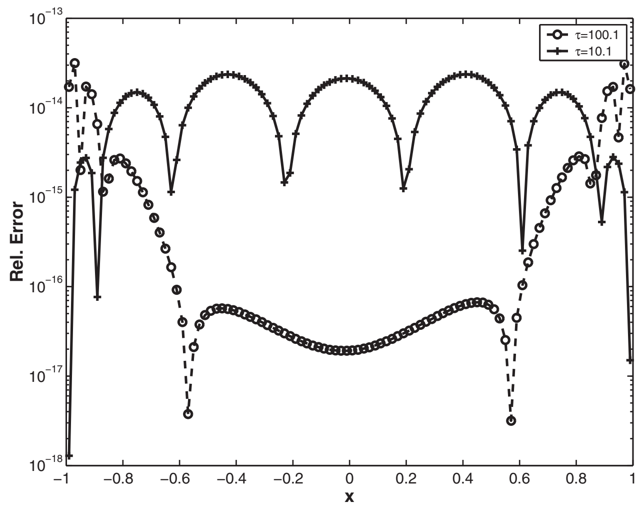

FIG. 6. *Relative errors in the computation of* $P^{-m+1}_{-1/2+i\tau}(x)$ *obtained by comparing* (4.5), (4.6) *against Maple. In this figure,* $\tau = 10.1, 100.1$ *and* $m = 40$.

term recurrence relation (3.5). For this, when backward recursion is stable ($x > 0$), we compute $P^{-m-1}_{-1/2+i\tau}(x)$ and $P^{-m}_{-1/2+i\tau}(x)$ from asymptotics and obtain $P^{-m+1}_{-1/2+i\tau}(x)$ by applying (3.5). This value is tested against the direct computation of $P^{-m+1}_{-1/2+i\tau}(x)$ from asymptotic expansion (4.1). For $x < 0$, we proceed similarly but in the opposite direction of recursion ($P^{-m-1}_{-1/2+i\tau}(x)$ is obtained from $P^{-m}_{-1/2+i\tau}(x)$ and $P^{-m+1}_{-1/2+i\tau}(x)$). Figure 5 illustrates this test as a function of m.

The variation with respect to x of the accuracy of the asymptotic expansion is illustrated in Figure 6. The figure shows the relative errors obtained by comparing the scaled forms of expansion (4.5), (4.6) against Maple. The values of the parameters are $m = 40$ and $\tau = 10.1, 100.1$.

5. Quadrature methods for $x \in [-1, 1]$. Next, we present a numerical quadrature method which can be used, for example, for computing starting values of the recursion when $x \in (-1, 0]$; see section 3.

As explained extensively in [7, Chapter 5], knowing the details of saddle-point contours of integrals defining special functions, we can use this information for constructing efficient quadrature methods for evaluating these integrals. We derive an integral that can be used for all nonnegative values of the parameters μ and τ (large or small) and for all $x \in [-1, 1]$. The basic idea consists in integrating through the saddle point in the direction of the steepest descent direction; in this way, the oscillations (stronger as τ is larger) become damped also when τ is large; this gives a fairly uniform accuracy in the numerical computation of the resulting integral representation by means of the trapezoidal rule.

We use the integral in (8.2) that is used for obtaining the asymptotic expansion of the previous section. This integral has a saddle point at t_0 given in (8.8), and we integrate along the horizontal line through t_0. That is, we write $t = s + is_0$, where $is_0 = t_0$ and denote the integral in (8.3) with I. We obtain

$$I = e^{-\mu\phi(t_0)} \int_{-\infty}^{\infty} e^{-\mu\psi(s)} \frac{ds}{\sqrt{x + \cosh(s + is_0)}}, \tag{5.1}$$

where $\phi(t_0)$ is given in (4.3) and

$$\psi(s) = \ln\left(1 + \frac{2\cos(s_0)\sinh^2(s/2)}{x + \cos(s_0)} + i\frac{\sin(s_0)\sinh(s)}{x + \cos(s_0)}\right) - i\beta s, \tag{5.2}$$

where

$$\cos(s_0) = \frac{x\left(1 - p\beta^2\right)}{p\left(1 + \beta^2\right)}, \quad \sin(s_0) = \frac{\beta x(p+1)}{p\left(1 + \beta^2\right)}, \tag{5.3}$$

and β and p are as in (4.2).

We write $\psi(s)$ in the form $\psi(s) = \psi_r(s) + i\psi_i(s)$, where

$$\begin{aligned} \psi_r(s) &= \tfrac{1}{2}\ln\left(1 + \frac{4(1+\beta^2)}{1+p}\sigma^2 + \frac{4(1+\beta^2)\left(1+p^2\beta^2\right)}{(1+p)^2}\sigma^4\right), \\ \psi_i(s) &= \arctan\frac{\beta(1+p)\sinh s}{1 + p + (1 - p\beta^2)\sinh^2(s/2)} - \beta s, \end{aligned} \tag{5.4}$$

where $\sigma = \sinh(\frac{1}{2}s)$. We have, as $s \to 0$,

$$\begin{aligned} \psi_r(s) &= \frac{1+\beta^2}{2(1+p)}s^2 + \frac{(1+\beta^2)(p - 2 + 3\beta^2\left(p^2 - 2\right)}{24(1+p)^2}s^4 + \mathcal{O}\left(s^6\right), \\ \psi_i(s) &= \frac{\beta(1+\beta^2)(p-2)}{6(1+p)}s^3 + \mathcal{O}\left(s^5\right). \end{aligned} \tag{5.5}$$

After these preparations, we obtain the desired integral representation

$$I = 2e^{-\mu\phi(t_0)}\sqrt{\frac{p(1+\beta^2)}{x(p+1)}} \int_0^{\infty} e^{-\left(\mu+\frac{1}{2}\right)\psi_r(s)} \cos\left(\left(\mu + \tfrac{1}{2}\right)\psi_i(s)\right) ds. \tag{5.6}$$

An efficient quadrature method can be based on the trapezoidal rule. For details, we refer to [7, Chapter 5].

For $x \sim -1$, we have $p \sim -1$, and the representation should be modified. As explained in section 4 (see (4.5)), the quantity $\phi(t_0)$ also becomes singular, but a suitable scaling of the conical function may take care of the factor $\exp(-\mu\phi(t_0))$. The factor $\sqrt{p+1}$ in (5.6) can be handled, for example, by introducing the new variable of integration $w = \sinh(s/2)/\sqrt{1+p}$. Then

$$\frac{ds}{dw} = 2\sqrt{\frac{1+p}{1+(1+p)w^2}}, \tag{5.7}$$

and in the new integral, we can use $x \sim -1$ without any problem. The new integral is not fast convergent, but acceleration methods can be used as described in [7, section 5.4.2].

6. Uniform asymptotic expansion for $x \geq 1$. To distinguish between the previous cases with $x \in [-1, 1]$, we now write $z = x$, $z \geq 1$. This is the area where zeros occur, and, for describing the transition between the monotonic behavior and oscillatory behavior, an expansion in terms of elementary functions is no longer adequate. We have used a representation in terms of the modified Bessel function $K_{i\tau}(\mu\zeta)$, which, for μ positive, reads (for details, we refer to section 9, Appendix B)

$$\begin{aligned} P^{-\mu}_{-\frac{1}{2}+i\tau}(z) = {} & \frac{2\Gamma(\frac{1}{2}+\mu)\,\left(z^2-1\right)^{\mu/2}\,e^{-\mu\lambda}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)}\;\Phi(\zeta) \\ & \times \left[A_\mu(\beta,\zeta)K_{i\tau}(\mu\zeta) - B_\mu(\beta,\zeta)K'_{i\tau}(\mu\zeta)\right], \end{aligned} \tag{6.1}$$

where

$$\lambda = \frac{1}{2}\left(\ln\frac{z^2-1}{\beta^2+1} + \beta\arccos\frac{1-\beta^2}{1+\beta^2}\right), \tag{6.2}$$

$$\beta = \frac{\tau}{\mu}, \quad \Phi(\zeta) = \left(\frac{\zeta^2-\beta^2}{1+\beta^2\,(1-z^2)}\right)^{\frac{1}{4}}, \tag{6.3}$$

and the functions $A_\mu(\beta,\zeta)$ and $B_\mu(\beta,\zeta)$ have the expansions

$$A_\mu(\beta,\zeta) \sim \sum_{n=0}^{\infty}\frac{A_n(\beta,\zeta)}{\mu^n}, \quad B_\mu(\beta,\zeta) \sim \sum_{n=0}^{\infty}\frac{B_n(\beta,\zeta)}{\mu^n}. \tag{6.4}$$

For describing the parameter ζ and the coefficients of the expansions, we use two different z-intervals. Let

$$z_c = \frac{\sqrt{1+\beta^2}}{\beta}. \tag{6.5}$$

6.1. The monotonic case: $1 \leq z \leq z_c$. In this case, the quantity $\zeta \geq \beta$ is given by the implicit equation

$$2\left[\sqrt{\zeta^2-\beta^2} - \beta\arccos(\beta/\zeta)\right] = \ln\frac{p+1}{p-1} - \beta\arccos\frac{\beta^2p^2-1}{\beta^2p^2+1}, \tag{6.6}$$

where p is given by

$$p = \frac{z}{\sqrt{1+\beta^2(1-z^2)}}. \tag{6.7}$$

For the numerical inversion of this equation, that is, computing ζ when z is given, we refer to section 6.1.1 below.

The first few coefficients $A_n(\beta,\zeta), B_n(\beta,\zeta)$ in (6.4) are

$$A_0(\beta,\zeta)=1,\quad B_0(\beta,\zeta)=0,\quad A_1(\beta,\zeta)=\frac{\beta^2}{24(1+\beta^2)}, \tag{6.8}$$

$$B_1(\beta,\zeta)=-\frac{\left(5\beta^2\left(W^3p^3-1-\beta^2\right)+3W^2\left(Wp\left(1-\beta^2\right)-1-\beta^2\right)\right)\zeta}{24W^4(1+\beta^2)}, \tag{6.9}$$

where p is given in (6.7) and

$$W=\sqrt{\zeta^2-\beta^2}. \tag{6.10}$$

Observe that function $\Phi(\zeta)$ given in (6.3) can be written as $\Phi(\zeta)=\sqrt{pW/z}$. This function is analytic at the point $\zeta=\beta$ and can be defined across this point by changing the sign under the square roots in (6.7) and (6.10), simultaneously.

6.1.1. Numerical inversion of the implicit equation (6.6). The left-hand side in (6.6) vanishes at $\zeta=\beta$ and is a strictly increasing function of $\zeta\in[\beta,\infty)$; the derivative with respect to ζ is $2\sqrt{1-\beta^2/\zeta^2}$. The right-hand side becomes $+\infty$ as $z\downarrow 1$ and vanishes at $z=z_c$; it is a strictly decreasing function of $z\in[1,z_c]$, the derivative with respect to z being $-2\sqrt{1+\beta^2(1-z^2)}/(z^2-1)$. It follows that, from (6.6), a unique value ζ can be obtained when values of z and β are given. Conversely, we can obtain a unique value z when values of ζ and β are given.

The derivatives are similar to those used in the Liouville transformation in [3, equation (4.5)], after using in our analysis $\zeta=\sqrt{\eta}$ and $z^2=1+1/\xi$.

An explicit analytical inversion of (6.6) is not possible, but one can find initial approximations which guarantee convergence of Newton's method. We rewrite the equation as

$$\sqrt{\gamma^2-1}-\arccos(1/\gamma)=\Omega, \tag{6.11}$$

where $\gamma=\zeta/\beta$, $\Omega=f(p,\beta)/(2\beta)$, and $f(p,\beta)$ is the right-hand side of (6.6). For large γ, we expand the left-hand side and get, after inverting the asymptotic series (see [7, Chapter 7]), the approximation

$$\gamma\sim\Omega+\frac{\pi}{2}-\frac{1}{2\Omega}+\frac{\pi}{4\Omega^2}+\mathcal{O}\left(\Omega^{-3}\right). \tag{6.12}$$

This approximation is, of course, better as Ω is large but can be used as a starting value for $\Omega\ge 1$.

For smaller values of Ω, which leads to smaller values of γ, we can expand the left-hand side of (6.11) in powers of $\gamma-1$. With the leading term of the expansion, we get

$$\gamma\sim 1+\left(\frac{3\Omega}{2\sqrt{2}}\right)^{2/3}. \tag{6.13}$$

This approximation gives a convenient starting value for Newton's method when $0\le\Omega\le 1$.

6.2. The oscillatory case: $z \geq z_c$. In this case, the quantity $\zeta \in [0, \beta]$ is given by the implicit equation

$$2\left[\sqrt{\beta^2 - \zeta^2} - \beta \operatorname{arccosh}(\beta/\zeta)\right] = 2 \operatorname{arccot} q - \beta \ln \frac{\beta q + 1}{\beta q - 1}, \tag{6.14}$$

where q is given by

$$q = \frac{z}{\sqrt{\beta^2(z^2 - 1) - 1}}. \tag{6.15}$$

The coefficients $A_0(\beta,\zeta), B_0(\beta,\zeta), A_1(\beta,\zeta)$ are as in (6.8), whereas the coefficient $B_1(\beta,\zeta)$ is given by

$$B_1(\beta,\zeta) = -\frac{\left(5\beta^2\left(V^3 q^3 - 1 - \beta^2\right) + 3V^2\left(Vq\left(1-\beta^2\right) - 1 - \beta^2\right)\right)\zeta}{24V^4(1+\beta^2)}, \tag{6.16}$$

where $V = \sqrt{\beta^2 - \zeta^2}$. In fact, this $B_1(\beta,\zeta)$ is the same as the one in (6.9), with proper interpretation of the square roots in W, p and V, q. See the remark at the end of section 6.1. The function $\Phi(\zeta)$ given in (6.3) can also be written as $\Phi(\zeta) = \sqrt{qV/z}$. More details on the relation between p and W for $z \sim z_c$ (that is, for small values of W) will be given in section 9.5.

6.2.1. Numerical inversion of the implicit equation (6.14). The derivative with respect to ζ of the left-hand side of (6.14) is $2\sqrt{\beta^2/\zeta^2 - 1}$ and that of the right-hand side equals $-2\sqrt{\beta^2(z^2-1)-1}/(z^2-1)$. It follows that (6.14) defines a unique value ζ when values of z and β are given and conversely, we can obtain a unique value z when values of ζ and β are given.

Again, (6.14) cannot be explicitly inverted, but accurate enough values for the Newton method can be determined. Writing the equation as

$$\sqrt{\gamma^2 - 1} - \operatorname{arccosh}(1/\gamma) = \Lambda, \tag{6.17}$$

where $\gamma = \zeta/\beta$, $\Lambda = \frac{g(p,\beta)}{2\beta}$, and $g(p,\beta)$ is the right-hand side of (6.14) we get, by expanding the left-hand side of the equation in powers of $\gamma - 1$, that for values of γ close to 1, the following approximation can be used:

$$\gamma \simeq 1 - \left(\frac{-3\Lambda}{2\sqrt{2}}\right)^{2/3}. \tag{6.18}$$

The approximation is sufficient for $-0.5 \leq \Lambda \leq 0$, while for $\Lambda < -0.5$ (and smaller γ), it is better to expand the left-hand side and invert the expansion. The following approximation can then be obtained after two resubstitutions:

$$\gamma \sim 2e^{\Lambda - 1}. \tag{6.19}$$

6.3. Numerical test of the expansions for $x > 1$. Figure 7 shows, as a function of x and for two values of τ, the relative error in the computation of $P^{-m}_{-1/2+i\tau}(x)$ by using (6.1) and Maple. We have used expansion (6.4) with terms $0 \leq k \leq 3$. The oscillations in the relative error for $\tau = 10.1$ can be explained by the loss of relative accuracy near the zeros of the function. Indeed, conical functions oscillate for $x > 1$ (and have an infinite number of zeros). For the case $\tau = 1.1$, we don't observe this type of accuracy loss because the smallest zero is approximately $x_0 \simeq 1129.1432$.

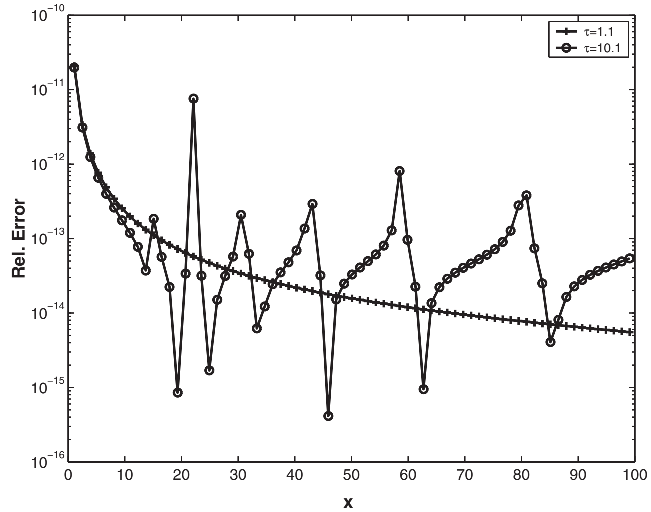

FIG. 7. *Relative error in the computation of* $P^{-m+1}_{-1/2+i\tau}(x)$ *for* $m = 100$ *and* $\tau = 1.1, 10.1$.

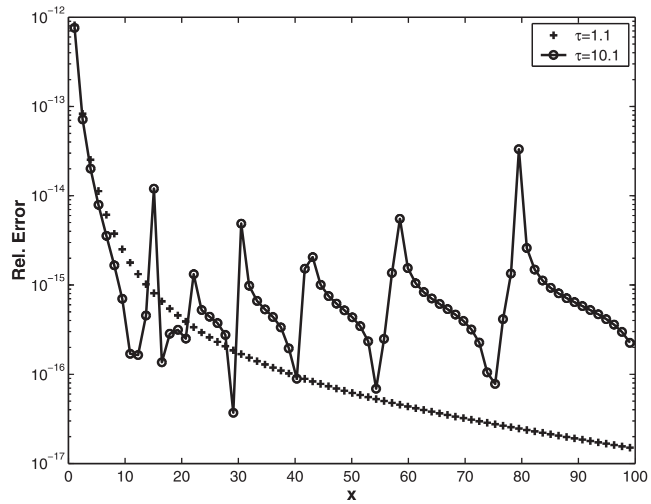

FIG. 8. *Relative error in the computation of* $P^{-m+1}_{-1/2+i\tau}(x)$ *by using* (6.1) *and* (3.5) *for* $m = 100$ *and* $\tau = 1.1, 10.1$.

As in the case $x \in [-1, 1]$, the asymptotic expansion has been also tested by checking three-term recurrence relation (3.5). Figure 8 shows, as a function of x and for the same two values of τ used in Figure 7, the relative error in the computation of $P^{-m+1}_{-1/2+i\tau}(x)$ by using (6.1) and (3.5). The results are consistent with those of Figure 7.

Finally, Figure 9 shows, as a function of x, the minimum values of m for which the use of (6.1) allows one to get single precision (10^{-8}) in the computation of $P^{-m}_{-1/2+i\tau}(x)$ for $\tau = 1.1, 10.1$.

7. Computational scheme. In section 3, it was shown that $P^{\mu}_{-1/2+i\tau}(x)$ is minimal as $\mu \to -\infty$ when $x > 0$ and dominant when $x < 0$; at $x = 0$, it is neither

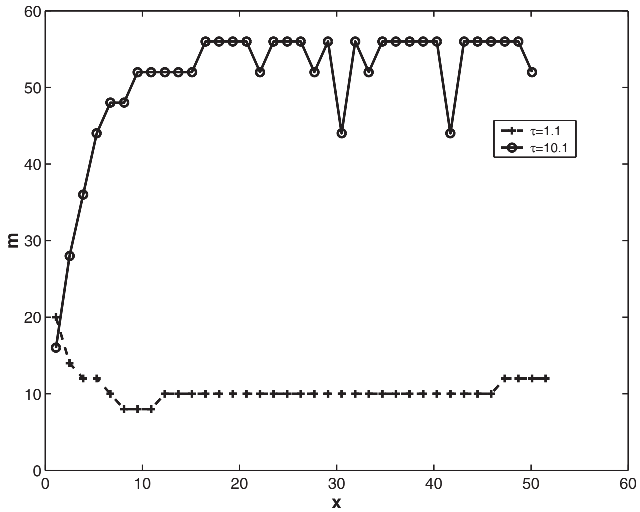

FIG. 9. *Minimum values of m for which the use of* (6.1) *allows one to get a precision better than 10^{-8} in the computation of $P^{-m}_{-1/2+i\tau}(x)$ for $\tau = 1.1, 10.1$.*

minimal nor dominant and at $x = 1$, the computation through recurrence is undefined (and unnecessary).

Therefore, for computing $P^{\mu}_{-1/2+i\tau}(x)$, $x > 0$, $\mu \leq 0$, recursion starting from large $|\mu|$, $\mu < 0$, and decreasing $|\mu|$ is possible. And for $P^{m}_{-1/2+i\tau}(x)$, $m \in \mathbb{N}$, relation (2.5) can be used, or alternatively, backward recursion is possible starting with large positive m. The starting values for these recurrences are provided by the uniform asymptotic expansions discussed in sections 4 and 6; the values of m which can be used for starting the recursion can be extracted from the information of the type described Figures 3, 4, and 9.

When $x \in (-1, 0)$, the recurrent scheme changes, and conical functions $P^{\mu}_{-1/2+i\tau}(x)$ should be computed in the direction of increasing $|\mu|$. Fortunately, methods for computing conical function for small μ are already available. And this, together with backward recursion and asymptotics for positive x, gives a complete scheme of stable computation.

For negative x and $\mu = m \in \mathbb{N}$, one can start from the values $P^{0}_{-1/2+i\tau}(x)$ and $P^{1}_{-1/2+i\tau}(x)$ (which are computed in [10]) and use recurrence (3.5) for computing for $m > 1$; analogously, one can start with two negative values of the order of small magnitude and use the recurrence for computing values $P^{\mu}_{-1/2+i\tau}(x)$ for $\mu < 0$ and large $|\mu|$. For computing $P^{m}_{-1/2+i\tau}(x)$ for small m, Kölbig [10] proposed using power series in terms of Gauss functions. One possibility is to use (2.1), which should be computed by other means close to $x = -1$ in order to avoid bad behavior of the Gauss series (with argument $z \simeq 1$). Kölbig suggested rational approximations for the Gauss function. Another possibility is to compute the initial values for the recurrence by means of integral representations; see section 5. Connection formula (2.3) is also useful for computations when μ is not an integer and x is close to -1.

In summary, the stable scheme of computation for positive integer or negative real orders μ is as follows:

1. If $x > 0$, compute $P^{\mu}_{-1/2+i\tau}(x)$ for large values of $|\mu|$ and use recursion in the direction of decreasing $|\mu|$.

2. If $x < 0$, compute $P^{\mu}_{-1/2+i\tau}(x)$ for small values of $|\mu|$ and use recursion in the direction of increasing $|\mu|$.

8. Appendix A: Details of the uniform asymptotic expansion for $x \in [-1, 1]$. We give the details of asymptotic expansion (4.1). Dunster [3, p. 326] has given an expansion for $x \in [0, 1]$ by using the differential equation of the conical functions. We derive a similar expansion for $P_\nu^{-\mu}(x)$, $\nu = -1/2 + i\tau$, by using an integral representation and conclude that the expansion is valid for $x \in [-1, 1]$, after suitable scaling of the conical function.

Our starting point is the integral representation

$$P_\nu^{-\mu}(x) = \frac{\sqrt{2/\pi}\,\Gamma\left(\frac{1}{2}+\mu\right)\left(1-x^2\right)^{\mu/2}}{\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_0^\infty (x+\cosh t)^{-\mu-\frac{1}{2}} \cos(\tau t)\, dt, \tag{8.1}$$

where $\nu = -1/2 + i\tau$, which is valid for $x \in (-1, 1)$ and $\mu > -\frac{1}{2}$. This representation is given in [12, p. 188]. See also [14, p. 35].

The integrand is even, and we have

$$P_\nu^{-\mu}(x) = \frac{\Gamma\left(\frac{1}{2}+\mu\right)\left(1-x^2\right)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^\infty (x+\cosh t)^{-\mu-\frac{1}{2}} e^{i\tau t}\, dt, \tag{8.2}$$

which we write in the form

$$P_\nu^{-\mu}(x) = \frac{\Gamma\left(\frac{1}{2}+\mu\right)\left(1-x^2\right)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^\infty e^{-\mu\phi(t)} \frac{dt}{\sqrt{x+\cosh t}}, \tag{8.3}$$

where

$$\phi(t) = \ln(x+\cosh t) - i\beta t, \quad \beta = \frac{\tau}{\mu}. \tag{8.4}$$

The integral has a saddle point where $\phi'(t) = 0$. That is, we have to solve the equation

$$\frac{\sinh t}{x+\cosh t} - i\beta = 0. \tag{8.5}$$

By using the exponential representation of the hyperbolic functions, we obtain, for the solution t_0, the relation

$$e^{t_0} = \frac{i\sqrt{1+\beta^2(1-x^2)} - \beta x}{\beta + i}. \tag{8.6}$$

We have taken the + sign for the square root because we need the solution that gives $t_0 \sim 0$ if $\beta \to 0$. We write this in the form

$$e^{t_0} = x\frac{1-p\beta^2 + i\beta(1+p)}{p(\beta^2+1)}, \quad p = \frac{x}{\sqrt{1+\beta^2(1-x^2)}}. \tag{8.7}$$

We have $-1 \le p \le 1$ for $-1 \le x \le 1$. The right-hand side of the first equation in (8.7) has absolute value equal to unity. Hence, t_0 is equal to the phase of that complex number, and we have

$$t_0 = i \arctan\frac{\beta(1+p)}{1-p\beta^2} = i \arccos\frac{x\left(1-p\beta^2\right)}{p(1+\beta^2)}, \tag{8.8}$$

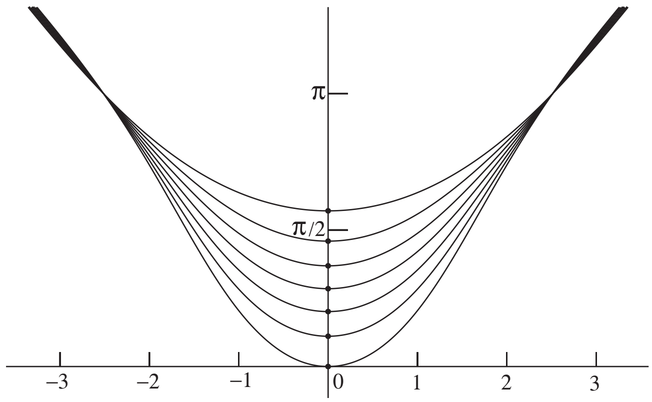

FIG. 10. *Saddle-point contours governed by* (8.10) *for* $\beta = 1.25$ *and* $x = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$. *The contour through the origin is for* $x = -1$*, the highest contour is for* $x = 1$.

where we assume (for the arctan) that $1 - p\beta^2 \ge 0$, otherwise we add π to the arctan. We prefer the notation in terms of the arccos function because the standard definition of the arctan function does not give the phase outside the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

There are more saddle points, all on the imaginary axis, but t_0 is the relevant saddle. There are also singularities on the imaginary axis, where $x + \cosh t = 0$, that is, at $t_s = i \arccos(-x)$ (and at other places, but t_s is the relevant singularity). We have

$$0 < \Im t_0 < \Im t_s \tag{8.9}$$

for all $x \in (-1, 1)$ and $\beta \ge 0$.

We can shift the path of integration in (8.3) upwards, through the saddle at t_0, and deform the path along the saddle point contour through t_0 defined by

$$\Im\phi(t) = \Im\phi(t_0), \tag{8.10}$$

where $\Im\phi(t_0) = 0$. See Figure 10.

The quantity $\phi(t_0)$ is given in (4.3); furthermore, we have

$$\phi''(t_0) = \frac{\beta^2 + 1}{p + 1}. \tag{8.11}$$

We transform

$$\phi(t) - \phi(t_0) = \tfrac{1}{2}\phi''(t_0) w^2, \tag{8.12}$$

and we assume that $\operatorname{sign} \Re(t - t_0) = \operatorname{sign}(w)$ when t is on the saddle-point contour and the corresponding w on the real axis. This gives for the integral in (8.3)

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)} \frac{dt}{\sqrt{x + \cosh t}} = e^{-\mu\phi(t_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\mu\phi''(t_0) w^2} f(w)\, dw, \tag{8.13}$$

where

$$f(w) = \frac{dt}{dw} \frac{1}{\sqrt{x + \cosh t}}, \quad \frac{dt}{dw} = \phi''(t_0) \frac{w}{\phi'(t)}. \tag{8.14}$$

We expand

$$f(w) = \sum_{k=0}^{\infty} f_k w^k \tag{8.15}$$

and substitute this in (8.13). This gives the asymptotic expansion

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)} \frac{dt}{\sqrt{x+\cosh t}} \sim e^{-\mu\phi(t_0)} \sum_{k=0}^{\infty} f_{2k} \Gamma\left(k+\tfrac{1}{2}\right) \left(\tfrac{1}{2}\mu\phi''(t_0)\right)^{-k-\frac{1}{2}}. \tag{8.16}$$

This can be written in the form

$$\int_{-\infty}^{\infty} e^{-\mu\phi(t)} \frac{dt}{\sqrt{x+\cosh t}} \sim \sqrt{\frac{2\pi p}{x\mu}} e^{-\mu\phi(t_0)} \sum_{k=0}^{\infty} \frac{u_k(\beta,p)}{\mu^k}, \tag{8.17}$$

where p/x is well defined when $x=0$ (see (8.7)) and

$$u_k(\beta,p) = \frac{2^k\, \left(\frac{1}{2}\right)_k}{\phi''(t_0)^k} \frac{f_{2k}}{f_0}, \quad k=0,1,2,\ldots. \tag{8.18}$$

Finally, by using (8.3), we obtain the expansion in (4.1) with the first coefficients given in (4.4).

8.1. Interpretation of the expansion at $x = \pm 1$. The expansion in (4.1) remains valid at the endpoints $x=\pm 1$, after properly scaling with the powers of $1-x^2$. For $x=-1$, we obtain from (2.1)

$$\lim_{x\downarrow -1} (1+x)^{\mu/2} P_\nu^{-\mu}(x) = \frac{2^{\mu/2}\,\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{2}-i\tau\right)\Gamma\left(\mu+\frac{1}{2}+i\tau\right)}, \tag{8.19}$$

where we have used

$${}_2F_1\left({a,\ b \atop c}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b)>0. \tag{8.20}$$

Taking the same limit in (4.1) and comparing the two results, we obtain

$$\frac{\sqrt{\mu}\,\Gamma(\mu)}{\Gamma\left(\mu+\frac{1}{2}\right)} \sim \sum_{k=0}^{\infty} \frac{u_k(\beta,-1)}{\mu^k}, \tag{8.21}$$

where the first few coefficients are

$$u_0(\beta,-1)=1, \quad u_1(\beta,-1)=\tfrac{1}{8}, \quad u_2(\beta,-1)=\tfrac{1}{128}. \tag{8.22}$$

These terms corresponds with the expansion first terms in the expansion of the ratio of gamma functions given in [1, equation 6.1.47].

Next, again from (2.1),

$$\lim_{x\uparrow 1} (1-x)^{-\mu/2} P_\nu^{-\mu}(x) = \frac{2^{-\mu/2}}{\Gamma(1+\mu)} \tag{8.23}$$

and comparing the results of a similar limit in (4.1), we obtain

$$B\left(\mu+\tfrac{1}{2}-i\tau, \mu+\tfrac{1}{2}+i\tau\right) \sim \sqrt{\pi/\mu}\, 2^{-2\mu} \left(1+\beta^2\right)^{\mu} e^{-2\tau\arctan\beta} \sum_{k=0}^{\infty} \frac{u_k(\beta,1)}{\mu^k}, \tag{8.24}$$

where $B(p,q)$ is the beta integral

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \quad \Re p, q > 0. \tag{8.25}$$

By using standard saddle-point methods, the expansion in (8.24) can be obtained directly from this integral representation with $p = \mu + \frac{1}{2} - i\tau$, $q = \mu + \frac{1}{2} + i\tau$.

9. Appendix B: Details of the uniform asymptotic expansion for $z \in [1,\infty)$. We give details of asymptotic expansion (6.1) that holds for large values of μ, uniformly with respect to $z \in [1,\infty)$ and $\tau \ge 0$. A similar expansion has been given by Dunster [3, p. 325]. His method is based on the differential equation for the conical functions, and we are using an integral representation. This gives a more direct procedure for obtaining the coefficients in (6.4). A similar approach has been used in [18, section 5].

We start with the same integral as in the previous section (see (8.3)):

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(z) = \frac{\Gamma\left(\frac{1}{2}+\mu\right)\left(z^2-1\right)^{\mu/2}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^{\infty} e^{-\mu\phi(t)} \frac{dt}{\sqrt{z+\cosh t}}, \tag{9.1}$$

where

$$\phi(t) = \ln(z+\cosh t) - i\beta t, \quad \beta = \frac{\tau}{\mu}, \quad \nu = -\tfrac{1}{2} + i\tau. \tag{9.2}$$

First, we observe that the expansion given in (4.1) remains valid for $1 \le z \le z_c - \delta$, where δ is a small positive number and z_c is given in (6.5). We have only to replace $(1-x^2)^{\mu/2}$ with $(z^2-1)^{\mu/2}$ and x by z, also in the quantity p defined in (8.7). The equation for the saddle points

$$\phi'(t) = \frac{\sinh t}{z+\cosh t} - i\beta = 0 \tag{9.3}$$

has two coalescing solutions when $1 + \beta^2(1-z^2) = 0$, that is, when $z = z_c$, which also become important in section 4 when we would have considered $x > 1$. The value z_c becomes large as $\beta = \tau/\mu$ becomes small. So, when $\tau \ll \mu$, we can still use expansion (4.1) for a large z-interval, but the expansion becomes invalid when z approaches z_c.

9.1. The saddle points. As remarked above, when we repeat the saddle-point analysis of section 4, we observe that (9.3) with $z \ge 1$ has two saddle points $t_\pm$ that coincide when $z = z_c$, where z_c is given in (6.5). In this section, we allow $z \sim z_c$, however, in the analysis, we consider two cases: $1 \le z \le z_c$ and $z_c \le z$.

9.1.1. The monotonic case: $1 \le z \le z_c$. We have for the two saddle points the relations

$$e^{t_+} = z\frac{1 - p\beta^2 + i\beta(1+p)}{p\,(\beta^2+1)}, \quad e^{t_-} = z\frac{1 + p\beta^2 + i\beta(1-p)}{-p\,(\beta^2+1)}, \tag{9.4}$$

where p is given in (6.7).

Again, the right-hand sides have modulus equal to unity, and we obtain

$$t_+ = i\arccos\frac{z(1-p\beta^2)}{p\,(1+\beta^2)}, \quad t_- = i\arccos\frac{-z(1+p\beta^2)}{p\,(1+\beta^2)}. \tag{9.5}$$

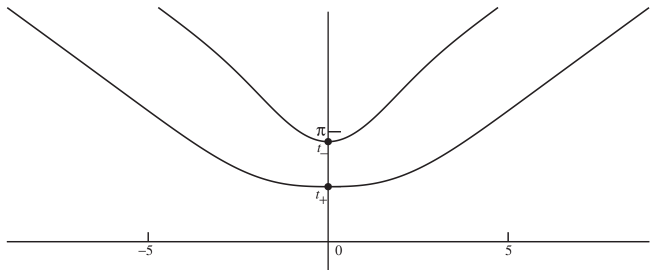

FIG. 11. *Saddle-point contours through the saddle points given in* (9.5) *for* $\beta = \frac{3}{4}$ *and* $z = \frac{4}{3}$.

For $\beta = 0$, we have $t_+ = 0$ and $t_- = i\pi$. For $z = 1$, we have $t_+ = i \arccos(1-\beta^2)/(1+\beta^2)$ and again $t_- = i\pi$. This value of t_- follows correctly from the second equation in (9.4), but it is, in fact, not a correct saddle point. Namely, for $z = 1$, (9.3) becomes $\tanh \frac{1}{2}t = i\beta$, with solution

$$t_+ = 2i \arctan \beta = i \arctan \frac{2\beta}{1-\beta^2} = i \arccos \frac{(1-\beta^2)}{(1+\beta^2)}, \tag{9.6}$$

but the solution $t_- = i\pi$ is not obtained now. In fact, the saddle point t_- vanishes as $z \downarrow 1$.

In Figure 11, we show the saddle-point contours for $\beta = \frac{3}{4}$. For this value of β, the saddle points coalesce when z equals $z_c = \frac{5}{3}$. We take $z = \frac{4}{3}$.

9.1.2. The oscillatory case: $z_c \le z$. We use the relations for the saddle point in (9.4) replacing p with $-ip = q$, where q is given in (6.15) and obtain

$$e^{t_\pm} = z\frac{(q\beta \pm 1)(i-\beta)}{q(\beta^2+1)} = z\frac{q\beta \pm 1}{q\sqrt{\beta^2+1}} e^{i(\pi - \arctan(1/\beta))}, \tag{9.7}$$

where we can replace $\pi - \arctan(1/\beta)$ with $\arccos(-\beta/\sqrt{1+\beta^2})$, and we obtain for the two saddle points

$$t_\pm = \ln \frac{z(q\beta \pm 1)}{q\sqrt{\beta^2+1}} + i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}. \tag{9.8}$$

We see that the saddle points have the same imaginary parts. We have

$$t_+ + t_- = 2i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}. \tag{9.9}$$

When $z \to \infty$, we have

$$t_\pm \sim \pm \ln \frac{2z\beta}{\sqrt{1+\beta^2}} + i \arccos \frac{-\beta}{\sqrt{1+\beta^2}}. \tag{9.10}$$

9.2. Integral representation for $K_{ia}(x)$. The modified Bessel function $K_{i\tau}(\mu\zeta)$ that is used in [3, p. 325] (with a slightly different notation) has the integral representation

$$K_{i\tau}(\mu\zeta) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-\mu\psi(w)}\, dw, \quad \psi(w) = \zeta \cosh w - i\beta w, \tag{9.11}$$

where $\beta = \tau/\mu$. We assume that $\zeta > 0$ and $\tau \geq 0$; μ is again a large positive parameter. The equation for the saddle points reads $\zeta \sinh w = i\beta$. When $\beta \leq \zeta$, we have the saddle points

$$w_+ = i \arcsin \frac{\beta}{\zeta}, \quad w_- = i\pi - i \arcsin \frac{\beta}{\zeta}. \tag{9.12}$$

When $\beta \geq \zeta$, we have

$$w_+ = \tfrac{1}{2}\pi i + \operatorname{arccosh}\frac{\beta}{\zeta}, \quad w_- = \tfrac{1}{2}\pi i - \operatorname{arccosh}\frac{\beta}{\zeta}. \tag{9.13}$$

There are more saddle points, but the given $w_\pm$ are relevant in our analysis.

We see that for $\beta/\zeta \leq 1$, saddle points $w_\pm$ lie on the imaginary axis and when $\beta/\zeta \geq 1$, they become complex and lie on the horizontal line with $\Im w_\pm = \frac{1}{2}\pi$.

Comparing the location of the saddle points $w_\pm$ for all ratios β/ζ with that of the saddle points $t_\pm$ for all ratios $z\beta/\sqrt{1+\beta^2}$, we observe that the pattern for the saddle points is quite similar in both cases. In both cases, saddle points coalesce and to obtain an asymptotic expansion that is valid for large μ when $t_+ \sim t_-$ or $w_+ \sim w_-$, Airy functions can be used. See [21] for more information on this topic from uniform asymptotic analysis for integrals and for examples. See [4, 5] for using Airy-type expansions in numerical algorithms for the modified Bessel functions of pure imaginary order, and see [7] for an extensive treatment of this type of expansion for computing several types of special functions.

In the present case, we transform integral (9.1) into an integral that is similar to that in (9.11) but with an extra function in the integrand. In this way, we cover the case of coalescing saddle points but also the large-scale pattern of the saddle-point behavior.

9.3. The transformation and the expansion. To obtain a representation of the conical P-function in terms of modified Bessel function $K_{i\tau}(\mu\zeta)$, we use the following transformation of the $t-$ variable in (9.1) to the w-variable in (9.11) by writing

$$\phi(t) = \psi(w) + \lambda, \tag{9.14}$$

where λ does not depend on t or w and should be determined, together with ζ in $\psi(w)$. This gives

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(z) = \frac{\Gamma\left(\frac{1}{2}+\mu\right)\,\left(z^2-1\right)^{\mu/2}\, e^{-\mu\lambda}}{\sqrt{2\pi}\,\Gamma(\mu-\nu)\Gamma(1+\mu+\nu)} \int_{-\infty}^{\infty} e^{-\mu\psi(w)} f(w)\, dw, \tag{9.15}$$

where

$$f(w) = \frac{dt}{dw}\,\frac{1}{\sqrt{z+\cosh t}}. \tag{9.16}$$

When we replace $f(w)$ in (9.15) with a constant, the integral becomes the modified Bessel function given in (9.11). An asymptotic expansion can be obtained by using integration by parts. We put in the first step

$$f_0(w) = A_0(\beta,\zeta) + B_0(\beta,\zeta)\cosh w + \psi'(w) g_0(w), \quad f_0(w) = f(w), \tag{9.17}$$

where $A_0(\beta,\zeta)$ and $B_0(\beta,\zeta)$ follow from substituting $w = w_+$ and $w = w_-$, respectively. That is,

$$\begin{aligned} A_0(\beta,\zeta) &= \frac{f_0(w_-)\cosh w_+ - f_0(w_+)\cosh w_-}{\cosh w_+ - \cosh w_-}, \\ B_0(\beta,\zeta) &= \frac{f_0(w_+) - f_0(w_-)}{\cosh w_+ - \cosh w_-}. \end{aligned} \tag{9.18}$$

We denote the integral in (9.15) by J and replace $f(w)$ by the right-hand side of (9.17). This gives, by (9.11) and using integration by parts,

$$J = 2A_0(\beta,\zeta)K_{i\tau}(\mu\zeta) - 2B_0(\beta,\zeta)K'_{i\tau}(\mu\zeta) + \frac{1}{\mu}\int_{-\infty}^{\infty} e^{-\mu\psi(w)} f_1(w)\,dw, \tag{9.19}$$

where $f_1(w) = g_0'(w)$. This procedure can be continued, and we can obtain representation (6.1), where the coefficients $A_n(\beta,\zeta)$ and $B_n(\beta,\zeta)$ of (6.4) follow from

$$f_n(w) = A_n(\beta,\zeta) + B_n(\beta,\zeta)\cosh w + \psi'(w)g_n(w), \quad n = 0,1,2,\ldots, \tag{9.20}$$

that is, from

$$\begin{aligned} A_n(\beta,\zeta) &= \frac{f_n(w_-)\cosh w_+ - f_n(w_+)\cosh w_-}{\cosh w_+ - \cosh w_-}, \\ B_n(\beta,\zeta) &= \frac{f_n(w_+) - f_n(w_-)}{\cosh w_+ - \cosh w_-}. \end{aligned} \tag{9.21}$$

The functions $f_n(w)$ follow from $f_{n+1}(w) = g_n'(w)$, $n \ge 0$.

9.4. Determination of $\boldsymbol{\lambda}$ and $\boldsymbol{\zeta}$. To determine λ and ζ, we prescribe, for the mapping in (9.14), that the saddle points in the t-plane should correspond with those in the w-plane. That is,

$$\phi(t_+) = \psi(w_+) + \lambda, \quad \phi(t_-) = \psi(w_-) + \lambda. \tag{9.22}$$

This gives

$$\zeta(\cosh w_+ - \cosh w_-) - i\beta(w_+ - w_-) = \phi(t_+) - \phi(t_-), \tag{9.23}$$

where $w_\pm$ also depend on ζ; see (9.12) and (9.13). When ζ is determined, one of the equations in (9.22) can be used to determine λ. Because we have two different representations of the saddle points in (9.12) and (9.13) (and similar in (9.5) and (9.8)), we obtain two different representations for ζ.

9.4.1. The monotonic case: $\boldsymbol{1 \le z \le z_c}$, $\boldsymbol{\beta \le \zeta}$. In this case, we have

$$\zeta\cosh w_+ = \sqrt{\zeta^2-\beta^2}, \quad \zeta\cosh w_- = -\sqrt{\zeta^2-\beta^2}, \tag{9.24}$$

and the left-hand side of (9.23) can be written as

$$2\left[\sqrt{\zeta^2-\beta^2} - \beta\arccos(\beta/\zeta)\right]. \tag{9.25}$$

For the right-hand side, we use (9.2) and (9.5). We have

$$z + \cosh t_+ = \frac{z(p+1)}{p(1+\beta^2)}, \quad z + \cosh t_- = \frac{z(p-1)}{p(1+\beta^2)}, \tag{9.26}$$

where p is given in (6.7). Hence,

$$\phi(t_+) - \phi(t_-) = \ln \frac{p+1}{p-1} - \beta \arccos \frac{\beta^2 p^2 - 1}{\beta^2 p^2 + 1}. \tag{9.27}$$

Combining the two sides, we have (6.6).

Next, we determine λ from (9.22). Either equation can be used, but it is better to add the equations and exploit the symmetry. We easily find the value given in (6.2).

9.4.2. The oscillatory case: $z_c \le z,\ \zeta \le \beta$. Using (9.13), we obtain for the left-hand side of (9.23)

$$2i \left[\sqrt{\beta^2 - \zeta^2} - \beta \operatorname{arccosh}(\beta/\zeta) \right]. \tag{9.28}$$

For the right-hand side, we solve (9.3). We have

$$\begin{aligned} z + \cosh t_+ &= \frac{z + i\sqrt{\beta^2(z^2-1)-1}}{\beta^2+1} = \sqrt{\frac{z^2-1}{\beta^2+1}}\, e^{i \operatorname{arccot} q}, \\ z + \cosh t_- &= \frac{z - i\sqrt{\beta^2(z^2-1)-1}}{\beta^2+1} = \sqrt{\frac{z^2-1}{\beta^2+1}}\, e^{-i \operatorname{arccot} q}, \end{aligned} \tag{9.29}$$

where q is given in (6.15). This gives, for the right-hand side we obtain, using (9.8):

$$\phi(t_+) - \phi(t_-) = 2i \operatorname{arccot} q - i\beta \ln \frac{\beta q + 1}{\beta q - 1}. \tag{9.30}$$

Combining the two sides, we obtain (6.14).

9.5. Representation of the coefficients for $\zeta \approx \beta$. Function $\Phi(\zeta)$ given in (6.3) is analytic for all $\zeta \ge 0$, in particular, at $\zeta = \beta$ (and for complex values). We give expansions that can be used for representing $\Phi(\zeta)$ and the coefficients of the asymptotic expansions in (6.4).

From (6.6), we can obtain several expansions. First, we mention

$$z = z_c + \sum_{k=1}^{\infty} z_k (\zeta - \beta)^k, \tag{9.31}$$

where the first two coefficients are

$$z_1 = \frac{-1}{\beta^2(1+\beta^2)^{1/6}}, \quad z_2 = \frac{7 + 8\beta^2 + 3(1+\beta^2)^{2/3}}{10\beta^3(1+\beta^2)^{5/6}}. \tag{9.32}$$

We can use the expansion in (6.3) and obtain

$$\Phi^4(\zeta) = \frac{\beta^2}{(1+\beta^2)^{1/3}} + \frac{2\beta\left(3 + 2\beta^2 + 2\left(1+\beta^2\right)^{2/3}\right)}{5(1+\beta^2)}(\zeta - \beta) + \mathcal{O}\left((\zeta-\beta)^2\right). \tag{9.33}$$

For representing the coefficients $A_n(\beta,\zeta), B_n(\beta,\zeta)$ of (6.4), it is convenient to expand $1/p$ in terms of powers of W. See (6.9). We write

$$\frac{1}{p} = \sum_{k=1}^{\infty} p_k W^k \tag{9.34}$$

and obtain, again from (6.6), $p_2 = p_4 = \cdots = 0$. Let

$$\gamma = \frac{1}{(1+\beta^2)^{1/3}}. \tag{9.35}$$

Then, the first few odd coefficients are

$$\begin{aligned} p_1 &= \gamma, \quad p_3 = -\frac{\gamma^3\left(2\gamma^2+2\gamma+1\right)}{5\left(\gamma^2+\gamma+1\right)}, \\ p_5 &= -\frac{\gamma^5\left(37\gamma^4+74\gamma^3+69\gamma^2+27\gamma+3\right)}{175\left(\gamma^2+\gamma+1\right)^2}. \end{aligned} \tag{9.36}$$

For the coefficient $B_1(\beta,\zeta)$ given in (6.9), we have

$$\begin{aligned} B_1(\beta,\zeta) = &-\frac{\gamma\zeta\left(9\gamma^5+9\gamma^4+9\gamma^3+4\gamma+4\right)}{280\left(\gamma^2+\gamma+1\right)} \\ &+\frac{\gamma^3\zeta\left(98\gamma^7+196\gamma^6+213\gamma^5+83\gamma^4-47\gamma^3-96\gamma^2-76\gamma-56\right)}{12600(\gamma^2+\gamma+1)^2}W^2 \\ &+\mathcal{O}\left(W^4\right). \end{aligned} \tag{9.37}$$

Observe that we have not expanded the quantity ζ that appears in all terms as a linear factor. This expansion holds for $1 \le z \le z_c$, that is, $\zeta \ge \beta$. When $z \ge z_c$, we have the same expansion with W^2 replaced with $-W^2$, throughout.

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