

Nonlinear electron and spin transport in semiconductor superlattices

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Abstract

Nonlinear charge transport in strongly coupled semiconductor superlattices is described by Wigner-Poisson kinetic equations involving one or two minibands. Electron-electron collisions are treated within the Hartree approximation whereas other inelastic collisions are described by a modified BGK (Bhatnagar-Gross-Krook) model. The hyperbolic limit is such that the collision frequencies are of the same order as the Bloch frequencies due to the electric field and the corresponding terms in the kinetic equation are dominant. In this limit, spatially nonlocal drift-diffusion balance equations for the miniband populations and the electric field are derived by means of the Chapman-Enskog perturbation technique. For a lateral superlattice with spin-orbit interaction, electrons with spin up or down have different energies and their corresponding drift-diffusion equations can be used to calculate spin-polarized currents and electron spin polarization. Numerical solutions show stable self-sustained oscillations of the current and the spin polarization through a voltage biased lateral superlattice thereby providing an example of superlattice spin oscillator.

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I. INTRODUCTION

Semiconductor superlattices are essential ingredients in fast nanoscale oscillators, quantum cascade lasers and infrared detectors. Quantum cascade lasers are used to monitor environmental pollution in gas emissions, to analyze breath in hospitals and in many other industrial applications⁵. A superlattice (SL) is a convenient approximation to a quasi-one-dimensional crystal that was originally proposed by Esaki and Tsu to observe Bloch oscillations, i.e., the periodic coherent motion of electrons in a miniband in the presence of an applied electric field. Fig. 1(a) shows a simple realization of a N -period SL. Each period of length l consists of two layers of semiconductors with different energy gaps but with similar lattice constants. The SL lengths in the lateral directions, L_y and L_z are much larger than l , typically tens of microns compared to about ten nanometers. The energy profile of the conduction band of this SL can be modeled as a succession of square quantum wells and barriers along the x direction (Kronig-Penney model) and, for a n-doped SL, we do not have to consider the valence band. A different quasi-1D crystal called a lateral superlattice (LSL) is shown in Fig. 1(b). In this case, a periodic structure is formed on the top surface of a quantum well (QW), so that L_z is of the order of l and $L_y \gg l$. The wave functions of a single electron in the conduction band of a SL can be expanded in terms of 1D Bloch wave functions times plane waves

$$\frac{1}{\sqrt{S}} e^{ik_y y} \psi(z) e^{ik_x x} u_\nu(x, k), \quad (1.1)$$

$$\psi(z) = \begin{cases} e^{ik_z z}, & \text{for a SL,} \\ \psi_n(z), & \text{for a LSL,} \end{cases} \quad (1.2)$$

where ν is the miniband index and n is the energy level of the quantum well in the case of a LSL. The function $u_\nu(x, k)$ is l -periodic in x and $2\pi/l$ -periodic in k . S is the area of the lateral cross section, equal to $L_y L_z$ for a rectangular cross section.

Many interesting nonlinear phenomena have been observed in voltage biased SL comprising finitely many periods, including self-oscillations of the current through the SL due to motion of electric field pulses, multistability of stationary charge and field profiles, and so on⁵. It is important to distinguish between strongly and weakly coupled SLs depending on the coupling between their component QWs. Roughly speaking, if barriers are narrow, QWs are strongly coupled and we can use the electronic states (1.1) as a convenient basis in

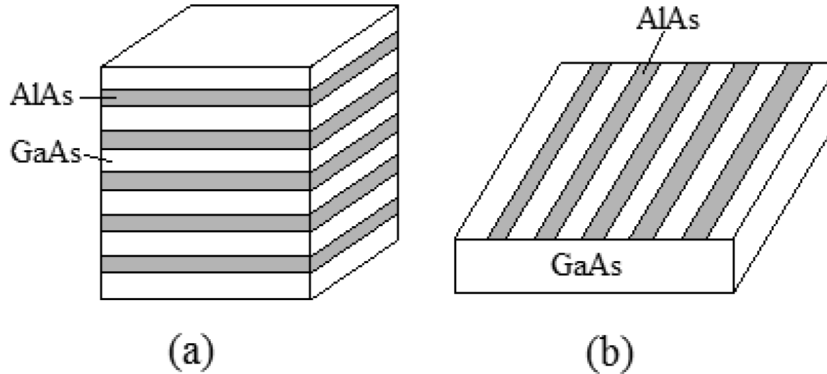


FIG. 1: (a) Schematic drawing of a superlattice. (b) A lateral superlattice.

a quantum kinetic description. The resulting reduced balance equations for electron density and electric field are partial differential equations (which may be nonlocal, as we shall see in this paper). On the other hand, for SLs having wide barriers, their QWs are weakly coupled and the electronic states of a single well provide a good basis in a quantum kinetic description, replacing the Bloch functions $e^{ikx}u_\nu(x, k)$ in (1.1). In this case, the balance equations are spatially discrete and phenomena such as multistability of stationary field profiles, formation and pinning of electric field domains, etc are theoretically predicted and observed in experiments. See the review⁵. Another promising field of applications is spintronics. Electrons in SLs having at least one period doped with magnetic impurities and subject to a static magnetic field can be distinguished by their spin because the magnetic field splits each miniband in two having different spin-dependent energy¹⁹. Recently a SL of this type has been proposed as a spin oscillator producing spin polarized oscillatory currents and able to inject polarized electrons in a contact⁶. Alternatively, materials displaying strong spin-orbit effects can be used as spintronic devices without having to apply magnetic fields; cf. the case of the LSL considered in Ref.¹⁴. In this paper, we will show that a LSL can be used as a spin oscillator.

This paper presents systematic derivations of quantum balance equations for SLs with two populated minibands, and it shows that their numerical solutions may predict space and time-dependent nonlinear phenomena occurring in these materials. Our methods can be used in 3D crystals, but their application to 1D structures such as SLs and LSLs leads to simpler equations that are less costly to solve. Although nonlinear charge transport in SLs

has been widely studied in the last decade (see the reviews^{5,17,21}), systematic derivations of tractable balance equations for miniband populations and electric field are scarce. One reason is that quantum kinetic equations are nonlocal in space and their collision terms may be nonlocal in space and time^{10,21}. Using them to analyze space and time-dependent phenomena such as wave propagation or self-sustained oscillations is problematic. In fact, only extremely simple solutions of general quantum kinetic equations (such as thermal equilibrium, disturbances thereof due to weak external fields and so on) are known, theoretical analysis of these equations is lacking and numerical solutions describing spatio-temporal phenomena are not available. One way to proceed is to adopt simple collision models similar to the Bhatnagar-Gross-Krook (BGK) collision model for classical kinetic theory¹. We discuss in this paper how to implement a BGK collision model for a quantum kinetic equation that is simple to handle yet keeps an important quantum feature such as the broadening of energy levels⁴. Once we have a quantum kinetic equation for a sufficiently general SL having two minibands, we implement a Chapman-Enskog perturbation procedure to derive the sought balance equations and solve them numerically for realistic SL configurations.

Previous to this work, Lei and coworkers derived quantum hydrodynamic equations describing SL having only one miniband^{15,16}. They use a closure assumption to close a hierarchy of moment equations. For the case of quantum particles in an arbitrary external three-dimensional potential, Degond and Ringhofer⁸ have used a similar procedure to derive balance equations. They close the system of moment equations by means of a local equilibrium density obtained by maximizing entropy. The Chapman-Enskog method has been used to derive drift-diffusion equations for single-miniband SLs described by semiclassical³ and quantum kinetic equations⁴. Earlier, Cercignani, Gamba and Levermore used the Chapman-Enskog method to derive balance equations for a semiclassical BGK-Poisson kinetic description of a semiconductor with one parabolic band under strong external bias⁷.

The rest of this paper is as follows. In Section II, we review the simpler case of nonlinear electron transport in a strongly coupled n-doped SL having only one populated miniband⁴. Starting with a kinetic equation for the Wigner function, we use the Chapman-Enskog perturbation method to derive balance equations for the electron density and the electric field. When these equations are solved numerically for a dc voltage biased SL with finitely many QWs and realistic parameter values, stable self-sustained oscillations of the current through the SL are found among their solutions, in agreement with experimental observations⁴. Sec-

tions III to V contain the main results of the present work. In Section III, we describe a SL having two populated minibands by proposing a kinetic equation for the Wigner matrix. In Section IV, we derive balance equations for the miniband electron populations and the electric field, using an appropriate Chapman-Enskog method and a tight-binding approximation to obtain explicit formulas. The case of a LSL having strong Rashba spin-orbit interaction¹⁸ is important for spintronic applications and has been considered in Section V. We derive and solve numerically the resulting balance equations. Novel self-sustained oscillations of the spin current and polarization are obtained for appropriate values of the parameters. Finally Section VI contains our conclusions and some technical matters are relegated to the Appendix.

II. SINGLE MINIBAND SUPERLATTICE

The Wigner-Poisson-Bhatnagar-Gross-Krook (WPBGK) system for 1D electron transport in the lowest miniband of a strongly coupled SL is:

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{i}{\hbar} \left[\mathcal{E} \left(k + \frac{1}{2i} \frac{\partial}{\partial x} \right) - \mathcal{E} \left(k - \frac{1}{2i} \frac{\partial}{\partial x} \right) \right] f \\ + \frac{ie}{\hbar} \left[W \left(x + \frac{1}{2i} \frac{\partial}{\partial k}, t \right) - W \left(x - \frac{1}{2i} \frac{\partial}{\partial k}, t \right) \right] f \\ = Q[f] \equiv -\nu_{en} (f - f^{FD}) - \nu_{imp} \frac{f(x, k, t) - f(x, -k, t)}{2}, \end{aligned} \quad (2.1)$$

$$\varepsilon \frac{\partial^2 W}{\partial x^2} = \frac{e}{l} (n - N_D), \quad (2.2)$$

$$n(x, t) = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f(x, k, t) dk = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} f^{FD}(k; n(x, t)) dk, \quad (2.3)$$

$$f^{FD}(k; n) = \frac{m^* k_B T}{\pi \hbar^2} \int_{-\infty}^{\infty} \ln \left[1 + \exp \left(\frac{\mu - E}{k_B T} \right) \right] \frac{\sqrt{2} \Gamma^3 / \pi}{[E - \mathcal{E}_1(k)]^4 + \Gamma^4} dE. \quad (2.4)$$

Here f , n , N_D , $\mathcal{E}(k)$, d_B , d_W , $l = d_B + d_W$, W , ε , m^* , k_B , T , Γ , ν_{en} , ν_{imp} and $-e < 0$ are the one-particle Wigner function, the 2D electron density, the 2D doping density, the miniband dispersion relation, the barrier width, the well width, the SL period, the electric potential, the SL permittivity, the effective mass of the electron in the lateral directions, the Boltzmann constant, the lattice temperature, the energy broadening of the equilibrium distribution due to collisions¹² (page 28 ss), the frequency of the inelastic collisions responsible for energy relaxation, the frequency of the elastic impurity collisions and the electron charge, respectively.

The left-hand side of Eq. (2.1) can be straightforwardly derived from the Schrödinger-Poisson equation for the wave function in the miniband using the definition of the 1D Wigner function⁴:

$$f(x, k, t) = \frac{2l}{S} \sum_{j=-\infty}^{\infty} \int_{\mathbb{R}^2} \langle \psi^\dagger(x + jl/2, y, z, t) \psi(x - jl/2, y, z, t) \rangle e^{ijk_l} d\mathbf{x}_\perp \quad (2.5)$$

(the second quantized wave function $\psi(x, \mathbf{x}_\perp, t) = \sum_{q, \mathbf{q}_\perp} a(q, q_\perp, t) \phi_q(x) e^{i\mathbf{q}_\perp \cdot \mathbf{x}_\perp}$, $\mathbf{x}_\perp = (y, z)$, is a superposition of the Bloch states corresponding to the miniband and S is the SL cross section⁴). The right hand side in Eq. (2.1) is the sum of $-\nu_e (f - f^{FD})$, which represents energy relaxation towards a 1D effective Fermi-Dirac (FD) distribution $f^{FD}(k; n)$ (local equilibrium), and $-\nu_i [f(x, k, t) - f(x, -k, t)]/2$, which accounts for impurity elastic collisions³. For simplicity, the collision frequencies ν_e and ν_i are fixed constants. Exact and FD distribution functions have the same electron density, thereby preserving charge continuity as in the classical BGK collision models¹. The chemical potential μ is a function of n resulting from solving equation (2.3) with the integral of the collision-broadened 3D Fermi-Dirac distribution over the lateral components of the wave vector $(k, \mathbf{k}_\perp) = (k, k_y, k_z)$:

$$f^{FD}(k; n) = \int_{-\infty}^{\infty} \frac{D_\Gamma(E - \mathcal{E}_1(k))}{1 + \exp\left(\frac{E - \mu}{k_B T}\right)} dE, \quad (2.6)$$

$$D_\Gamma(E) = \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} \delta_\Gamma\left(\frac{\hbar^2 \mathbf{k}_\perp^2}{2m^*} - E\right) d\mathbf{k}_\perp = \frac{m^*}{\pi \hbar^2} \int_0^\infty \delta_\Gamma(E_\perp - E) dE_\perp. \quad (2.7)$$

Using the residue theorem for a line-width:

$$\delta_\Gamma(E) = \frac{\sqrt{2} \Gamma^3 / \pi}{\Gamma^4 + E^4}, \quad (2.8)$$

(2.7) yields

$$\begin{aligned} D_\Gamma(E) = & \frac{m^*}{\pi \hbar^2} \left\{ 1 + \frac{1}{4\pi} \ln \left[\frac{E^2 + \sqrt{2} \Gamma E + \Gamma^2}{E^2 - \sqrt{2} \Gamma E + \Gamma^2} \right] \right. \\ & - \frac{\theta(\sqrt{2}|E| - \Gamma)}{2\pi} \left[2\pi - \arctan\left(\frac{\Gamma}{\sqrt{2}|E| + \Gamma}\right) - \arctan\left(\frac{\Gamma}{\sqrt{2}|E| - \Gamma}\right) \right] \\ & - \frac{\theta(\Gamma - \sqrt{2}|E|)}{2\pi} \left[\pi + \arctan\left(\frac{\Gamma}{\sqrt{2}E + \Gamma}\right) - \arctan\left(\frac{\Gamma}{\Gamma - \sqrt{2}E}\right) \right] \\ & \left. - \frac{\theta(\sqrt{2}E - \Gamma)}{2\pi} \left[\arctan\left(\frac{\Gamma}{\sqrt{2}E + \Gamma}\right) + \arctan\left(\frac{\Gamma}{\sqrt{2}E - \Gamma}\right) \right] \right\}, \end{aligned} \quad (2.9)$$

which is equivalent to Eq. (2.4)²². Here $\theta(E)$ is the Heaviside unit step function. As $\Gamma \rightarrow 0+$, the line-width (2.8) tends to the delta function $\delta(E)$, $D_\Gamma(E)$ tends to the 2D

density of states, $D(E) = m^* \theta(E) / (\pi \hbar^2)$, and f^{FD} tends to the 3D Fermi-Dirac distribution function integrated over the lateral wave vector \mathbf{k}_\perp . In Ref.⁴, a Lorentzian line-width was used instead of (2.8) and the integral over E in (2.6) extended from 0 to ∞ . The integral with the Lorentzian function is not convergent in $E = -\infty$, which is why we prefer using convolution with the “super-Lorentzian” function (2.8) in this work. The integration in (2.7) cannot be carried out explicitly for other standard line-widths such as a Gaussian or a hyperbolic secant. This unnecessarily complicates the numerical integration of the balance equations we will obtain later. Note that, following Ignatov and Shashkin¹¹, we have not included the effects of the electric potential in our Fermi-Dirac distribution. These model equations can be improved by including scattering processes with change of lateral momentum and an electric field-dependent local equilibrium. However the resulting model could only be treated numerically and the qualitative features of our derivation and of the nonlocal drift-diffusion equation would be lost in longer formulas.

A different way to introduce a quantum BGK collision model is to define a local equilibrium density matrix operator by minimizing quantum entropy (defined with the opposite sign of the convention that is usual in physics) under constraints giving the electron density and energy density in terms of the density matrix. The resulting expression involves an inverse Wigner transform and another transform is needed to deduce the local equilibrium Wigner function f^{FD} entering the BGK formula⁸. This f^{FD} is nonlocal in space and can only be found by solving some partial differential equation⁸. As a model for quantum collisions^{10,21}, the resulting quantum BGK model is not realistic, in the same way as the original BGK model is not a realistic model for classical collisions. Moreover, the implicit manner in which the model is defined defeats the main asset of the classical BGK collision model: its simplicity, that makes it possible to obtain results analytically. Thus we prefer to introduce a BGK model that can be handled more easily and still incorporates quantum effects. The most important quantum effect affecting the collision term is the broadening of energy levels due to scattering, $\Gamma \approx \hbar/\tau$ (where τ is the lifetime of the level)¹², and this is taken phenomenologically into account by the convolution with the line-width function (2.8) in (2.6). In the semiclassical limit “ $\hbar \rightarrow 0$ ”, $\Gamma \rightarrow 0$ and we recover the semiclassical FD distribution.

The WPBGK system (2.1) to (2.4) should be solved for a Wigner function which is $2\pi/l$ -periodic in k and satisfies appropriate initial and boundary conditions. It is convenient to

derive the charge continuity equation and a nonlocal Ampère's law for the current density. The Wigner function f is periodic in k ; its Fourier expansion is

$$f(x, k, t) = \sum_{j=-\infty}^{\infty} f_j(x, t) e^{ijkl}. \quad (2.10)$$

Defining $F = \partial W / \partial x$ (minus the electric field) and the average

$$\langle F \rangle_j(x, t) = \frac{1}{jl} \int_{-jl/2}^{jl/2} F(x + s, t) ds, \quad (2.11)$$

it is possible to obtain the following equivalent form of the Wigner equation⁴

$$\frac{\partial f}{\partial t} + \sum_{j=-\infty}^{\infty} \frac{ijl}{\hbar} e^{ijkl} \left(\mathcal{E}_j \frac{\partial}{\partial x} \langle f \rangle_j + e \langle F \rangle_j f_j \right) = Q[f]. \quad (2.12)$$

Here the nonzero Fourier coefficients of the dispersion relation are simply $\mathcal{E}_0 = \Delta/2$ and $\mathcal{E}_{\pm 1} = -\Delta/4$ for the tight-binding dispersion relation $\mathcal{E}(k) = \Delta(1 - \cos kl)/2$ (Δ is the miniband width), which yields a miniband group velocity $v(k) = \frac{\Delta l}{2\hbar} \sin kl$. Integrating this equation over k yields the charge continuity equation

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} \sum_{j=1}^{\infty} \frac{2jl}{\hbar} \langle \text{Im}(\mathcal{E}_{-j} f_j) \rangle_j = 0. \quad (2.13)$$

Here we can eliminate the electron density by using the Poisson equation and then integrate over x , thereby obtaining the nonlocal Ampère's law for the total current density $J(t)$:

$$\varepsilon \frac{\partial F}{\partial t} + \frac{2e}{\hbar} \sum_{j=1}^{\infty} j \langle \text{Im}(\mathcal{E}_{-j} f_j) \rangle_j = J(t). \quad (2.14)$$

To derive the QDDE, we shall assume that the electric field contribution in Eq. (2.12) is comparable to the collision terms and that they dominate the other terms (*the hyperbolic limit*)³. Let v_M and F_M be the electron velocity and field positive values at which the (zeroth order) drift velocity reaches its maximum. In this limit, the time t_0 it takes an electron with speed v_M to traverse a distance $x_0 = \varepsilon F_M l / (e N_D)$, over which the field variation is of order F_M , is much longer than the mean free time between collisions, $\nu_e^{-1} \sim \hbar / (e F_M l) = t_1$. We therefore define the *small parameter* $\lambda = t_1 / t_0 = \hbar v_M N_D / (\varepsilon F_M^2 l^2)$ and formally multiply the first two terms on the left side of (2.1) or (2.12) by $\lambda^{3,4}$. The result is

$$\lambda \left(\frac{\partial f}{\partial t} + \sum_{j=-\infty}^{\infty} \frac{ijl}{\hbar} e^{ijkl} \mathcal{E}_j \frac{\partial}{\partial x} \langle f \rangle_j \right) = Q[f] - \sum_{j=-\infty}^{\infty} \frac{iejl}{\hbar} e^{ijkl} \langle F \rangle_j f_j. \quad (2.15)$$

The solution of Eq. (2.15) for $\lambda = 0$ is calculated in terms of its Fourier coefficients as

$$f^{(0)}(k; F) = \sum_{j=-\infty}^{\infty} \frac{(1 - ij\mathcal{F}_j/\tau_e) f_j^{FD}}{1 + j^2 \mathcal{F}_j^2} e^{ijkl}, \quad (2.16)$$

where $\mathcal{F}_j = \langle F \rangle_j / F_M$, $F_M = \frac{\hbar}{el} \sqrt{\nu_e(\nu_e + \nu_i)}$ and $\tau_e = \sqrt{(\nu_e + \nu_i)/\nu_e}$.

The Chapman-Enskog ansatz for the Wigner function is⁴:

$$f(x, k, t; \lambda) = f^{(0)}(k; F) + \sum_{m=1}^{\infty} f^{(m)}(k; F) \lambda^m, \quad (2.17)$$

$$\varepsilon \frac{\partial F}{\partial t} + \sum_{m=0}^{\infty} J^{(m)}(F) \lambda^m = J(t). \quad (2.18)$$

The coefficients $f^{(m)}(k; F)$ depend on the ‘slow variables’ x and t only through their dependence on the electric field and the electron density. The electric field obeys a reduced evolution equation (2.18) in which the functionals $J^{(m)}(F)$ are chosen so that the $f^{(m)}(k; F)$ are bounded and $2\pi/l$ -periodic in k . After we keep the desired number of terms and set $\lambda = 1$, Eq. (2.18) is the QDDE provided by our perturbation procedure.

Differentiating the Ampère’s law (2.18) with respect to x , we obtain the charge continuity equation. Moreover the compatibility condition

$$\int_{-\pi/l}^{\pi/l} f^{(m)}(k; n) dk = \frac{2\pi}{l} f_0^{(m)} = 0, \quad m \geq 1, \quad (2.19)$$

is obtained by inserting the expansion (2.17) into (2.3). Inserting (2.17) and (2.18) in (2.15), we find the hierarchy:

$$\mathcal{L}f^{(1)} = -\frac{\partial f^{(0)}}{\partial t} \Big|_0 + \sum_{j=-\infty}^{\infty} \frac{ijl\mathcal{E}_j e^{ijkl}}{\hbar} \frac{\partial}{\partial x} \langle f^{(0)} \rangle_j \quad (2.20)$$

$$\mathcal{L}f^{(2)} = -\frac{\partial f^{(1)}}{\partial t} \Big|_0 + \sum_{j=-\infty}^{\infty} \frac{ijl\mathcal{E}_j e^{ijkl}}{\hbar} \frac{\partial}{\partial x} \langle f^{(1)} \rangle_j - \frac{\partial f^{(0)}}{\partial t} \Big|_1, \quad (2.21)$$

and so on. Here

$$\mathcal{L}u(k) \equiv \frac{ie}{\hbar} \sum_{j=-\infty}^{\infty} jl \langle F \rangle_j u_j e^{ijkl} + \left(\nu_e + \frac{\nu_i}{2} \right) u(k) - \frac{\nu_i}{2} u(-k), \quad (2.22)$$

and the subscripts 0 and 1 in the right hand side of these equations mean that $\varepsilon \partial F / \partial t$ is replaced by $J - J^{(0)}(F)$ and by $-J^{(1)}(F)$, respectively.

The condition (2.19) implies that

$$\int_{-\pi/l}^{\pi/l} \mathcal{L}f^{(m)} dk = 0, \quad (2.23)$$

for $m \geq 1$. Using this, the solvability conditions for the linear hierarchy of equations yield

$$J^{(m)} = \frac{2e}{\hbar} \sum_{j=1}^{\infty} j \langle \text{Im}(\mathcal{E}_{-j} f_j^{(m)}) \rangle_j, \quad (2.24)$$

which can also be obtained by insertion of Eq. (2.17) in (2.14).

Particularized to the case of the tight-binding dispersion relation and $\Gamma = 0$ in the Fermi-Dirac distribution (2.4), the leading order of the Ampère's law (2.18) is

$$\varepsilon \frac{\partial F}{\partial t} + \frac{ev_M}{l} \langle n \mathcal{M} V(\mathcal{F}) \rangle_1 = J(t), \quad (2.25)$$

$$V(\mathcal{F}) = \frac{2\mathcal{F}}{1 + \mathcal{F}^2}, \quad v_M = \frac{\Delta l \mathcal{I}_1(M)}{4\hbar\tau_e \mathcal{I}_0(M)}, \quad \mathcal{M} \left(\frac{n}{N_D} \right) = \frac{\mathcal{I}_1(\tilde{\mu}) \mathcal{I}_0(M)}{\mathcal{I}_1(M) \mathcal{I}_0(\tilde{\mu})}, \quad (2.26)$$

$$\mathcal{I}_m(s) = \int_{-\pi}^{\pi} \cos(mk) \ln(1 + e^{s - \delta + \delta \cos k}) dk, \quad (2.27)$$

provided $\mathcal{F} \equiv \mathcal{F}_1$, $\delta = \Delta/(2k_B T)$ and $\tilde{\mu} \equiv \mu/(k_B T)$. Here M (calculated graphically in Fig. 1 of Ref.³) is the value of the dimensionless chemical potential $\tilde{\mu}$ at which (2.3) holds with $n = N_D$. The drift velocity $v_M V(\mathcal{F})$ has the Esaki-Tsu form with a peak velocity that becomes $v_M \approx \Delta l \mathcal{I}_1(\delta)/[4\hbar\tau_e \mathcal{I}_0(\delta)]$ in the Boltzmann limit¹¹ ($\mathcal{I}_n(\delta)$ is the modified Bessel function of the n th order).

To find the first-order correction in (2.18), we first solve (2.20) and find $J^{(m)}$ for $m = 1$. The calculation yields the first correction to Eq. (2.25) (here ' means differentiation with respect to n)⁴:

$$\varepsilon \frac{\partial F}{\partial t} + \frac{ev_M}{l} \mathcal{N} \left(F, \frac{\partial F}{\partial x} \right) = \varepsilon \left\langle D \left(F, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2} \right) \right\rangle_1 + \langle A \rangle_1 J(t), \quad (2.28)$$

$$A = 1 + \frac{2ev_M}{\varepsilon F_M l (\nu_e + \nu_i)} \frac{1 - (1 + 2\tau_e^2) \mathcal{F}^2}{(1 + \mathcal{F}^2)^3} n \mathcal{M}, \quad (2.29)$$

$$\mathcal{N} = \langle n V \mathcal{M} \rangle_1 + \langle (A - 1) \langle n V \mathcal{M} \rangle_1 \rangle_1 - \frac{\Delta l \tau_e}{F_M \hbar (\nu_e + \nu_i)} \left\langle \frac{B}{1 + \mathcal{F}^2} \right\rangle_1, \quad (2.30)$$

$$D = \frac{\Delta^2 l^2}{8\hbar^2 (\nu_e + \nu_i) (1 + \mathcal{F}^2)} \left(\frac{\partial^2 \langle F \rangle_1}{\partial x^2} - \frac{4\hbar v_M \tau_e C}{\Delta l} \right), \quad (2.31)$$

$$B = \left\langle \frac{4\mathcal{F}_2 n \mathcal{M}_2}{(1 + 4\mathcal{F}_2^2)^2} \frac{\partial \langle F \rangle_2}{\partial x} \right\rangle_1 + \mathcal{F} \left\langle \frac{n \mathcal{M}_2 (1 - 4\mathcal{F}_2^2)}{(1 + 4\mathcal{F}_2^2)^2} \frac{\partial \langle F \rangle_2}{\partial x} \right\rangle_1 - \frac{4\hbar v_M (1 + \tau_e^2) \mathcal{F} (n \mathcal{M})'}{\Delta l \tau_e (1 + \mathcal{F}^2)} \left\langle n \mathcal{M} \frac{1 - \mathcal{F}^2}{(1 + \mathcal{F}^2)^2} \frac{\partial \langle F \rangle_1}{\partial x} \right\rangle_1, \quad (2.32)$$

$$C = \left\langle \frac{(n \mathcal{M}_2)'}{1 + 4\mathcal{F}_2^2} \frac{\partial^2 F}{\partial x^2} \right\rangle_1 - 2\mathcal{F} \left\langle \frac{(n \mathcal{M}_2)' \mathcal{F}_2}{1 + 4\mathcal{F}_2^2} \frac{\partial^2 F}{\partial x^2} \right\rangle_1 + \frac{8\hbar v_M (1 + \tau_e^2) (n \mathcal{M})' \mathcal{F}}{\Delta l \tau_e (1 + \mathcal{F}^2)} \left\langle \frac{(n \mathcal{M})' \mathcal{F}}{1 + \mathcal{F}^2} \frac{\partial^2 F}{\partial x^2} \right\rangle_1. \quad (2.33)$$

Here $\mathcal{M}_2(n/N_D) \equiv \mathcal{I}_2(\tilde{\mu}) \mathcal{I}_0(M)/[\mathcal{I}_1(M) \mathcal{I}_0(\tilde{\mu})]$. If the electric field and the electron density do not change appreciably over two SL periods, $\langle F \rangle_j \approx F$, the spatial averages can be ignored, and the *nonlocal* QDDE (2.28) becomes the *local* generalized DDE (GDDE) obtained from the semiclassical theory³. The boundary conditions for the QDDE (2.28) (which contains triple spatial averages) need to be specified on the intervals $[-2l, 0]$ and $[Nl, Nl + 2l]$, not just at the points $x = 0$ and $x = Nl$, as in the case of the parabolic GDDE. Similarly, the initial condition has to be defined on the extended interval $[-2l, Nl + 2l]$. For realistic values of the parameters representing a strongly coupled SL under dc voltage bias, the numerical solution of the QDDE yields a stable self-sustained oscillation of the current⁴ in quantitative agreement with experiments²⁰. Details of the numerical procedure can be found in⁹.

III. WIGNER DESCRIPTION OF A TWO-MINIBAND SUPERLATTICE

We shall consider a 2×2 Hamiltonian $\mathbf{H}(x, -i\partial/\partial x)$, in which¹³

$$\begin{aligned} \mathbf{H}(x, k) &= [h_0(k) - eW(x)]\boldsymbol{\sigma}_0 + \vec{h}(k) \cdot \vec{\boldsymbol{\sigma}} \\ &\equiv \begin{pmatrix} (\alpha + \gamma)(1 - \cos kl) - eW(x) + g & -i\beta \sin kl \\ i\beta \sin kl & (\alpha - \gamma)(1 - \cos kl) - eW(x) - g \end{pmatrix}. \end{aligned} \quad (3.1)$$

Here

$$\begin{aligned} h_0(k) &= \alpha(1 - \cos kl), & h_1(k) &= 0, \\ h_2(k) &= \beta \sin kl, & h_3(k) &= \gamma(1 - \cos kl) + g, \end{aligned} \quad (3.2)$$

and

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.3)$$

are the Pauli matrices.

The Hamiltonian (3.1) corresponds to the simplest 2×2 Kane model in which the quadratic and linear terms $(kl)^2/2$ and kl are replaced by $(1 - \cos kl)$ and $\sin kl$, respectively. For a SL with two minibands, $2g$ is the miniband gap and $\alpha = (\Delta_1 + \Delta_2)/4$ and $\gamma = (\Delta_1 - \Delta_2)/4$, provided Δ_1 and Δ_2 are the miniband widths. In the case of a LSL, $g = \gamma = 0$, and $h_2\boldsymbol{\sigma}_2$ corresponds to the precession term in the Rashba spin-orbit interaction¹⁴. The other term, the intersubband coupling, depends on the momentum in

the y direction and we have not included it here. Small modifications of (3.1) represent a single miniband SL with dilute magnetic impurities in the presence of a magnetic field B : $g = \gamma = h_2 = 0$, and $h_1 = \beta(B)^{19}$. As in the case of a single miniband SL, $W(x)$ is the electric potential.

The energy minibands $\mathcal{E}^\pm(k)$ are the eigenvalues of the free Hamiltonian $\mathbf{H}_0(k) = h_0(k)\boldsymbol{\sigma}_0 + \vec{h}(k) \cdot \vec{\boldsymbol{\sigma}}$ and are given by

$$\mathcal{E}^\pm(k) = h_0(k) \pm |\vec{h}(k)|. \quad (3.4)$$

The corresponding spectral projections are

$$\mathbf{P}^\pm(k) = \frac{\boldsymbol{\sigma}_0 \pm \vec{\nu}(k) \cdot \vec{\boldsymbol{\sigma}}}{2}, \quad \text{where} \quad \vec{\nu}(k) = \vec{h}(k)/|\vec{h}(k)|, \quad (3.5)$$

so that we can write

$$\mathbf{H}_0(k) = \mathcal{E}^+(k)\mathbf{P}^+(k) + \mathcal{E}^-(k)\mathbf{P}^-(k). \quad (3.6)$$

We shall now write the WPBGK equations for the Wigner matrix written in terms of the Pauli matrices:

$$\mathbf{f}(x, k, t) = \sum_{i=0}^3 f^i(x, k, t)\boldsymbol{\sigma}_i = f^0(x, k, t)\boldsymbol{\sigma}_0 + \vec{f}(x, k, t) \cdot \vec{\boldsymbol{\sigma}}. \quad (3.7)$$

The Wigner components are real and can be related to the coefficients of the Hermitian Wigner matrix by

$$\begin{aligned} f_{11} &= f^0 + f^3, & f_{12} &= f^1 - if^2, \\ f_{21} &= f^1 + if^2, & f_{22} &= f^0 - f^3. \end{aligned} \quad (3.8)$$

Hereinafter we shall use the equivalent notations

$$f = \begin{pmatrix} f^0 \\ f^1 \\ \vec{f} \\ f^3 \end{pmatrix} = \begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix}. \quad (3.9)$$

The populations of the minibands with energies \mathcal{E}^\pm are given by the moments:

$$n^\pm(x, t) = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} \left[f^0(x, k, t) \pm \vec{\nu}(k) \cdot \vec{f}(x, k, t) \right] dk, \quad (3.10)$$

and the total electron density is $n^+ + n^-$. After some algebra, we can obtain the following WPBGK equations for the Wigner components

$$\frac{\partial f^0}{\partial t} + \frac{\alpha}{\hbar} \sin kl \Delta^- f^0 + \vec{b} \cdot \Delta^- \vec{f} - \Theta f^0 = Q^0[f], \quad (3.11)$$

$$\frac{\partial \vec{f}}{\partial t} + \frac{\alpha}{\hbar} \sin kl \Delta^- \vec{f} + \vec{b} \Delta^- f^0 + \vec{\omega} \times \vec{f} - \Theta \vec{f} = \vec{Q}[f], \quad (3.12)$$

$$\varepsilon \frac{\partial^2 W}{\partial x^2} = \frac{e}{l} (n^+ + n^- - N_D), \quad (3.13)$$

whose right hand sides contain collision terms to be described later. Here

$$(\Delta^\pm u)(x, k) = u(x + l/2, k) \pm u(x - l/2, k), \quad (3.14)$$

$$\vec{\omega} = \vec{\omega}_0 + \vec{\omega}_1, \quad (3.15)$$

$$\vec{\omega}_0 = \frac{2g}{\hbar} (0, 0, 1), \quad (3.16)$$

$$\vec{\omega}_1 = \frac{1}{\hbar} (0, \beta \sin kl \Delta^+, 2\gamma - \gamma \cos kl \Delta^+), \quad (3.17)$$

$$\vec{b} = \frac{1}{\hbar} (0, \beta \cos kl, \gamma \sin kl), \quad (3.18)$$

$$\Theta f^i(x, k, t) = \sum_{j=-\infty}^{\infty} \frac{e j l}{i \hbar} \langle F(x, t) \rangle_j e^{i j k l} f_j^i(x, t). \quad (3.19)$$

Our collision model contains two terms: a BGK term which tries to send the miniband Wigner function to its local equilibrium and a scattering term from the miniband with higher energy to the lowest miniband:

$$Q^0[f] = -\frac{f^0 - \Omega^0}{\tau}, \quad (3.20)$$

$$\vec{Q}[f] = -\frac{\vec{f} - \vec{\Omega}}{\tau} - \frac{\vec{\nu} f^0 + \vec{f}}{\tau_{sc}}, \quad (3.21)$$

$$\Omega^0 = \frac{\phi^+ + \phi^-}{2}, \quad \vec{\Omega} = \frac{\phi^+ - \phi^-}{2} \vec{\nu}, \quad (3.22)$$

$$\phi^\pm(k; n^\pm) = \frac{m^* k_B T}{\pi \hbar^2} \int_{-\infty}^{\infty} \frac{\sqrt{2} \Gamma^3 / \pi}{\Gamma^4 + [E - \mathcal{E}^\pm(k)]^4} \ln \left(1 + e^{\frac{\mu^\pm - E}{k_B T}} \right) dE, \quad (3.23)$$

$$n^\pm = \frac{l}{2\pi} \int_{-\pi/l}^{\pi/l} \phi^\pm(k; n^\pm) dk. \quad (3.24)$$

The chemical potentials of the minibands, μ^+ and μ^- are calculated in terms of n^+ and n^- respectively, by inserting (3.23) in (3.24) and solving the resulting equations. Our collision model should enforce charge continuity. To check this, we first calculate the time derivative

of n^\pm using (3.10) to (3.12):

$$\begin{aligned} \frac{\partial n^\pm}{\partial t} + \frac{\alpha l \Delta^-}{2\pi \hbar} \int_{-\pi/l}^{\pi/l} \sin kl (f^0 \pm \vec{v} \cdot \vec{f}) dk + \frac{l \Delta^-}{2\pi} \int_{-\pi/l}^{\pi/l} (\vec{b} \cdot \vec{f} \pm \vec{v} \cdot \vec{b} f^0) dk \\ \pm \frac{l \Delta^-}{2\pi} \int_{-\pi/l}^{\pi/l} \vec{v} \cdot \vec{\omega} \times \vec{f} dk \mp \frac{l \Delta^-}{2\pi} \int_{-\pi/l}^{\pi/l} \vec{v} \cdot \Theta \vec{f} dk \\ = \frac{l \Delta^-}{2\pi} \int_{-\pi/l}^{\pi/l} (Q^0[f] \pm \vec{v} \cdot \vec{Q}[f]) dk = \mp \frac{n^\pm}{\tau_{sc}}, \end{aligned} \quad (3.25)$$

where we have employed $\int \Theta f^0 dk = 0$. Then we obtain:

$$\frac{\partial}{\partial t} (n^+ + n^-) + \Delta^- \left[\frac{l}{\pi} \int_{-\pi/l}^{\pi/l} \left(\frac{\alpha}{\hbar} \sin kl f^0 + \vec{b} \cdot \vec{f} \right) dk \right] = 0. \quad (3.26)$$

Noting that $\Delta^- u(x) = l \partial \langle u(x) \rangle_1 / \partial x$, we see that this equation corresponds to charge continuity. Differentiating in time the Poisson equation (3.13), using (3.26) in the result and integrating with respect to x , we get the following nonlocal Ampère's law for the balance of current:

$$\varepsilon \frac{\partial F}{\partial t} + \left\langle \frac{el}{\pi} \int_{-\pi/l}^{\pi/l} \left(\frac{\alpha}{\hbar} \sin kl f^0 + \vec{b} \cdot \vec{f} \right) dk \right\rangle_1 = J(t). \quad (3.27)$$

Here the space independent function $J(t)$ is the total current density. Since the Wigner components are real, we can rewrite (3.27) in the following equivalent form:

$$\varepsilon \frac{\partial F}{\partial t} - \frac{2e}{\hbar} \langle \alpha \text{Im} f_1^0 - \beta \text{Re} f_1^2 + \gamma \text{Im} f_1^3 \rangle_1 = J(t). \quad (3.28)$$

IV. DERIVATION OF BALANCE EQUATIONS BY THE CHAPMAN-ENSKOG METHOD

In this Section, we shall derive the reduced balance equations for our two-miniband SL using the Chapman-Enskog method. First of all, we should decide the order of magnitude of the terms in the WPBGK equations (3.11) and (3.12) in the hyperbolic limit. Recall that in this limit, the collision frequency $1/\tau$ and the Bloch frequency $eF_M l/\hbar$ are of the same order, about 10 THz for the SL of Section II. Typically, $2g/\hbar$ is of the same order, so that the term containing $\vec{\omega}_0$ should also balance the BGK collision term. What about the other terms?

The scattering time τ_{sc} is much longer than the collision time τ , and we shall consider $\tau/\tau_{sc} = O(\lambda) \ll 1$. Moreover, the gap energy is typically much larger than the miniband

widths or the spin-orbit coefficient and a rich dominant balance is obtained by assuming that β/g and γ/g are of order λ . Then we can expand the unit vector $\vec{\nu}$ as follows:

$$\vec{\nu} = (0, 0, 1) + \frac{\lambda\beta}{g} \sin kl (0, 1, 0) - \lambda^2 \left[\frac{\beta\gamma}{g^2} \sin kl (1 - \cos kl) (0, 1, 0) + \frac{\beta^2 \sin^2 kl}{2g^2} (0, 0, 1) \right] + O(\lambda^3). \quad (4.1)$$

In this expansion, we have inserted the book-keeping parameter λ which is set equal to 1 at the end of our calculations (cf. Section II). From (3.11) and (3.12), we can write the scaled WPBGK equations as follows:

$$\mathbb{L}f - \Omega = -\lambda \left(\tau \frac{\partial f}{\partial t} + \Lambda f \right). \quad (4.2)$$

Here the operators \mathbb{L} and Λ are defined by

$$\mathbb{L}f = f - \tau \Theta f + \delta_1 \begin{pmatrix} 0 \\ -f^2 \\ f^1 \\ 0 \end{pmatrix}, \quad (4.3)$$

$$\Lambda f = \delta_2 \begin{pmatrix} 0 \\ \vec{f} + \vec{\nu} f^0 \end{pmatrix} + \frac{\alpha\tau}{\hbar} \sin kl \Delta^- f + \Delta^- \begin{pmatrix} \tau \vec{b} \cdot \vec{f} \\ \tau \vec{b} f^0 \end{pmatrix} + \begin{pmatrix} 0 \\ \tau \vec{\omega}_1 \times \vec{f} \end{pmatrix}, \quad (4.4)$$

where

$$\delta_1 = \frac{2g\tau}{\hbar}, \quad \delta_2 = \frac{\tau}{\tau_{\text{sc}}}. \quad (4.5)$$

The expansion of $\vec{\nu}$ in powers of λ gives rise to a similar expansion of Ω and Λ .

To derive the reduced balance equations, we use the following Chapman-Enskog ansatz:

$$f(x, k, t; \lambda) = f^{(0)}(k; n^+, n^-, F) + \sum_{m=1}^{\infty} f^{(m)}(k; n^+, n^-, F) \lambda^m, \quad (4.6)$$

$$\varepsilon \frac{\partial F}{\partial t} + \sum_{m=0}^{\infty} J_m(n^+, n^-, F) \lambda^m = J(t), \quad (4.7)$$

$$\frac{\partial n^{\pm}}{\partial t} = \sum_{m=0}^{\infty} A_m^{\pm}(n^+, n^-, F) \lambda^m. \quad (4.8)$$

The functions A_m^{\pm} and J_m are related through the Poisson equation (3.13), so that

$$A_m^+ + A_m^- = -\frac{l}{e} \frac{\partial J_m}{\partial x}. \quad (4.9)$$

Inserting (4.6) to (4.8) into (4.2), we get

$$\mathbb{L}f^{(0)} = \Omega_0, \quad (4.10)$$

$$\mathbb{L}f^{(1)} = \Omega_1 - \tau \frac{\partial f^{(0)}}{\partial t} \Big|_0 - \Lambda_0 f^{(0)}, \quad (4.11)$$

$$\mathbb{L}f^{(2)} = \Omega_2 - \tau \frac{\partial f^{(1)}}{\partial t} \Big|_0 - \Lambda_0 f^{(1)} - \tau \frac{\partial f^{(0)}}{\partial t} \Big|_1 - \Lambda_1 f^{(0)}, \quad (4.12)$$

and so on. The subscripts 0 and 1 in the right hand side of these equations mean that we replace $\varepsilon \partial F / \partial t|_m = J\delta_{0m} - J_m$, $\partial n^\pm / \partial t|_m = A_m^\pm$. Moreover, inserting (4.1) and (4.6) into (3.10) yields the following compatibility conditions:

$$f_0^{(1)0} = 0, \quad f_0^{(1)3} = \frac{\beta}{g} \text{Im} f_1^{(0)2}, \quad (4.13)$$

$$f_0^{(2)0} = 0, \quad (4.14)$$

$$f_0^{(2)3} = \frac{\beta}{g} \text{Im} f_1^{(1)2} + \frac{\beta^2}{4g^2} (f_0^{(0)3} - \text{Re} f_2^{(0)3}) - \frac{\beta\gamma}{g^2} \text{Im} \left(f_1^{(0)2} - \frac{f_2^{(0)2}}{2} \right),$$

etc.

To solve (4.10) for $f^{(0)} \equiv \varphi$, we first note that

$$-\tau \Theta \varphi = \sum_{j=-\infty}^{\infty} i\vartheta_j \varphi_j e^{ijkl}, \quad (4.15)$$

$$\vartheta_j \equiv \frac{\tau e j l}{\hbar} \langle F \rangle_j. \quad (4.16)$$

Then (4.10), (3.22) and (4.1) yield

$$\varphi_j^0 = \frac{\phi_j^+ + \phi_j^-}{2} \frac{1 - i\vartheta_j}{1 + \vartheta_j^2}, \quad \varphi_j^1 = \varphi_j^2 = 0, \quad \varphi_j^3 = \frac{\phi_j^+ - \phi_j^-}{2} \frac{1 - i\vartheta_j}{1 + \vartheta_j^2}, \quad (4.17)$$

where we have used that the Fourier coefficients

$$\phi_j^\pm = \frac{l}{\pi} \int_0^{\pi/l} \cos(jkl) \phi^\pm dk, \quad (4.18)$$

are real because ϕ^\pm are even functions of k . Similarly, the solution of (4.11) is $f^{(1)} \equiv \psi$ with

$$\begin{aligned} \psi_j^m &= r_j^m \frac{1 - i\vartheta_j}{1 + \vartheta_j^2} \quad (m = 0, 3), \\ \psi_j^1 &= \frac{(1 + i\vartheta_j) r_j^1 + \delta_1 r_j^2}{(1 + i\vartheta_j)^2 + \delta_1^2}, \\ \psi_j^2 &= \frac{(1 + i\vartheta_j) r_j^2 - \delta_1 r_j^1}{(1 + i\vartheta_j)^2 + \delta_1^2}. \end{aligned} \quad (4.19)$$

Here r is the right hand side of (4.11). The balance equations can be found in two ways. We can calculate A_m^\pm for $m = 0, 1$ by using the compatibility conditions (4.13) and (4.14) in Equations (4.11) and (4.12), respectively. More simply, we can insert the solutions (4.17) and (4.19) in the balance equations (3.25) and in the Ampère's law (3.27). The result is:

$$\frac{\partial n^\pm}{\partial t} + \Delta^- D_\pm(n^+, n^-, F) = \pm R(n^+, n^-, F), \quad (4.20)$$

$$\varepsilon \frac{\partial F}{\partial t} + \frac{e}{\hbar} \left\langle [\alpha(\phi_1^+ + \phi_1^-) + \gamma(\phi_1^+ - \phi_1^-)] \frac{\vartheta_1}{1 + \vartheta_1^2} \right\rangle_1 + \frac{2e}{\hbar} [\beta \text{Re}\langle \psi_1^2 \rangle_1 - \alpha \text{Im}\langle \psi_1^0 \rangle_1 - \gamma \text{Im}\langle \psi_1^3 \rangle_1] = J, \quad (4.21)$$

$$D_\pm = \frac{\alpha \pm \gamma}{\hbar} \left[\frac{\phi_1^\pm \vartheta_1}{1 + \vartheta_1^2} - \text{Im}(\psi_1^0 \pm \psi_1^3) \right] + \frac{\beta}{\hbar} \text{Re}\psi_1^2 \pm \frac{\beta^2 \vartheta_2}{4g\hbar} \frac{\phi_2^+ + \phi_2^-}{1 + \vartheta_2^2}, \quad (4.22)$$

$$R = -\frac{\delta_2 n^+}{\tau} - \frac{\beta^2 \vartheta_2^2 (\phi_2^+ - \phi_2^-)}{8g^2 \tau (1 + \vartheta_2^2)} + \frac{\beta}{g\tau} \vartheta_1 \text{Re}\psi_1^2 + \frac{\beta}{\hbar} (2 - \Delta^+) \text{Im}\psi_1^1. \quad (4.23)$$

Appendix A justifies this second and more direct method by showing that equivalent expressions are obtained from the compatibility conditions. Note that Eq. (4.21) can be obtained from (4.20) and the Poisson equation.

V. SPINTRONICS: QUANTUM DRIFT-DIFFUSION EQUATIONS FOR A LATERAL SUPERLATTICE WITH RASHBA SPIN-ORBIT INTERACTION

In the simpler case of a LSL with the precession term of Rashba spin-orbit interaction (but no intersubband coupling), we can obtain explicit rate equations for n^\pm by means of the Chapman-Enskog method. In the Hamiltonian (3.1), we have $\gamma = g = 0$, so that $h_3 = 0$ and $\vec{\nu} = (0, 1, 0)$. However, the Fermi-Dirac distribution is different from (2.6) for a LSL. We have to replace E_n instead of $\hbar^2 k_z^2 / (2m^*)$, sum over n for all populated QW energy levels and integrate over k_y only. Provided only E_1 is populated, we obtain the following expression instead of (3.23):

$$\phi^\pm(k; n^\pm) = \int_{-\infty}^{\infty} \frac{D_\Gamma(E - \mathcal{E}^\pm(k) - E_1)}{1 + \exp\left(\frac{E - \mu^\pm}{k_B T}\right)} dE, \quad (5.1)$$

where the broadened density of states is

$$D_\Gamma(E) = \frac{1}{2\pi L_z} \int_{-\infty}^{\infty} dk_y \delta_\Gamma\left(\frac{\hbar^2 k_y^2}{2m^*} - E\right) = \frac{\sqrt{2m^*}}{2\pi\hbar L_z} \int_0^\infty dE_y \frac{\delta_\Gamma(E_y - E)}{\sqrt{E_y}}. \quad (5.2)$$

Note that (5.2) becomes the 1D density of states $D(E) = \sqrt{2m^*}\theta(E)/(2\pi\hbar L_z\sqrt{E})$ as $\Gamma \rightarrow 0+$. We have not included a factor 2 in (5.2) because all the electrons in each of the minibands (with energies $\mathcal{E}^\pm(k)$) have the same spin. Inserting (2.8) in (5.2) and using the residue theorem to evaluate the integral, we obtain

$$D_\Gamma(E) = \frac{\sqrt{m^*}}{4\pi\hbar L_z} \quad (5.3)$$

$$\times \left[\frac{\sqrt{\sqrt{E^2 + \sqrt{2}\Gamma E + \Gamma^2} + E + \frac{\Gamma}{\sqrt{2}}} - \sqrt{\sqrt{E^2 + \sqrt{2}\Gamma E + \Gamma^2} - E - \frac{\Gamma}{\sqrt{2}}}}{\sqrt{E^2 + \sqrt{2}\Gamma E + \Gamma^2}} \right.$$

$$\left. + \frac{\sqrt{\sqrt{E^2 - \sqrt{2}\Gamma E + \Gamma^2} + E - \frac{\Gamma}{\sqrt{2}}} + \sqrt{\sqrt{E^2 - \sqrt{2}\Gamma E + \Gamma^2} - E + \frac{\Gamma}{\sqrt{2}}}}{\sqrt{E^2 - \sqrt{2}\Gamma E + \Gamma^2}} \right].$$

As $E \rightarrow +\infty$, $D_\Gamma(E) \sim \sqrt{2m^*}/(2\pi\hbar L_z\sqrt{E})$, whereas $D_\Gamma(E) = O(|E|^{-5/2})$ as $E \rightarrow -\infty$. Then the convolution integral (5.1) is convergent.

In the present case, minibands correspond to electrons with spin up or down which have different energy. Scattering between minibands is the same as in (3.21), $-(\vec{\nu}f^0 + \vec{f})/\tau_{\text{sc}}$ which yields $\partial n^\pm/\partial t + \dots = \mp n^\pm/\tau_{\text{sc}}$ in (3.25), only if the chemical potential of the miniband with lowest energy, μ^- , is less than the minimum energy of the other miniband, $\mathcal{E}_{\min}^+ = \min_k \mathcal{E}^+(k)$. Otherwise ($\mu^- > \mathcal{E}_{\min}^+$), the scattering term should be $-2\vec{f}/\tau_{\text{sc}}$, which yields $\partial n^\pm/\partial t + \dots = \mp(n^+ - n^-)/\tau_{\text{sc}}$ in (3.25), thereby trying to equalize n^+ and n^- ; cf. Ref.¹⁹.

Now we shall derive the balance equations in the hyperbolic limit using the Chapman-Enskog method as in Section IV. In the scaled WPBGK equations (4.2), the operators \mathbb{L} and Λ are

$$\mathbb{L}f = f - \tau \Theta f, \quad (5.4)$$

$$\Lambda f = \delta_2 \begin{pmatrix} 0 \\ 2\vec{f} + (\vec{\nu}f^0 - \vec{f})\theta(\mathcal{E}_{\min}^+ - \mu^-) \end{pmatrix} + \frac{\alpha\tau}{\hbar} \sin kl \Delta^- f \quad (5.5)$$

$$+ \frac{\beta\tau}{\hbar} \cos kl \Delta^- \begin{pmatrix} f^2 \\ 0 \\ f^0 \\ 0 \end{pmatrix} + \frac{\beta\tau}{\hbar} \sin kl \Delta^+ \begin{pmatrix} 0 \\ f^3 \\ 0 \\ -f^1 \end{pmatrix},$$

where δ_2 is given by (4.5), $\theta(x)$ is the Heaviside unit step function and $\Omega^0 = (\phi^+ + \phi^-)/2$,

$\vec{\Omega} = (0, 1, 0) (\phi^+ - \phi^-)/2$. The hierarchy of equations (4.10) - (4.12) is simply

$$\mathbb{L}f^{(0)} = \Omega, \quad (5.6)$$

$$\mathbb{L}f^{(1)} = -\tau \frac{\partial f^{(0)}}{\partial t} \Big|_0 - \Lambda f^{(0)}, \quad (5.7)$$

$$\mathbb{L}f^{(2)} = -\tau \frac{\partial f^{(1)}}{\partial t} \Big|_0 - \Lambda f^{(1)} - \tau \frac{\partial f^{(0)}}{\partial t} \Big|_1, \quad (5.8)$$

and so on. The compatibility and solvability conditions are:

$$f_0^{(m)0} = f_0^{(m)2} = 0 \implies (\mathbb{L}f^{(m)0})_0 = (\mathbb{L}f^{(m)2})_0 = 0, \quad m \geq 1. \quad (5.9)$$

The solution $f^{(0)} \equiv \varphi$ of (5.6) is

$$\varphi_j^0 = \frac{\phi_j^+ + \phi_j^-}{2} \frac{1 - i\vartheta_j}{1 + \vartheta_j^2}, \quad \varphi_j^1 = \varphi_j^3 = 0, \quad \varphi_j^2 = \frac{\phi_j^+ - \phi_j^-}{2} \frac{1 - i\vartheta_j}{1 + \vartheta_j^2}, \quad (5.10)$$

where we have used that the Fourier coefficients ϕ_j^\pm are real because ϕ^\pm are even functions of k . Similarly, the solution of (5.7) is $f^{(1)} \equiv \psi$ with

$$\psi_j^m = r_j^m \frac{1 - i\vartheta_j}{1 + \vartheta_j^2} \quad (m = 0, 2), \quad \psi_j^1 = \psi_j^3 = 0. \quad (5.11)$$

Here r is the right hand side of (5.7). The balance equations can be found in two ways. We can calculate A_m^\pm for $m = 0, 1$ by using the solvability conditions (5.9) in Equations (5.7) and (5.8), respectively. More simply, we can insert the solutions (5.10) and (5.11) in the balance equations (3.25) and in the Ampère's law (3.27). In both cases, the result is:

$$\frac{\partial n^\pm}{\partial t} + \Delta^- D_\pm(n^+, n^-, F) = \mp R(n^+, n^-, F), \quad (5.12)$$

$$\varepsilon \frac{\partial F}{\partial t} + e \langle D_+ + D_- \rangle_1 = J, \quad (5.13)$$

$$D_\pm = -\frac{\alpha}{\hbar} \Delta^- \text{Im}(\varphi_1^0 \pm \varphi_1^2 + \psi_1^0 \pm \psi_1^2) \pm \frac{\beta}{\hbar} \Delta^- \text{Re}(\varphi_1^0 \pm \varphi_1^2 + \psi_1^0 \pm \psi_1^2), \quad (5.14)$$

$$R = \frac{n^+ - n^- \theta(\mu^- - \mathcal{E}_{\min}^+)}{\tau_{sc}}. \quad (5.15)$$

A straightforward calculation of (5.14) yields

$$\begin{aligned}
D_{\pm} = & \frac{(\alpha\vartheta_1 \pm \beta)\phi_1^{\pm}}{\hbar(1+\vartheta_1^2)} \mp \frac{\tau(\phi_1^+ - \phi_1^-)[2\alpha\vartheta_1 \pm \beta(1-\vartheta_1^2)]}{2\hbar\tau_{sc}(1+\vartheta_1^2)^2} \\
& + \frac{[2\alpha\vartheta_1 \pm \beta(1-\vartheta_1^2)]\alpha\tau}{\hbar^2(1+\vartheta_1^2)^2} \frac{\partial\phi_1^{\pm}}{\partial n^{\pm}} \left[\Delta^- \left(\frac{\alpha\vartheta_1 \pm \beta}{\hbar(1+\vartheta_1^2)} \phi_1^{\pm} \right) \pm \frac{\hbar}{\alpha\tau_{sc}}(n^+ - n^-) \right] \\
& + \frac{\alpha(3\vartheta_1^2 - 1) \pm \beta\vartheta_1(3 - \vartheta_1^2)}{\hbar(1+\vartheta_1^2)^3} \frac{l\tau^2\phi_1^{\pm}}{\hbar\varepsilon} \left(\frac{J}{e} - \left\langle \left\langle \frac{\alpha(\phi_1^+ + \phi_1^-)\vartheta_1}{\hbar(1+\vartheta_1^2)} \right\rangle \right\rangle_1 \right. \\
& \left. - \left\langle \left\langle \frac{\beta(\phi_1^+ - \phi_1^-)}{\hbar(1+\vartheta_1^2)} \right\rangle \right\rangle_1 \right) - \frac{(\alpha^2 + \beta^2)\tau}{2\hbar^2(1+\vartheta_1^2)} \Delta^- n^{\pm} \\
& + \frac{\tau}{2\hbar^2(1+\vartheta_1^2)} \left[(\alpha^2 - \beta^2 \mp 2\alpha\beta\vartheta_1) \Delta^- \left(\frac{\phi_2^{\pm}}{1+\vartheta_2^2} \right) \right. \\
& \left. + [(\beta^2 - \alpha^2)\vartheta_1 \mp 2\alpha\beta] \Delta^- \left(\frac{\vartheta_2\phi_2^{\pm}}{1+\vartheta_2^2} \right) \right].
\end{aligned} \tag{5.16}$$

We have numerically solved the system of equations (5.12) - (5.16), with the following boundary conditions in the interval $-2l \leq x \leq 0$:

$$\varepsilon \frac{\partial F}{\partial t} + \sigma F = J, \tag{5.17}$$

$$n^+ = n^- = \frac{N_D}{2}, \tag{5.18}$$

whereas in the collector $Nl \leq x \leq N(l+2)$, (5.17) and

$$\frac{\partial n^{\pm}}{\partial x} = 0 \tag{5.19}$$

hold. We have used the following values of the parameters: $\alpha = \Delta_1/2 = 8$ meV, $\beta = 2.63$ meV, $d_W = 3.1$ nm, $d_B = 1.96$ nm, $l = d_W + d_B = 5.06$ nm, $L_z = 3.1$ nm, $T = 5$ K, $\tau = 5.56 \times 10^{-14}$ s, $\tau_{sc} = 5.56 \times 10^{-13}$ s, $N_D = 4.048 \times 10^{10}$ cm $^{-2}$, $m^* = (0.067d_W + 0.15d_B)m_0/l$, $V = 3$ V, $N = 110$. We have used a large conductivity of the injecting contact $\sigma = 11.78 \Omega^{-1}\text{m}^{-1}$. With these values, we select the following units to present graphically our results: $F_M = \hbar/(e\ell\tau) = 23.417$ kV/cm, $x_0 = \varepsilon F_M l/(eN_D) = 19.4$ nm, $t_0 = \hbar/\alpha = 0.082$ ps, $J_0 = \alpha e N_D/(2\hbar) = 3.94 \times 10^4$ A/cm 2 .

Fig. 2(b) - (d) illustrates the resulting stable self-sustained current oscillations. They are due to the periodic formation of a pulse of the electric field at the cathode $x = 0$ and its motion through the LSL. Fig. 2(b) depicts the pulse when it is far from the contacts and the corresponding spin polarization is shown in Fig. 2(d). It is interesting to consider the influence of the broadening Γ and the Fermi-Dirac statistics on the oscillations. At high temperatures, Boltzmann statistics and a semiclassical approximation should provide

a good description. The semiclassical approximation is equivalent to dropping all spatial averages in our previous formulas. Since $x_0 \gg l$, the effect of dropping spatial averages should be rather small. Using Boltzmann statistics yields explicit formulas for μ^\pm in terms of n^\pm . In fact, we only have to replace $e^{(\mu^\pm - E)/(k_B T)}$ instead of the 3D Fermi distribution $[1 + e^{(E - \mu^\pm)/(k_B T)}]^{-1}$ in Eq. (5.1). Using the relation (3.24) between n^\pm and ϕ^\pm , we obtain

$$\phi^\pm = n^\pm \frac{\pi \exp\left(\frac{\alpha \cos kl \mp \beta |\sin kl|}{k_B T}\right)}{\int_0^\pi dK \exp\left(\frac{\alpha \cos K \mp \beta \sin K}{k_B T}\right)}, \quad (5.20)$$

and therefore,

$$\phi_j^\pm = n^\pm \frac{\int_0^\pi dK \cos(jK) \exp\left(\frac{\alpha \cos K \mp \beta \sin K}{k_B T}\right)}{\int_0^\pi dK \exp\left(\frac{\alpha \cos K \mp \beta \sin K}{k_B T}\right)}, \quad (5.21)$$

for $j = 0, 1, \dots$. Similar relations hold for the case of a SL with Boltzmann statistics in the tight-binding approximation.

The results are shown in Fig. 2. Fig. 2(a) depicts the relation between electron current and field for a spatially uniform stationary solution with $n^\pm = N_D/2$. We observe that all curves are similar. However the curves for $\Gamma = 0$ and $\Gamma = 1$ meV are close while the curve for $\Gamma = 5$ meV has dropped noticeably. The shapes of $J(t)$ for $\Gamma = 0$ and $\Gamma = 1$ meV in Fig. 2(b) are close and quite different from that for $\Gamma = 5$ meV. If we look at the corresponding field profiles in Fig. 2(c) and (d), for $\Gamma = 0$ and $\Gamma = 1$ meV the oscillations of the current are caused by the periodic nucleation of a pulse of the electric field at $x = 0$ and its motion towards the end of the LSL. The pulse far from the contacts shown in Fig. 2(c) is larger in the case of $\Gamma = 0$ than for $\Gamma = 1$ meV. In the case of $\Gamma = 5$ meV (not shown), the pulse created at $x = 0$ becomes attenuated and it disappears before arriving at $x = Nl$. This seems to indicate that the lowest voltage at which there exist stable self-sustained current oscillations is an increasing function of Γ : If we fix the voltage at 3 V and increase Γ , the critical voltage threshold to have stable oscillations approaches our fixed voltage of 3 V. Then the observed oscillations are smaller and the field profiles correspond to waves that vanish before reaching the end of the device, as it also occurs in models of the Gunn effect in bulk semiconductors².

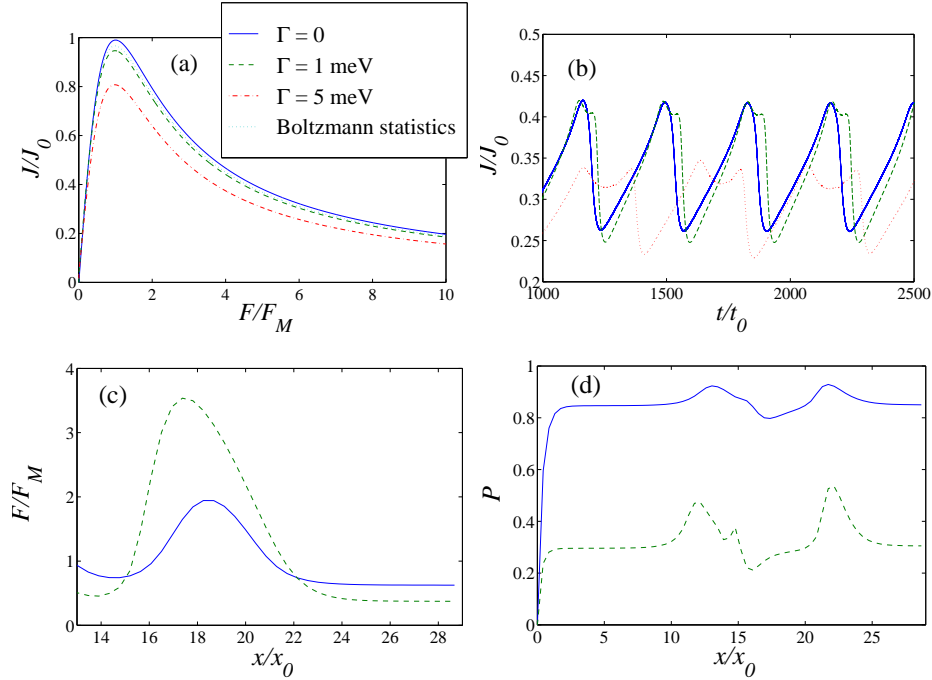


FIG. 2: (a) Electron current vs field in a spatially uniform stationary state for different values of the broadening Γ using the Fermi-Dirac distribution and for the Boltzmann distribution without broadening. (b) Total current density vs time, and the (c) electric field and (d) spin polarization profiles during current self-oscillations for $\Gamma = 0$ (solid line) and 1 meV (dashed line). Parameter values are $N = 110$, $N_D = 4.048 \times 10^{10} \text{ cm}^{-2}$, $d_B = 1.96 \text{ nm}$, $L_z = d_W = 3.1 \text{ nm}$, $l = 5.06 \text{ nm}$, $\tau = 0.0556 \text{ ps}$, $\tau_{sc} = 0.556 \text{ ps}$, $V = 3 \text{ V}$, $\sigma = 11.78 \Omega^{-1} \text{ m}^{-1}$, $T = 5 \text{ K}$, $m\alpha = 8 \text{ meV}$, $\beta = 2.63 \text{ meV}$. With these values, $\Delta_1 = 16 \text{ meV}$, $x_0 = 19.4 \text{ nm}$, $t_0 = 0.082 \text{ ps}$, $J_0 = 3.94 \times 10^4 \text{ A/cm}^2$.

VI. CONCLUSIONS

We have presented a Wigner-Poisson-BGK system of equations with a collision broadened local Fermi-Dirac distribution for strongly coupled SLs having only one populated miniband. In the hyperbolic limit in which the collision and Bloch frequencies are of the same order and dominate all other frequencies, the Chapman-Enskog perturbation method yields a quantum drift-diffusion equation for the field. Numerical solutions of this equation exhibit self-sustained oscillations of the current due to recycling and motion of charge dipole domains⁴.

For strongly coupled SLs having two populated minibands, we have introduced a peri-

odic version of the Kane Hamiltonian and derived the corresponding Wigner-Poisson-BGK system of equations. The collision model comprises two terms, a BGK term trying to bring the Wigner matrix closer to a broadened Fermi-Dirac local equilibrium at each miniband, and a scattering term that brings down electrons from the upper to the lower miniband. By using the Chapman-Enskog method, we have derived quantum drift-diffusion equations for the miniband populations which contain generation-recombination terms. As it should be, the recombination terms vanish if there is no inter-miniband scattering and the off-diagonal terms in the Hamiltonian are zero. These terms may represent a Rashba spin-orbit interaction for a lateral superlattice. For a lateral superlattice under dc voltage bias in the growth direction, numerical solutions of the corresponding quantum drift-diffusion equations show self-sustained current oscillations due to periodic recycling and motion of electric field pulses. The periodic changes of the spin polarization and spin polarized current indicate that this system acts as a spin oscillator.

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APPENDIX A: BALANCE EQUATIONS FROM COMPATIBILITY CONDITIONS

We know that $\varphi^1 = \varphi^2 = 0$ from (4.11). Then the compatibility conditions (4.13) and (4.14) become

$$\psi_0^0 = 0, \quad \psi_0^3 = 0, \quad (\text{A.1})$$

$$f_0^{(2)0} = 0, \quad f_0^{(2)3} = \frac{\beta}{g} \text{Im} \psi_1^2 + \frac{\beta^2}{4g^2} (\varphi_0^3 - \text{Re} \varphi_2^3), \quad (\text{A.2})$$

Equations (A.1) imply that $(\mathbb{L}\psi)_0^m = 0$ for $m = 0, 3$ in (4.11). Since $\varphi_0^0 = (n^+ + n^-)/2$ and $\varphi_0^3 = (n^+ - n^-)/2$, these conditions yield

$$\begin{aligned} \frac{\tau}{2} \frac{\partial(n^+ + n^-)}{\partial t} \Big|_0 - \frac{\alpha\tau}{\hbar} \Delta^- \text{Im} \varphi_1^0 - \frac{\gamma\tau}{\hbar} \Delta^- \text{Im} \varphi_1^3 &= 0, \\ \frac{\tau}{2} \frac{\partial(n^+ - n^-)}{\partial t} \Big|_0 + \delta_2 n^+ - \frac{\alpha\tau}{\hbar} \Delta^- \text{Im} \varphi_1^3 - \frac{\gamma\tau}{\hbar} \Delta^- \text{Im} \varphi_1^0 &= 0, \end{aligned}$$

wherefrom we obtain

$$A_0^\pm = \mp \frac{n^+}{\tau_{sc}} + \frac{\alpha \pm \gamma}{\hbar} \Delta^- \text{Im}(\varphi_1^0 \pm \varphi_1^3). \quad (\text{A.3})$$

Let us now calculate A_1^\pm . Equations (A.2) imply $(\mathbb{L}f^{(2)})_0^0 = 0$ and $(\mathbb{L}f^{(2)})_0^3 = f_0^{(2)3}$ given by (A.2) in (4.12). After a little algebra, we find

$$\begin{aligned} A_1^\pm &= \frac{\alpha \pm \gamma}{\hbar} \Delta^- \text{Im}(\psi_1^0 \pm \psi_1^3) - \frac{\beta}{\hbar} (\Delta^- \text{Re}\psi_1^2 \pm \Delta^+ \text{Im}\psi_1^1) \\ &\mp \frac{\beta}{g\tau} \text{Im}\psi_1^2 \pm \frac{\beta^2}{8g^2\tau} [2\text{Re}\varphi_2^3 + \phi_2^+ - \phi_2^- - 2(n^+ - n^-)]. \end{aligned} \quad (\text{A.4})$$

We will now transform (A.4) in an equivalent form by eliminating $\text{Re}\varphi_2^3$ and $\text{Im}\psi_1^2$ in favor of $\text{Re}\varphi_2^3$ and $\text{Im}\psi_1^2$, respectively. Eq. (4.10) implies that $(1 + i\vartheta_2)\varphi_2^3 = (\phi_2^+ - \phi_2^-)/2$, and therefore,

$$\text{Re}\varphi_2^3 = \vartheta_2 \text{Im}\varphi_2^3 + \frac{\phi_2^+ - \phi_2^-}{2}. \quad (\text{A.5})$$

Similarly, Eq. (4.11) implies that $(1 + i\vartheta_1)\psi_1^2 + \delta_1 \psi_1^1 = r_1^2$, and therefore,

$$\text{Im}\psi_1^2 = -\vartheta_1 \text{Re}\psi_1^2 - \delta_1 \text{Im}\psi_1^1 + \text{Im}r_1^2. \quad (\text{A.6})$$

The right hand side of (4.11) yields

$$r_1^2 = \frac{\beta}{2g} \left(\frac{1 - e^{-i2kl}}{2i} (\phi^+ - \phi^-) \right)_0 - \frac{\beta\tau}{\hbar} \Delta^- \left(\frac{1 + e^{-i2kl}}{2} \varphi^0 \right),$$

wherefrom

$$\text{Im}r_1^2 = \frac{\beta}{4g} (\phi_2^+ - \phi_2^- - n^+ + n^-) - \frac{\beta\tau}{2\hbar} \Delta^- \text{Im}\varphi_2^0. \quad (\text{A.7})$$

Inserting (A.5), (A.6) and (A.7) in (A.4), we obtain the equivalent form:

$$\begin{aligned} A_1^\pm &= \frac{\alpha \pm \gamma}{\hbar} \Delta^- \text{Im}(\psi_1^0 \pm \psi_1^3) - \frac{\beta}{\hbar} (\Delta^- \text{Re}\psi_1^2 \pm \Delta^+ \text{Im}\psi_1^1) \\ &\pm \frac{2\beta}{\hbar} \text{Im}\psi_1^1 \pm \frac{\beta}{g\tau} \vartheta_1 \text{Re}\psi_1^2 \pm \frac{\beta^2}{4g^2\tau} \vartheta_2 \text{Im}\varphi_2^3 \pm \frac{\beta^2}{2\hbar g} \Delta^- \text{Im}\varphi_2^0. \end{aligned} \quad (\text{A.8})$$

Inserting (A.3) and this expression in (4.8) and using (4.17), yield (4.20), (4.22) and (4.23).

Up to order λ^2 , we have thus proven the following statement:

By using the compatibility conditions in the hierarchy of equations (4.11), (4.12), we obtain the same balance equations for n^\pm as by direct substitution of the solutions of the

hierarchy into equations (3.25) (which arise from integration of the kinetic equation over k).

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