

Pointwise Green function bounds and long-time stability of large-amplitude noncharacteristic boundary layers

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Abstract

Using pointwise semigroup techniques of Zumbrun–Howard and Mascia–Zumbrun, we obtain sharp global pointwise Green function bounds for noncharacteristic boundary layers of arbitrary amplitude. These estimates allow us to analyze linearized and nonlinearized stability of noncharacteristic boundary layers of one-dimensional systems of conservation laws, showing that both are equivalent to a numerically checkable Evans function condition. Our results extend to the large-amplitude case results obtained for small amplitudes by Matsumura, Nishihara and others using energy estimates.

1 Introduction

Boundary layers appear in many physical settings, such as gas dynamics, MHD, and rotating fluids; see, for example, the physical discussion in [SGKO]. In this paper, we study the stability of boundary layers assuming that the boundary layer solution is noncharacteristic which means that signals are transmitted into or out of but not along the boundary. Specifically, we consider a boundary layer, or stationary solution,

$$(1.1) \quad u = \bar{u}(x), \quad \lim_{z \rightarrow +\infty} \bar{u}(z) = u_+, \quad \bar{u}(0) = u_0$$

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of a system of conservation laws on the quarter-plane

$$(1.2) \quad u_t + f(u)_x = (B(u)u_x)_x, \quad x, t > 0,$$

$u, f \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times n}$, with initial data $u(x, 0) = g(x)$ and Dirichlet boundary condition

$$(1.3) \quad u(0, t) = h(t).$$

A fundamental question is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small perturbation of \bar{u} remains close to \bar{u} , or converges time-asymptotically to \bar{u} , under the evolution of (1.2).

Long-time stability of boundary layers has been considered for scalar equations in [LN, LY] and for the equations of isentropic gas dynamics in [MN, KNZ]. The latter results, obtained by energy estimates, apply to arbitrary amplitude layers of “expansive inflow” type analogous to rarefaction waves, but only to small-amplitude layers of “compressive inflow or outflow” layers analogous to shock waves or “expansive outflow” type. For general symmetric hyperbolic-parabolic systems, stability of small-amplitude non-characteristic boundary layers has been shown in multi-dimensions for strictly parabolic systems in [GG], and in one dimension for partially parabolic (“real viscosity”) systems in [R1]. Here, in the spirit of results obtained for shock waves in [ZH, MZ3, MZ4], we show for general strictly parabolic systems of conservation laws that linearized and nonlinear stability are equivalent to a generalized spectral stability condition phrased in terms of the Evans function associated with the linearized equations about the wave, *independent of the amplitude of the boundary layer in question*.

The Evans condition is readily checkable numerically, and in some cases analytically; see [Br1, Br2, BrZ, BDG, HuZ, BHRZ, HLZ]. In particular, stability of small-amplitude uniformly noncharacteristic boundary layers has been shown for general hyperbolic–parabolic systems in multi-dimensions in [GMWZ1] using elementary Evans function arguments (convergence to the constant layer). An exhaustive numerical study for isentropic gas layers in one dimension has been carried out in [CHNZ], with the conclusion of stability for arbitrary amplitudes.

Our method of analysis is by pointwise Green function methods like those used in [ZH, MZ3, MZ4], and especially [HZ], to analyze the stability of

viscous shock layers. Similar results have been obtained for the related small-viscosity-limit problem in [GR, MeZ, GMWZ2]. In particular, we point to the analysis of Grenier and Rousset [GR] as using pointwise Green function estimates very similar to those that we use here, though adapted for different purposes.

1.1 Equations and assumptions

Consider a *viscous boundary layer*, a standing-wave solution (1.1) of a general parabolic system of conservation laws (1.2). Assume, similarly as in the treatment of the viscous shock case in [HZ]:

$$(H0) \quad f, B \in C^3.$$

$$(H1) \quad \operatorname{Re} \sigma(B) > 0.$$

$$(H2) \quad \sigma(f'(u_+)) \text{ real, distinct, and nonzero.}$$

$$(H3) \quad \operatorname{Re} \sigma(-ikf'(u_+) - k^2B(u_+)) < -\theta k^2 \text{ for all real } k, \text{ some } \theta > 0.$$

$$(H4) \text{ Solution } \bar{u} \text{ is unique.}$$

Here, (H2)(iii) corresponds to noncharacteristicity.

1.2 Linearized stability and the Evans function

After linearizing 1.2 about the stationary solution \bar{u} , we obtain the linearized equation

$$(1.4) \quad u_t = Lu := -(Au)_x + (Bu_x)_x, \quad A, B \in C^2,$$

where

$$(1.5) \quad B := B(\bar{u})$$

and

$$(1.6) \quad Au := dF(\bar{u})u - dB(\bar{u})(u, \bar{u}_x).$$

Definition 1.1. *The boundary layer \bar{u} is said to be linearly asymptotically stable, if $u(\cdot, t)$ approaches 0 as $t \rightarrow \infty$, for any solution u of (1.4) with initial data bounded in in some specified norm.*

We define the following *stability criterion*, where $D(\lambda)$ described below, denotes the Evans function associated with the linearized operator L about the layer, an analytic function analogous to the characteristic polynomial of a finite-dimensional operator, whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of L :

(1.7) There exist no zeroes of $D(\cdot)$ in the nonstable half-plane $Re\lambda \geq 0$.

As discussed, e.g., in [R2], under assumptions (H0)–(H4), this is equivalent to *strong spectral stability*, $\sigma(L) \subset \{Re\lambda < 0\}$, (ii) *transversality* of \bar{u} as a solution of the connection problem in the associated standing-wave ODE, and *hyperbolic stability* of an associated boundary value problem obtained by formal matched asymptotics. Here and elsewhere σ denotes spectrum of a linearized operator or matrix.

Our first main result is as follows.

Theorem 1.2. *Assuming (H0)–(H4), linearized asymptotic $L^1 \cap L^p \rightarrow L^p$ stability, $p > 1$, is equivalent to (1.7).*

Theorem 1.2 is obtained as a consequence of the following detailed, point-wise bounds on the Green function $G(x, t; y)$ of the linearized evolution equations (1.4) with homogeneous boundary conditions (more properly speaking, a distribution), defined by:

- (i) $(\partial_t - L_x)G = 0$ in the distributional sense, for all $x, y, t > 0$;
- (ii) $G(x, t; y) \rightarrow \delta(x - y)$ as $t \rightarrow 0$;
- (iii) $G(0, t; y) \equiv 0$, for all $y, t > 0$.

Denote by

$$(1.8) \quad a_1^+ < a_2^+ < \cdots < a_n^+$$

the eigenvalues of of the limiting convection matrix $A_+ := df(u_+)$.

Then, our second main result is as follows.

Theorem 1.3. *Assuming (H0)–(H4) and stability condition (1.7),*

$$(1.9) \quad \begin{aligned} & |\partial_x^\gamma \partial_y^\alpha G(x, t; y)| \leq C e^{-\eta(|x-y|+t)} \\ & + C(t^{-|\alpha|/2} + |\alpha|e^{-\eta|y|} + |\gamma|e^{-\eta|x|}) \left(\sum_{k=1}^n t^{-1/2} e^{-(x-y-a_k^- t)^2/Mt} \right. \\ & \left. + \sum_{a_k^+ < 0, a_j^+ > 0} \chi_{\{|a_k^+ t| \geq |y|\}} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^+|))^2/Mt} \right), \end{aligned}$$

$0 \leq |\alpha|, |\gamma| \leq 1$, for some $\eta, C, M > 0$, where x^+ denotes the positive/negative part of x , indicator function $\chi_{\{|a_k^- t| \geq |y|\}}$ is 1 for $|a_k^- t| \geq |y|$ and 0 otherwise.

1.3 Nonlinear stability

Definition 1.4. *The boundary layer \bar{u} is said to be nonlinearly asymptotically stable if $\tilde{u}(\cdot, t)$ exists for all $t \geq 0$ and approaches \bar{u} as $t \rightarrow \infty$, for any solution \tilde{u} of (1.2) with initial data sufficiently close in some norm to the original layer \bar{u} .*

Denoting by

$$(1.10) \quad a_1^+ < a_2^+ < \cdots < a_n^+$$

the eigenvalues of the limiting convection matrix $A_+ := df(u_+)$, define

$$(1.11) \quad \theta(x, t) := \sum_{a_j^+ > 0} (1+t)^{-1/2} e^{-|x-a_j^+ t|^2/Lt},$$

$$(1.12) \quad \psi_1(x, t) := \chi(x, t) \sum_{a_j^+ > 0} (1+|x|+t)^{-1/2} (1+|x-a_j^+ t|)^{-1/2},$$

and

$$(1.13) \quad \psi_2(x, t) := (1 - \chi(x, t))(1 + |x - a_n^+ t| + t^{1/2})^{-3/2},$$

where $\chi(x, t) = 1$ for $x \in [0, a_n^+ t]$ and $\chi(x, t) = 0$ otherwise and $L > 0$ is a sufficiently large constant. For simplicity, take B identically constant. Then, our third and final main result is as follows.

Theorem 1.5. *Assuming (H0)–(H4), $B \equiv \text{constant}$, and the linear stability condition (1.7), the profile \bar{u} is nonlinearly asymptotically stable with respect to perturbations g, h in initial and boundary data satisfying*

$$|g(x)| \leq E_0(1 + |x|)^{-3/2}, \quad |h(t)| \leq E_0(1 + |t|)^{-3/2}, \quad |h'(t)| \leq E_0(1 + |t|)^{-1}$$

for E_0 sufficiently small. More precisely,

$$(1.14) \quad |\tilde{u}(x, t) - \bar{u}(x)| \leq CE_0(\theta + \psi_1 + \psi_2)(x, t),$$

where \tilde{u} denotes the solution of (1.2) with initial data $\tilde{g} = \bar{u} + g$ and boundary data $\tilde{h} = u_0 + h$.

Remark 1.6. *Pointwise bound (1.14) yields as a corollary the sharp L^p decay rate*

$$(1.15) \quad |\tilde{u}(x, t) - \bar{u}(x)|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty.$$

1.4 Discussion and open problems

The case of boundary layers is quite analogous to the undercompressive shock case; in particular, pointwise estimates as in [HZ] appear to be necessary to close the one-dimensional analysis by a linearized semigroup approach suitable for large-amplitude layers. (On the other hand, small-amplitude stability has been established using energy estimates in, e.g., [MN, GG].) A new feature of the present analysis as compared to those of [HZ, HR, HRZ] is the admission of perturbations in boundary as well as initial data. Open problems are extensions to systems with physical (partial) or quasilinear viscosity and to multi-dimensional boundary layers.

2 The Evans Function

Before starting the analysis, we review the basic Evans function methods and gap/conjugation lemma.

2.1 The gap/conjugation lemma

Consider a family of first order ODE systems on the half-line:

$$(2.1) \quad \begin{aligned} W' &= \mathbb{A}(x, \lambda)W, & \lambda \in \Omega & \text{ and } x > 0, \\ \mathbb{B}(\lambda)W &= 0, & \lambda \in \Omega & \text{ and } x = 0. \end{aligned}$$

These systems of ODEs should be considered as a generalized eigenvalue equation, with λ representing frequency. We assume that the boundary matrix \mathbb{B} is analytic in λ and that the coefficient matrix \mathbb{A} is analytic in λ as a function from Ω into $L^\infty(x)$, C^K in x , and approaches exponentially to a limit $\mathbb{A}_+(\lambda)$ as $x \rightarrow \infty$, with uniform exponentially decay estimates

$$(2.2) \quad |(\partial/\partial x)^k(\mathbb{A} - \mathbb{A}_+)| \leq C_1 e^{-\theta|x|/C_2}, \quad \text{for } x > 0, 0 \leq k \leq K,$$

$C_j, \theta > 0$, on compact subsets of Ω . Now we can state a refinement of the “Gap Lemma” of [GZ, KS], relating solutions of the variable-coefficient ODE to the solutions of its constant-coefficient limiting equations

$$(2.3) \quad Z' = \mathbb{A}_+(\lambda)Z$$

as $x \rightarrow +\infty$.

Lemma 2.1 (Conjugation Lemma [MeZ]). *Under assumption (2.2), there exists locally to any given $\lambda_0 \in \Omega$ a linear transformation $P_+(x, \lambda) = I + \Theta_+(x, \lambda)$ on $x \geq 0$, Φ_+ analytic in λ as functions from Ω to $L^\infty[0, +\infty)$, such that:*

(i) $|P_+|$ and their inverses are uniformly bounded, with

$$(2.4) \quad |(\partial/\partial\lambda)^j(\partial/\partial x)^k\Theta_+| \leq C(j)C_1C_2e^{-\theta|x|/C_2} \quad \text{for } x > 0, 0 \leq k \leq K+1,$$

$j \geq 0$, where $0 < \theta < 1$ is an arbitrary fixed parameter, and $C > 0$ and the size of the neighborhood of definition depend only on θ, j , the modulus of the entries of \mathbb{A} at λ_0 , and the modulus of continuity of \mathbb{A} on some neighborhood of $\lambda_0 \in \Omega$.

(ii) The change of coordinates $W := P_+Z$ reduces (2.1) on $x \geq 0$ to the asymptotic constant-coefficient equations (2.3). Equivalently, solutions of (2.1) may be conveniently factorized as

$$(2.5) \quad W = (I + \Theta_+)Z_+,$$

where Z_+ are solutions of the constant-coefficient equations, and Θ_+ satisfy bounds.

Proof. As described in [MZ3], for $j = k = 0$ this is a straightforward corollary of the gap lemma as stated in [Z.3], applied to the “lifted” matrix-valued ODE

$$P' = \mathbb{A}_+P - P\mathbb{A} + (\mathbb{A} - \mathbb{A}_+)P$$

for the conjugating matrices P_+ . The x -derivative bounds $0 < k \leq K+1$ then follow from the ODE and its first K derivatives. Finally, the λ -derivative bounds follow from standard interior estimates for analytic functions. \square

Definition 2.2. *Following [AGJ], we define the domain of consistent splitting for the ODE system $W' = \mathbb{A}(x, \lambda)W$ as the (open) set of λ such that the limiting matrix \mathbb{A}_+ is hyperbolic (has no center subspace) and the boundary matrix \mathbb{B} is full rank, with $\dim S_+ = \text{rank } \mathbb{B}$.*

Lemma 2.3. *On any simply connected subset of the domain of consistent splitting, there exist analytic bases $\{v_1, \dots, v_k\}^+$ and $\{v_{k+1}, \dots, v_N\}^+$ for the subspaces S_+ and U_+ defined in Definition 2.2.*

Proof. By spectral separation of U_+ , S_+ , the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [Kat], pp. 99–102. \square

Corollary 2.4. *By the Conjugation Lemma, on the domain of consistent splitting, the stable manifold of solutions decaying as $x \rightarrow +\infty$ of (2.1) is*

$$(2.6) \quad \mathcal{S}^+ := \text{span} \{P_+v_1^+, \dots, P_+v_k^+\},$$

where $W_+^j := P_+v_j^+$ are analytic in λ and C^{K+1} in x for $\mathbb{A} \in C^K$.

2.2 Definition of the Evans Function

On any simply connected subset of the domain of consistent splitting, let $W_1^+, \dots, W_k^+ = P_+v_1^+, \dots, P_+v_k^+$ be the analytic basis described in Corollary 2.4 of the subspace \mathcal{S}^+ of solutions W of (2.1) satisfying the boundary condition $W \rightarrow 0$ at $+\infty$. Then, the *Evans function* for the ODE systems $W' = \mathbb{A}(x, \lambda)W$ associated with this choice of limiting bases is defined as the $k \times k$ Gramian determinant

$$(2.7) \quad \begin{aligned} D(\lambda) &:= \det \left(\mathbb{B}W_1^+, \dots, \mathbb{B}W_k^+ \right)_{|x=0, \lambda} \\ &= \det \left(\mathbb{B}P_+v_1^+, \dots, \mathbb{B}P_+v_k^+ \right)_{|x=0, \lambda}. \end{aligned}$$

Remark 2.5. *Note that D is independent of the choice of P_+ as, by uniqueness of stable manifolds, the exterior products (minors) $P_+v_1^+ \wedge \dots \wedge P_+v_k^+$ are uniquely determined by their behavior as $x \rightarrow +\infty$.*

Proposition 2.6. *Both the Evans function and the subspace \mathcal{S}^+ are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for λ within this region, equation (2.1) admits a nontrivial solution $W \in L^2(x > 0)$ if and only if $D(\lambda) = 0$.*

Proof. Analyticity follows by uniqueness, and local analyticity of P_+ , v_k^+ . Noting that the first $P_+v_j^+$ are a basis for the stable manifold of (2.1) at $x \rightarrow +\infty$, we find that the determinant of $\mathbb{B}P_+v_j^+$ vanishes if and only if $\mathbb{B}(\lambda)$ has nontrivial kernel on $\mathcal{S}_+(\lambda, 0)$, whence the second assertion follows. \square

Remark 2.7. *In this case that the ODE system describes an eigenvalue equation associated with an ordinary differential operator L , Proposition 2.6 implies that eigenvalues of L agree in location with zeroes of D . (Indeed, they agree also in multiplicity; see [GJ1, GJ2]; Lemma 6.1, [ZH]; or Proposition 6.15 of [MZ3].)*

When $\ker \mathbb{B}$ has an analytic basis $v_{k+1}^0, \dots, v_{N-k}^0$, for example, in the commonly occurring case, as here, that $\mathbb{B} \equiv \text{constant}$, we have the following useful alternative formulation. This is the version that we will use in our analysis of the Green function and Resolvent kernel.

Proposition 2.8. *Let $v_{k+1}^0, \dots, v_{N-k}^0$ be an analytic basis of $\ker \mathbb{B}$, normalized so that $\det(\mathbb{B}^*, v_{k+1}^0, \dots, v_N^0) \equiv 1$. Then, the solutions W_j^0 of (2.1) determined by initial data $W_j^0(\lambda, 0) = v_j^0$ are analytic in λ and C^{K+1} in x , and*

$$(2.8) \quad D(\lambda) := \det \left(W_1^+, \dots, W_k^+, W_{k+1}^0, \dots, W_N^0 \right)_{|x=0, \lambda}.$$

Proof. Analyticity/smoothness follow by analytic/smooth dependence on initial data/parameters. By the chosen normalization, and standard properties of Gramian determinants, $D(\lambda) = \det \left(W_1^+, \dots, W_k^+, v_{k+1}^0, \dots, v_N^0 \right)_{|x=0, \lambda}$, yielding (2.8). \square

3 Construction of the Resolvent kernel

In this section we construct the explicit form of the resolvent kernel, which is nothing more than the Green function $G_\lambda(x, y)$ associated with the elliptic operator $(L - \lambda I)$, where

$$(3.1) \quad (L - \lambda I)G_\lambda(\cdot, y) = \delta_y I, \quad G_\lambda(0, y) \equiv 0.$$

Let Λ be the region of consistent splitting for L . It is an established fact (see [He]) that the resolvent $(L - \lambda I)^{-1}$ and the Green function $G_\lambda(x, y)$ are

meromorphic in λ on Λ , with isolated poles of finite order. G_λ in fact admits a meromorphic extension to a sector

$$(3.2) \quad \Omega_\theta = \{\lambda : \operatorname{Re}(\lambda) \geq -\theta_1 - \theta_2 |\operatorname{Im}(\lambda)|\}, \quad \theta_1, \theta_2 > 0.$$

Writing the associated eigenvalue equation in the form of a first-order system (2.1), we obtain

$$(3.3) \quad W' = \mathbb{A}(\lambda, x)W, \quad \mathbb{B}W(0) = 0,$$

where

$$W = \begin{pmatrix} w \\ w' \end{pmatrix} \in \mathbb{C}^{2n}, \quad \mathbb{A} = \begin{pmatrix} 0 & I \\ \lambda B^{-1} + A'B^{-1} & AB^{-1} - B'B^{-1} \end{pmatrix}$$

and $\mathbb{B} \equiv \text{constant}$ is the rank- n projection onto the first coordinate w of W , with kernel spanned by the constant basis $v_{n+j}^0 = e_{n+j}$, $j = 1, \dots, n$ and e_j the j th standard basis element.

Denote by

$$(3.4) \quad \Phi^0 = (\phi_1^0(x; \lambda) \ \cdots \ \phi_n^0(x; \lambda)) = (W_1^0 \ \cdots \ W_n^0)$$

and

$$(3.5) \quad \Phi^+ = (\phi_1^+(x; \lambda) \ \cdots \ \phi_n^+(x; \lambda)) = (W_{n+1}^+ \ \cdots \ W_{2n}^+) = (P_+ v_1^+ \ \cdots \ P_+ v_k^+)$$

the matrices whose columns span the subspaces of solutions of (2.1) decaying at $x = 0, +\infty$ respectively, denoting (analytically chosen) complementary subspaces by

$$(3.6) \quad \Psi^0 = (\psi_1^0(x; \lambda) \ \cdots \ \psi_n^0(x; \lambda)) = (W_{n+1}^0 \ \cdots \ W_{2n}^0)$$

and

$$(3.7) \quad \Psi^+ = (\psi_1^+(x; \lambda) \ \cdots \ \psi_n^+(x; \lambda)) = (W_1^+ \ \cdots \ W_n^+).$$

As described in the previous subsection, eigenfunctions decaying at both $0, +\infty$ occur precisely when the subspaces $\text{span } \Phi^0$ and $\text{span } \Phi^+$ intersect, i.e., at zeros of the Evans function defined in (2.8):

$$(3.8) \quad D_L(\lambda) := \det(\Phi^0, \Phi^+)|_{x=0} = (\phi_1^0 \wedge \cdots \wedge \phi_n^0 \wedge \phi_1^+ \wedge \cdots \wedge \phi_n^+)|_{x=0}.$$

Lemma 3.1 ([GZ, ZH]). *For $\theta_1, \theta_2 > 0$ sufficiently small, D_L is locally analytic on sector Ω_θ as defined in (3.2).*

Proof. Direct calculation showing that the domain Λ of consistent splitting is contained in $\Omega_\theta - B(0, r)$ for $r > 0$ arbitrary and θ sufficiently small, with v_j^+ extending analytically to $B(0, r)$. \square

Lemma 3.2. *Let $H_\lambda(x, y)$ denote the Green function for the adjoint operator $(L - \lambda I)^*$ on the half-plane $x \geq 0$. Then $G_\lambda(x, y) = H_\lambda^*(x, y)$. In particular, for $x \neq y$, the matrix $z = G_\lambda(x, \cdot)$ satisfies*

$$(3.9) \quad (z'B)' = -z'A + z\lambda.$$

Proof. Standard duality argument; see [ZH] for operators on the whole line. \square

Considering (3.9) as an ODE system for the vector $Z = (z, z')$, it becomes

$$(3.10) \quad Z' = Z\tilde{\mathbb{A}}(\lambda, x),$$

where

$$(3.11) \quad \tilde{\mathbb{A}} = \begin{pmatrix} 0 & \lambda B^{-1} - A'B^{-1} \\ I & -AB^{-1} - B'B^{-1} \end{pmatrix}.$$

Lemma 3.3 ([ZH]). *Z is a solution of (3.11) if and only if $Z\mathcal{S}W \equiv \text{constant}$ for any solution W of (2.1), where $\mathcal{S} = \begin{pmatrix} -A & B \\ -B & 0 \end{pmatrix}$.*

Proof. Direct computation/comparison with 0 of $(Z\mathcal{S}W)'$; see [ZH]. \square

Using Lemma 3.3, we can define dual bases \tilde{W}_j^0 and \tilde{W}_j^+ by the relations

$$(3.12) \quad \tilde{W}_j^{0,+} \mathcal{S}W_k^{0,+} = \delta_k^j.$$

Likewise, $\tilde{\mathbb{A}}_{0,+}$ can be defined as

$$(3.13) \quad \tilde{\mathbb{A}}_{0,+} = \begin{pmatrix} 0 & \lambda B_{0,+}^{-1} \\ I & -A_{0,+}B_{0,+}^{-1} \end{pmatrix}.$$

We define also the dual subspaces

$$(3.14) \quad \tilde{\Phi}^0 = (\tilde{\phi}_1^0(x; \lambda) \ \cdots \ \tilde{\phi}_n^0(x; \lambda)) = (\tilde{W}_{n+1}^0 \ \cdots \ \tilde{W}_{2n}^0),$$

$$(3.15) \quad \tilde{\Phi}^+ = (\tilde{\phi}_1^+(x; \lambda) \cdots \tilde{\phi}_n^+(x; \lambda)) = (\tilde{W}_1^+ \cdots \tilde{W}_n^+),$$

$$(3.16) \quad \tilde{\Psi}^0 = (\tilde{\psi}_1^0(x; \lambda) \cdots \tilde{\psi}_n^0(x; \lambda)) = (\tilde{W}_1^0 \cdots \tilde{W}_n^0),$$

$$(3.17) \quad \tilde{\Psi}^+ = (\tilde{\psi}_1^+(x; \lambda) \cdots \tilde{\psi}_n^+(x; \lambda)) = (\tilde{W}_{n+1}^+ \cdots \tilde{W}_{2n}^+).$$

With these preparations, the construction of the Resolvent kernel goes exactly as in the construction performed in [ZH, MZ3] on the whole line.

Lemma 3.4. *We have the representation*

$$(3.18) \quad \begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \begin{cases} \Phi^+(\lambda, x)M^+(\lambda)\tilde{\Psi}^0(\lambda, y) & \text{for } x > y, \\ \Phi^0(\lambda, x)M^0(\lambda)\tilde{\Psi}^+(\lambda, y) & \text{for } x < y, \end{cases}$$

where $M^{0,+}$ are to be determined.

Proof. See [ZH] Lemma 4.6. □

Using Lemma 3.4, we find the explicit coordinate-free representation for $x > y$:

$$(3.19) \quad \begin{pmatrix} G_\lambda & G_{\lambda_y} \\ G_{\lambda_x} & G_{\lambda_{xy}} \end{pmatrix} = \mathcal{F}^{z \rightarrow x} \Pi_+(z) \mathcal{S}^{-1}(z) \tilde{\Pi}_0(z) \tilde{\mathcal{F}}^{z \rightarrow y},$$

where

$$(3.20) \quad \Pi_+(y) = (\Phi^+(y), 0)(\Phi^+(y), \Phi^-(y))^{-1},$$

$$(3.21) \quad \tilde{\Pi}_0(y) = \begin{pmatrix} \tilde{\Psi}^0(y) \\ \tilde{\Psi}^+(y) \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\Psi}^0(y) \\ 0 \end{pmatrix},$$

$$(3.22) \quad \mathcal{F}^{z \rightarrow x} = (\Phi^+(x), \Phi^0(x))(\Phi^+(z), \Phi^0(z))^{-1},$$

$$(3.23) \quad \tilde{\mathcal{F}}^{z \rightarrow y} = \begin{pmatrix} \tilde{\Psi}^0(z) \\ \Phi^+(z) \end{pmatrix} \begin{pmatrix} \Psi^0(y) \\ \Psi^+(y) \end{pmatrix}^{-1},$$

and similarly for $x < y$.

Corollary 3.5. *The resolvent kernel may be expressed as*

$$(3.24) \quad G_\lambda(x, y) = \begin{cases} (I_n, 0)\Phi^+(x; \lambda)M^+(\lambda)\tilde{\Psi}^{0*}(y; \lambda)(I_n, 0)^{tr} & x > y, \\ -(I_n, 0)\Phi^0(x; \lambda)M^0(\lambda)\tilde{\Psi}^{+*}(y; \lambda)(I_n, 0)^{tr} & x < y, \end{cases}$$

where

$$(3.25) \quad M(\lambda) := \text{diag}(M^+(\lambda), M^0(\lambda)) = \Phi^{-1}(z; \lambda)\bar{\mathcal{S}}^{-1}(z)\tilde{\Psi}^{-1*}(z; \lambda).$$

4 Low-frequency bounds

Our goal in this section is the estimation of the resolvent kernel in the critical regime $|\lambda| \rightarrow 0$, i.e., the large time behavior of the Green function G , or global behavior in space and time. We are basically following the same treatment as that carried out for viscous shock waves of strictly parabolic conservation laws in [ZH, MZ3]; we refer to those references for details. In the low frequency case the behavior is essentially governed by the equation

$$(4.1) \quad U_t = L_+ U := -A_+ U_x + B_+ U_{xx}$$

Proposition 4.1. *Assuming (H0)-(H4), let K be the order of the pole of G_λ at $\lambda = 0$ and r be sufficiently small that there are no other poles in $B(0, r)$. Then for $\lambda \in \Omega_\theta$ such that $|\lambda| \leq r$ and for $x > y > 0$ we have*

$$(4.2) \quad \begin{pmatrix} G_\lambda & G_{\lambda y} \\ G_{\lambda x} & G_{\lambda xy} \end{pmatrix} = \sum_{j,k} d_{jk}(\lambda)\phi_j^+(x)\tilde{\psi}_k^+(y) + \sum_{j,k} (\lambda)\phi_k^+(x)\tilde{\phi}_k^+(y),$$

where $d_{jk}(\lambda) = \mathcal{O}(\lambda^{-K})$ and $e_{jk}(\lambda) = \mathcal{O}(\lambda^{1-K})$ are scalar meromorphic functions, moreover $K \leq$ order of vanishing of the Evans function $D(\lambda)$ at $\lambda = 0$.

Proof. See [ZH] Proposition 7.1 for the first statement and theorem 6.3 for the second statement linking order K of the pole to multiplicity of the zero of the Evans Function. \square

Lemma 4.2. *Assuming (H0)-(H4), for $|\lambda|$ sufficiently small, the eigenvalue equation $(L_+ - \lambda)W = 0$ associated with the limiting, constant-coefficient operator L_+ has a basis of $2n$ solutions $\bar{W}_j^+ = e^{\mu_j^+(\lambda)x}V_j(\lambda)$ where μ_j^+ and V_j^+ are analytic in λ , consisting of n fast modes*

$$(4.3) \quad \begin{aligned} \mu_j^+ &= \gamma_j^+ + \mathcal{O}(\lambda), \\ V_j^+ &= S_j^+ + \mathcal{O}(\lambda), \end{aligned}$$

where γ_j^+ , S_j^+ are eigenvalues and associated right eigenvectors of $B_+^{-1}A_+$, and n slow modes

$$(4.4) \quad \begin{aligned} \mu_{r+j}^+(\lambda) &:= -\lambda/a_j^+ + \lambda^2\beta_j^+/a_j^{+3} + \mathcal{O}(\lambda^3), \\ V_{r+j}^+(\lambda) &:= r_j^+ + \mathcal{O}(\lambda), \end{aligned}$$

where a_j^+ , l_j^+ , r_j^+ are eigenvalues and left and right eigenvectors of $A_+ := dF(u_+)$, and $\beta_j^+ := l_j^+ B_+ r_j^+ > 0$ with $B_+ := B(u_+)$. The same is true for the adjoint eigenvalue equation

$$(L^+ - \lambda)^* Z = 0,$$

i.e, it has a basis of solutions

$$\tilde{W}_j^+ = e^{-\mu_j^+(\lambda)x} \tilde{V}_j(\lambda)$$

with

$$(4.5) \quad \tilde{V}_j^+(\lambda) = \tilde{T}_j^+ + \mathcal{O}(\lambda),$$

$$(4.6) \quad \tilde{V}_{r+j}^+(\lambda) = l_j^+ + \mathcal{O}(\lambda),$$

\tilde{V}^+ analytic in λ .

Proof. See [MZ3]. □

Proposition 4.3. Assume (H0)-(H4) and (1.7), then, for $r > 0$ sufficiently small, the Resolvent kernel G_λ associated with the linearized evolution equation

$$(4.7) \quad U_t = L_+ U := -A_+ U_x + B_+ U_{xx}$$

satisfies, for $0 \leq y \leq x$:

$$(4.8) \quad \begin{aligned} |\partial_x^\gamma \partial_y^\alpha G_\lambda(x, t; y)| &\leq C(|\lambda|^\gamma + e^{-\theta|x|})(|\lambda|^\alpha + e^{-\theta|y|}) \left(\sum_{a_k^+ > 0} |e^{(-\lambda/a_k^+ + \lambda^2\beta_k^+/a_k^{+3})(x-y)}| \right. \\ &\quad \left. + \sum_{a_k^+ < 0, a_j^+ > 0} |e^{(-\lambda/a_j^+ + \lambda^2\beta_j^+/a_j^{+3})x + (\lambda/a_k^+ - \lambda^2\beta_k^+/a_k^{+3})y}| \right), \end{aligned}$$

$0 \leq |\alpha|, |\gamma| \leq 1$, $\theta > 0$, with similar bounds for $0 \leq x \leq y$. Moreover, each term in the summation on the righthand side of (4.8) bounds a separately analytic function.

Proof. By 1.8 D does not vanish on $Re(\lambda) \geq 0$, hence, by continuity, on $|\lambda| \leq r$. Thus, according to (4.2), all $|d_{jk}(\lambda)|$ are uniformly bounded on $|\lambda| \leq r$, and so it is enough to find estimates for fast and slow modes ϕ_j^+ , $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. By using (3.5) we find:

$$(4.9) \quad \begin{pmatrix} \phi_j^+ \\ \partial_x \phi_j^+ \end{pmatrix} = e^{\mu_j(\lambda)x} P^+ \begin{pmatrix} V_j \\ \mu_j V_j \end{pmatrix} = e^{\mu_j(\lambda)x} (I + \Theta) \begin{pmatrix} V_j \\ \mu_j V_j \end{pmatrix}$$

and similarly for $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. Now using (2.4) and the fact, by (4.4), that $e^{\mu_j(\lambda)x}$ is of order $e^{-|\theta x|}$ for fast modes and order $e^{-\lambda/a_j^+ + \lambda^2 \beta_j^+ / a_j^+{}^3 + \mathcal{O}(\lambda^3)}$ for slow modes, substituting this and corresponding dual estimates in (4.9) and grouping terms, we obtain the result. \square

5 High frequency bounds

To analyze the high frequency behavior of the Green function of the boundary layer, we first establish some bounds for the projection terms in the Green function, using the symmetric formula

$$(5.1) \quad \begin{pmatrix} G_\lambda(x, y) & \partial_y G_\lambda(x, y) \\ \partial_x G_\lambda(x, y) & \partial_x \partial_y G_\lambda(x, y) \end{pmatrix} = \begin{cases} \mathcal{F}^{y \rightarrow x} \Pi_+(x) \mathcal{S}^{-1}(y) & \text{if } x > y, \\ \tilde{\mathcal{F}}^{x \rightarrow y} \tilde{\Pi}_+(x) \mathcal{S}^{-1}(x) & \text{if } x < y. \end{cases}$$

By setting $\bar{x} = |\lambda^{\frac{1}{2}}|x$, $\bar{\lambda} = \frac{\lambda}{|\lambda|}$, $\bar{B}(\bar{x}) = B(\frac{\bar{x}}{\lambda^{\frac{1}{2}}})$, $\bar{w}(\bar{x}) = w(\frac{x}{\lambda^{\frac{1}{2}}})$ in the eigenvalue equation $Lw = \lambda w$ associated with (1.4) we obtain

$$(5.2) \quad \bar{W}' = \bar{\mathbb{B}}\bar{W} + \mathcal{O}(|\lambda^{-\frac{1}{2}}|)\bar{W}$$

where

$$(5.3) \quad \bar{\mathbb{B}} = \begin{pmatrix} 0 & I \\ \bar{\lambda}\bar{B} & 0 \end{pmatrix}$$

and $\bar{\mathbb{B}}' = \mathcal{O}(|\lambda^{-\frac{1}{2}}|)$ and $|\bar{\lambda}| = 1$. Since $\bar{\mathbb{B}}(\lambda, \bar{x})$ varies within a compact set, then there are C^1 eigenprojections P_0 and P_+ with property $|P_+| = \mathcal{O}(|\lambda^{-\frac{1}{2}}|)$ and $|P_0| = \mathcal{O}(|\lambda^{-\frac{1}{2}}|)$ taking \bar{W} onto the stable and unstable subspace. By using the two new coordinates $Y_+ = P_+\bar{W}$ and $Y_0 = P_0\bar{W}$, we obtain

$$(5.4) \quad \begin{pmatrix} Y_+ \\ Y_0 \end{pmatrix}' = \begin{pmatrix} A_+ & 0 \\ 0 & A_0 \end{pmatrix} \begin{pmatrix} Y_+ \\ Y_0 \end{pmatrix} + \mathcal{O}(|\lambda^{-\frac{1}{2}}|) \begin{pmatrix} y \\ y \end{pmatrix}.$$

Equivalently, we can find continuous invertible transformations Q_+ , Q_0 such that $E_+ = Q_+ A_+ Q_+^{-1}$ and $E_0 = Q_0 A_0 Q_0^{-1}$ where

$$(5.5) \quad \operatorname{Re}(E) := \frac{1}{2}(E_+ + E_+^*) < -\beta^{-\frac{1}{2}}I.$$

in the sense of quadratic forms.

Again by coordinate change $Z_+ = Q_+ Y_+$, $Z_0 = Q_0 Y_0$ we find

$$(5.6) \quad \begin{pmatrix} z_+ \\ z_0 \end{pmatrix}' = \begin{pmatrix} E_+ & 0 \\ 0 & E_0 \end{pmatrix} \begin{pmatrix} z_+ \\ z_0 \end{pmatrix} + \mathcal{O}(|\lambda^{-\frac{1}{2}}|) \begin{pmatrix} z \\ z \end{pmatrix}$$

where

$$(5.7) \quad \frac{|\bar{w}|}{C} \leq |z| \leq C|\bar{w}|.$$

From this we find by energy estimate that

$$(5.8) \quad (|z_+|^2)' < -2\beta^{-\frac{1}{2}}|z_+|^2$$

and hence

$$(5.9) \quad \frac{|z_+(x)|}{|z_+(y)|} \leq e^{-\tilde{\beta}^{-\frac{1}{2}}|x-y|}$$

for any solution z_+ decaying at ∞ , where $\tilde{\beta} < \beta$ and thus

$$(5.10) \quad \frac{|z(x)|}{|z(y)|} \leq e^{-\beta^{-\frac{1}{2}}|x-y|}$$

for $x > y$, provided that $|\lambda|$ is sufficiently large. From this we obtain

$$(5.11) \quad \frac{|\bar{W}(x)|}{|\bar{W}(y)|} \leq C^2 e^{-\beta^{-\frac{1}{2}}|x-y|}$$

where C is as in (5.7). Applying a symmetric argument for the adjoint equation, we obtain the following lemma.

Lemma 5.1. *On the manifolds Φ_+ and $\tilde{\Psi}_+$ defined in (3.5) and (3.17), for λ sufficiently large, within the sector $\Omega_\theta = \{\lambda : \operatorname{Re}(\lambda) \geq -\theta_1 - \theta_2|\operatorname{Im}(\lambda)|\}$, $\theta_1, \theta_2 > 0$, we have in rescaled coordinates \bar{x} , for some uniform $C > 0$,*

$$(5.12) \quad |\mathcal{F}^{y \rightarrow x}|, |\tilde{\mathcal{F}}^{x \rightarrow y}| \leq C e^{-\frac{|y-x|}{C}}$$

for $x > y$ and $x < y$ respectively.

Lemma 5.2. *In rescaled coordinates \bar{x} , $\bar{\lambda}$, for the projection terms $\Pi_+(y)$ and $\tilde{\Pi}_+(x)$, the projection along Φ_0 onto Φ_+ , for λ sufficiently large,*

$$(5.13) \quad |\Pi_+(y)|, |\tilde{\Pi}_+(x)| < C$$

for some uniform $C > 0$.

Proof. Choosing the coordinates $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in \mathbb{C}^{2n}$ where $W^j = \begin{pmatrix} W_1^j \\ W_2^j \end{pmatrix}$, we show for small enough ϵ and fixed $c > 0$ such that $\frac{|W_2^j|}{|W_1^j|} \leq c\epsilon$ and $\frac{|W_1^j|}{|W_2^j|} < c$ the projection along $E = \text{span}(W^1 \dots W^n)$ onto $F = \text{span}(W^{n+1} \dots W^{2n})$

$$(5.14) \quad \Pi := (w^1 \dots w^{2n})(O_n, I_n)(w^1 \dots w^{2n})^{-1}$$

satisfies

$$(5.15) \quad |\Pi| \leq C_2(c, \epsilon)$$

To show this without loss of generality we assume that

$$(5.16) \quad (w^{n+1} \dots w^{2n}) = \begin{pmatrix} I_n \\ \mathcal{O}(\epsilon) \end{pmatrix} (w^1 \dots w^n) = \begin{pmatrix} \mathcal{O}(1) \\ I_n \end{pmatrix}.$$

Now it is sufficient to show that

$$(5.17) \quad |(w^1, \dots, w^{2n})| \leq C_2(c, \epsilon).$$

But, this amounts to showing that

$$(5.18) \quad \left| \begin{pmatrix} M & I_n \\ I_n & O \end{pmatrix} + \mathcal{O}(\epsilon) \right|^{-1} \leq C,$$

which amounts to showing that

$$(5.19) \quad \left| \begin{pmatrix} M & I_n \\ I_n & O \end{pmatrix}^{-1} \right| \leq C_2(c),$$

where $|M| \leq c$. But, this is easy to show because

$$\begin{pmatrix} M & I_n \\ I_n & O \end{pmatrix}^{-1} = \begin{pmatrix} I_n & -M \\ O & I_n \end{pmatrix},$$

and so

$$\left| \begin{pmatrix} M & I_n \\ I_n & O \end{pmatrix}^{-1} \right| \leq 1 + |M| \leq C.$$

□

Proposition 5.3. *Assume (H0)-(H4) and (1.7). Then, for $R > 0$ sufficiently large, the Resolvent kernel G_λ associated with the linearized evolution equation (4.7) satisfies, for $c, C > 0$ and $0 \leq |\alpha|, |\gamma| \leq 1$:*

$$(5.20) \quad |\partial_x^\gamma \partial_y^\alpha G_\lambda(x, y)| \leq C |\lambda|^{\frac{|\alpha|+|\gamma|-1}{2}} e^{-\sqrt{\lambda} \frac{|y-x|}{c}}$$

Proof. Recalling the coordinate-free representation (5.1) and combining with (5.12) and (5.13), we find that the Green function \bar{G}_λ in rescaled coordinates $\bar{x}, \bar{\lambda}$ satisfies

$$(5.21) \quad |\partial_{\bar{x}}^\gamma \partial_{\bar{y}}^\alpha \bar{G}_\lambda(\bar{x}, \bar{y})| \leq C e^{-\frac{|\bar{y}-\bar{x}|}{c}},$$

whence (5.20) follows in the original coordinates. \square

Remark 5.4. *The argument of Lemma 5.2 is the key new ingredient in the resolvent estimates for the boundary layer case as compared to the analysis on the whole line carried out for viscous shock layers in [ZH], making essential use of compatibility of the boundary condition with high-frequency behavior. On the whole line, there is no such requirement and high-frequency stability is automatic.*

6 Pointwise Green function bounds

With the pointwise bounds established on the resolvent kernel G_λ , we obtain pointwise bounds on the Green function through the inverse Laplace transform formula by a simplified version of the stationary-phase arguments used in [ZH] for the shock case, repeated here for completeness.

Proof of Theorem 1.3. By sectoriality of L , we have the inverse Laplace transform representation (see [ZH]):

$$(6.1) \quad G(t; x, y) = \int_\Gamma e^{\lambda t} G_\lambda(x, y) d\lambda.$$

Let $\theta_1 > 0, \theta_2 > 0$ be chosen sufficiently small, in particular so small as to satisfy the hypotheses of all previous assertions. By assumption (1.7), the large- $|\lambda|$ bounds on the Resolvent kernel, and analyticity of the Evans function $D_L(\lambda)$, it follows that G_λ has finitely many poles in Ω_θ (corresponding to roots of D_L), each with strictly negative real part. Choosing θ_1, θ_2

still smaller, if necessary, we can thus arrange that G_λ is analytic on Ω_θ . It follows from Cauchy's Theorem that

$$(6.2) \quad G(x, t; y) = \int_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda,$$

for any contour Γ that can be expressed as $\Gamma = \partial(\Omega_\theta \setminus \mathcal{S})$ for $\mathcal{S} \subset \mathbb{C}$ open.

Case I. $|x - y|/t$ large. We first treat the trivial case that $|x - y|/t \geq S$, S sufficiently large, the regime in which standard short-time parabolic theory applies. Set

$$(6.3) \quad \bar{\alpha} := \frac{|x - y|}{2\beta t}, \quad R := \beta \bar{\alpha}^2,$$

where β is as in (5.5), and consider again the representation of G , that is

$$(6.4) \quad G(x, t; y) = \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda t} G_\lambda(x, y) d\lambda,$$

where $\Gamma_1 := \partial B(0, R) \cap \bar{\Omega}_\theta$ and $\Gamma_2 := \partial \Omega_\theta \setminus B(0, R)$. Note that the intersection of Γ with the real axis is $\lambda_{min} = R = \beta \bar{\alpha}^2$. \square

By the large $|\lambda|$ estimates of Proposition 5.3, we have for all $\lambda \in \Gamma_1 \cup \Gamma_2$ that

$$|G_\lambda(x, y)| \leq C \frac{e^{-\sqrt{|\lambda|} \frac{|y-x|}{c}}}{\sqrt{|\lambda|}}.$$

Further, we have

$$(6.5) \quad \operatorname{Re} \lambda \leq R(1 - \eta \omega^2), \quad \lambda \in \Gamma_1, \quad \operatorname{Re} \lambda \leq \operatorname{Re} \lambda_0 - \eta(|\operatorname{Im} \lambda| - |\operatorname{Im} \lambda_0|), \quad \lambda \in \Gamma_2$$

for R sufficiently large, where ω is the argument of λ and λ_0 and λ_0^* are the two points of intersection of Γ_1 and Γ_2 , for some $\eta > 0$ independent of $\bar{\alpha}$. Combining these estimates, we obtain

$$(6.6) \quad \begin{aligned} \left| \int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda \right| &\leq \int_{\Gamma_1} C |\lambda|^{-\frac{1}{2}} e^{\operatorname{Re} \lambda t - \beta^{-\frac{1}{2}} |\lambda|^{-\frac{1}{2}} |x-y|} d\lambda \\ &\leq C e^{-\beta \bar{\alpha}^2 t} \int_{-L}^{+L} R^{-\frac{1}{2}} e^{-\beta R \eta \omega^2 t} R d\omega \leq C t^{-\frac{1}{2}} e^{-\beta \bar{\alpha}^2 t}. \end{aligned}$$

Likewise,

$$\begin{aligned}
(6.7) \quad \left| \int_{\Gamma_2} e^{\lambda t} G_\lambda d\lambda \right| &\leq \int_{\Gamma_2} C |\lambda|^{-\frac{1}{2}} C e^{Re\lambda t - \beta^{-\frac{1}{2}} |\lambda|^{-\frac{1}{2}} |x-y|} d\lambda \\
&\leq C e^{Re(\lambda_0)t - |\beta|^{-\frac{1}{2}} |\lambda_0|^{-\frac{1}{2}} |x-y|} \int_{\Gamma_2} |\lambda|^{-\frac{1}{2}} e^{(Re\lambda - Re\lambda_0)t} |d\lambda| \\
&\leq C e^{-\beta \bar{\alpha}^2 t} \int_{\Gamma_2} |Im \lambda|^{-\frac{1}{2}} e^{-\eta |Im \lambda - Im \lambda_0| t} |d Im \lambda| \\
&\leq C t^{-\frac{1}{2}} e^{-\beta \bar{\alpha}^2 t}.
\end{aligned}$$

Combining these last two estimates, we have

$$(6.8) \quad |G(x, t; y)| \leq C t^{-\frac{1}{2}} e^{-\frac{\beta \bar{\alpha}^2 t}{2}} e^{-\frac{(x-y)^2}{8\beta t}} \leq C t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{(x-y)^2}{8\beta t}},$$

for $\eta > 0$ independent of $\bar{\alpha}$. Observing that $\frac{|x-at|}{2t} \leq \frac{|x-y|}{t} \leq \frac{2|x-at|}{t}$ for any bounded a , for $\frac{|x-y|}{t}$ sufficiently large, we find that this contribution may be absorbed in any summand $t^{-\frac{1}{2}} e^{-\frac{(x-y-a_k^+ t)^2}{Mt}}$.

Case II. $|x-y|/t$ bounded. We now turn to the critical case that $|x-y|/t \leq S$. A few remarks are in order at the outset. Our goal is to bound $|G|$ by terms of form $C t^{-1/2} e^{-\bar{\alpha}^2 t/M}$, where $\bar{\alpha} := (x - a_j^+(t - |y/a_k^+|))/2t$ or $\bar{\alpha} := (x - y - a_k^+ t)/2t$ are now *uniformly bounded*, by

$$(6.9) \quad |x-y|/2t + \max_j \{|a_j^+|\}/2 \leq S/2 + \max |a_j^+|/2.$$

Thus, in particular, *contributions of order $t^{-1/2} e^{-\eta t}$, $\eta > 0$, can be absorbed in any summand $t^{-1/2} e^{-(x-y-a_k^+ t)^2/Mt}$* if we take M sufficiently large. Likewise, for G_x and G_y , contributions of order $t^{-1} e^{-\eta t}$ can be absorbed. We will use this observation repeatedly.

In contrast to the previous case of large characteristic speed $|x-y|/t \geq S$, we are not trying to show rapid time-exponential decay. Rather, we are trying to show that the rate of exponential decay of the solution does not degrade too rapidly as $\bar{\alpha} \rightarrow 0$: precisely, that it vanishes to order $\bar{\alpha}^2$ and no more. Thus, the crucial part of our analysis will be for small $\bar{\alpha}$. All other situations can be estimated crudely as described just above.

Let r be sufficiently small that the small- $|\lambda|$ bounds hold on $B(0, r)$. Next, choose θ_1 and θ_2 still smaller than before, if necessary, so that $\Omega_\theta \setminus B(0, r) \subset \Lambda$. This implies that $\partial\Omega_\theta \cap B(0, r) \neq \emptyset$, giving the configuration pictured in the

Figure. Similarly as in the previous case, define $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the portion of the circle $\partial B(0, r)$ contained in $\bar{\Omega}_\theta$, and Γ_2 is the portion of $\partial\Omega_\theta$ outside $B(0, r)$.

$$(6.10) \quad G(x, t; y) = \int_{\Gamma_1} e^{\lambda t} G_\lambda(x, y) d\lambda + \int_{\Gamma_2} e^{\lambda t} G_\lambda(x, y) d\lambda,$$

we separately estimate the terms \int_{Γ_1} and \int_{Γ_2} .

Large- and medium- λ estimates. The \int_{Γ_2} term is straightforward. The points λ_0, λ_0^* where Γ_1 meets Γ_2 satisfy $Re(\lambda_0) = -\eta < 0$. Moreover, combining the results low frequency case, we have the bound $|G_\lambda| \leq C|\lambda|^{-\frac{1}{2}}$ for $\lambda \in \Gamma_2$. Thus, we have

$$(6.11) \quad \begin{aligned} \left| \int_{\Gamma_2} e^{\lambda t} G_\lambda d\lambda \right| &\leq C e^{-Re \lambda_0 t} \int_{\Gamma_2} |Im \lambda|^{-\frac{1}{2}} e^{-\eta |Im \lambda - Im \lambda_0| t} |d Im \lambda| \\ &\leq C t^{-\frac{1}{2}} e^{-\eta t}. \end{aligned}$$

This contribution can be absorbed as described above. An analogous computation using $|G_{\lambda_x}|, |G_{\lambda_y}| \leq C|\lambda|^{-1}$ shows that the Γ_2 contribution to G_x and G_y is $O(t^{-1}e^{-\eta t})$, and can likewise be absorbed.

Small $|\lambda|$ estimates. It remains to estimate the critical term $\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda$. This we will estimate in different ways, depending on the size of t .

Bounded time. For t bounded, we can use the medium- λ bounds $|G_\lambda|, |G_{\lambda_x}|, |G_{\lambda_y}| \leq C$ to obtain $|\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda| \leq C_2 |\Gamma_1|$. This contribution is order $Ce^{-\eta t}$ for bounded time, hence can be absorbed.

Large time. For t large, we must instead estimate $\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda$ using the small- $|\lambda|$ expansions. First, observe that, all coefficient functions $d_{jk}(\lambda)$ are uniformly bounded (since $|\lambda|$ is bounded in this case).

Expanding $G = \int_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda$ as

$$\begin{pmatrix} G & G_x \\ G_y & G_{xy} \end{pmatrix} = \int_{\Gamma} e^{\lambda t} \begin{pmatrix} G_\lambda & G_{\lambda_x} \\ G_{\lambda_y} & G_{\lambda_{xy}} \end{pmatrix} d\lambda$$

we estimate the \int_{Γ_1} contributions to G, G_x and G_y simultaneously.

Case II(i). ($0 < y < x$). By our low-frequency estimates, we have

$$(6.12) \quad \int_{\Gamma} e^{\lambda t} \begin{pmatrix} G_{\lambda} & G_{\lambda x} \\ G_{\lambda y} & G_{\lambda xy} \end{pmatrix} d\lambda = \int_{\Gamma} \sum_{j,k} e^{\lambda t} \phi_j^+(x) d_{jk} \tilde{\psi}_k^+(y) d\lambda \\ + \int_{\Gamma} \sum_{j,k} e^{\lambda t} \psi_k^+(x) \tilde{\psi}_k^+(y) d\lambda,$$

where each d_{jk} is analytic, hence bounded. We estimate separately each of the terms

$$\int_{\Gamma_1} e^{\lambda t} \phi_j^+(x) d_{jk} \tilde{\psi}_k^+(y) d\lambda$$

on the righthand side of (6.12). Estimates for terms

$$\int_{\Gamma} \sum_{j,k} e^{\lambda t} \psi_k^+(x) \tilde{\psi}_k^+(y) d\lambda$$

go similarly.

Case II(ia). First, consider the critical case $a_j^+ > 0$, $a_k^+ < 0$. For this case,

$$|\phi_{j(x)}^+ d_{jk} \tilde{\psi}_k^+(y)| \leq C e^{Re(\rho_j^+ x - \nu_k^+ y)},$$

where

$$\begin{cases} \nu_k^+(\lambda) = -\lambda/a_k^+ + \lambda^2 \beta_k^+ / (a_k^+)^3 + \mathcal{O}(\lambda^3) \\ \rho_j^+(\lambda) = -\lambda/a_j^+ + \lambda^2 \beta_j^+ / (a_j^+)^3 + \mathcal{O}(\lambda^3). \end{cases}$$

Set

$$\bar{\alpha} = \frac{a_k^+ x / a_j^+ - y - a_k^+ t}{2t}, \quad p := \frac{\beta_j^+ a_k^+ x / (a_j^+)^3 - \beta_k^+ y / (a_k^+)^2}{t} > 0.$$

Define Γ'_{1a} to be the portion contained in Ω_{θ} of the hyperbola

$$(6.13) \quad Re(\rho_j^+ x - \nu_k^+ y) + \mathcal{O}(\lambda^3)(|x| + |y|) \\ = (1/a_k^+) Re[\lambda(-a_k^+ x/a_j^+ + y) + \lambda^2(x\beta_j^+ a_k^+ / (a_j^+)^3 - y\beta_k^- / (a_k^+)^2)] \\ \equiv \text{constant} \\ = (1/a_k^-)[(\lambda_{min}(-a_k^- x/a_j^+ + y) + \lambda_{min}^2(x\beta_j^+ a_k^+ / (a_j^+)^3 - y\beta_k^+ / (a_k^+)^2)],$$

where

$$(6.14) \quad \lambda_{min} := \begin{cases} \frac{\bar{\alpha}}{p} & \text{if } |\frac{\bar{\alpha}}{p}| \leq \epsilon \\ \pm \epsilon & \text{if } \frac{\bar{\alpha}}{p} \gtrless \epsilon \end{cases}$$

Denoting by λ_1, λ_1^* , the intersections of this hyperbola with $\partial\Omega_\theta$, define Γ'_{1_b} to be the union of $\lambda_1\lambda_0$ and $\lambda_0^*\lambda_1^*$, and define $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$. Note that $\lambda = \bar{\alpha}/p$ minimizes the left hand side of (6.13) for λ real. Note also that p is bounded for $\bar{\alpha}$ sufficiently small, since $\bar{\alpha} \leq \epsilon$ implies that

$$(|a_k^+x/a_j^+| + |y|)/t \leq 2|a_k^+| + 2\epsilon$$

i.e. $(|x| + |y|)/t$ is controlled by $\bar{\alpha}$.

With these definitions, we readily obtain that

$$(6.15) \quad \begin{aligned} \operatorname{Re}(\lambda t + \rho_j^+ x - \nu_k^+ y) &\leq -(t/a_k^-)(\bar{\alpha}^2/4p) - \eta \operatorname{Im}(\lambda)^2 t \\ &\leq -\bar{\alpha}^2 t/M - \eta \operatorname{Im}(\lambda)^2 t, \end{aligned}$$

for $\lambda \in \Gamma'_{1_a}$ (note: here, we have used the crucial fact that $\bar{\alpha}$ controls $(|x| + |y|)/t$, in bounding the error term $\mathcal{O}(\lambda^3)(|x| + |y|)/t$ arising from expansion). Likewise, we obtain for any q that

$$(6.16) \quad \int_{\Gamma'_{1_a}} |\lambda|^q e^{\operatorname{Re}(\lambda t + \rho_j^+ x - \nu_k^- y)} d\lambda \leq C t^{-\frac{1}{2} - \frac{q}{2}} e^{-\bar{\alpha}^2 t/M},$$

for suitably large $C, M > 0$ (depending on q). Observing that

$$\bar{\alpha} = (a_k^+/a_j^+)(x - a_j^+(t - |y/a_k^+|))/2t,$$

we find that the contribution of (6.16) can be absorbed in the described bounds for $t \geq |y/a_k^-|$. At the same time, we find that $\bar{\alpha} \geq x > 0$ for $t \leq |y/a_k^+|$, whence

$$\bar{\alpha} \geq (x - y - a_j^+ t)/Mt + |x|/M,$$

for some $\epsilon > 0$ sufficiently small and $M > 0$ sufficiently large.

This gives

$$e^{-\bar{\alpha}^2/p} \leq e^{-(x-y-a_k^+t)^2/Mt} e^{-\eta|x|}$$

provided $|x|/t > a_j^+$, a contribution which can again be absorbed. On the other hand, if $t \leq |x/a_j^+|$, we can use the dual estimate

$$(6.17) \quad \begin{aligned} \bar{\alpha} &= (-y - a_k^+(t - |x/a_j^+|))/2t \\ &\geq (x - y - a_k^+ t)/Mt + |y|/M, \end{aligned}$$

together with $|y| \geq |a_k^- t|$, to obtain

$$e^{-\bar{\alpha}^2/p} \leq e^{-(x-y-a_j^+ t)^2/Mt} e^{-\eta|y|},$$

a contribution that can likewise be absorbed.

Case II(ib). In case $a_j^+ < 0$ or $a_k^+ > 0$, terms $|\varphi_j^+| \leq C e^{-\eta|x|}$ and $|\tilde{\psi}_j^+| \leq C e^{-\eta|y|}$ are strictly smaller than those already treated in Case II(ia), so may be absorbed in previous terms.

Case II(ii) ($0 < x < y$). The case $0 < x < y$ can be treated very similarly to the previous one; see [ZH] for details. This completes the proof of Case II, and the theorem.

7 Nonlinear Analysis

Introducing the perturbation variable

$$(7.1) \quad u(x, t) := \tilde{u}(x, t) - \bar{u}(x),$$

we obtain

$$(7.2) \quad u_t - Lu = Q(u)_x,$$

where the second-order Taylor remainder satisfies

$$(7.3) \quad Q(u) := f(\bar{u} + u) - f(\bar{u}) - df(\bar{u})u = \mathcal{O}(|u|^2)$$

so long as $|u|$ remains bounded.

Lemma 7.1 (Integral formulation). *Under the assumptions of Theorem 1.5, there exists a classical solution of (7.2) for $0 < t \leq T$, $T > 0$, continuous in $L^\infty(x)$ at $t = 0$, extending for all $t > 0$ such that $u(\cdot, t)$ remains sufficiently small in $L^1 \cap L^\infty$, given by*

$$(7.4) \quad \begin{aligned} u(x, t) = & \int_0^\infty G(x, t; y) g(y) dy + \int_0^t G_y(x, t-s; 0) Bh(s) ds \\ & - \int_0^t \int_0^\infty G_y(x, t-s; y) Q(u)(y, s) dy ds. \end{aligned}$$

Proof. From Lemma (3.2) and the inverse Laplace representation (6.1) we find that $G(x, t - s; y)$ considered as a function of y, s satisfies the adjoint equation

$$(7.5) \quad (\partial_s - L_y)^* G^*(x, t - \cdot; \cdot) = 0,$$

or

$$(7.6) \quad -G_s - (GA)_y + GA_y = (G_y B)_y.$$

Likewise, reviewing the construction of the resolvent, we find $G_\lambda(x, 0) \equiv 0$, yielding

$$(7.7) \quad G(x, t - s; 0) \equiv 0.$$

That is, $G^*(x, t - \cdot; \cdot)$ is the Green function for the adjoint equation, as may alternatively be seen directly by a duality argument analogous to the proof of Lemma (3.2).

Thus, integrating G against (7.2), integrating by parts, and using the fact that $G = 0$ and $u = h$ on the boundary $y = 0$, we obtain for any classical solution of (7.2) that

$$(7.8) \quad \begin{aligned} & \int_0^t \int_0^\infty G(x, t - s; y) Q(u(y, s))_y dy ds = \\ & \int_0^t \int_0^\infty G(x, t - s; y) (\partial_s - L_y) u(y, s) dy ds \\ & = \int_0^t \int_0^\infty ((\partial_s - L_y)^* G^*)^*(x, t - s; y) u(y, s) dy ds \\ & + u(x, t) - \int_0^\infty G(x, t; y) g(y) dy - \int_0^t G_y(x, t - s; 0) B h(s) ds, \end{aligned}$$

from which we obtain (7.4) by rearranging and integrating by parts the term $\int_0^t \int_0^\infty G(x, t - s; y) Q(u(y, s))_y dy ds$.

Indeed, (7.4) may be taken as the definition of a weak solution in $L^\infty(x, t)$. (One can see using convolution identities that this agrees with the usual definition in terms of integration against test functions $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.) Existence of weak solutions can be obtained by a standard contraction mapping/continuation argument using the convolution bounds of Lemmas 7.2–7.4 below; we omit the details, since we shall carry out quite similar but more difficult estimates in the proof of stability. Smoothness of solutions may then be obtained by a bootstrapping argument as sketched in Appendix A. \square

To establish stability, we use the following lemmas proved in [HZ].

Lemma 7.2 (Linear estimates [HZ]). *Under the assumptions of Theorem 1.5,*

$$(7.9) \quad \int_0^{+\infty} |G(x, t; y)|(1 + |y|)^{-3/2} dy \leq C(\theta + \psi_1 + \psi_2)(x, t),$$

for $0 \leq t \leq +\infty$, some $C > 0$.

Lemma 7.3 (Nonlinear estimates [HZ]). *Under the assumptions of Theorem 1.5,*

$$(7.10) \quad \int_0^t \int_0^{+\infty} |G_y(x, t - s; y)|\Psi(y, s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x, t),$$

for $0 \leq t \leq +\infty$, some $C > 0$, where

$$(7.11) \quad \begin{aligned} \Psi(y, s) := & (1 + s)^{1/2} s^{-1/2} (\theta + \psi_1 + \psi_2)^2(y, s) \\ & + (1 + s)^{-1} (\theta + \psi_1 + \psi_2)(y, s). \end{aligned}$$

We require also the following estimate accounting boundary effects.

Lemma 7.4 (Boundary estimate). *Under the assumptions of Theorem 1.5,*

$$(7.12) \quad \left| \int_0^t G_y(x, t - s; 0) B h(s) ds \right| \leq C E_0 (\theta + \psi_1 + \psi_2)(x, t),$$

for $0 \leq t \leq +\infty$, some $C > 0$.

Proof. The estimate on \int_0^{t-1} , where $G_y(x, t - s; 0)$ is nonsingular, follows readily by estimates similar to but somewhat simpler than those of Lemma (7.3), which we therefore omit.

To bound the singular part \int_{t-1}^t , we integrate (7.6) in y from 0 to $+\infty$, recalling that $G(x, t - s; 0) \equiv 0$, to obtain

$$(7.13) \quad G_y B = - \int_0^{+\infty} A_y(y) G(x, t - s; y) dy - \int_0^{+\infty} G_s(x, t - s; y) dy.$$

Substituting in the lefthand side of (7.12), and integrating by parts in s , we obtain

$$\begin{aligned}
(7.14) \quad \int_{t-1}^t G_y B h(s) ds &= \int_0^1 \left(\int_0^{+\infty} A_y(y) G(x, \tau; y) dy \right) h(t - \tau) d\tau \\
&\quad - \int_0^1 \left(\int_0^{+\infty} G(x, \tau; y) dy \right) h'(t - \tau) d\tau \\
&\quad + \left(\int_0^{+\infty} G(x, 1; y) dy \right) h(t - 1),
\end{aligned}$$

which by $\int |G| dy \leq C$ has norm bounded by $\max_{0 \leq \tau \leq 1} (|h| + |h'|)(t - \tau)$.

Combining this with the more straightforward estimate

$$\begin{aligned}
(7.15) \quad \left| \int_{t-1}^t G_y(x, t; 0) B h(s) ds \right| &\leq \int_0^1 |G_y(x, \tau; 0)| B h(s) ds \\
&\leq C \max_{0 \leq \tau \leq 1} |h(t - \tau)| \int_0^1 \tau^{-1} e^{-|x|^2/C\tau} d\tau \\
&= C |x|^{-2} \max_{0 \leq \tau \leq 1} |h(t - \tau)| \\
&\quad \times \int_0^1 (|x|^2/\tau) e^{-|x|^2/C\tau} d\tau \\
&\leq C \max_{0 \leq \tau \leq 1} |h(t - \tau)| |x|^{-2},
\end{aligned}$$

we find that the contribution from \int_{t-1}^t has norm bounded by

$$\max_{0 \leq \tau \leq 1} (|h| + |h'|)(t - \tau) (1 + |x|)^{-2}.$$

Combining this estimate with the one for \int_0^{t-1} , we obtain (7.12). \square

With these preparations, the proof of stability is straightforward.

Proof of Theorem 1.5. Define

$$(7.16) \quad \zeta(t) := \sup_{y, 0 \leq s \leq t} |u|(\theta + \psi_1 + \psi_2)^{-1}(y, t).$$

We will establish:

Claim. For all $t \geq 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$(7.17) \quad \zeta(t) \leq C_2(E_0 + \zeta(t)^2).$$

From this result, provided $E_0 < 1/4C_2^2$, we have that $\zeta(t) \leq 2C_2E_0$ implies $\zeta(t) < 2C_2E_0$, and so we may conclude by continuous induction that

$$(7.18) \quad \zeta(t) < 2C_2E_0$$

for all $t \geq 0$. (By Lemma 7.1 and standard short-time estimates, $u \in C^0(x)$ exists and ζ remains continuous so long as ζ remains bounded by some uniform constant, hence (7.18) is an open condition. From (7.18) and the definition of ζ in (7.16) we then obtain the bounds of (1.14). Thus, it remains only to establish the claim above.

Proof of Claim. We must show that $u(\theta + \psi_1 + \psi_2)^{-1}$ is bounded by $C(E_0 + \zeta(t)^2)$, for some $C > 0$, all $0 \leq s \leq t$, so long as ζ remains sufficiently small. By (7.16), we have for all $t \geq 0$ and some $C > 0$ that

$$(7.19) \quad |u(x, t)| \leq \zeta(t)(\theta + \psi_1 + \psi_2)(x, t),$$

and therefore

$$(7.20) \quad |Q(u)(y, s)| \leq C\zeta(t)^2\Psi(y, s)$$

with Ψ as defined in (7.11), for $0 \leq s \leq t$. Combining (7.20) with representation (7.4) and applying Lemmas 7.2–7.4, we obtain

$$(7.21) \quad \begin{aligned} |u(x, t)| &\leq \int_0^\infty |\tilde{G}(x, t; y)||g(y)| dy + \left| \int_0^t G_y(x, t-s; 0)Bh(s) ds \right| \\ &\quad + \int_0^t \int_0^\infty |\tilde{G}_y(x, t-s; y)||Q(u)(y, s)| dy ds \\ &\leq E_0 \int_0^\infty |\tilde{G}(x, t; y)|(1+|y|)^{-3/2} dy \\ &\quad + \left| \int_0^t G_y(x, t-s; 0)Bh(s) ds \right| \\ &\quad + C\zeta(t)^2 \int_0^t \int_0^\infty |\tilde{G}_y(x, t-s; y)|\Psi(y, s) dy ds \\ &\leq C(E_0 + \zeta(t)^2)(\theta + \psi_1 + \psi_2)(x, t). \end{aligned}$$

Dividing by $(\theta + \psi_1 + \psi_2)(x, t)$, we obtain (7.17) as claimed. This completes the proof of the claim, and the theorem. \square

A Smoothness of solutions

In this appendix, we briefly sketch the proof that weak solutions defined by (7.4) are necessarily smooth, classical solutions as well, by indicating how to get the necessary derivative bounds.

Time-derivative. Rewriting the second, boundary term, on the right-hand side of (7.4) using its convolution structure, as

$$\int_0^t G_y(x, \tau; 0) Bh(t - \tau) d\tau,$$

and differentiating in t , we obtain

$$G_y(x, t; 0) Bh(0) + \int_0^t G_y(x, \tau; 0) Bh'(t - \tau) d\tau,$$

for which the first term is bounded and smooth for $x, t > 0$, and the second by the same estimate as in (7.15) is bounded by

$$C|x|^{-2} \int_0^t |h'(t - \tau)| d\tau \leq C|x|^{-2} \log(1 + t).$$

Differentiating the first and third terms with respect to t and integrating the third term by parts in y yields

$$(A.1) \quad \int_0^\infty G(x, t; y) g(y) dy - \int_{t/2}^t \int_0^\infty G_y(x, t - s; y) Q(u)_s(y, s) dy ds \\ - \int_0^{t/2} \int_0^\infty G_{yt}(x, t - s; y) Q(u)(y, s) dy ds,$$

from which, in combination with the boundary estimate already performed, we may readily obtain a short-time bound $|u_t| \leq C|x|^{-2}t^{-1}$ by Picard iteration.

Spatial-derivatives. Likewise, differentiating (7.14) with respect to x , we may bound the x -derivative of the boundary term $\int_0^t G_y B ds$ by

$$C \int_0^t \tau^{-1/2} (|h'| + |h|)(t - \tau) d\tau \leq C \log(1 + t).$$

Differentiating the first and third terms of the righthand side of (7.4) with respect to x and integrating the third term by parts in y yields

$$(A.2) \quad \int_0^\infty G(x, t; y)g(y) dy - \int_{t/2}^t \int_0^\infty G_x(x, t - s; y)Q(u)_y(y, s) dy ds \\ - \int_0^{t/2} \int_0^\infty G_{yx}(x, t - s; y)Q(u)(y, s) dy ds,$$

from which, in combination with the boundary estimate already performed, we obtain a short-time bound $|u_x| \leq Ct^{-1/2}$ by Picard iteration. From the bounds on $|u_t|$ and $|u_x|$, finally, we obtain bounds on $|u_{xx}|$ by the equation satisfied by u .

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