A GENERAL LOWER BOUND FOR POTENTIALLY H-GRAPHIC SEQUENCES

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Abstract. We consider a variation of the classical Turán-type extremal problem as introduced by Erdős *et al.* in [7]. Let π be an n-element graphic sequence, and $\sigma(\pi)$ be the sum of the terms in π, that is the degree sum. Let H be a graph. We wish to determine the smallest m such that any n-term graphic sequence π having $\sigma(\pi) \geq m$ has some realization containing H as a subgraph. Denote this value m by $\sigma(H, n)$. For an arbitrarily chosen H, we construct a graphic sequence $\pi^*(H, n)$ such that $\sigma(\pi^*(H, n)) + 2 \leq \sigma(H, n)$. Furthermore, we conjecture that equality holds in general, as this is the case for all choices of H where $\sigma(H, n)$ is currently known. We support this conjecture by examining those graphs that are the complement of triangle-free graphs, and showing that the conjecture holds despite the wide variety of structure in this class. We will conclude with a brief discussion of a connection between potentially H-graphic sequences and H-saturated graphs of minimum size.

Keywords: Degree sequence, Potentially graphic sequence, H-saturated graph.

1. Introduction

A good reference for any undefined terms is [1]. Let G be a simple undirected graph, without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively and let $d(v)$ denote the degree of a vertex v. Let $\overline{G}$ denote the complement of G. Denote the complete graph on t vertices and the complete bipartite graph with partite sets of size r and s by K_t and $K_{r,s}$, respectively. Additionally, let K_s^t denote the complete balanced multipartite graph with t partite sets of size s. Given any two graphs G and H, their join, denoted $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G),\ h \in V(H)\}$. Additionally, let $\alpha(G)$ denote the independence number of G. If H is a subgraph of G, we will write $H \subset G$, and if H is an induced subgraph of G, we will write $H < G$.

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is called *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* π, and we will write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, we may write $\pi = ({d_1}^{\mu_1}, \ldots, {d_t}^{\mu_t})$.

For a given graph H, a sequence π is said to be *potentially H-graphic* if there is some realization of π which contains H as a subgraph. Additionally, let $\sigma(\pi)$ denote the sum of the terms of π. Define $\sigma(H, n)$ to be the smallest integer m so that every n-term graphic sequence π with $\sigma(\pi) \geq m$ is potentially H-graphic. In this paper, given an arbitrary H, we construct a graphic sequence $\pi^*(H, n)$ such that $\sigma(\pi^*(H, n)) + 2 \leq \sigma(H, n)$. We then show that equality holds for all graphs H that are the complement of a triangle-free graph. There have been numerous papers, including but certainly not limited to [5], [3], [4], [7], [9], [11], [12], [14], [15], [16], [17] and [18], that consider the potential problem for specific graphs or narrow families of graphs. It is our hope that the ideas and results presented in this paper will facilitate a broader consideration of problems of this type.

2. A Short History

In this section, we present the extremal sequences for two classes of graphs: complete graphs and complete balanced bipartite graphs. Our goal is to motivate the general constructions in the next section.

2.1. $H = K_t$.

In [7] Erdős, Jacobson and Lehel conjectured that $\sigma(K_t, n) = (t-2)(2n-t+1)+2$. The conjecture rises from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$. It is easy to observe that this graph contains no K_t, is the unique realization of the sequence $((n-1)^{t-2}, (t-2)^{n-t+2})$, and has degree sum $(t-2)(2n-t+1)$. The cases $t = 3, 4$ and 5 were proved separately (see respectively [7], [12] and [15], and [16]), and Li, Song and Luo [17] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. A purely graph-theoretic proof was given in [10] and also as a corollary to the main result in [4].

2.2. $H = K_{s,s}$.

The following results appears in [12] and [18]. Here E_1, E_2, E_3 and E_4 are somewhat technical numerical classes which, based on the parity of n and s, assure that the given degree sums are even.

Theorem 2.1.
- *If s is an odd, positive integer and $n \geq 4s^2 + 3s - 8$, then*

$$\sigma(K_{s,s}, n) = \begin{cases} (\frac{5}{2}s - \frac{5}{2})n - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{7}{8} & if (s,n) \in E_3 \\ (\frac{5}{2}s - \frac{5}{2})n - \frac{11}{8}s^2 + \frac{5}{2}s + \frac{15}{8} & if (s,n) \in E_4. \end{cases} \tag{1}$$

- *If s is an even, positive integer and $n \geq 4s^2 - s - 6$, then*

$$\sigma(K_{s,s}, n) = \begin{cases} (\frac{5}{2}s - 2)n - \frac{11}{8}s^2 + \frac{5}{4}s + 2 & if (s,n) \in E_1 \\ (\frac{5}{2}s - 2)n - \frac{11}{8}s^2 + \frac{5}{4}s + 1 & if (s,n) \in E_2. \end{cases} \tag{2}$$

In order to establish a lower bound on $\sigma(K_{s,s}, n)$ the authors present several sequences dependent on the parities of s and n.

(i) If s is odd and $(s,n) \in E_3$, then

$$\pi(K_{s,s},n) = ((n-1)^{s-1}, 2s-2, 2s-3, \ldots, \frac{3}{2}s+\frac{3}{2}, \frac{3}{2}s+\frac{1}{2}, (\frac{3}{2}s-\frac{1}{2})^{\frac{s}{2}+\frac{3}{2}}, (\frac{3}{2}s-\frac{3}{2})^{n-2s}, \frac{3}{2}s-\frac{5}{2}). \quad (3)$$

(ii) If s is odd and $(s,n) \in E_4$, then

$$\pi(K_{s,s},n) = ((n-1)^{s-1}, 2s-2, 2s-3, \ldots, \frac{3}{2}s+\frac{3}{2}, \frac{3}{2}s+\frac{1}{2}, (\frac{3}{2}s-\frac{1}{2})^{\frac{s}{2}+\frac{3}{2}}, (\frac{3}{2}s-\frac{3}{2})^{n-2s+1}). \quad (4)$$

(iii) If s is even and $(s,n) \in E_1$, then

$$\pi(K_{s,s},n) = ((n-1)^{s-1}, 2s-2, 2s-3, \ldots, \frac{3}{2}s+1, \frac{3}{2}s, (\frac{3}{2}s-1)^{n-\frac{3}{2}s+2)}). \quad (5)$$

(iv) If s is even and $(s,n) \in E_2$, then

$$\pi(K_{s,s},n) = ((n-1)^{s-1}, 2s-2, 2s-3, \ldots, \frac{3}{2}s+1, \frac{3}{2}s, (\frac{3}{2}s-1)^{n-\frac{3}{2}s+1}, (\frac{3}{2}s-2)). \quad (6)$$

Each of these sequences can be realized by the join of K_{s-1} and some graph H'. This H' has no vertices of degree s, one vertex of degree $s-1$, two vertices of degree $s-2$ and so on. More generally, no choice of H' contains x_1 vertices of degree x_2, where $x_1 + x_2 = s+1$. This implies that H' cannot possibly contain a copy of K_{x_1,x_2}. However, if any of these sequences were to be potentially $K_{s,s}$-graphic, at least $s+1$ of the vertices in a copy of $K_{s,s}$ would have to be chosen from H'. These vertices in turn, would comprise some K_{x_1,x_2} where $x_1 + x_2 = s+1$.

3. A General Lower Bound

We assume that H has no isolated vertices and furthermore that n is sufficiently large relative to $|V(H)|$. We define the quantities

$$u(H) = |V(H)| - \alpha(H) - 1,$$

and

$$d(H) = min\{\Delta(F) : F < H, |V(F)| = \alpha(H) + 1\}.$$

Consider the following sequence,

$$\widehat{\pi}(H,n) = ((n-1)^{u(H)}, (u(H) + d(H) - 1)^{n-u(H)}). \quad (7)$$

If this sequence is not graphic, that is if $n-u(H)$ and $d(H)-1$ are both odd, we reduce the smallest term by one. To see that this will result in a graphic sequence, we make two observations. First, $(d(h)-1)$-regular graphs of order $n-u(H) \geq d(H)$ exist whenever $d(H)-1$ and $n-u(H)$ are not both odd. If n and $d(H)-1$ are both odd, it is not difficult to show that the sequence $((d(H)-1)^{n-u(H)-1}, d(H)-2)$ is graphic

Every realization of $\widehat{\pi}(H,n)$ is a complete graph on $u(H)$ vertices, joined to a graph, call it G', that is either $(d(H)-1)$-regular or nearly so. Note that the subgraph induced by any $\alpha(H)+1$ vertices of H has maximum degree at least $d(H)$. Thus, no realization of $\widehat{\pi}(H,n)$ could possibly contain a copy of H, as at least $\alpha(H)+1$ vertices of such a subgraph would have to lie in G'.

The degree sum of (7) is

$$\sigma(\widehat{\pi}(H,n)) = n(2u(H)+d(H)-1) - u(H)(u(H)+d(H)), \tag{8}$$

and if both $n-u(H)$ and $d(H)-1$ are odd, the sum will be one smaller.

To gain some additional insight, we will consider first the case $H = K_t$. Then $u(K_t) = t-2$ and $d(K_t) = 1$, so that

$$\widehat{\pi}(K_t,n) = ((n-1)^{t-2},(t-2)^{n-t+2}).$$

This is exactly the extremal sequence put forth to establish the lower bound for $\sigma(K_t,n)$. Similarly, the extremal sequences used to determine $\sigma(kK_2,n)$, $\sigma(C_{2k+1},n)$ and $\sigma(K_1+kK_2,n)$ are precisely $\widehat{\pi}(kK_2,n)$, $\widehat{\pi}(C_{2k+1},n)$ and $\widehat{\pi}(K_1+kK_2,n)$, respectively (see [12],[14] and[11]). However, $\sigma(\widehat{\pi}(K_{s,s},n))$ is asymptotically equivalent to, but smaller than $\sigma(K_{s,s},n)$. Along these lines, we are able to refine the sequence given above.

For convenience, let $d = d(H), u = u(H)$ and $\alpha = \alpha(H)$ and let $v_i(H)$ denote the number of vertices of degree i in H. For all $i, d \le i \le \alpha$ we define the quantity m_i to be the minimum number of vertices of degree at least i over all induced subgraphs F of H with $|V(F)| = \alpha+1$ and $\sum_{j=i}^{\alpha} v_j(F) > 0$ and 0 if no such subgraphs exist. The quantities n_i, $d \le i \le \alpha$, are defined recursively such that $n_d = m_d - 1$ and either $n_i = \min\{m_i - 1, n_{i-1}\}$ if $m_i \ge 1$ or $n_i = 0$ if $m_i = 0$. Finally, we define $\delta_{\alpha-1} = n_{\alpha-1}$ and for $d \le i \le \alpha-2$ we define $\delta_i = n_i - n_{i+1}$. We do not define δ_α, as any induced subgraph composed of a maximum independent set and an additional vertex has at most one vertex of degree α, and as such n_α is always 0.

We now consider the following sequence:

$$\pi^*(H,n) = ((n-1)^u,(u+\alpha-1)^{\delta_{\alpha-1}},(u+\alpha-2)^{\delta_{\alpha-2}},\ldots(u+d)^{\delta_d},(u+d-1)^{n-u-\Sigma\delta_i}). \tag{9}$$

The sequence π^* is constructed so that it contains n_i terms that are at least $u+i$ and δ_i terms that are exactly u_i.

If this sequence is not graphic, then we will reduce the smallest term which is strictly greater than $u(H)$ in the sequence by one and redefine $\pi^*(H,n)$ to be this graphic sequence instead. The following is the main result of this paper.

Theorem 3.1. *Given a graph H, with $u(H)$ and $d(H)$ as above, and n sufficiently large then,*

$$\sigma(H,n) \ge \max\{\sigma(\pi^*(H^*,n)) + 2 \mid H^* \subseteq H\}. \tag{10}$$

Proof. Let H^* be the subgraph of H that realizes the maximum above. Let G be any realization of $\pi^*(H^*,n)$. We show that G does not contain a copy of H^*.

Note that this degree sequence implies that G is a copy of $K_{u(H^*)}$ joined to another graph G^* on $n-u(H^*)$ vertices. Assume that there is a copy of H^* contained in G. There are at least $\alpha(H^*)+1$ vertices from G^* that must belong to this copy of H. Let H^{**} denote the subgraph of H^* induced by these $\alpha(H^*)+1$ vertices. Notice, however, no $\alpha(H^*)+1$ vertices of G^* have sufficient degree to contain a copy of any H^{**}. In particular, if $\sum_{j\geq\ell} v_j(H^{**}) > 0$ then H^{**} contains at least m_ℓ vertices of degree ℓ or greater. By our construction, there are at most $n_\ell \leq m_\ell - 1$ vertices of degree at least ℓ in G^*. This contradicts the assumption that $H^{**} \subseteq G^*$. Thus, G contains no copy H^* and hence no copy of H. $\square$

Theorem 3.1 requires that we examine all subgraphs of H. To see that this is necessary, we consider the split graph $K_t + \overline{K_s}$ with a pendant vertex v adjacent to one of the vertices in the independent set of order s. For this choice of H, $\alpha(H) = s$ and hence $u(H) = (s+t+1) - s - 1 = t$ and $d(H) = 1$. However, if we remove v, the pendant vertex, and consider the split graph, we can see that $u(K_t + \overline{K_s}) = t-1$ but any $s+1$-vertex subgraph of $K_t + \overline{K_s}$ must contain some vertex from the K_t, implying that $d(K_t + \overline{K_s}) = s$. Therefore, if we choose $s \geq 3$, $\sigma(\pi^*(K_t + \overline{K_s}, n)) \geq \sigma(\pi^*(H, n))$.

The reader should note that for any values of n and s, $\pi^*(K_{s,s}, n)$ is exactly those sequences given in (3)-(6). Additionally, given values of n, s and t, $\pi^*(K_s^t, n)$ matches the extremal sequences given in [23].

We conjecture that equality holds in Theorem 3.1.

Conjecture 1. *Let H be any graph, and let n be a sufficiently large integer. Then*

$$\sigma(H, n) = \max\{\sigma(\pi^*(H^*, n)) + 2 \mid H^* \subseteq H\}. \tag{11}$$

We also pose the weaker conjecture, that the bound put forth is asymptotically correct.

Conjecture 2. *Let H be any graph, and let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, H)$ such that for any $n > n_0$*

$$\sigma(H, n) \leq \max\{(n(2u(H^*) + d(H^*) - 1 + \epsilon) \mid H^* \subseteq H\}. \tag{12}$$

Conjectures 1 and 2 have been verified for a wide variety of graphs. This includes, but is not limited to: complete graphs and unions of complete graphs [7], [9], [12], [15], [16], [17], complete bipartite graphs [3],[12], [18], complete multipartite graphs [5], [20], matchings [12], cycles [14], (generalized) friendship graphs [2], [9], [11], and split graphs [4] At this time we know of no subgraph for which these conjectures do not hold for sufficiently large n.

While Conjecture 1 seems challenging, we feel that there is a good chance that Conjecture 2 could be verified. In the following section, we will verify Conjecture 1 for a broad class of graphs.

4. Complements of Triangle-Free Graphs

We now turn our attention to graphs H of order $k \geq 3$ with $\alpha(H) = 2$, or those graphs that are the complement of a triangle-free graph. The main result of this section is as follows.

Theorem 4.1. *Let H be any graph of order k with $\alpha(H) = 2$. Then*

$$\sigma(H, n) = \sigma(\pi^*(H, n)) + 2.$$

Any graph H in this class has $u(H) = k - 3$ and $d(H) \leq 2$. We prove Theorem 4.1 by considering the cases $d(H) = 1$ and $d(H) = 2$ separately. In each case we construct a graph $H(d)$ that contains H as a subgraph and show that $\sigma(H(d), n) = \sigma(\pi^*(H, n)) + 2$. This implies that $\max\{\sigma(\pi^*(H^*, n)) + 2 \mid H^* \subseteq H\} = \sigma(\pi^*(H, n)) + 2$.

The following result from [4] will be very useful.

Theorem 4.2. *If $n \geq 3s + 2t^2 + 3t - 3$ then*

$$\sigma(K_s + \overline{K}_t, n) = \begin{cases} (t + 2s - 3)n - (s - 1)(s + t - 1) + 2 & \text{if } t \text{ or } n - s \text{ is odd.} \\ (t + 2s - 3)n - (s - 1)(s + t - 1) + 1 & \text{if } t \text{ and } n - s \text{ are even.} \end{cases}$$

It is not difficult to see that if $d(H) = 2$ then H is isomorphic to $K_k - tK_2$, where k is the order of H and t is some positive integer that is at most $\frac{k}{2}$. Let H be a graph of order $k \geq 3$ with $\alpha(H) = 2$ and $d(H) = 2$ and let $n \geq k$ be an integer. Then, by (9), we have.

(i) If $n \equiv k - 3 \ (mod\ 2)$ then
$$\pi^*(H, n) = ((n - 1)^{k-3}, (k - 2)^{n-k+3}) \tag{13}$$
(ii) If $n \not\equiv k - 3 \ (mod\ 2)$ then
$$\pi^*(H, n) = ((n - 1)^{k-3}, (k - 2)^{n-k+2}, k - 3) \tag{14}$$

Proposition 4.3. *Let H be a graph of order k with $\alpha(H) = 2$ and $d(H) = 2$, and let n be a sufficiently large integer. Then*

$$\sigma(H, n) = \sigma(\pi^*(H, n)) + 2 = n(2k - 5) - k^2 + 4k - 1 - m,$$

where $m = n - k + 3 \ (mod\ 2)$.

Proof. The fact that $\sigma(H, n) \geq \sigma(\pi^*(H, n)) + 2$ follows from Theorem 3.1. Note that any H with $\alpha(H) = 2$ and $d(H) = 2$ is a subgraph of $K_{k-2} + \overline{K}_2$ so that $\sigma(H, n) \leq \sigma(K_{k-2} + \overline{K}_2, n)$. Theorem 4.2 implies

$$\sigma(K_{k-2} + \overline{K}_2, n) = n(2k - 5) - k^2 + 4k - 1 + m = \sigma(\pi^*(H, n)) + 2.$$

The proposition follows. □

Those graphs H with $\alpha(H) = 2$ and $d(H) = 1$ have a considerably wider variety of structures. Any graph H in this class is the complement of a triangle-free graph G that is not a matching. The disjoint union of two cliques falls into this class, as does $K_k - tP_3$ and many other graphs of varying densities. We are able to verify Conjecture 1 for this diverse class of graphs. Our first observation is that any graph

H with $\alpha(H) = 2$ and $d(H) = 1$ must contain $K_2 \cup K_1$ as an induced subgraph, as this is the only graph on 3 vertices with maximum degree 1. This also immediately implies that $m_d = m_1 = 2$. Therefore, if H is any graph of order k with $\alpha(H) = 2$ and $d(H) = 1$ and $n \geq k$ is an integer, then (9) implies that

$$\pi^*(H, n) = \left((n-1)^{k-3}, (k-3)^{n-k+3}\right). \tag{15}$$

The following lemma from [12] will be useful in the next proof.

Lemma 4.4. *If π is a graphical sequence with a realization G containing H as a subgraph, then there is a realization G' of π containing H with the vertices of H having the $|V(H)|$ largest degrees of π.*

We now show that Conjecture 1 holds when $\alpha(H) = 2$ and $d(H) = 1$.

Proposition 4.5. *Let H be a graph of order k with $\alpha(H) = 2$ and $d(H) = 1$, and let n be a sufficiently large integer. Then*

$$\sigma(H, n) = \sigma(\pi^*(H, n)) + 2 = n(2k-6) - k^2 + 5k - 4$$

Proof. Let π be a nonincreasing, n-term graphic sequence with $\sigma(\pi) \geq n(2k-6) - k^2+5k-4$. Note that if n is sufficiently large, $\sigma(\pi) \geq \sigma(K_{k-1}, n) \geq \sigma(K_{k-3}+\overline{K}_3, n)$. We will show that π has a realization containing $K_{k-3} + (K_2 \cup K_1)$ and, as we have previously observed that H must contain an induced copy of $K_2 \cup K_1$, a copy of H.

Let G be a realization of π that contains a copy of $K_{k-3} + \overline{K}_3$ on the k vertices of highest degree in G. Such a realization exists by Lemma 4.4. Let S denote this subgraph, F denote the complete subgraph of order $k-3$ and let I denote the independent set of order 3 in S, so that $S = F + I$. We can assume that F is comprised of the $k-3$ vertices of highest degree in G. If not, there are vertices x in I and y in F such that $d(y) < d(x)$. We wish to create a realization of G containing a copy of $K_{k-3} + \overline{K}_3$ on the k vertices of highest degree such that x is in F and y is in I. If x is adjacent to all the other vertices in S, we can simply exchange the roles of x and y. If x was not adjacent to exactly one vertex in I, say v, then as $d(x) > d(y)$ there is some vertex w outside of S that is adjacent to x but not to y. We will create a new realization of π by adding the edges yw and xv and deleting the edges yv and xw. The case where x is not adjacent to exactly two vertices in I is handled similarly. Repeating this process allows us to create a realization of π containing $K_{k-3} + \overline{K}_3 = F + I$ in which the $k-3$ highest degree vertices of G lie in F.

Let x_1 and x_2 be the vertices in I having the highest degrees, and note that $\sigma(\pi) \geq \sigma(K_{k-1}, n)$ implies $d(x_1)$ and $d(x_2)$ are both at least $k-2$. If there is any edge in the subgraph induced by I, then G contains a copy of $K_{k-3} + (K_2 \cup K_1)$ and we are done. Therefore, we may assume that I is an independent set. Let N_1 and N_2 denote $N(x_1) \setminus S$ and $N(x_2) \setminus S$, respectively, and note that both of these sets are nonempty since $d(x_1)$ and $d(x_2)$ are both at least $k-2$. If y_1 and y_2 are distinct vertices in N_1 and N_2, respectively, then we may assume that y_1 and y_2 are adjacent. If they are not, then we would exchange the edges x_1y_1 and x_2y_2 for the nonedges x_1x_2 and y_1y_2, creating an edge in I and completing the proof.

The goal of the next part of this proof is to show that we may assume that there is some vertex v in F such that $d(v) \leq 4k$.

Consider first the case where $N_2 \subseteq N_1$ ($N_1 \subseteq N_2$ is handled identically) and let w be a vertex in N_2. If $|N_1 \setminus N_2| > k$ then $d(w) > d(x_2)$ since w is adjacent to every vertex in $N_1 \setminus N_2$. We therefore assume that $|N_1 \setminus N_2| \leq k$. Also note that $N_1 \cap N_2$ is a clique, and hence contains at most $k-2$ vertices. There is some vertex v in F that is not adjacent to w, otherwise $d(w) > d(x_1)$, which contradicts our choice of G. Let y be a neighbor of v that does not lie in $S \cup N_1 \cup N_2$. If no such y exists, then clearly $d(v) \leq 4k$. We claim that wy is an edge of G, lest we could exchange the edges x_1w, x_2w and yv for the nonedges wv, wy and x_1x_2 (see Figure 1), creating an edge in I. However, if the degree of v is more than $4k$ there are at least $k-1$ such choices for y. This implies that $d(w) \geq k + |N_1| > d(x_1)$, which contradicts our choice of G. Thus we may assume that $d(v) \leq 4k$.

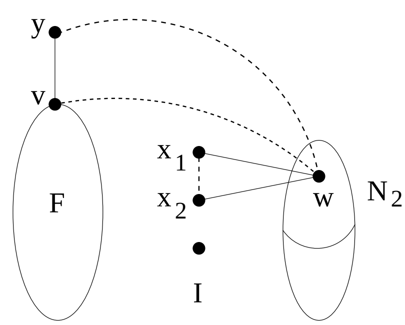

FIGURE 1. $N_2 \subseteq N_1$

Assume now that there is some vertex w_1 in $N_1 \setminus N_2$ and some vertex w_2 in $N_2 \setminus N_1$. We first show that $N_1 \cup N_2$ is complete. To accomplish this, we need only show that for any w_1' in $N_1 \setminus N_2$, w_1w_1' is an edge of G (or symmetrically, if w_2' is an element of $N_2 \setminus N_1$ then w_2w_2' is an edge in G). If not, we can exchange the edges x_1w_1, x_1w_1' and x_2w_2 for the nonedges w_1w_1', x_1w_2 and x_1x_2, creating an edge in I and completing the proof. Thus, since $N_1 \cup N_2$ is complete we may assume that $|N_1 \cup N_2| \leq k-1$. Again, there is some v in F such that w_2 is not adjacent to v, lest $d(w_2) > d(x_2)$. Let y be any neighbor of v not in $S \cup N_1 \cup N_2$. Then w_1 is adjacent to y or else we could exchange the edges yv, x_1w_1 and x_2w_2 for the nonedges yw_1, vw_2 and x_1x_2 (see Figure 2), creating an edge in I. If $d(v) > 3k$, then there are at least k such choices for y, implying that $d(w_1) \geq k + |N_1 \cup N_2| - 1 > d(x_1)$, a contradiction.

Hence, we may assume that there is some vertex v in F such that $d(v) \leq 4k$. As a result, there are at most $(k-4)(n-1)+4k$ edges adjacent to vertices in F, at most $12k$ edges adjacent to vertices in I and, as both N_1 and N_2 have at most $4k$ vertices each, at most $4k(8k) = 32k^2$ edges adjacent to vertices in $N_1 \cup N_2$. This is at most $(k-4)n+32k^2+15k+4$ edges. However, there are at least $\sigma(\pi)/2 = (k-3+o(1))n$ edges in G, so for n sufficiently large there is some edge yz in G such that y is not adjacent to any w_1 in N_1 and z is not adjacent to any w_2 in N_2, where w_1 and $w_@$ may be the same vertex. We can therefore exchange the edges x_1w_1, x_2w_2 and

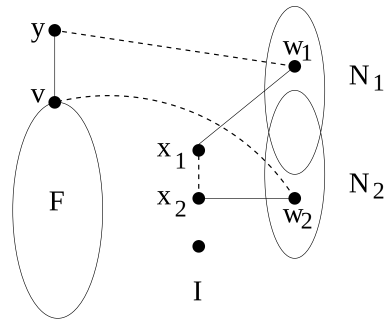

FIGURE 2. $N_2 \not\subseteq N_1$ and $N_1 \not\subseteq N_2$

yz for the nonedges w_1y, w_2z and x_1x_2, creating an edge in I, and completing the proof. □

Propositions 4.3 and 4.5 together imply Theorem 4.1. As mentioned above, there is quite a wide variety to the structures of those graphs H having independence number 2, and yet we have demonstrated that $\sigma(H, n)$ for this class depends only on the value of $d(H)$, as suggested by Conjecture 1.

5. H-Saturated Graphs

Here we describe the relationship of $\sigma(H, n)$ to another extremal function $sat(n, H)$. We begin with the relevant terminology and results.

A graph G is said to be H-saturated if G contains no copy of H as a subgraph and for any edge e not in G, $G+e$ does contain a copy of H. The problem of determining the *minimum* number of edges in an H-saturated graph, denoted $sat(n, H)$, was first considered in 1963 by Erdős, Hajnal and Moon [6] for $H = K_t$. They determined that $sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}$, which arises from consideration of the split graph $K_{t-2} + \overline{K}_{n-t+2}$. The best known upper bound for an arbitrary graph H is given by the following result of Kászonyi and Tuza [13].

Theorem 5.1 ([13])**.** *Let $u(H)$ be as defined above, and set*

$$s(H) = min\{e(H^*)|\alpha(H^*) = \alpha(H),\ |V(H^*)| = \alpha(H) + 1, H^* \subseteq H\}$$

then,

$$sat(n, H) \leq n(u(H) + \frac{s(H)-1}{2}) - \frac{u(H)\big(u(H) + s(H)\big)}{2}. \tag{16}$$

The reader should note that the bound given in Theorem 5.1 reflects the number of edges in the join of $K_{u(H)}$ and a graph which is (nearly) $(s-1)$-regular. Comparing Theorem 5.1 to the construction of $\pi^*(H, n)$, we note that $d(H) \leq s(H)$ and hence that if $i \geq s(H)$, $n_i = 0$. Theorem 5.1 and Theorem 3.1 immediately imply the following result.

Theorem 5.2. *Given a graph H, if there exists an $H' \subseteq H$ with $2u(H') + d(H') - 1 \geq 2u(H) + s(H) - 1$ then for n sufficiently large we have*

$$2sat(n, H) < \sigma(H, n). \tag{17}$$

In particular, this result holds if $d(H) = s(H)$.

We strongly believe that the conclusion of Theorem 5.2 holds in general, even though the hypothesis does not. Therefore, we conjecture the following.

Conjecture 3. *Let H be a graph and let n be a sufficiently large integer. Then*

$$2sat(n, H) < \sigma(H, n).$$

As the problem of determining $sat(n, H)$ has proven difficult over time, we are not able to confirm Conjecture 3 in as many cases as Conjectures 1 and 2. We know that Conjecture 3 holds for complete graphs [6], [7], tK_p and certain generalized friendship graphs [8], C_4 [12], [22],[24], and $K_{1,t}$ [13].

6. Conclusion

In light of Theorem 4.1, it may be interesting to individually consider classes of graphs with fixed independence number. This may be a fruitful direction, although the diversity in the structures of the $\alpha(H) + 1$ vertex induced subgraphs of such graphs rapidly increases. We feel that this line of investigation would move us closer to the goal of verifying either of Conjectures 1 and 2.

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