SIAM J. DISCRETE MATH.
Vol. 23, No. 2, pp. 940–958

ON BLOCK SEQUENCES OF STEINER QUADRUPLE SYSTEMS WITH ERROR CORRECTING CONSECUTIVE UNIONS*

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Dedicated to Professor Sanpei Kageyama on the occasion of his retirement from Hiroshima University

Abstract. Motivated by applications in combinatorial group testing for consecutive positives, we investigate a block sequence of a maximum packing MP(t, k, v) which contains the blocks exactly once such that the collection of all blocks together with all unions of two consecutive blocks of this sequence forms an error correcting code with minimum distance d. Such a sequence is usually called a block sequence with consecutive unions having minimum distance d, and denoted by BSCU$(t, k, v|d)$. In this paper, we show that the necessary conditions for the existence of BSCU$(3, 4, v|4)$s of Steiner quadruple systems, namely, $v \equiv 2, 4 \pmod 6$ and $v \geq 4$, are also sufficient, excepting $v = 8, 10$.

Key words. BSCU, CSCU, CSCU-CQS, CSCU-GDD

AMS subject classifications. 92D20, 94C12, 05B05

DOI. 10.1137/08071750X

1. Introduction. Let V be a finite set of v elements and let $\mathcal{X}$ be a collection of k-subsets of V with $|\mathcal{X}| = m$. Let $S = [x_0, x_1, \ldots, x_{m-1}]$ be a sequence of the elements in $\mathcal{X}$. The indices of the elements x_i of S are considered modulo m. Define $y_i = x_i \cup x_{i+1}$ for each i, $0 \leq i \leq m-1$. The sequence S is called a *cyclic sequence of* $\mathcal{X}$ *with consecutive unions having minimum distance* d, denoted as CSCU$(k, v|d)$, if $C = \{x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}\}$ has minimum distance d. Note that the distance between any two sets x and y is defined as $d(x, y) = |(x \cup y) \setminus (x \cap y)|$. Furthermore, a CSCU$(k, v|d)$ is said to be *maximal* if the number of elements in $\mathcal{X}$ is maximum for given k, v, and d, denoted as MCSCU$(k, v|d)$.

The concept of an MCSCU is motivated by the applications in *combinatorial group testing for consecutive positives*. Group testing was proposed by Dorfman [3] in 1940s to do large scale blood testing economically, and new applications of group testing have been found recently such as *DNA library screening*, being error-prone, in which it is desired to determine the set of all clones containing a specific sequence of nucleotides in an economical and correct way. A clone is *positive* if it contains the specific sequence and *negative* otherwise. One chooses arbitrarily a subset of clones called a *group* or *pool* and tests all clones in the pool in one stroke by some chemical analysis. The pool is *positive* when it contains at least one positive clone and *negative* otherwise. Colbourn [1] developed some strategy for group testing when the clones are *linearly*

*Received by the editors March 3, 2008; accepted for publication (in revised form) February 11, 2009; published electronically May 20, 2009.
http://www.siam.org/journals/sidma/23-2/71750.html
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ordered, and the positive clones form a *consecutive subset* of the set of all clones, the typical example being the problem of locating a *sequence-tagged site* (or STS) among ordered clones. Jimbo and his collaborators [16, 15, 17, 18] improved Colbourn's strategy by considering the error detecting and correcting capability of group testing which is essential in view of applications such as DNA library screening. In particular, Momihara and Jimbo [16, 15] suggested using MCSCUs of a combinatorial structure called *t-packings* to correct *false negative* or *false positive* clones in the pool outcomes. For more details of such applications, we refer the reader to [1, 4, 16, 15, 17, 18, 19] and references therein.

A *t-packing* of order v, block size k, briefly $\mathrm{P}(t,k,v)$, is an ordered pair $(V,\mathcal{B})$, where V is a finite set of v elements called *points*, and $\mathcal{B}$ is a set of k-subsets of V called *blocks*, such that each t-tuple of distinct points of V is contained in at most one block of $\mathcal{B}$. In particular, a $\mathrm{P}(t,k,v)$ is said to be *maximal*, denoted $\mathrm{MP}(t,k,v)$, if the number of blocks is maximum for given t,k, and v. For $v \equiv 2,4 \pmod 6$, an $\mathrm{MP}(3,4,v)$ is also called a *Steiner quadruple system*, briefly $\mathrm{SQS}(v)$. The existence of an $\mathrm{SQS}(v)$ for every admissible v is proved by Hanani [5], Hartman [7, 8], and Lenz [12].

It is known (see [16]) that a $\mathrm{CSCU}(k,v|d)$ of $\mathcal{B}$ is maximal if $\mathcal{B}$ is the block set of an $\mathrm{MP}(\lfloor k-d/2\rfloor+1,k,v)$. A $\mathrm{CSCU}(k,v|d)$ of $\mathcal{B}$ which is the block set of an $\mathrm{MP}(t,k,v)$ is also called a *block sequence of $\mathcal{B}$ with consecutive unions having minimum distance d*, briefly $\mathrm{BSCU}(t,k,v|d)$.

In the case of $d=2$, Müller and Jimbo [18] showed that there exists a $\mathrm{BSCU}(k,k,v|2)$ for every $v \geq v_k$ for the pairs of parameters k and v_k, $(k,v_k)=(2,6),(3,8),(4,11),(5,12),(6,17)$, and $(7,19)$, without introducing the notion of block sequences of t-packings. In the case of $d=3$, Momihara and Jimbo [16] showed the existence of a $\mathrm{BSCU}(2,3,v|3)$ for every $v \geq 10$. For the case of $d=4$, it is clear that a $\mathrm{BSCU}(3,4,v|4)$ forms an $\mathrm{MCSCU}(4,v|4)$. Momihara and Jimbo [15] recently showed the existence of a $\mathrm{BSCU}(3,4,v|4)$ for 47 small values $v \leq 500$ using the following two constructions.

THEOREM 1.1. (see [15]). *Let v be an integer satisfying $v \equiv 2,4 \pmod 6$ and $v \geq 14$.*

(1) *If there exists a* $\mathrm{BSCU}(3,4,v|4)$, *then there exists a* $\mathrm{BSCU}(3,4,2v|4)$ *which contains a sub*-$\mathrm{BSCU}(3,4,v|4)$.

(2) *If there exists a* $\mathrm{BSCU}(3,4,v|4)$, *then there exists a* $\mathrm{BSCU}(3,4,3v-2|4)$ *which contains a sub*-$\mathrm{BSCU}(3,4,v|4)$.

It is not difficult to see [15] that if there exists a $\mathrm{BSCU}(3,4,v|4)$, then every two consecutive blocks must be disjoint. Furthermore, there does not exist a $\mathrm{BSCU}(3,4,v|4)$ for $v \leq 11$ except for $v=4$, in which there is only one block. We call such a $\mathrm{BSCU}(3,4,4|4)$ *trivial*.

In this paper, we write $\mathrm{BSCU}(3,4,v|4)$ of the block sets of Steiner quadruple systems as $\mathrm{BSCU}(v)$ for brevity. The necessary conditions for the existence of a $\mathrm{BSCU}(v)$ are $v \equiv 2,4 \pmod 6$ and $v \geq 4$. In the following sections, we will prove that the above necessary conditions are also sufficient except for $v=8,10$. Our main tools are the recursive constructions used in the 3-design theory (see [9, 10, 11] for detailed information).

2. Recursive constructions. Let v,s be two nonnegative integers, t be a positive integer, and K be a set of positive integers. A *candelabra t-system* (or t-CS) of order v and block sizes from K, denoted by $\mathrm{CS}(t,K,v)$, is a quadruple $(X,S,\mathcal{G},\mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements (called *points*);

(2) S is an s-subset (called the *stem* of the *candelabra*) of X;

(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets (called *groups* or *branches*) of $X \setminus S$, which partition $X \setminus S$;

(4) $\mathcal{A}$ is a collection of subsets (called *blocks*) of X, each of cardinality from K;

(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$ for all i is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$ for all i is contained in any block of $\mathcal{A}$.

By the *group type* (or *type*) of a t-CS $(X, S, \mathcal{G}, \mathcal{A})$ we mean the list $(|G| \mid G \in \mathcal{G} : |S|)$ of group sizes and stem size, where the stem size is separated from the group sizes by a colon. If a t-CS has n_i groups of size g_i, $1 \le i \le r$, and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ to denote the group type. When $t = 3$ and $K = \{4\}$, such a system is usually called a *candelabra quadruple system* and denoted for short by $\mathrm{CQS}(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$.

A *holey quadruple system* of order v with a hole of order s, denoted by $\mathrm{HSQS}(v : s)$, is a triple $(X, S, \mathcal{A})$ where X is a set of v elements (called *points*), S is an s-subset of X, and $\mathcal{A}$ is a collection of 4-subsets (called *blocks*) of X such that every 3-subset T of X with $T \not\subseteq S$ is contained in a unique block of $\mathcal{A}$ and no 3-subset of S is contained in any block of $\mathcal{A}$.

Let $(X, S, \mathcal{G}, \mathcal{A})$ be a $\mathrm{CS}(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ with $S = \{\infty_1, \infty_2, \ldots, \infty_s\}$, where $s \ge 1$. For $1 \le i \le s$, let $\mathcal{B}_i = \{A \setminus \{\infty_i\} \mid \infty_i \in A \in \mathcal{A}\}$ and $\mathcal{T} = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$. Then the $(s+3)$-tuple $(X \setminus S, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ is called an *s-fan design*. If block sizes of $\mathcal{B}_i$, $1 \le i \le s$, and $\mathcal{T}$ are from K_i and $K_{\mathcal{T}}$, respectively, then the s-fan design is denoted by $s\text{-FG}(3, (K_1, \ldots, K_s, K_{\mathcal{T}}), \sum_{i=1}^{r} n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$.

A *group divisible t-design* of order v and block sizes from K, denoted by $\mathrm{GDD}(t, K, v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*);

(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;

(3) $\mathcal{B}$ is a collection of subsets (called *blocks*) of X, each of cardinality from K, such that each block intersects any given group in at most one point; and

(4) every t-subset T of X from t distinct groups is contained in a unique block of $\mathcal{B}$.

The *type* of $\mathrm{GDD}(t, K, v)$ is defined as the list $(|G| \mid G \in \mathcal{G})$. In this paper, we consider only $\mathrm{GDD}(3, 4, v)$ of type T and always write $\mathrm{GDD}(T)$ for brevity.

THEOREM 2.1. (see [14]). For $u > 3, u \ne 5$, a $\mathrm{GDD}(g^u)$ exists if and only if ug is even and $g(u-1)(u-2)$ is divisible by 3. For $u = 5$, a $\mathrm{GDD}(g^5)$ exists if g is divisible by 4 or 6.

A $\mathrm{CSCU}(4, v|4)$ of $\mathcal{B}$ which is the block set of a $\mathrm{CQS}(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ will be denoted by $\mathrm{CSCU\text{-}CQS}(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ in this paper. Similarly, we can define CSCU-HSQS, CSCU-GDD, etc.

Now we apply the fundamental constructions in the 3-design theory, where “filling in holes” and the “weighting method” are always useful (see [9]). First, we may think of one CSCU (the master design) as a cycle which can be cut off at any place. Next, we view the sequence of the other cut-off CSCU (the subdesign) as a segment and insert it into some cut place of the master design to form a bigger cycle. Then we calculate the number of the places in the master design where the obtained bigger cycle is also a CSCU. If this number is positive, then the construction succeeds. We explain it in detail as follows.

For any k-subset sequence $S = [x_0, x_1, \ldots, x_{m-1}]$ with length m, define

$$\sigma^j(S) = [x_j, x_{j+1}, \ldots, x_{m-1}, x_0, \ldots, x_{j-1}],$$

$$\overline{S} = \{x_0 \cup x_1, x_1 \cup x_2, \ldots, x_{m-2} \cup x_{m-1}\},$$

and

$$\widehat{S} = \{x_0 \cup x_1, x_1 \cup x_2, \ldots, x_{m-2} \cup x_{m-1}, x_{m-1} \cup x_0\}.$$

Let U, V be two finite sets with $|U| = u$ and $|V| = v$, where U is not necessarily disjoint with V. Let $S = [b_0, b_1, \ldots, b_{p-1}]$ be a CSCU$(4, u|4)$ of $\mathcal{B}$ which is a collection of 4-subsets of U with $p = |\mathcal{B}|$, and let $T = [a_0, a_1, \ldots, a_{q-1}]$ be a CSCU$(4, v|4)$ of $\mathcal{A}$ which is a collection of 4-subsets of V with $q = |\mathcal{A}|$. It is clear that $|b \cap b'| \le 2$ and $|a \cap a'| \le 2$ for any distinct $b, b' \in \mathcal{B}$ and $a, a' \in \mathcal{A}$. We may assume that, for any $b \in \mathcal{B}$, we always have $|b \cap V| \le 2$. Then, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we always have $|b \cap a| \le 2$. We view S as a cycle, cut T between a_0 and a_{q-1}, and keep the order fixed. We insert $T = [a_0, a_1, \ldots, a_{q-1}]$ into S between b_{i-1} and b_i for some i, $0 \le i \le p-1$, and denote the bigger cycle $[a_0, a_1, \ldots, a_{q-1}, b_i, b_{i+1}, \ldots, b_{i-1}]$ by $S_i = [T, \sigma^i(S)]$. Let $M = \{i \mid S_i \text{ is a CSCU}(4, w|4) \text{ of } \mathcal{B} \cup \mathcal{A},\ 0 \le i \le p-1\}$, where $w = |U \cup V|$, and let $|M| = m$. If $m > 0$, then we obtain a bigger CSCU$(4, w|4)$ from the two small CSCUs. Next, we estimate the value of m.

Let $C = \mathcal{A} \cup \mathcal{B} \cup \overline{T} \cup \overline{\sigma^i(S)} \cup D$, where $D = \{a_0 \cup b_{i-1}, a_{q-1} \cup b_i\}$. We check the distance between any two elements of C. First, we consider the case that $a_0 \cap b_{i-1} = \emptyset$ and $a_{q-1} \cap b_i = \emptyset$. In this case, we have the following conclusions:
Since T is a CSCU of $\mathcal{A}$, we have

Case 1: $d(a, a') \ge 4$ for any $a, a' \in \mathcal{A}$;
Case 2: $d(c, c') \ge 4$ for any $c, c' \in \overline{T}$; and
Case 3: $d(a, c) \ge 4$ for any $a \in \mathcal{A}$ and $c \in \overline{T}$.

Since S is a CSCU of $\mathcal{B}$, we have

Case 4: $d(b, b') \ge 4$ for any $b, b' \in \mathcal{B}$;
Case 5: $d(c, c') \ge 4$ for any $c, c' \in \overline{\sigma^i(S)}$; and
Case 6: $d(b, c) \ge 4$ for any $b \in \mathcal{B}$ and $c \in \overline{\sigma^i(S)}$.

Since $|a_0 \cap b_i| \le 2$, $|a_{q-1} \cap b_{i-1}| \le 2$, and $b_{i-1} \cap b_i = \emptyset$, we know that $d(b_{i-1}, b_i) = 8$, $d(a_0 \cup b_{i-1}, b_i) \ge 6$, and also

Case 7: $d(a_0 \cup b_{i-1}, a_{q-1} \cup b_i) \ge 4$.

Since $|a \cap b| \le 2$, we have

Case 8: $d(a, b) \ge 4$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Since $|a| = 4$, $|b| = 4$, and $|c| = 8$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \overline{T} \cup \overline{\sigma^i(S)} \cup D$, we have

Case 9: $d(a, c) \ge 4$ for any $a \in \mathcal{A}$ and $c \in \overline{\sigma^i(S)} \cup D$; and
Case 10: $d(b, c) \ge 4$ for any $b \in \mathcal{B}$ and $c \in \overline{T} \cup D$.

Since $|b \cap V| \le 2$ for any $b \in \mathcal{B}$, we have

Case 11: $d(c, c') \ge 4$ for $c \in \overline{T}$ and $c' \in \overline{\sigma^i(S)} \cup D$.

Under the assumption that $a_0 \cap b_{i-1} = \emptyset$ and $a_{q-1} \cap b_i = \emptyset$, we still need to consider the values of $d(c, c')$ for any $c \in \overline{\sigma^i(S)}$ and $c' \in D$. Note that we should also check the distance between any two elements of C in the case that $a_0 \cap b_{i-1} \ne \emptyset$ or $a_{q-1} \cap b_i \ne \emptyset$.

Let $N(a_{q-1}) = \{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k \ne \emptyset \text{ or } d(a_{q-1} \cup b_k, c) < 4 \text{ for some } c \in \widehat{S}\}$ and $n(a_{q-1}) = |N(a_{q-1})|$. Also let $\alpha(a_{q-1}) = |\{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k \ne \emptyset\}|$.

Then $n(a_{q-1}) = \alpha(a_{q-1}) + |\{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k = \emptyset,\ \text{and } d(a_{q-1} \cup b_k, c) < 4 \text{ for some } c \in \widehat{S}\}|$. In order to estimate $n(a_{q-1})$, we consider the case that $a_{q-1} \cap b_k = \emptyset$. It is clear that for any index $0 \le l \le p-1$, $|(b_l \cup b_{l+1}) \cap a_{q-1}| \le 4$.

If there exists an index l such that $|(b_l \cup b_{l+1}) \cap a_{q-1}| = 4$, i.e., $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 2$, then if $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$, we should have $|(b_l \cup b_{l+1}) \cap b_k| \ge 3$. In the case that $\mathcal{B}$ is the block set of some 3-packing of order u, there is at most one such k that $|(b_l \cup b_{l+1}) \cap b_k| = 4$, that is, $b_k = (b_l \cup b_{l+1}) \setminus a_{q-1}$, or there are at most 4 such k that $|(b_l \cup b_{l+1}) \cap b_k| = 3$, that is, b_k are obtained by choosing any three points from the four points in $(b_l \cup b_{l+1}) \setminus a_{q-1}$ and the other one from the other points of U, which implies that $|\{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k = \emptyset,\ \text{and } d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| \le 4$.

If there exists an index l such that $|(b_l \cup b_{l+1}) \cap a_{q-1}| = 3$, i.e., $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 1$, or $|b_l \cap a_{q-1}| = 1$ and $|b_{l+1} \cap a_{q-1}| = 2$, then we should have $|(b_l \cup b_{l+1}) \cap b_k| = 4$ if $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$. In the case that $\mathcal{B}$ is the block set of some 3-packing of order u, there is at most one k such that $|(b_l \cup b_{l+1}) \cap b_k| = 4$, that is, b_k is obtained by choosing four points from $(b_l \cup b_{l+1}) \setminus a_{q-1}$. If there is another k' such that $|(b_l \cup b_{l+1}) \cap b_{k'}| = 4$, then $|b_k \cap b_{k'}| \ge 3$ because $|(b_l \cup b_{l+1}) \setminus a_{q-1}| = 5$, which is a contradiction. In this case, we have $|\{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k = \emptyset,\ \text{and } d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| \le 1$.

If $|(b_l \cup b_{l+1}) \cap a_{q-1}| \le 2$, then we can easily check that there is no k such that $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$, that is, $|\{k \mid 0 \le k \le p-1,\ a_{q-1} \cap b_k = \emptyset,\ \text{and } d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| = 0$.

Therefore, if we define $\gamma(a_{q-1}) = |\{l \mid 0 \le l \le p-1,\ |(b_l \cup b_{l+1}) \cap a_{q-1}| = 4\}|$, and $\delta(a_{q-1}) = |\{l \mid 0 \le l \le p-1,\ |(b_l \cup b_{l+1}) \cap a_{q-1}| = 3\}|$, then under the condition that $a_{q-1} \cap b_k = \emptyset$, there are at most $4\gamma(a_{q-1}) + \delta(a_{q-1})$ k such that $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$. So we have $n(a_{q-1}) \le \alpha(a_{q-1}) + 4\gamma(a_{q-1}) + \delta(a_{q-1})$. From the definition of $\gamma(a_{q-1})$, we know that the existence of one such index l in $\gamma(a_{q-1})$ would imply both $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 2$. Also, from the definition of $\delta(a_{q-1})$, the existence of one such index l in $\delta(a_{q-1})$ would imply $|b_l \cap a_{q-1}| = 2$ or $|b_{l+1} \cap a_{q-1}| = 2$, but not both. Keeping in mind the possibilities of occurrences of consecutive blocks in $\Xi = \{k \mid 0 \le k \le p-1,\ |b_k \cap a_{q-1}| = 2\}$ and one block in Ξ followed by one block in $\{k \mid 0 \le k \le p-1,\ |b_k \cap a_{q-1}| = 1\}$, we can know that these would imply $2\gamma(a_{q-1}) + \delta(a_{q-1}) < 2\beta(a_{q-1})$, since γ and δ are mutually exclusive, where $\beta(a_{q-1}) = |\Xi|$.

Similarly, we can analyze the set $N(a_0) = \{k \mid 0 \le k \le p-1,\ a_0 \cap b_{k-1} \ne \emptyset,\ \text{or } d(a_0 \cup b_{k-1}, c) < 4 \text{ for some } c \in \widehat{S}\}$, where $n(a_0) = |N(a_0)|$.

Then from the definitions of M, $N(a_0)$, and $N(a_{q-1})$, we immediately have that $M \supseteq Z_p \setminus (N(a_0) \cup N(a_{q-1}))$ and $m \ge p - n(a_0) - n(a_{q-1}) + |N(a_0) \cap N(a_{q-1})| \ge p - n(a_0) - n(a_{q-1}) + |E|$, where $E \subseteq N(a_0) \cap N(a_{q-1})$.

THEOREM 2.2. *Suppose that there are both a CSCU-HSQS$(u : v)$ and a BSCU(v). Then there is a BSCU(u) when $u \ge 44$ and $u > v$.*

Proof. Let $S = [b_0, b_1, \ldots, b_{p-1}]$ be a CSCU-HSQS$(u : v)$ on U and $T = [a_0, a_1, \ldots, a_{q-1}]$ be a BSCU(v) on V with $V \subset U$. By the definition of an HSQS$(u : v)$, we know that for any of its blocks, say, b, we always have $|b \cap V| \le 2$. We view S as a cycle, cut T between a_0 and a_{q-1}, and keep the order fixed. Next, insert $T = [a_0, a_1, \ldots, a_{q-1}]$ into S between b_{i-1} and b_i for some i, $0 \le i \le p-1$, and denote the resultant cycle $[a_0, a_1, \ldots, a_{q-1}, b_i, b_{i+1}, \ldots, b_{i-1}]$ by $S_i = [T, \sigma^i(S)]$. Using the same notation as above, we prove the theorem as follows.

Since T is a BSCU(v), we have $a_{q-1} \cap a_0 = \emptyset$ since they are consecutive. From the balanced property of t-designs, we also have $n(a_0) = n(a_{q-1})$. Then $m \geq p - n(a_0) - n(a_{q-1}) \geq p - 2(\alpha(a_{q-1}) + 4\gamma(a_{q-1}) + \delta(a_{q-1})) > p - 2\alpha(a_{q-1}) - 8\beta(a_{q-1})$. Here $p = u(u-1)(u-2)/24 - v(v-1)(v-2)/24$, $\alpha(a_{q-1}) = 2(u-1)(u-2)/3 - 3(u-2) - 2(v-1)(v-2)/3 + 3(v-2)$, and $\beta(a_{q-1}) = 3(u-v)$. Then we have $m > p - 2\alpha(a_{q-1}) - 8\beta(a_{q-1}) > 0$ when $u \geq 44$ and $u > v$. This means that there is a BSCU(u) when $u \geq 44$ and $u > v$. $\square$

THEOREM 2.3. *Suppose that there are both a CSCU-CQS*($m^n : s$) *and a CSCU-HSQS*($m + s : s$). *Then there are both a CSCU-HSQS*($mn + s : m + s$) *and a CSCU-HSQS*($mn + s : s$) *when* $mn \geq 44$, $m + 2s \geq 5$, *and* $m \geq 2$.

Proof. Let $(X, S, \{G_1, \ldots, G_n\}, \mathcal{B})$ be the CQS($m^n : s$). Then we construct an HSQS($m + s : s$) on $S \cup G_k$, $1 \leq k \leq n$, with S as the hole to obtain the desired HSQS($mn+s : m+s$) (or HSQS($mn+s : s$), respectively). Let $S_0 = [b_0, b_1, \ldots, b_{p-1}]$ be a CSCU-CQS($m^n : s$) and $T_k = [a_0^k, a_1^k, \ldots, a_{q-1}^k]$ be a CSCU-HSQS($m + s : s$) on $S \cup G_k$. Note that for each block $b \in \mathcal{B}$, we have $|b \cap (S \cup G_k)| \leq 2$. View S_0 as a cycle, and cut each T_k between a_0^k and a_{q-1}^k. Then we insert each T_k into S_0 between b_{i_k-1} and b_{i_k} one by one. Here, we require that $i_k \neq i_{k'}$ if $k \neq k'$.

Using the same notation, we have that $p = m^2 n(n-1)(m + mn + 3s - 3)/24$. By counting the number r_x of blocks in $\mathcal{B}$ containing a point $x \in X$, and the assumption that $m + 2s \geq 5$, we know that $r_x \leq m(n-1)(mn + m + 2s - 3)/6$. By counting the number $r_{x,y}$ of blocks in $\mathcal{B}$ containing a pair of distinct points $\{x, y\}$ of X, and the assumption that $m \geq 2$, we also know that $r_{x,y} \geq m(n-1)/2$. Then we have $\max\{\alpha(a_0^k), \alpha(a_{q-1}^k)\} \leq \alpha = 4 \times m(n-1)(mn + m + 2s - 3)/6 - 6 \times m(n-1)/2$ and $\max\{\beta(a_0^k), \beta(a_{q-1}^k)\} \leq \beta = 6 \times m(n-1)/2$ for any $1 \leq k \leq n$.

First, since $m_1 \geq p - \alpha(a_0^1) - 4\beta(a_0^1) - \alpha(a_{q-1}^1) - 4\beta(a_{q-1}^1) \geq p - 2\alpha - 8\beta \geq 1$, there exists one i_1, $0 \leq i_1 \leq p - 1$, such that $S_1 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots]$ is a CSCU. Here, S_1 is obtained by inserting T_1 into S_0 between b_{i_1-1} and b_{i_1}.

Next, we want to insert T_2 into S_1 between b_{i_2-1} and b_{i_2}, where $0 \leq i_2 \leq p-1$ and $i_2 \neq i_1$, so that $S_2 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots, b_{i_2-1}, T_2, b_{i_2}, \ldots]$ is a CSCU. Since $|b \cap (S \cup G_2)| \leq 2$ for each block $b \in T_1 \cup \mathcal{B}$, in order to estimate m_2, the number of suitable places that we can properly insert T_2 into S_1, we need only to compute the numbers of the consecutive unions $c \in \widehat{S_1} = \overline{\sigma^{i_1}(S_0)} \cup \{a_{q-1}^1 \cup b_{i_1}, a_0^1 \cup b_{i_1-1}\} \cup \overline{T_1}$ such that $|c \cap a_j^2| = 3$ and 4, $j = 0, q-1$, respectively, for the reason that $m_2 \geq p' - n'(a_0^2) - n'(a_{q-1}^2) \geq p - 1 - (\alpha'(a_0^2) + 4\gamma'(a_0^2) + \delta'(a_0^2)) - (\alpha'(a_{q-1}^2) + 4\gamma'(a_{q-1}^2) + \delta'(a_{q-1}^2))$, where $\alpha'(a_j^2) = |\{k \mid 0 \leq k \leq p-1,\ b_k \cap a_j^2 \neq \emptyset\}| = \alpha(a_j^2)$, $\gamma'(a_j^2) = |\{l \mid |(c_l \cup c_{l+1}) \cap a_j^2| = 4\}|$, $\delta'(a_j^2) = |\{l \mid |(c_l \cup c_{l+1}) \cap a_j^2| = 3\}|$ for $j = 0$ and $q - 1$, and $c_l \cup c_{l+1} \in \widehat{S_1}$. It is easy to know that there are no such unions in $\overline{T_1}$. We then consider the unions in $\{a_{q-1}^1 \cup b_{i_1}, a_0^1 \cup b_{i_1-1}\} \cup \overline{\sigma^{i_1}(S_0)}$. For the unions in $\overline{\sigma^{i_1}(S_0)}$, we know that $4\gamma(a_j^2) + \delta(a_j^2) < 4\beta(a_j^2)$ holds for $j = 0, q-1$. For the unions in $\{a_{q-1}^1 \cup b_{i_1}, a_0^1 \cup b_{i_1-1}\}$, since $|(a_0^1 \cup a_{q-1}^1) \cap a_j^2| \leq 2$ for $j = 0, q-1$, and $a_0^1 \cap a_{q-1}^1 = \emptyset$, we know that the only possible cases are the following: (1) both $a_{q-1}^1 \cup b_{i_1}$ and $a_0^1 \cup b_{i_1-1}$ intersect a_j^2 at 3 elements; (2) exactly one of $a_{q-1}^1 \cup b_{i_1}$ and $a_0^1 \cup b_{i_1-1}$ intersects a_j^2 at 4 elements, and the other at less than 3 points; (3) exactly one of $a_{q-1}^1 \cup b_{i_1}$ and $a_0^1 \cup b_{i_1-1}$ intersects a_j^2 at 3 elements, and the other at less than 3 points. In any case, we have $m_2 \geq p - 1 - 2\alpha - 8\beta - 2 \times 4 \geq 1$. This means that there exists at least one such index $i_2 \neq i_1$ so that $S_2 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots, b_{i_2-1}, T_2, b_{i_2}, \ldots]$ is a CSCU, where S_2 is obtained by inserting T_2 into S_1 between b_{i_2-1} and b_{i_2}.

Suppose we have inserted T_k into S_{k-1} for $k = 1, \ldots, n-1$, and let S_k denote the obtained CSCU. We want to insert T_{k+1} into S_k between $b_{i_{k+1}-1}$ and $b_{i_{k+1}}$ where $0 \le i_{k+1} \le p-1$ and $i_{k+1} \ne i_l$ for any $1 \le l \le k$. Similarly, we need only care about the unions in $\widehat{S_0} \cup \{a^1_{q-1} \cup b_{i_1}, \ldots, a^k_{q-1} \cup b_{i_k}, a^1_0 \cup b_{i_1-1}, \ldots, a^k_0 \cup b_{i_k-1}\}$. Then we have $m_{k+1} \ge p-k-2\alpha-8\beta-8k$. It is easy to check that $m_k \ge 1$ for any $1 \le k \le n$. So there exist n distinct indices $0 \le i_1, i_2, \ldots, i_n \le p-1$ such that when we insert each T_k into S_{k-1} between b_{i_k-1} and b_{i_k}, the obtained sequence is a CSCU-HSQS$(mn+s : s)$ when $1 \le k \le n$ or a CSCU-HSQS$(mn+s : m+s)$ when $1 \le k \le n-1$. $\square$

For a CQS $(X, S, \mathcal{G}, \mathcal{B})$, we may view S as a special group, that is, let $S \in \mathcal{G}$, and we will write CQS $(X, \mathcal{G}, \mathcal{B})$ for convenience. If a block of size k intersects each group in at most one point, we say it is *k-partite* (see [9]). For any design $(X, \mathcal{G}, \mathcal{B})$, GDD, or CQS, let P be a permutation on X. For each $G \in \mathcal{G}$, if $P(G) = G$, then the design $(X, \mathcal{G}, P(\mathcal{B}))$ is isomorphic to $(X, \mathcal{G}, \mathcal{B})$. For a point $x \in X$, denote by $\overline{G}_x$ the group containing x. For a block $B \in \mathcal{B}$, let $P_B = \{\prod_{x\in B}(x\ y) \mid y \in \overline{G}_x$ and $(x\ y)$ is a transposition$\}$. Note that each permutation in P_B permutes each point of B to a point in the same group and leaves any other point invariant.

THEOREM 2.4. *Let $(X, \mathcal{G}, \mathcal{B}_1, \ldots, \mathcal{B}_e, \mathcal{T})$ be an e-FG$(3, (K_1, \ldots, K_e, K_{\mathcal{T}}), v)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$. Suppose that there exist a CQS$(m^{k_1} : s_1)$ for any $k_1 \in K_1$, a GDD$(m^{k_i} s_i^1)$ for any $k_i \in K_i$ with $2 \le i \le e$, and a GDD(m^k) for any $k \in K_{\mathcal{T}}$. Then there exists a CQS $((mg_1)^{n_1}(mg_2)^{n_2} \ldots (mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$. Furthermore, if*

(1) *the block set of each ingredient design can be arranged into a CSCU, and for any $A \in \mathcal{B}_1$, the ingredient CQS$(m^{|A|} : s_1)$ contains a 4-partite block, and*

(2) *the master e-fan design has two disjoint blocks $b, b' \in \mathcal{T}$ if $e = 0$, or $b \in \mathcal{T}$ and $b' \in \mathcal{B}_1$ if $e \ne 0$,*

then there exists a CSCU-CQS $((mg_1)^{n_1}(mg_2)^{n_2} \ldots (mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$ when $m \ge \max\{5, s_i \mid 1 \le i \le e\}$ and $s_i \ne 1$ for each $1 \le i \le e$.

Proof. Let $I_l = \{0, 1, \ldots, l-1\}$ for any positive integer l and $I_0 = \emptyset$. Denote $G_x = \{x\} \times I_m$ for $x \in X$, and $S_j = \{\infty_j\} \times I_{s_j}$ for $1 \le j \le e$, where $\{\infty_j \mid 1 \le j \le e\} \cap X = \emptyset$. We construct the desired design on $X' = (X \times I_m) \cup S$ with the group set $\mathcal{G}' = \{G \times I_m \mid G \in \mathcal{G}\}$ and the stem $S = S_1 \cup S_2 \cup \cdots \cup S_e$. Clearly, $(X \times I_m) \cap S = \emptyset$.

For each block $A \in \mathcal{B}_1$, construct a CSCU-CQS$(m^{|A|} : s_1)$ on $X_A = (A \times I_m) \cup S_1$ having $\{G_x \mid x \in A\}$ as its group set, S_1 as its stem, and $\mathcal{A}_A$ as its block set. Denote $G_A = \{G_x \mid x \in A\} \cup \{S_1\}$.

For each block $A \in \mathcal{B}_j$, $2 \le j \le e$, construct a CSCU-GDD$(m^{|A|} s_j^1)$ on $X_A = (A \times I_m) \cup S_j$ having $G_A = \{G_x \mid x \in A\} \cup \{S_j\}$ as its group set and $\mathcal{A}_A$ as its block set.

For each block $A \in \mathcal{T}$, construct a CSCU-GDD$(m^{|A|})$ on $X_A = A \times I_m$ having $G_A = \{G_x \mid x \in A\}$ as its group set and $\mathcal{A}_A$ as its block set.

Let $\mathcal{B} = (\cup_{1\le i\le e} \mathcal{B}_i) \cup \mathcal{T}$. Then $\cup_{A\in\mathcal{B}} \mathcal{A}_A$ is the block set of a CQS $((mg_1)^{n_1}(mg_2)^{n_2} \ldots (mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$. We try to find a CSCU of $\cup_{A\in\mathcal{B}} \mathcal{A}_A$.

First, by our assumption, when $e \ne 0$, we can arrange $\mathcal{B}$ into a sequence $\mathcal{S}' = [b_0, b_1, \ldots, b_{p-1}]$ where the blocks of $\mathcal{B}_1$ are consecutive with $b_{p-2} \in \mathcal{B}_1$ being the tail-end, $b_{p-1} \in \mathcal{T}$, and $b_{p-2} \cap b_{p-1} = \emptyset$; when $e = 0$, we simply let $b_{p-2} \cap b_{p-1} = \emptyset$. Next, we replace each block b_i by a cut CSCU $T_i = [a^i_0, a^i_1, \ldots, a^i_{q_i-1}]$ of $\mathcal{A}_{b_i}$, where a^i_0 and $a^i_{q_i-1}$ are the two ends, and $q_i = |\mathcal{A}_{b_i}|$, $0 \le i \le p-1$. By the hypothesis and the definition of a GDD, without loss of generality, we may assume that a^i_0 intersects each group in G_A in at most one point. Now we have the following claim.

CLAIM. *There exists a set of permutations $\{\sigma_k \in P_{a^k_0} \mid 0 \le k \le p-1\}$ such that in the cyclic sequence $\mathcal{S} = [\sigma_0(T_0), \sigma_1(T_1), \ldots, \sigma_{p-1}(T_{p-1})]$, we have $\sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cap$*

$\sigma_k(a_0^k) = \emptyset$ *and* $d(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l), \sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cup \sigma_k(a_0^k)) \geq 4$ *for any* $0 \leq k, l \leq p-1$, *and* $|k-l| \geq 2$.

We use a recursive method to prove this claim. Denote $\Gamma_0 = \{\sigma \in P_{a_0^0} \mid a^{p-1}_{q_{p-1}-1} \cap \sigma(a_0^0) = \emptyset\} \subseteq P_{a_0^0}$. From the assumptions on a_0^i and b_{p-1}, we know that $|a_0^0 \cap a^{p-1}_{q_{p-1}-1}| \leq 2$. We consider all possible intersections of a_0^0 and $a^{p-1}_{q_{p-1}-1}$. Let $a_0^0 = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $a^{p-1}_{q_{p-1}-1} = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. We first consider the case that $x_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$. If $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 0$, then $|\Gamma_0| = m^4$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 1$, then $|\Gamma_0| = (m-1)m^3$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 2$, then $|\Gamma_0| = (m-1)^2m^2$. Next we consider the case that $x_i = \infty_j$ for a unique $1 \leq i \leq 4$ and a unique $1 \leq j \leq e$, which implies that $s_j \geq 2$. If $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 0$, then $|\Gamma_0| = s_j m^3$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 1$, then $|\Gamma_0| = s_j(m-1)m^2$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 2$, then $|\Gamma_0| = s_j(m-1)^2 m$. So we know that $|\Gamma_0| \geq \min\{m^4, (m-1)m^3, (m-1)^2m^2, s_jm^3, s_j(m-1)m^2, s_j(m-1)^2m \mid s_j \geq 2\} \geq 1$. Choose $\sigma_0 \in \Gamma_0$ and let $\mathcal{S}_0 = \langle \sigma_0(T_0) \rangle$ be a noncyclic sequence of $\sigma_0(T_0)$, that is, $\langle \sigma_0(T_0) \rangle$ is exactly the same as $[\sigma_0(T_o)]$ except that $\sigma(a_0^0)$ is not considered as a successor of $\sigma(a^0_{q_0-1})$.

Similarly, we denote $\Gamma_1 = \{\sigma \in P_{a_0^1} \mid \sigma_0(a^0_{q_0-1}) \cap \sigma(a_0^1) = \emptyset\} \subseteq P_{a_0^1}$. Again, we consider all possible intersections of a_0^1 and $\sigma(a^0_{q_0-1})$. Let $a_0^1 = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $\sigma_0(a^0_{q_0-1}) = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and all y_i are distinct, then $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-1)^2m^2$ or s_jm^3 or $s_j(m-1)m^2$ or $s_j(m-1)^2m$, with $s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and exactly two of y_i are equal, then $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-2)m^3$ or $(m-1)^2m^2$ or $(m-2)(m-1)m^2$ or s_jm^3 or $s_j(m-1)m^2$ or $s_j(m-2)m^2$ or $s_j(m-1)^2m$ or $s_j(m-2)(m-1)m$, with $s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and $y_{i_1} = y_{i_2}$ and $y_{i_3} = y_{i_4}$, but these two values are not the same, then $|\Gamma_1| = m^4$ or $(m-2)m^3$ or $(m-2)^2m^2$ or s_jm^3 or $s_j(m-2)m^2$ or $s_j(m-2)^2m$, with $s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i = \infty_j$ for a unique $1 \leq i \leq 4$ and a unique $1 \leq j \leq e$, then all y_i should be distinct, and $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-1)^2m^2$ or s_im^3 or $s_i(m-1)m^2$ or $(s_j-1)m^3$ or $s_i(m-1)^2m$ or $(s_j-1)(m-1)m^2$, with $s_i \geq 2$, $s_j \geq 2$, and $1 \leq i \neq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. In any case, we know that $|\Gamma_1| \geq 1$. Let $\mathcal{S}_1 = \langle \sigma_0(T_0), \sigma_1(T_1) \rangle$, where $\sigma_1 \in \Gamma_1$.

Suppose that there exists a set of permutations $\{\sigma_k \in P_{a_0^k} \mid 0 \leq k \leq i-1 < p-2\}$ such that $\sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cap \sigma_k(a_0^k) = \emptyset$ and $d(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l), \sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cup \sigma_k(a_0^k)) \geq 4$ for any $0 \leq k, l \leq i-1$, and $|k-l| \geq 2$. Let $\mathcal{S}_{i-1} = \langle \sigma_0(T_0), \sigma_1(T_1), \ldots, \sigma_{i-1}(T_{i-1}) \rangle$.

For $k = i$, we try to find a permutation $\sigma_i \in P_{a_0^i}$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a_0^i) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a_0^i), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l)) \geq 4$ for any $0 \leq l < i-1$.

Let $a_0^i = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. Denote $\Gamma_i = \{\sigma \in P_{a_0^i} \mid \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma(a_0^i) = \emptyset\} \subseteq P_{a_0^i}$. We first divide the problem into two possible cases.

(a) Suppose that $\{x_1,x_2,x_3,x_4\}\cap\{y_1,y_2,y_3,y_4\}=\{\infty_j\}$ for some j, $1\le j\le e$. Then $s_j\ge 2$. For convenience, let $x_4=y_4=\infty_j$. Then $b_i,b_{i-1}\in\mathcal{B}_j$ and $|b_i\cap b_{i-1}|\le 1$. In a way similar to the above analysis, we can prove that $|\Gamma_i|\ge(s_j-1)(m-1)m^2\ge 1$. Now we choose $\sigma_{i,0}\in\Gamma_i$, which satisfies that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cap\sigma_{i,0}(a^i_0)=\emptyset$. If there exists an index l, $0\le l<i-1$, such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))<4$, then exactly one of the two blocks $\{b_{l-1},b_l\}$ belongs to $\mathcal{B}_j$. The reason is explained below. If $b_{l-1},b_l\in\mathcal{B}_j$, then since $b_{i-1},b_i\in\mathcal{B}_j$, we know that $|(b_i\cup b_{i-1})\cap(b_{l-1}\cup b_l)|\le|b_i\cap b_{l-1}|+|b_{i-1}\cap b_{l-1}|+|b_i\cap b_l|+|b_{i-1}\cap b_l|\le 4$, and hence $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\le 6$, which is impossible, for in this case we would have $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\ge 4$. On the other hand, if $b_{l-1},b_l\notin\mathcal{B}_j$, then $(\infty_j,l''_4)\notin\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0)$ for any $l''_4\in I_{s_j}$, so $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\le 6$, which is again impossible. Then there are two cases to be considered: $b_{l-1}\in\mathcal{B}_j$, $b_l\notin\mathcal{B}_j$ and $b_l\in\mathcal{B}_j$, $b_{l-1}\notin\mathcal{B}_j$. We first assume that $b_{l-1}\in\mathcal{B}_j$ and $b_l\notin\mathcal{B}_j$. Then clearly $\sigma_l(a^l_0)\cap S_j=\emptyset$. Since we have supposed that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))<4$, i.e., $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\ge 7$, we should have that one of $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cap(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))|$ and $|\sigma_l(a^l_0)\cap(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))|$ equals 4 and the other at least 3. Since $b_{l-1}\in\mathcal{B}_j$, then $|b_{l-1}\cap b_i|\le 1$ and $|b_{l-1}\cap b_{i-1}|\le 1$, i.e., $|(b_{l-1}\cup\{\infty_j\})\cap(b_i\cup b_{i-1}\cup\{\infty_j\})|\le 3$, which implies that $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cap(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))|\le 3$. Therefore, it is necessary that $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cap(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))|=3$ and $|\sigma_l(a^l_0)\cap(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))|=4$. Then we can let $\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})=\{(x_1,\sigma_{i,0}(l_1)),(y_1,l'_1),(\infty_j,\diamondsuit),(\star,*)\}$ and $\sigma_l(a^l_0)=\{(x_2,\sigma_{i,0}(l_2)),(x_3,\sigma_{i,0}(l_3)),(y_2,l'_2),(y_3,l'_3)\}$, where $\diamondsuit\in\{\sigma_{i,0}(l_4),l'_4\}$. If $\diamondsuit=\sigma_{i,0}(l_4)$, no permutation $\sigma\in\Gamma_i$ satisfies that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\ge 7$ except $\sigma=\sigma_{i,0}$. If $\diamondsuit=l'_4$, then all the permutations $\sigma_{i,1}\in\Gamma_i$ which change $(\infty_j,\sigma_{i,0}(l_4))$ to every element in $\{\infty_j\}\times(Z_{s_j}\setminus\{l'_4\})$ and fix the other three points in $\sigma_{i,0}(a^i_0)$ satisfy that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,1}(\sigma_{i,0}(a^i_0)))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\ge 7$. So for such a pair (b_{l-1},b_l), there are at most (s_j-1) $\sigma\in\Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))<4$. Now we compute the number of such pairs (b_{l-1},b_l) or, equivalently, the number of such b_l. There are $\binom{6}{3}-2=18$ triples in $\{x_1,x_2,x_3,y_1,y_2,y_3\}$ excluding the two triples $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$. Since each triple occurs in exactly one block of $\mathcal{B}$, each block of $\mathcal{B}$ contains exactly 4 triples, $|b_l\cap\{x_1,x_2,x_3\}|=2$, and $|b_l\cap\{y_1,y_2,y_3\}|=2$, we know that there are at most $\lfloor 18/4\rfloor=4$ such b_l. From the assumption, we have $|\Gamma_i|\ge(m-1)m^2(s_j-1)>4(s_j-1)$, which implies that there exists at least one permutation $\sigma_i\in\Gamma_i$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cap\sigma_i(a^i_0)=\emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_i(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\ge 4$ for any $0\le l<i-1$. For the case that $b_l\in\mathcal{B}_j$ and $b_{l-1}\notin\mathcal{B}_j$, we can also prove, in the same fashion as above, that the same assertion holds.

(b) Suppose that $\{x_1,x_2,x_3,x_4\}\cap\{y_1,y_2,y_3,y_4\}\ne\{\infty_j\}$ for any j, $1\le j\le e$. We further divide this case into two possible subcases.

(b.1) $y_1=y_2=x_1$ and $y_3=x_2$, i.e., $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})=\{(x_1,l'_1),(x_1,l'_2),(x_2,l'_3),(y_4,l'_4)\}$. If $y_4=x_2$, then $|\Gamma_i|\ge\min\{(m-2)^2m^2,(m-2)^2ms_j\mid s_j\ge 2\}\ge 1$. If $y_4\ne x_2$, then $|\Gamma_i|\ge\min\{(m-2)(m-1)m^2,(m-2)(m-1)ms_j\mid s_j\ge 2\}\ge 1$. In any case, we know that $|\Gamma_i|\ge 1$. Assume that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cap\sigma_{i,0}(a^i_0)=\emptyset$. We now prove that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for $0 \leq l \leq i-1$. If $\{x_1, x_2\} \subset b_l$, then $|b_l \cap (b_{i-1} \cup b_i)| = 2$. Since a^l_0 is 4-partite, we know that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap \sigma_l(a^l_0)| \leq 2$, which makes $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$. If $\{x_1, x_2\} \subset b_{l-1}$, then since $y_1 = y_2 = x_1$, we know that $b_{i-1} \in \mathcal{B}_1$ and thus $b_{l-1} \notin \mathcal{B}_1$, implying that $a^{l-1}_{q_{l-1}-1}$ is a block in some ingredient GDD and therefore is 4-partite. So $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap \sigma_{l-1}(a^{l-1}_{q_{l-1}-1})| \leq 2$, which ensures that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$. We still need to consider the case when $\{x_1, x_2\} \not\subset b_{l-1}$ and $\{x_1, x_2\} \not\subset b_l$. If $y_4 = x_2$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap \sigma_l(a^l_0)| \leq 2$, which makes again $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$. If $y_4 \neq x_2$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap \sigma_l(a^l_0)| \leq 3$ and $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap \sigma_{l-1}(a^{l-1}_{q_{l-1}-1})| \leq 3$, which also ensures that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$.

(b.2) All cases except (b.1), that is, $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \in \{\{(x_1, l'_1), (x_1, l'_2), (y_3, l'_3), (y_4, l'_4)\}, \{(x_1, l'_1), (x_2, l'_2), (y_3, l'_3), (y_4, l'_4)\}, \{(x_1, l'_1), (y_2, l'_2), (y_3, l'_3), (y_4, l'_4)\}, \{(y_1, l'_1), (y_2, l'_2), (y_3, l'_3), (y_4, l'_4)\}\}$, where $y_i \neq x_j$ for any i anf j. A tedious calculation shows that $|\Gamma_i| \geq \min\{(m-2)m^3, (m-2)m^2 s_j, (m-1)^2 m^2, (m-1)^2 m s_j, (m-1)m^3, (m-1)m^2 s_j, m^4, m^3 s_j \mid s_j \geq 2\} \geq 2(m-2)m^2 \geq 1$. Choose $\sigma_{i,0} \in \Gamma_i$. Then $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_{i,0}(a^i_0) = \emptyset$. If there exists an index l, $0 \leq l < i-1$, such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$. Let $R = \{r \subset X' \mid r \subset \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), r \not\subset \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}), r \not\subset \sigma_{i,0}(a^i_0)$, and $|r| = 3\}$; then $|R| \leq \binom{4}{2} \times \binom{4}{1} \times 2 = 48$. Suppose that there are t_1 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| = 8$ and t_2 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| = 7$. For the former case, each block in $\{\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}), \sigma_l(a^l_0)\}$ contains four triples from R. Even if there is one point in one of these two blocks with its second component changed, we still have $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$, that is, $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)), (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. So there are at most $\max\{4(m-1)+1, 3(m-1)+(s_j-1)+1 \mid s_j \geq 2\} = 4m-3$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. For the latter case, one block in $\{\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}), \sigma_l(a^l_0)\}$ contains four triples and the other one contains only one triple from R. Even if the second component of the uncommon point is changed in the block which contains only one triple from R, we still have $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$, that is, $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)), (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. So there are at most $\max\{m, s_j \mid s_j \geq 2\} = m$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. Here, $8t_1 + 5t_2 \leq |R| \leq 48$. Since $t_1 \leq |R|/8 \leq 6$, we have $t_1(4m-3) + t_2 m \leq t_1(4m-3) + (48-8t_1)m/5 = t_1(2.4m-3) + 9.6m \leq 6(2.4m-3) + 9.6m = 24m-18$. From the assumption that $m \geq 5$, we have $2(m-2)m^2 > 24m-18$, which implies that there exists at least one permutation $\sigma_i \in \Gamma_i$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a^i_0) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $0 \leq l < i-1$.

Then we set $\mathcal{S}_i = \langle \mathcal{S}_{i-1}, \sigma_i(T_i) \rangle$ for $1 \leq i \leq p-2$.

When $k = p-1$, we want to find a permutation $\sigma_{p-1} \in P_{a^{p-1}_0}$ such that $\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0) = \emptyset$, $\sigma_{p-1}(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}$, and $d(\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $1 \leq l < p-2$. By assumption,

$b_{p-2} \cap b_{p-1} = \emptyset$. This implies that $\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0) = \emptyset$ for any $\sigma_{p-1} \in P_{a^{p-1}_0}$. Denote $\Gamma_{p-1} = \{\sigma \in P_{a^{p-1}_0} \mid \sigma(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}\}$. Since $b_{p-1} \in \mathcal{T}$ is replaced by the cut CSCU $T_{p-1} = [a^{p-1}_0, a^{p-1}_1, \ldots, a^{p-1}_{q_{p-1}-1}]$, then, as we said in section 1, we have that $a^{p-1}_0 \cap a^{p-1}_{q_{p-1}-1} = \emptyset$. This, together with $\sigma(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}$, shows that $|\Gamma_{p-1}| \geq (m-1)^4 \geq 1$. Similar to the proof in $(b.2)$, we can prove that there are at most $(24m-18)$ $\sigma \in \Gamma_i$ such that $d(\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma(a^{p-1}_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$ for some l, $1 \leq l < p-2$. From the assumption, we have $(m-1)^4 > 24m-18$. Thus we have proved the existence of σ_{p-1}.

Now we have finished the proof of the claim. For convenience, we use T_k to denote $\sigma_k(T_k)$ for $0 \leq k \leq p-1$. Then $\mathcal{S} = [T_0, T_1, \ldots, T_{p-1}]$ satisfies the conditions that $a^{k-1}_{q_{k-1}-1} \cap a^k_0 = \emptyset$ and $d(a^{l-1}_{q_{l-1}-1} \cup a^l_0, a^{k-1}_{q_{k-1}-1} \cup a^k_0) \geq 4$ for any $0 \leq k, l \leq p-1$, and $|k-l| \geq 2$. Next, we will prove $\mathcal{S}$ is actually a CSCU.

To do this, we should check the distance between any two elements of $C = (\bigcup_{i=0}^{p-1} \mathcal{A}_{b_i}) \cup (\bigcup_{i=0}^{p-1} \overline{T_i}) \cup (\bigcup_{i=0}^{p-1} \{a^{i-1}_{q_{i-1}-1} \cup a^i_0\})$. Elements of C are classified into three types.

Type I: $a \in \mathcal{A}_{b_i}$ for some i, $0 \leq i \leq p-1$. If $b_i \in \mathcal{B}_1$, we say that a belongs to Type $\mathrm{I_{CQS}}$; otherwise, a belongs to Type $\mathrm{I_{GDD}}$.

Type II: $c \in \overline{T_i}$ for some i, $0 \leq i \leq p-1$. If $b_i \in \mathcal{B}_1$, we say that c belongs to Type $\mathrm{II_{CQS}}$; otherwise, c belongs to Type $\mathrm{II_{GDD}}$.

Type III: $c \in \bigcup_{i=0}^{p-1} \{a^{i-1}_{q_{i-1}-1} \cup a^i_0\}$.

Since the resultant design is a CQS, we easily know the following:

Case 1: $d(a, a') \geq 4$ for any two distinct a, a' from Type I;

Case 2: $d(a, c) \geq 4$ for any a from Type I and c from Types II, III, respectively.
Since each T_i, $0 \leq i \leq p-1$, is a CSCU, we have

Case 3: $d(c, c') \geq 4$ for any $c, c' \in \overline{T_i}$, $0 \leq i \leq p-1$, from Type II.
For each $a \in \mathcal{A}_{b_i}$ from Type $\mathrm{I_{GDD}}$, $|a \cap X_{b_j}| \leq 2$ when $a \notin \mathcal{A}_{b_j}$, so

Case 4: $d(c, c') \geq 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_{GDD}}$ and $c' \in \overline{T_j}$ from Type II.
For each $a \in \mathcal{A}_{b_i}$ from Type $\mathrm{I_{CQS}}$, we know that $b_i \in \mathcal{B}_1$ and $|a \cap X_{b_j}| \leq 2$ when $b_j \in \mathcal{B}_1$ and $a \notin \mathcal{A}_{b_j}$, so

Case 5: $d(c, c') \geq 4$ for any c, c' from Type $\mathrm{II_{CQS}}$.
If $b_i \notin \mathcal{B}_1$, then by the definition of a GDD and our special arrangement of $\mathcal{B}$ into $\mathcal{S}' = [b_0, b_1, \ldots, b_{p-1}]$, we know that $|a \cap a'| \leq 1$ for any $a \in \mathcal{A}_{b_i}$ and $a' \in \mathcal{A}_{b_{i-1}}$, so

Case 6: $d(c, a^{i-1}_{q_{i-1}-1} \cup a^i_0) \geq 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_{GDD}}$.
If $b_i \in \mathcal{B}_1$, then $|a \cap X_{b_i}| \leq 2$ for any $a \notin \mathcal{A}_{b_i}$, so

Case 7: $d(c, a^{i-1}_{q_{i-1}-1} \cup a^i_0) \geq 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_{CQS}}$.
Since a^j_0 is 4-partite, we know that $|a^j_0 \cap X_{b_i}| \leq 2$ for any $1 \leq i \neq j \leq p-1$, and then

Case 8: $d(c, a^{j-1}_{q_{j-1}-1} \cup a^j_0) \geq 4$ for any $c \in \overline{T_i}$ and $1 \leq i \neq j \leq p-1$.
Since $a^i_0 \cap a^i_{q_i-1} = \emptyset$ and $|a^{i-1}_{q_{i-1}-1} \cap a^i_{q_i-1}| \leq 2$, we have

Case 9: $d(a^{i-1}_{q_{i-1}-1} \cup a^i_0, a^i_{q_i-1} \cup a^{i+1}_0) \geq 4$, $0 \leq i \leq p-1$.
From the property of $\mathcal{S}$, we know that

Case 10: $d(a^{i-1}_{q_{i-1}-1} \cup a^i_0, a^{l-1}_{q_{l-1}-1} \cup a^l_0) \geq 4$ for $|i-l| \geq 2$.

Then we have proved that $\mathcal{S}$ is in fact a CSCU. $\square$

THEOREM 2.5. *In Theorem* 2.4, *if we change the condition* (1) *to be*

(1′) *the block set of each ingredient design can be arranged into a CSCU with two consecutive* 4*-partite blocks,*

then there exists a CSCU-CQS $((mg_1)^{n_1}(mg_2)^{n_2} \ldots (mg_r)^{n_r} : \sum_{1 \leq i \leq e} s_i)$ *when* $m \geq \max\{4, s_i \mid 1 \leq i \leq e\}$ *and* $s_i \neq 1$ *for each* $1 \leq i \leq e$.

Proof. Since the proof is similar to that of Theorem 2.4, we will look at only those places which are different from Theorem 2.4.

First, without loss of generality, we may assume that both a_0^i and $a_{q_i-1}^i$ of $T_i = [a_0^i, a_1^i, \ldots, a_{q_i-1}^i]$ are 4-partite for any i, $0 \le i \le p-1$.

Remember that in the proof of Theorem 2.4, we need $m = 5$ only in Case (b.2). So we can omit the proof for all cases except for (b.2). We divide Case (b.2) into two subcases.

(b.2.1) If $x_i = \infty_j$ for some i and j, $1 \le i \le 4$, $1 \le j \le e$, then $|\Gamma_i| \ge \min\{(m-1)^2 m s_j, (m-1)m^2 s_j, m^3 s_j \mid s_j \ge 2\} = (m-1)^2 m s_j \ge 1$. Assume that $\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cap \sigma_{i,0}(a_0^i) = \emptyset$. If there exists an index l, $0 \le l < i-1$, such that $d(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_{i,0}(a_0^i), \sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l)) < 4$, then, as we knew already in case (b.2) of Theorem 2.4, $|(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_{i,0}(a_0^i)) \cap (\sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l))| \ge 7$. Again, let $R = \{r \subset X' \mid r \subset \sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_{i,0}(a_0^i), r \not\subset \sigma_{i-1}(a_{q_{i-1}-1}^{i-1}), r \not\subset \sigma_{i,0}(a_0^i), \text{ and } |r| = 3\}$, and then we know that $|R| \le \binom{4}{2} \times \binom{4}{1} \times 2 = 48$. Suppose again that there are t_1 l's such that $|(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_{i,0}(a_0^i)) \cap (\sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l))| = 8$ and t_2 l's such that $|(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_{i,0}(a_0^i)) \cap (\sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l))| = 7$. For the former case, just as in case (b.2) of Theorem 2.4, we can prove that there are at most $3(m-1)+(s_j-1)+1 = 3m+s_j-3$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma(a_0^i), \sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l)) < 4$. Similarly, for the latter case, we can prove that there are at most $\max\{m, s_j\} = m$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma(a_0^i), \sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l)) < 4$. Since $8t_1 + 5t_2 \le |R| \le 48$, we have $t_1 \le |R|/8 \le 6$, and $t_1(3m+s_j-3)+t_2 m \le t_1(3m+s_j-3)+(48-8t_1)m/5 = t_1(1.4m+s_j-3)+9.6m \le 6(1.4m+s_j-3)+9.6m = 18m+6s_j-18$. From the assumptions that $m \ge 4$ and $s_j \ge 2$, we have $(m-2)m^2 s_j > 18m + 6s_j - 18$, which implies that there exists at least one permutation $\sigma_i \in \Gamma_i$ such that $\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cap \sigma_i(a_0^i) = \emptyset$ and $d(\sigma_{i-1}(a_{q_{i-1}-1}^{i-1}) \cup \sigma_i(a_0^i), \sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l)) \ge 4$ for any $0 \le l < i-1$.

(b.2.2) If $x_i \ne \infty_j$ for any i and j, $1 \le i \le 4$, $1 \le j \le e$, then the proof is exactly the same as that for (b.2.1) except that s_j is replaced by m. $\square$

COROLLARY 2.6. *Let $m \ge 4$ and $gn \ge 16$. Assume there exists a GDD$(3,4,gn)$ of type g^n. If there exists a CSCU-GDD$(3,4,4m)$ of type m^4, then there exists a CSCU-GDD$(3,4,gnm)$ of type $(mg)^n$.*

Proof. In a GDD$(3,4,gn)$ of type g^n, the number of blocks is $\lambda_0 = \frac{g^3 n(n-1)(n-2)}{24}$, the number of blocks containing one point is $\lambda_1 = \frac{g^2(n-1)(n-2)}{6}$, and the number of blocks containing two distinct points is $\lambda_2 = \frac{g(n-2)}{2}$. There exist two disjoint blocks if and only if $\lambda_0 > \binom{4}{1}(\lambda_1 - 1) - \binom{4}{2}(\lambda_2 - 1) + 1$. This inequality is satisfied provided that $gn \ge 16$. Then apply Theorem 2.5 with $e = 0$. $\square$

THEOREM 2.7. *There exists a CSCU-GDD$(3,4,4g)$ of type g^4 for any $g \ge 5$.*

Proof. Let $X = Z_4 \times Z_g$. We build a GDD$(3,4,4g)$ of type g^4 on X with the group set $\mathcal{G} = \{\{i\} \times Z_g \mid i \in Z_4\}$ and the block set $\mathcal{B} = \{\alpha(i,j,k) = \{(0,i),(1,i+j),(2,k),(3,k+j)\} \mid i,j,k \in Z_g\}$. Let $T(j,k) = \langle \alpha(0,j,k), \alpha(1,j,k+1), \ldots, \alpha(g-1, j, k+g-1)\rangle$, $T_j = \langle T(j,j), T(j,j+1), \ldots, T(j,j-1)\rangle$, and $S = [T_0, T_1, \ldots, T_{g-1}]$. It is clear that $S = \mathcal{B}$ if we view S as a block set. We will check that S is in fact a CSCU.

It is easy to check that any two consecutive blocks in S are disjoint and $d(\alpha(i,j,k), \alpha(i',j',k')) \ge 4$ for any distinct (i,j,k) and (i',j',k'). Let c_t be the union of two consecutive blocks. Then $d(\alpha(i,j,k), c_t) \ge 4$ for any $c_t \in \widehat{S}$. Thus we need only consider the distance between any two unions. We separate the unions into the following three types.

Type I: $c_1(i,j,k) = \alpha(i,j,k) \cup \alpha(i+1,j,k+1), 0 \le i \le g-2, 0 \le j,k \le g-1$.

Type II: $c_2(j,k) = \alpha(g-1,j,k-2) \cup \alpha(0,j,k), 0 \le j,k \le g-1$.

Type III: $c_3(j) = \alpha(g-1,j-1,j-3) \cup \alpha(0,j,j), 0 \le j \le g-1$.

We should check that any two unions from these three types have distance more than or equal to 4. Let n_q be the number of points in $c_1(i,j,k) \cap c_1(i',j',k')$ with the first coordinate being q, where $q \in Z_4$. Then $n_q \le 2$ for any $q \in Z_4$. If there are at least two n_q's of the $c_1(i,j,k) \cap c_1(i',j',k')$ having value no more than 1, then $|c_1(i,j,k) \cap c_1(i',j',k')| \le 6$, which means that $d(c_1(i,j,k), c_1(i',j',k')) \ge 4$.

Case a: Two unions from Type I, say, $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_1(i',j',k') = \{(0,i'),(0,i'+1),(1,i'+j'),(1,i'+j'+1),(2,k'),(2,k'+1),(3,k'+j'),(3,k'+j'+1)\}$. We will show that $|c_1(i,j,k) \cap c_1(i',j',k')| \le 6$ for any distinct (i,j,k) and (i',j',k'). Note the fact that if $l \ne l'$ and $g \ge 5$, then $|\{l,l+1\} \cap \{l',l'+1\}| \le 1$. Since each of the three parameters $\{i,j,k\}$ is related to two different first coordinates, it is easy to check that at least two of the n_q's have value no more than 1. The details are listed below.

(1) When $i \ne i'$, $j \ne j'$, and $k \ne k'$, we have $n_0 \le 1$ and $n_2 \le 1$.
(2) When $i \ne i'$, $j \ne j'$, and $k = k'$, then $k+j \ne k'+j'$, so $n_0 \le 1$ and $n_3 \le 1$.
(3) When $i \ne i'$, $j = j'$, and $k \ne k'$, we have $n_0 \le 1$ and $n_2 \le 1$.
(4) When $i \ne i'$, $j = j'$, and $k = k'$, then $i+j \ne i'+j'$, so $n_0 \le 1$ and $n_1 \le 1$.
(5) When $i = i'$, $j \ne j'$, and $k \ne k'$, then $i+j \ne i'+j'$, so $n_1 \le 1$ and $n_2 \le 1$.
(6) When $i = i'$, $j \ne j'$, and $k = k'$, then $i+j \ne i'+j'$ and $k+j \ne k'+j'$, so $n_1 \le 1$ and $n_3 \le 1$.
(7) When $i = i'$, $j = j'$, and $k \ne k'$, then $k+j \ne k'+j'$, so $n_2 \le 1$, and $n_3 \le 1$.

Case b: Two unions from Type I and Type II, respectively, say, $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_2(j',k') = \{(0,g-1),(0,0),(1,j'-1),(1,j'),(2,k'-2),(2,k'),(3,k'+j'-2),(3,k'+j')\}$. Since $0 \le i \le g-2$ in $c_1(i,j,k)$, we know that $n_0 \le 1$. Since $g \ge 5$, we have $|\{k,k+1\} \cap \{k'-2,k'\}| \le 1$, i.e., $n_2 \le 1$. Then $d(c_1(i,j,k), c_2(j',k')) \ge 4$.

Case c: Two unions from Type I and Type III, respectively, say, $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$. Since $0 \le i \le g-2$ and $g \ge 5$, in a similar way, we can know that $n_0 \le 1$ and $n_1 \le 1$.

Case d: Two unions from Type II, say, $c_2(j,k) = \{(0,g-1),(0,0),(1,j-1),(1,j),(2,k-2),(2,k),(3,k+j-2),(3,k+j)\}$ and $c_2(j',k') = \{(0,g-1),(0,0),(1,j'-1),(1,j'),(2,k'-2),(2,k'),(3,k'+j'-2),(3,k'+j')\}$, where (j,k) and (j',k') are distinct. Similarly to Case a, we can show that there are at least two n_q's having value no more than 1.

Case e: Two unions from Type II and Type III, respectively, say, $c_2(j,k) = \{(0,g-1),(0,0),(1,j-1),(1,j),(2,k-2),(2,k),(3,k+j-2),(3,k+j)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$. It is readily checked that at least one of the following assertions holds:

(e.1): $n_1 \le 1$ and $n_2 \le 1$;
(e.2): $n_1 \le 1$ and $n_3 \le 1$.

Case f: Two unions from Type III, say, $c_3(j) = \{(0,g-1),(0,0),(1,j-2),(1,j),(2,j-3),(2,j),(3,2j-4),(3,2j)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$, where $j \ne j'$. Similarly to Case a, we can prove that there are at least two n_q's having value no more than 1. $\square$

3. Direct constructions. In this section, we directly construct some small CSCUs which will be used in the recursive constructions. In order to save space, we list only a few examples. The interested reader is referred to the authors or to the new results website for Handbook of Combinatorial Designs [2] maintained by Professor Jeff Dinitz of the University of Vermont for a copy of the detailed cyclic sequences of blocks.

LEMMA 3.1. *There exists a CSCU-CQS*$(g^n : s)$ *for each* $(g, n, s) \in \{(4, 4, 2), (4, 4, 4), (6, 3, 2), (6, 3, 4), (6, 5, 2), (6, 5, 4), (8, 3, 2), (12, 3, 2), (12, 3, 4), (12, 4, 2), (12, 4, 4)\}$.

Proof. We list only the sequence of a CSCU-CQS$(4^4 : 2)$ on the point set $X = Z_{18}$, with the group set $\mathcal{G} = \{\{i, 4+i, 8+i, 12+i\} \mid i \in Z_4\}$ and the stem $S = \{16, 17\}$.

$$\begin{aligned}
S =[&\{6,7,12,17\},\{4,10,11,16\},\{12,14,15,17\},\{2,5,7,16\},\{1,6,8,10\},\{2,7,9,17\},\{5,10,15,16\},\\
&\{0,1,2,14\},\{4,5,6,10\},\{1,3,14,16\},\{0,6,9,10\},\{1,2,15,17\},\{3,6,9,16\},\{1,7,10,17\},\\
&\{9,11,14,16\},\{3,5,6,17\},\{2,4,13,14\},\{1,6,11,16\},\{3,13,14,17\},\{1,7,11,12\},\{2,8,9,14\},\\
&\{0,7,11,13\},\{2,5,12,14\},\{4,6,7,8\},\{0,3,5,15\},\{6,10,12,13\},\{1,3,4,15\},\{5,7,8,11\},\\
&\{0,2,3,12\},\{4,7,9,11\},\{3,8,13,15\},\{0,10,11,12\},\{2,5,9,15\},\{3,4,8,10\},\{0,6,12,15\},\\
&\{2,4,8,11\},\{3,9,12,15\},\{1,2,7,13\},\{4,8,14,15\},\{1,3,6,13\},\{2,4,9,10\},\{0,7,12,14\},\\
&\{5,6,9,11\},\{1,10,13,15\},\{3,5,9,14\},\{0,2,10,13\},\{5,6,8,14\},\{0,1,3,7\},\{2,6,8,13\},\\
&\{5,7,9,10\},\{1,11,13,14\},\{2,5,8,10\},\{3,11,12,13\},\{0,2,5,6\},\{3,7,8,9\},\{1,4,10,14\},\\
&\{2,6,9,12\},\{3,4,5,11\},\{1,2,10,12\},\{0,3,9,11\},\{1,2,4,6\},\{8,10,13,14\},\{3,5,7,12\},\\
&\{0,6,13,14\},\{1,3,8,11\},\{0,5,10,14\},\{3,4,7,13\},\{9,10,12,14\},\{4,5,7,15\},\{1,6,12,14\},\\
&\{8,9,11,15\},\{0,3,4,6\},\{7,12,13,15\},\{4,6,9,14\},\{5,11,12,15\},\{0,3,8,14\},\{4,11,13,15\},\\
&\{0,7,8,10\},\{1,9,14,15\},\{0,2,4,7\},\{3,6,8,12\},\{0,1,11,15\},\{2,7,8,12\},\{0,4,11,14\},\\
&\{1,7,8,15\},\{3,4,12,14\},\{0,7,9,15\},\{4,6,11,12\},\{1,3,9,10\},\{8,11,12,14\},\{1,2,3,5\},\\
&\{0,4,10,15\},\{1,6,7,9\},\{0,2,8,15\},\{4,7,10,12\},\{1,2,9,11\},\{8,10,12,15\},\{2,3,9,13\},\\
&\{0,6,8,11\},\{5,13,14,15\},\{2,3,6,7\},\{1,5,10,11\},\{0,4,9,13\},\{10,11,14,15\},\{5,6,7,13\},\\
&\{2,4,12,15\},\{9,10,11,13\},\{1,5,8,12\},\{7,9,13,14\},\{0,1,4,5\},\{2,6,11,15\},\{3,7,10,14\},\\
&\{2,5,11,13\},\{1,4,9,12\},\{2,3,14,15\},\{0,5,8,13\},\{2,7,10,15\},\{0,1,12,13\},\{6,7,14,15\},\\
&\{4,5,8,9\},\{2,7,11,14\},\{1,5,6,15\},\{8,9,12,13\},\{3,6,11,14\},\{4,5,12,13\},\{6,7,10,11\},\\
&\{0,1,8,9\},\{3,6,10,15\},\{1,5,7,14\},\{6,9,13,15\},\{2,3,10,11\},\{4,5,14,16\},\{2,12,13,17\},\\
&\{0,1,10,16\},\{2,4,5,17\},\{8,9,10,16\},\{0,1,6,17\},\{3,5,10,13\},\{6,8,9,17\},\{12,13,14,16\},\\
&\{5,8,15,17\},\{1,4,7,16\},\{0,5,9,12\},\{1,4,8,13\},\{5,10,12,17\},\{0,2,9,16\},\{4,10,13,17\},\\
&\{1,2,8,16\},\{0,9,14,17\},\{4,6,13,16\},\{1,8,14,17\},\{5,6,12,16\},\{0,13,15,17\},\{7,9,12,16\},\\
&\{1,4,11,17\},\{0,3,13,16\},\{9,11,12,17\},\{3,5,8,16\},\{0,2,11,17\},\{1,12,15,16\},\{3,4,9,17\},\\
&\{0,6,7,16\},\{1,3,12,17\},\{0,5,11,16\},\{7,8,13,17\},\{4,9,15,16\},\{0,5,7,17\},\{6,8,15,16\},\\
&\{0,3,10,17\},\{8,11,13,16\},\{4,6,15,17\},\{7,8,14,16\},\{6,11,13,17\},\{3,10,12,16\},\{4,7,14,17\},\\
&\{2,11,12,16\},\{9,10,15,17\},\{2,3,4,16\},\{8,10,11,17\},\{0,14,15,16\},\{2,3,8,17\},\{7,10,13,16\},\\
&\{5,11,14,17\},\{2,13,15,16\}]. \qquad \square
\end{aligned}$$

LEMMA 3.2. *There exists a CSCU-GDD*(g^u) *for each* $(g, u) \in \{(3, 4), (4, 4), (4, 5), (6, 5), (6, 6)\}$.

Proof. We list only two examples here. First, we list the sequence of a CSCU-GDD(3^4) on the point set $X = Z_{12}$ with the group set $\mathcal{G} = \{\{i, 4+i, 8+i\} \mid i \in Z_4\}$.

$$\begin{aligned}
S =[&\{0,1,2,3\},\{4,5,6,7\},\{8,9,10,11\},\{0,1,6,7\},\{4,5,10,11\},\{8,9,2,3\},\{0,1,10,11\},\\
&\{4,5,2,3\},\{8,9,6,7\},\{0,5,10,3\},\{4,9,2,7\},\{8,1,6,11\},\{0,5,2,7\},\{4,9,6,11\},\\
&\{8,1,10,3\},\{0,5,6,11\},\{4,9,10,3\},\{8,1,2,7\},\{0,9,6,3\},\{4,1,10,7\},\{8,5,2,11\},\\
&\{0,9,10,7\},\{4,1,2,11\},\{8,5,6,3\},\{0,9,2,11\},\{4,1,6,3\},\{8,5,10,7\}].
\end{aligned}$$

Next, we list the sequence of a CSCU-GDD(4^4) on the point set $X = Z_{16}$ with the group set $\mathcal{G} = \{\{i, 4+i, 8+i, 12+i\} \mid i \in Z_4\}$.

$$\begin{aligned} S =[&\{0,1,2,3\},\{4,5,6,7\},\{8,9,10,11\},\{12,13,14,15\},\{0,1,6,7\},\{4,5,10,11\},\{8,9,14,15\},\\ &\{2,3,12,13\},\{0,1,10,11\},\{4,5,14,15\},\{2,3,8,9\},\{6,7,12,13\},\{2,3,4,5\},\{0,1,14,15\},\\ &\{6,7,8,9\},\{10,11,12,13\},\{0,2,5,7\},\{4,6,9,11\},\{8,10,13,15\},\{1,3,12,14\},\{0,5,6,11\},\\ &\{4,9,10,15\},\{3,8,13,14\},\{1,2,7,12\},\{0,5,10,15\},\{3,4,9,14\},\{2,7,8,13\},\{1,6,11,12\},\\ &\{2,4,7,9\},\{6,8,11,13\},\{1,10,12,15\},\{0,3,5,14\},\{4,6,13,15\},\{0,2,9,11\},\{1,3,8,10\},\\ &\{5,7,12,14\},\{0,6,9,15\},\{3,4,10,13\},\{1,7,8,14\},\{2,5,11,12\},\{0,3,9,10\},\{4,7,13,14\},\\ &\{1,2,8,11\},\{5,6,12,15\},\{2,4,11,13\},\{1,6,8,15\},\{3,5,10,12\},\{0,7,9,14\},\{1,3,4,6\},\\ &\{0,2,13,15\},\{5,7,8,10\},\{9,11,12,14\},\{0,3,6,13\},\{1,4,7,10\},\{5,8,11,14\},\{0,7,10,13\},\\ &\{5,9,12,15\},\{1,4,11,14\},\{2,5,8,15\},\{3,6,9,12\},\{1,2,4,15\},\{0,11,13,14\},\{3,5,6,8\},\\ &\{7,9,10,12\}]. \end{aligned}$$ □

LEMMA 3.3. *There exists a CSCU-HSQS$(v : s)$ for each $(v, s) \in \{(16, 4), (20, 8), (22, 10), (26, 10)\}$.*

Proof. Here we list only the sequence of a CSCU-HSQS$(16 : 4)$ on the point set $X = Z_{16}$ with the hole set $\{0, 1, 2, 3\}$.

$$\begin{aligned} S =[&\{3,4,11,12\},\{0,1,6,7\},\{8,9,10,11\},\{0,1,4,5\},\{2,3,6,7\},\{8,9,12,13\},\{0,2,4,6\},\\ &\{8,9,14,15\},\{0,2,5,7\},\{8,10,12,14\},\{0,3,4,7\},\{8,10,13,15\},\{0,3,5,6\},\{8,11,12,15\},\\ &\{4,5,6,7\},\{8,11,13,14\},\{2,3,4,5\},\{12,13,14,15\},\{1,3,5,7\},\{10,11,14,15\},\{1,3,4,6\},\\ &\{10,11,12,13\},\{1,2,5,6\},\{9,11,13,15\},\{1,2,4,7\},\{9,11,12,14\},\{0,1,10,15\},\{2,7,8,9\},\\ &\{0,1,11,14\},\{2,7,10,15\},\{0,1,12,13\},\{2,7,11,14\},\{9,10,12,15\},\{3,6,11,14\},\{0,1,8,9\},\\ &\{3,6,10,15\},\{2,7,12,13\},\{4,5,10,15\},\{3,6,8,9\},\{0,2,12,15\},\{4,5,8,9\},\{3,6,12,13\},\\ &\{4,5,11,14\},\{1,3,12,15\},\{9,10,13,14\},\{5,6,12,15\},\{0,2,8,10\},\{4,5,12,13\},\{0,2,9,11\},\\ &\{1,3,8,10\},\{0,2,13,14\},\{1,3,9,11\},\{4,7,8,10\},\{1,3,13,14\},\{4,7,9,11\},\{5,6,8,10\},\\ &\{2,4,9,13\},\{0,3,8,11\},\{4,7,13,14\},\{1,5,10,12\},\{0,3,9,13\},\{4,7,12,15\},\{5,6,13,14\},\\ &\{0,3,10,12\},\{2,4,8,11\},\{0,3,14,15\},\{5,6,9,11\},\{2,4,10,12\},\{1,5,9,13\},\{0,4,8,12\},\\ &\{1,5,14,15\},\{6,7,10,12\},\{2,4,14,15\},\{1,5,8,11\},\{6,7,9,13\},\{0,4,10,14\},\{6,7,8,11\},\\ &\{0,4,9,15\},\{2,6,11,13\},\{3,5,8,12\},\{0,4,11,13\},\{6,7,14,15\},\{3,5,11,13\},\{2,6,10,14\},\\ &\{3,5,9,15\},\{1,7,8,12\},\{3,5,10,14\},\{2,6,8,12\},\{1,7,10,14\},\{2,6,9,15\},\{0,5,8,13\},\\ &\{1,7,9,15\},\{0,5,12,14\},\{4,6,11,15\},\{0,5,9,10\},\{1,7,11,13\},\{4,6,12,14\},\{3,7,8,13\},\\ &\{0,5,11,15\},\{4,6,8,13\},\{3,7,11,15\},\{4,6,9,10\},\{3,7,12,14\},\{1,2,8,13\},\{3,7,9,10\},\\ &\{1,2,12,14\},\{0,6,13,15\},\{1,2,9,10\},\{0,6,8,14\},\{1,2,11,15\},\{0,6,9,12\},\{5,7,8,14\},\\ &\{1,4,13,15\},\{0,6,10,11\},\{5,7,13,15\},\{1,4,9,12\},\{5,7,10,11\},\{1,4,8,14\},\{5,7,9,12\},\\ &\{2,3,8,14\},\{1,4,10,11\},\{2,3,13,15\},\{0,7,9,14\},\{2,3,10,11\},\{0,7,8,15\},\{1,6,9,14\},\\ &\{0,7,11,12\},\{1,6,8,15\},\{2,5,10,13\},\{3,4,8,15\},\{1,6,10,13\},\{2,3,9,12\},\{0,7,10,13\},\\ &\{2,5,9,14\},\{3,4,10,13\},\{2,5,11,12\},\{3,4,9,14\},\{1,6,11,12\},\{2,5,8,15\}]. \end{aligned}$$ □

LEMMA 3.4. *There exists a BSCU(v) for each $v \in \{20, 22, 26, 34, 38\}$.*

Proof. Here we show only the existence of a BSCU(20). Take $X = Z_{19} \cup \{\infty\}$ as the point set. Let

$$\begin{aligned} A =[&\{0,4,5,6\},\{2,7,12,14\},\{1,3,4,9\},\{0,8,14,17\},\{\infty,1,7,9\},\{5,11,18,2\},\\ &\{0,1,3,7\},\{2,6,11,17\},\{\infty,1,10,13\},\{14,16,18,2\},\{7,15,0,6\},\{1,8,9,12\},\\ &\{5,6,13,15\},\{1,2,12,17\},\{\infty,8,9,13\}], \end{aligned}$$

and $S = [A, A+1, A+2, \ldots, A+18]$, where additions are taken modulo 19. Then S is the required BSCU(20). □

4. Results obtained by recursion. First, we list some known results on 3-designs. A *t-wise balanced design* (or t-BD) of order v and block sizes from K, denoted by $\mathrm{S}(t,K,v)$, is a pair $(X,\mathcal{B})$, where X is a set of v elements (called *points*), $\mathcal{B}$ is a collection of subsets (called *blocks*) of X, each of cardinality from K, such that every t-subset of X is contained in a unique block of $\mathcal{B}$. The set of all positive integers v such that an $\mathrm{S}(t,K,v)$ exists is denoted by $B_t(K)$.

THEOREM 4.1. (see [13]). There exists a CQS($6^n : 0$) for any positive integer n.

THEOREM 4.2. (see [5]). $B_3(\{4\}) = \{v > 0 \mid v \equiv 2,4 \pmod 6\}$.

THEOREM 4.3. (see [11]). $B_3(\{4,5,6\}) = \{v > 0 \mid v \equiv 0,1,2 \pmod 4 \text{ and } v \neq 9,13\}$.

LEMMA 4.4. (see [15]). There exists a BSCU(v) for $v \in \{14,16,32,46,56\}$.

LEMMA 4.5. *If there exists a CSCU-GDD*(g^n)*, then there exists a CSCU-GDD* ($(mg)^n$) *for any integer* $m \geq 3$.

Proof. Combining Lemma 3.2 with Theorem 2.7, we know that there exists a CSCU-GDD(m^4) for any $m \geq 3$.

Let $S = [b_0,\ldots,b_{q-1}]$ be the CSCU-GDD(g^n). For any $b_i \in S$, there is a CSCU-GDD(m^4), denoted S_i, on the point set $b_i \times I_m$ for any integer $m \geq 3$. Let $S' = [S_0,\ldots,S_{q-1}]$. Then it is easy to check that S' is a CSCU-GDD($(mg)^n$). ∎

LEMMA 4.6. *There exists a CSCU-CQS*($12^n : s$) *for each* $s \in \{8,10\}$ *and* $n \geq 4$.

Proof. For each $n \equiv 0,1 \pmod 3$ and $n \geq 4$, there exists an $S(3,4,2n+2)$ by Theorem 4.2. Deleting two points from this 3-BD yields a 2-FG$(3,(\{3\},\{3\},\{4\}),2n)$ of type 2^n. By counting the numbers of blocks in the $S(3,4,2n+2)$ containing t, where $t = 0,1,2$ common points, we can know that in the 2-FG$(3,(\{3\},\{3\},\{4\}),2n)$ of type 2^n, when $n \geq 4$, there exist two disjoint blocks with one of size 4 and the other of size 3. For each $s \in \{8,10\}$, applying Theorem 2.4 with a CSCU-CQS($6^3 : s-6$) and a CSCU-GDD(6^4), we obtain a CSCU-CQS($12^n : s$). Here, the ingredient designs come from Theorem 2.7 and Lemma 3.1.

For any $n \equiv 2 \pmod 3$ and $n \geq 5$, there is a CQS($6^{\frac{n+1}{3}} : 0$) by Theorem 4.1. For $n = 5,8,11$, it can be checked from the detailed construction in [13] for each of these CQS($6^{\frac{n+1}{3}} : 0$) that there exist two disjoint blocks a and b intersecting two groups, say, g_1 and g_2, in two points, respectively. So there are two points $y \in g_1$ and $z \in g_2$ not covered by a and b. Choose $x \in a \cap g_2$ and delete x, y. Then we obtain a 2-FG$(3,(\{3,5\},\{3,5\},\{4,6\}),10)$ of type 2^n with two disjoint blocks $a \setminus \{x\} \in \mathcal{B}_1$ and $b \in \mathcal{T}$. For $n \geq 14$, let x, y be two points from different groups g_x, g_y, respectively, and g be a group disjoint from a block containing x, y. By deleting x and y, we obtain a 2-FG$(3,(\{3,5\},\{3,5\},\{4,6\}),2n)$ of type 2^n with two disjoint blocks $g_x \setminus \{x\} \in \mathcal{B}_1$ and $g \in \mathcal{T}$. Then for each $s \in \{8,10\}$, by applying Theorem 2.4 with a CSCU-CQS($6^3 : s-6$), a CSCU-CQS($6^5 : s-6$), a CSCU-GDD(6^4), and a CSCU-GDD(6^6), we obtain a CSCU-CQS($12^n : s$), where the ingredient designs come from Theorem 2.7 and Lemmas 3.1 and 3.2. ∎

LEMMA 4.7. *There exists a BSCU*(v) *for each* $v \equiv 8,10 \pmod{12}$ *and* $v \geq 12$.

Proof. For each $v \in \{20,22,32,34,46\}$, there is a BSCU(v) by Lemmas 3.4 and 4.4. For $v = 44$, there is a BSCU(v) by applying Theorem 1.1.(1) with a BSCU(22).

For each $v \equiv 8,10 \pmod{12}$ and $v \geq 56$, there is a CSCU-CQS($12^n : s$) where $v = 12n+s$, $n \geq 4$, and $s \in \{8,10\}$ by Lemma 4.6. Then by applying Theorem 2.3 with a CSCU-HSQS($12+s : s$), we obtain a CSCU-HSQS($12n+s : 12+s$), and furthermore, by applying Theorem 2.2 with a BSCU($12+s$), we obtain a BSCU($12n+s$), where the ingredient CSCU-HSQS comes from Lemma 3.3. ∎

LEMMA 4.8. *There exists a CSCU-GDD*(12^u) *for each* $u \in \{5,6\}$.

Proof. From Lemma 3.2, we know that there exists a CSCU-GDD(4^5). Applying Lemma 4.5 with $m = 3$, we obtain a CSCU-GDD(12^5).

From Theorem 2.1, we know that there exists a GDD(3^6). Applying Corollary 2.6 with a CSCU-GDD(4^4) from Lemma 3.2, we obtain a CSCU-GDD(12^6). □

LEMMA 4.9. *There exists a CSCU-CQS*($12^n : s$) *for each* $n \in \{5, 8\}$ *and* $s \in \{2, 4\}$.

Proof. For each $n \in \{5, 8\}$, there is an S$(3, 5, 3n + 2)$ in [6]. Deleting two points gives a 2-FG$(3, (\{4\}, \{4\}, \{5\}), 3n)$ of type 3^n, which is also a 1-FG$(3, (\{4\}, \{4, 5\}), 3n)$ of type 3^n. By counting the numbers of blocks in the $S(3, 5, 3n+2)$ containing t, where $t = 0, 1, 2$ common points, we can know that in the 2-FG$(3, (\{4\}, \{4\}, \{5\}), 3n)$ of type 3^n, when $n = 5, 8$, there exist two disjoint blocks with one of size 5 and the other of size 4. For each $s \in \{2, 4\}$, by applying Theorem 2.5 with a CSCU-CQS($4^4 : s$), a CSCU-GDD(4^4), and a CSCU-GDD(4^5), which come from Lemmas 3.1 and 3.2, we obtain a CSCU-CQS($12^n : s$). □

LEMMA 4.10. *There exists a CSCU-CQS*($12^n : s$) *for* $s \in \{2, 4\}$ *and* $n \equiv 0, 1, 3 \pmod 4$, $n \geq 7$, $n \neq 8, 12$.

Proof. For each $n \equiv 0, 1, 3 \pmod 4$, $n \geq 7$, and $n \neq 8, 12$, there exists an $S(3, \{4, 5, 6\}, n + 1)$ $(X, \mathcal{B})$ by Theorem 4.3. Let x, y be two points of X, and $b_1, b_2, \ldots, b_w$ be the blocks in $\mathcal{B}$ containing both x and y. Then $\{b_1 \setminus \{x, y\}, b_2 \setminus \{x, y\}, \ldots, b_w \setminus \{x, y\}\}$ is a partition of $X \setminus \{x, y\}$, and $2 \leq |b_i \setminus \{x, y\}| \leq 4$ for $i = 1, 2, \ldots, w$. Let $u \in b_1 \setminus \{x, y\}$, $v \in b_2 \setminus \{x, y\}$, and b be a block containing both u and v. If $w \geq 7$, which would happen if $n \geq 27$, then there must exist one $b_i \setminus \{x, y\}$, say, $i = i_0$, which is disjoint with b. Deleting u from this 3-BD yields a 1-FG$(3, (\{3, 4, 5\}, \{4, 5, 6\}), n)$ of type 1^n with two disjoint blocks $b \setminus \{u\} \in \mathcal{B}_1$ and $b_{i_0} \in \mathcal{T}$. For each $n \equiv 1, 3 \pmod 6$, there exists an $S(3, 4, n+1)$ by Theorem 4.2. By counting the numbers of blocks in the $S(3, 4, n+1)$ containing t, where $t = 0, 1, 2$ common points, we can know that there exist two disjoint blocks b, b' when $n \geq 7$. Deleting one point $x \in b$ from this 3-BD yields a 1-FG$(3, (\{3\}, \{4\}), n)$ of type 1^n with two disjoint blocks $b \setminus \{x\} \in \mathcal{B}_1$ and $b' \in \mathcal{T}$. For $n = 16$, there exists an S$(3, 5, 17)$ from [6]. By the same method as above, we know that there exists a 1-FG$(3, (\{4\}, \{5\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. For $n = 20, 24$, there exist an S$(3, 6, 22)$ and an S$(3, 6, 26)$ from [6]. In a similar fashion, we can prove the existence of two disjoint blocks in each of these two Steiner systems. Deleting two points from one of these two disjoint blocks yields a 1-FG$(3, (\{4, 5\}, \{5, 6\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. For $n = 11, 17, 23$, just as in the proof of Lemma 4.6, we can know that there exist two disjoint blocks in the CQS($6^{\frac{n+1}{6}} : 0$). Deleting one point from one of these two disjoint blocks yields a 1-FG$(3, (\{3, 5\}, \{4, 6\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. Now for each $s \in \{2, 4\}$, by applying Theorem 2.4 with a CSCU-CQS($12^h : s$) and a CSCU-GDD(12^{h+1}) for each $h \in \{3, 4, 5\}$, we obtain a CSCU-CQS($12^n : s$). Here, the ingredient designs come from Theorem 2.7 and Lemmas 3.1, 4.8, and 4.9. □

LEMMA 4.11. *There exists a BSCU*$(12n + s)$ *for* $s \in \{2, 4\}$, $n \equiv 0, 1, 3 \pmod 4$, $n \geq 4$, *and* $n \neq 12$.

Proof. For each $n \equiv 0, 1, 3 \pmod 4$, $n \geq 4$, and $n \neq 12$, there exists a CSCU-CQS($12^n : s$) for $s \in \{2, 4\}$ by Lemmas 3.1, 4.9, and 4.10. Then by applying Theorem 2.3 with a CSCU-HSQS($12 + s : s$) and Theorem 2.2 with a BSCU($12 + s$), we obtain a BSCU($12n + s$). Here, the ingredient designs come from Theorem 1.1 and Lemmas 3.3 and 4.4, where the BSCU(14) in Theorem 1.1 is actually a CSCU-HSQS($12 + 2 : 2$). □

LEMMA 4.12. *There exists a BSCU*$(48n + 26)$ *for any* $n \geq 0$.

Proof. A BSCU(26) was shown in Lemma 3.4. For each integer $n \geq 1$, as was shown in the proof of Lemma 4.6, there exists a 2-FG$(3, (\{3\}, \{3\}, \{4\}), 2(3n + 1))$ of type 2^{3n+1} with two disjoint blocks, one being of size 4 and the other of size 3. Applying Theorem 2.4 with a CSCU-CQS$(8^3 : 2)$ and a CSCU-GDD(8^4), we obtain a CSCU-CQS$(16^{3n+1} : 10)$. Then by applying Theorem 2.3 with a CSCU-HSQS$(26 : 10)$ and Theorem 2.2 with a BSCU(26), we obtain a BSCU$(48n+26)$. Here, the ingredient designs come from Theorem 2.7 and Lemmas 3.1, 3.3, and 3.4. □

LEMMA 4.13. *There exists a BSCU*$(12n + s)$ *for* $n \in \{1, 3, 12\}$ *and* $s \in \{2, 4\}$.

Proof. For each $v \in \{14, 16, 38\}$, there is a BSCU(v) by Lemmas 3.4 and 4.4. For each $v \in \{40, 148\}$, there is a BSCU(v) by applying Theorem 1.1.(1) with a BSCU(u) for $u \in \{20, 74\}$ in Lemmas 3.4 and 4.12, respectively.

For $v = 146$, there exists an S$(3, 6, 26)$ in [6]. Deleting two points gives a 2-FG$(3, (\{5\}, \{5\}, \{6\}), 24)$ of type 4^6, which is also a 1-FG$(3, (\{5\}, \{5, 6\}), 24)$ of type 4^6. It can be easily shown that this 2-FG$(3, (\{5\}, \{5\}, \{6\}), 24)$ has two disjoint blocks with one of size 6 and the other of size 5. Applying Theorem 2.4 with a CSCU-CQS$(6^5 : 2)$, a CSCU-GDD(6^5), and a CSCU-GDD(6^6), we obtain a CSCU-CQS$(24^6 : 2)$. Then applying Theorem 2.3 with a CSCU-HSQS$(26 : 2)$ and Theorem 2.2 with a BSCU(26), we obtain a BSCU(146). Here, the ingredient designs come from Theorem 1.1 and Lemmas 3.1 and 3.2, where the BSCU(26) in Theorem 1.1 is actually a CSCU-HSQS$(26 : 2)$. □

LEMMA 4.14. *There exists a BSCU*(v) *for* $v \equiv 28 \pmod{48}$.

Proof. Combining Lemmas 4.11 and 4.13, we have the fact that there exists a BSCU$(12n + 2)$ for each $n \equiv 1 \pmod 2$. Then apply Theorem 1.1.(1). □

5. Concluding remarks. Combining Lemmas 4.7 and 4.11–4.14, we have the following conclusion.

THEOREM 5.1. *The necessary conditions for the existence of a BSCU*(v)*, namely,* $v \equiv 2, 4 \pmod 6$ *and* $v \geq 4$*, are also sufficient, with two exceptions* $v = 8, 10$.

Acknowledgments. The authors thank the two anonymous referees for their helpful comments and Professor Jeff Dinitz of the University of Vermont for posting our useful small CSCUs on the new results website for *Handbook of Combinatorial Designs, Second Edition* [2].

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