# Families of Sets with Intersecting Clusters 

William Y.C. Chen ${ }^{1}$<br>Center for Combinatorics, LPMC-TJKLC<br>Nankai University, Tianjin 300071, P. R. China<br>Jiuqiang Liu ${ }^{2}$<br>Department of Mathematics<br>Eastern Michigan University<br>Ypsilanti, MI 48197, USA<br>Larry X.W. Wang ${ }^{3}$<br>Center for Combinatorics, LPMC-TJKLC<br>Nankai University, Tianjin 300071, P. R. China<br>${ }^{1}$ chen@nankai.edu.cn, ${ }^{2} \mathrm{jliu} @$ emich.edu, ${ }^{3}$ wxw@cfc.nankai.edu.cn<br>In Memory of Professor Chao Ko


#### Abstract

A family of $k$-subsets $A_{1}, A_{2}, \ldots, A_{d}$ on $[n]=\{1,2, \ldots, n\}$ is called a $(d, c)$-cluster if the union $A_{1} \cup A_{2} \cup \cdots \cup A_{d}$ contains at most $c k$ elements with $c<d$. Let $\mathcal{F}$ be a family of $k$-subsets of an $n$-element set. We show that for $k \geq 2$ and $n \geq k+2$, if every ( $k, 2$ )-cluster of $\mathcal{F}$ is intersecting, then $\mathcal{F}$ contains no ( $k-1$ )-dimensional simplices. This leads to an affirmative answer to Mubayi's conjecture for $d=k$ based on Chvátal's simplex theorem. We also show that for any $d$ satisfying $3 \leq d \leq k$ and $n \geq \frac{d k}{d-1}$, if every ( $d, \frac{d+1}{2}$ )-cluster is intersecting, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star. This result is an extension of both Frankl's theorem and Mubayi's theorem.


Keywords: Clusters of subsets, Chvátal's simplex theorem, $d$-simplex, Erdős-Ko-Rado Theorem

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## 1 Introduction

This paper is concerned with the study of families of subsets with intersecting clusters. The first result is a proof of an important case of a conjecture recently proposed by Mubayi [7] on intersecting families with the aid of Chvátal's simplex theorem. The second result is an extension of both Frankl's theorem and Mubayi's
theorem. It should be noted that we have used these two theorems themselves as a starting point to prove this extension.

Let us review some notation and terminology. The set $\{1,2, \ldots, n\}$ is usually denoted by $[n]$ and the family of all $k$-subsets of a finite set $X$ is denoted by $X^{k}$ or $\binom{X}{k}$. A family $\mathcal{F}$ of sets is said to be intersecting if every two sets in $\mathcal{F}$ have a nonempty intersection. A family $\mathcal{F}$ of sets in $X^{k}$ is called a complete star if $\mathcal{F}$ consists of all $k$-subsets containing $x$ for some $x \in X$.

The classical Erdős-Ko-Rado (EKR) theorem [3] is stated as follows.

Theorem 1.1 (The EKR Theorem) Let $n \geq 2 k$ and let $\mathcal{F} \subseteq\binom{[n]}{k}$ be an intersecting family, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$. Furthermore, for $n>2 k$, the equality holds only when $\mathcal{F}$ is a complete star.

The following generalization of the EKR theorem is due to Frankl [4].

Theorem 1.2 (Frankl) Let $k \geq 2, d \geq 2$, and $n \geq d k /(d-1)$. Suppose that $\mathcal{F} \subseteq[n]^{k}$ such that every $d$ sets of $\mathcal{F}$ have a nonempty intersection. Then $|\mathcal{F}| \leq$ $\binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

The following conjecture due to Erdős on triangle free families implies Frankl's theorem for $d \geq 3$. Recall that a $d$-dimensional simplex, or a $d$-simplex for short, is defined to be a family of $d+1$ sets $A_{1}, A_{2}, \ldots, A_{d+1}$ such that every $d$ of them have a nonempty intersection, but $A_{1} \cap A_{2} \cap \cdots \cap A_{d+1}=\emptyset$. A 2-dimensional simplex is called a triangle. This conjecture has been proved by Mubayi and Verstraëte [9]

Conjecture 1.3 (Erdős) For $n \geq \frac{3 k}{2}$, if $\mathcal{F} \subseteq[n]^{k}$ contains no triangle, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

However, as generalization of Erdős' conjecture, Chvátal [1] proposed the following conjecture which remains open in general case.

Conjecture 1.4 (Chvátal's Simplex Conjecture) Let $k \geq d+1 \geq 3, n \geq$ $k(d+1) / d$, and $\mathcal{F} \subseteq[n]^{k}$. If $\mathcal{F}$ contains no d-dimensional simplex, then $|\mathcal{F}| \leq$ $\binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

Chvátal [1] has shown that it is true for $d=k-1$, which we call Chvátal's simplex theorem.

Theorem 1.5 (Chvátal's Simplex Theorem) For $n \geq k+2 \geq 5$, if $\mathcal{F} \subseteq[n]^{k}$ contains no ( $k-1$ )-dimensional simplices, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

Frankl and Füredi [5] have shown that Chvátal's conjecture holds for sufficiently large $n$.

Theorem 1.6 (Frankl and Füredi) For $k \geq d+2 \geq 4$, there exists $n_{0}$ such that for $n>n_{0}$, if $\mathcal{F} \subseteq[n]^{k}$ contains no d-dimensional simplices, then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

As will be seen, a recent conjecture proposed by Mubayi [7] is related to Chvátal's simplex theorem. Here we introduce the terminology of clusters of subsets. A family of $k$-subsets $A_{1}, A_{2}, \ldots, A_{d}$ of $[n]$ is called a $(d, c)$-cluster if $\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right| \leq c k$, where $c<d$ is a constant that may depend on $d$. A cluster is said to be intersecting if their intersection is nonempty.

Conjecture 1.7 (Mubayi's Conjecture) Let $k \geq d \geq 3$ and $n \geq d k /(d-1)$. Suppose that $\mathcal{F} \subseteq[n]^{k}$ such that every $(d, 2)$-cluster of $\mathcal{F}$ is intersecting i.e., for any $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F},\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right| \leq 2 k$ implies $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \neq \emptyset$. Then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

Mubayi [7] has shown that this conjecture holds for $d=3$ (Theorem 1.8). He has also proved that his conjecture holds for $d=4$ when $n$ is sufficiently large [8].

Theorem 1.8 (Mubayi) Let $k \geq 3$ and $n \geq \frac{3 k}{2}$. Suppose that $\mathcal{F} \subseteq[n]^{k}$ is a family such that every $(3,2)$-cluster $A_{1}, A_{2}, A_{3} \in \mathcal{F}$ is intersecting, then $|\mathcal{F}| \leq$ $\binom{n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

In this paper, we study the case $d=k$ of Mubayi's conjecture in connection with Chvátal's simplex theorem. We show that in this case the conditions for Mubayi's conjecture imply the nonexistence of any ( $k-1$ )-dimensional simplex. Therefore, Chvátal's simplex theorem leads to Mubayi's conjecture for $d=k$. As the main result of this paper, we present a theorem on families of subsets with intersecting clusters which can be viewed as an extension of both Frankl's Theorem (Theorem 1.2) and Mubayi's Theorem (Theorem 1.8).

## 2 Families of Subsets with Intersecting Clusters

In this section, we first consider a special case of Mubayi's conjecture for $k=d$. We show that this case can be deduced from Chvátal's simplex theorem (Theorem 1.5). Then we study families of $k$-subsets with intersecting $\left(d, \frac{d+1}{2}\right)$-clusters and obtain a theorem as an extension of both Frankl's theorem (Theorem 1.2) and Mubayi's theorem (Theorem 1.8). Our proof is based on the EKR Theorem and Frankl's Theorem. We will also use a similar strategy as in the proof of Mubayi's theorem [7].

Theorem 2.1 Let $k \geq 3$ and $n \geq k+2$. Suppose that $\mathcal{F} \subseteq[n]^{k}$ is a family of subsets of $[n]$ such that every $(k, 2)$-cluster is intersecting. Then $\mathcal{F}$ contains no ( $k-1$ )-dimensional simplices.

Proof. Suppose to the contrary that $A_{1}, A_{2}, \ldots, A_{k} \in \mathcal{F}$ form a ( $k-1$ )-dimensional simplex, namely, every $k-1$ of them have a nonempty intersection but

$$
\begin{equation*}
A_{1} \cap A_{2} \cap \cdots \cap A_{k}=\emptyset \tag{2.1}
\end{equation*}
$$

It follows that two distinct families $\left\{A_{i_{1}}, A_{i_{2}}, \ldots, A_{i_{k-1}}\right\}$ and $\left\{A_{j_{1}}, A_{j_{2}}, \ldots, A_{j_{k-1}}\right\}$ cannot have a common element, because the union of these two families equals $\left\{A_{1}, A_{2}, \ldots, A_{k}\right\}$. Without loss of generality, let

$$
i \in A_{1} \cap \cdots \cap A_{i-1} \cap A_{i+1} \cap \cdots \cap A_{k} .
$$

That is, $i$ belongs to every subset $A_{j}$ other than $A_{i}$. It follows that that $\{1, \ldots, i-$ $1, i+1, \ldots k\} \subset A_{i}$. Since $A_{i}$ is a $k$-subset, $A_{i}$ must contain an element in $\{k+1, \ldots, n\}$. So we have

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{k}\right| \leq 2 k
$$

This means that $\left\{A_{1}, A_{2} \ldots, A_{k}\right\}$ is a $(k, 2)$-cluster that is not intersecting, contradicting to the assumption of the theorem. So we conclude that $\mathcal{F}$ does not contain any $(k-1)$-dimensional simplex. This completes the proof.

The following theorem is the main result of this paper.

Theorem 2.2 Let $k \geq d \geq 3$ and $n \geq \frac{d k}{d-1}$. Suppose that $\mathcal{F} \subseteq[n]^{k}$ is a family of subsets of $[n]$ such that every $\left(d, \frac{d+1}{2}\right)$-cluster is intersecting (i.e., for any $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F},\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right| \leq \frac{d+1}{2} k$ implies that $\left.\cap_{i=1}^{d} A_{i} \neq \emptyset\right)$. Then $|\mathcal{F}| \leq\binom{ n-1}{k-1}$ with equality only when $\mathcal{F}$ is a complete star.

The next lemma gives an upper bound on the number of edges in a graph with intersecting clusters, and it will be used in the proof of Theorem 2.2.

Lemma 2.3 Let $n>d \geq 3$. Suppose that $\mathcal{F} \subseteq[n]^{2}$ is a family of 2 -subsets of $[n]$ such that every $\left(d, \frac{d+1}{2}\right)$-cluster is intersecting. Then $|\mathcal{F}| \leq n-1$ with equality only when $\mathcal{F}$ is a complete star.

Proof. Since $\mathcal{F}$ is a family of 2 -subsets, we may consider it as a graph $G$ with vertex set $[n]$. The conditions in the lemma imply that any $d$ edges $A_{1}, A_{2}, \ldots$, $A_{d}$ of $G$ either intersect at a common vertex or cover at least $d+2$ vertices (for $d=3, G$ does not contain any triangle because every (3,2)-cluster is intersecting).

We proceed by induction on $n$. For $n=d+1$, since any $d$ edges cover at most $n=d+1$ vertices, any $d$ edges of $G$ must intersect at a common vertex and thus
form a star. This implies that $|\mathcal{F}|=|E(G)| \leq d=n-1$ with equality only when $\mathcal{F}($ or $G)$ is a complete star.

Assume that $n \geq d+2$ and that the lemma holds for $n-1$. We first claim that $G$ must contain a vertex of degree one. Otherwise, every vertex of $G$ has degree at least two which implies that for every connected component $C$ of $G$ we have

$$
\begin{equation*}
|V(C)| \leq|E(C)| \tag{2.2}
\end{equation*}
$$

Let $C_{1}, C_{2}, \ldots, C_{m}$ be the connected components of $G$ ordered by the condition

$$
\left|E\left(C_{1}\right)\right| \geq\left|E\left(C_{2}\right)\right| \geq \cdots \geq\left|E\left(C_{m}\right)\right|
$$

We aim to find $d$ edges that form a non-intersecting $\left(d, \frac{d+1}{2}\right)$-cluster to reach a contradiction. Let us consider two cases.

Case 1. $\left|C_{1}\right| \geq d$. Since $C_{1}$ is not a star, it contains a path $P$ with three edges. Since $d \geq 3$, we can add $d-3$ edges to $P$ to obtained a connected subgraph $H$ of $C_{1}$. Let $A_{1}, A_{2}, \ldots, A_{d}$ be $d$ edges of $H$. Then we have

$$
\left|A_{1} \cup A_{2} \ldots \cup A_{d}\right|=|V(H)| \leq|E(H)|+1=d+1
$$

Since $H$ is not a star, we obtain $A_{1} \cap A_{2} \ldots \cap A_{d}=\emptyset$.
Case 2. $\left|C_{1}\right|<d$. Let $r \geq 1$ be the integer such that

$$
b=\sum_{i=1}^{r}\left|E\left(C_{i}\right)\right|<d \quad \text { and } \quad \sum_{i=1}^{r+1}\left|E\left(C_{i}\right)\right| \geq d
$$

It is clear that $C_{r+1}$ has at least $d-b$ edges. We now take any connected subgraph $H$ of $C_{r+1}$ with $d-b$ edges. Since $H$ is connected, we have

$$
\begin{equation*}
|E(H)| \geq|V(H)|-1 \tag{2.3}
\end{equation*}
$$

Let $A_{1}, A_{2}, \ldots, A_{d}$ be the $d$ edges in $C_{1}, C_{2}, \ldots, C_{r}, H$. From (2.2) and (2.3) it follows that

$$
\begin{aligned}
& \left|A_{1} \cup A_{2} \cdots \cup A_{d}\right| \\
& \quad=\left|V\left(C_{1}\right)\right|+\left|V\left(C_{2}\right)\right|+\cdots+\left|V\left(C_{r}\right)\right|+|V(H)| \\
& \quad \leq\left|E\left(C_{1}\right)\right|+\left|E\left(C_{2}\right)\right|+\cdots+\left|E\left(C_{r}\right)\right|+|E(H)|+1 \\
& \quad=d+1
\end{aligned}
$$

Noting that $C_{1}, C_{2}, \ldots, C_{r}$ and $H$ are disjoint, we have $A_{1} \cap A_{2} \cdots \cap A_{d}=\emptyset$.
In summary, we have reached the conclusion that $G$ has a vertex with degree one. Let $v$ be a vertex of degree one in $G$ and let $G^{\prime}$ be the induced graph obtained from $G$ by deleting the vertex $v$. Clearly, $G^{\prime}$ is a graph with $n-1$ vertices in which every $d$ edges $A_{1}, A_{2}, \ldots, A_{d}$ either intersect at a common vertex or cover
at least $d+2$ vertices. By the inductive hypothesis, we have $\left|E\left(G^{\prime}\right)\right| \leq n-2$ with equality only if $G^{\prime}$ is a complete star. Hence

$$
|\mathcal{F}|=|E(G)|=|E(C)|+1 \leq n-1
$$

with equality only if $\mathcal{F}$ (or $G$ ) is a complete star.
The following lemma is an extension of Lemma 3 of Mubayi [7]. While the proof of Mubayi relies on the EKR theorem, our proof is based on the above Lemma 2.3 and Frankl's theorem (Theorem 1.2). We will also use a similar framework as in the proof of Mubayi's theorem [7].

Lemma 2.4 Let $k \geq d \geq 2, t \geq 2$, and $2 \leq l \leq k$. Let $S_{1}, S_{2}, \ldots, S_{t}$ be pairwise disjoint $k$-subsets and $X=S_{1} \cup S_{2} \cup \cdots \cup S_{t}$. Suppose that $\mathcal{F}$ is a family of $l$-subsets of $X$ satisfying the conditions (1) $S_{i} \in \mathcal{F}$ for all $i$ if $l=k$; (2) For every $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}$ and $1 \leq i \leq t, A_{1} \cap A_{2} \cdots \cap A_{d} \cap S_{i}=\emptyset$ implies $\left|A_{1} \cup A_{2} \cdots \cup A_{d}-S_{i}\right|>\frac{d l}{2}$. Then we have $|\mathcal{F}|<\binom{t k-1}{l-1}$.

Proof. For $d=2$, the above lemma reduces to Lemma 3 in [7. So we may assume that $d \geq 3$. Let $n=|X|=t k$. We consider the following two cases.

Case 1. Assume $l=2$. We claim that any $\left(d, \frac{d+1}{2}\right)$-cluster of $\mathcal{F}$ is intersecting, namely, for any $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}$, we have either $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \neq \emptyset$ or $\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right| \geq d+2$. To this end, we assume that $A_{1} \cap A_{2} \cap \cdots \cap A_{d}=\emptyset$. This gives $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \cap S_{i}=\emptyset$ for any $S_{i}$. Since $X=\cup S_{i}$ is the ground set of $\mathcal{F}$, there exists $S_{m}$ such that $A_{1} \cap S_{m} \neq \emptyset$. As $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \cap S_{m}=\emptyset$ and $l=2$, in view of Condition 2 we get

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}-S_{m}\right|>d
$$

Furthermore, the condition $A_{1} \cap S_{m} \neq \emptyset$ yields

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right|>d+1 .
$$

So the claim holds.
Since $d \geq 3$, by Lemma 2.3, we find that $|\mathcal{F}| \leq n-1$, where $n=t k$. So it remains to show that it is impossible for $|\mathcal{F}|$ to reach the upper bound $n-$ 1. Assume that $|\mathcal{F}|=n-1$. Again, by Lemma $2.3, \mathcal{F}$ must be a complete star, namely, $\mathcal{F}$ consists of all 2 -subsets of $X$ for some $x$ in $X$. Without loss of generality, we may assume that $x \in S_{1}$. Let $A_{1}$ be a 2 -subset from $\mathcal{F}$ such that $A_{1} \subseteq S_{1}$. Since $d-1 \leq k$, we may choose $d-12$-subsets $A_{2}, A_{3}, \ldots, A_{d}$ such that $A_{i} \in \mathcal{F}$ and $A_{i}-x \subseteq S_{2}$ for $2 \leq i \leq d$. This implies that

$$
A_{1} \cap A_{2} \cap \cdots \cap A_{d} \cap S_{2}=\emptyset
$$

and

$$
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)-S_{2}\right|=2<d
$$

contradicting Condition (2). Thus we have $|\mathcal{F}|<n-1=t k-1$. So the lemma is proved for $l=2$.

Case 2. Assume $l \geq 3$. So we have $k \geq l \geq 3$. We use induction on $t$.
We first consider the case $t=2$, namely, $X=S_{1} \cup S_{2}$. We will show that $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \neq \emptyset$ for any $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}$. If this were not true, there would exist subsets $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}$ for which

$$
\begin{equation*}
A_{1} \cap A_{2} \cap \cdots \cap A_{d}=\emptyset \tag{2.4}
\end{equation*}
$$

Let $A=A_{1} \cup A_{2} \cup \cdots \cup A_{d}$. It is clear that $A$ contains at most $d l$ elements. Since $S_{1}$ and $S_{2}$ are disjoint, so are $A \cap S_{1}$ and $A \cap S_{2}$. Therefore, either $A \cap S_{1}$ or $A \cap S_{2}$ contains at most half of the elements in $A$. We may assume without loss of generality that

$$
\left|A \cap S_{1}\right| \leq \frac{d l}{2}
$$

Note that (2.4) implies $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \cap S_{2}=\emptyset$. Since $X=S_{1} \cup S_{2}$, we get

$$
\left|A-S_{2}\right|=\left|A \cap S_{1}\right| \leq \frac{d l}{2}
$$

contradicting Condition (2). Thus we deduce that $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \neq \emptyset$ for any $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}$. By Frankl's Theorem (Theorem 1.2) we obtain

$$
\begin{equation*}
|\mathcal{F}| \leq\binom{ 2 k-1}{l-1} \tag{2.5}
\end{equation*}
$$

Next we prove that the equality in (2.5) can never be reached. Let us assume that

$$
\begin{equation*}
|\mathcal{F}|=\binom{2 k-1}{l-1} \tag{2.6}
\end{equation*}
$$

Since $d \geq 3$, by Frankl's theorem, $\mathcal{F}$ is a complete star, that is, $\mathcal{F}$ consists of all $l$-subsets of $[2 k]$ containing an element $x$ for some $x$ in $[2 k]$. Without loss of generality, we may assume that $x \in S_{1}$. Thus $\mathcal{F}$ contains every subset $A_{i}$ which is either of the form $B \cup\{x\}$ for $B \in\left[S_{1}-x\right]^{l-1}$ or of the form $C \cup\{x\}$ for $C \in\left[S_{2}\right]^{l-1}$. Since $d \leq k$ and $3 \leq l \leq k$, we have

$$
d-1 \leq k \leq\binom{ k}{l-1}
$$

Now we may choose $A_{1} \in \mathcal{F}$ with $A_{1} \subseteq S_{1}$ and $d-1$ sets $A_{2}, A_{3}, \ldots, A_{d} \in \mathcal{F}$ with $A_{i}-x \subseteq S_{2}$ for each $i \geq 2$. Since $A_{1} \cap S_{2}=\emptyset, A_{1} \cap A_{2} \cdots \cap A_{d} \cap S_{2}=\emptyset$. Moreover, since $A_{i}-x \subseteq S_{2}$ for $i=2,3, \ldots, d$, we have

$$
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)-S_{2}\right|=\left|A_{1}\right|=l<\frac{d l}{2},
$$

contradicting Condition (2). It follows that $|\mathcal{F}|<\binom{2 k-1}{l-1}$ and hence the lemma is valid for $t=2$.

Next suppose that $t \geq 3$ and the result holds for $t-1$. We first show that there exists at most one set $S_{m}$ such that

$$
\left|\mathcal{F} \cap\left[S_{m}\right]^{l}\right| \geq \frac{d}{2}
$$

Suppose, to the contrary, that there exist two sets, say $S_{1}$ and $S_{2}$, such that

$$
\left|\mathcal{F} \cap\left[S_{i}\right]^{l}\right| \geq \frac{d}{2},
$$

for $i=1,2$. Then we have

$$
\left|\mathcal{F} \cap\left[S_{1}\right]^{l}\right|+\left|\mathcal{F} \cap\left[S_{2}\right]^{l}\right| \geq d
$$

Since $\left|\mathcal{F} \cap\left[S_{1}\right]^{l}\right| \geq \frac{d}{2} \geq 1$ and $\left|\mathcal{F} \cap\left[S_{2}\right]^{l}\right| \geq \frac{d}{2} \geq 1$, we are able to choose $d$ sets $A_{1}, A_{2}, \ldots, A_{d}$ from $\left(\mathcal{F} \cap\left[S_{1}\right]^{l}\right) \cup\left(\mathcal{F} \cap\left[S_{2}\right]^{l}\right)$ such that $A_{1} \subseteq S_{1}$ and $A_{2} \subseteq S_{2}$. Since $\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)\right| \leq d l$ and $S_{1} \cap S_{2}=\emptyset$, we have either

$$
\begin{equation*}
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right) \cap S_{1}\right| \leq \frac{d l}{2} \tag{2.7}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right) \cap S_{2}\right| \leq \frac{d l}{2} \tag{2.8}
\end{equation*}
$$

Without loss of generality, assuming that (2.7) is valid. We see that

$$
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)-S_{2}\right|=\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right) \cap S_{1}\right| \leq \frac{d l}{2}
$$

However, the choice of $A_{1}, A_{2}, \ldots, A_{d}$ ensures that $A_{1} \cap A_{2} \cdots \cap A_{d} \cap S_{2}=\emptyset$, contradicting Condition (2). This leads to the conclusion that there exists at most one set $S_{m}$ such that

$$
\left|\mathcal{F} \cap\left[S_{m}\right]^{l}\right| \geq \frac{d}{2}
$$

Without loss of generality, let us assume that $m=t$. Thus we have

$$
\left|\mathcal{F} \cap\left[S_{i}\right]^{l}\right| \leq \frac{d-1}{2},
$$

for $i=1, \ldots, t-1$. Set

$$
\mathcal{H}_{i}=\left\{F \in \mathcal{F}:\left|F \cap S_{i}\right|=l-1\right\}
$$

and

$$
\operatorname{deg}_{\mathcal{H}_{i}}(B)=\left|\left\{F \in \mathcal{H}_{i}: B \subset F\right\}\right|
$$

for each $1 \leq i \leq t$.
We claim that there exists at least one set $S_{i}(i \in\{1, \ldots, t\})$ such that

$$
\left|\mathcal{H}_{i}\right| \leq\binom{ k}{l-1} \quad \text { and } \quad\left|\mathcal{F} \cap\left[S_{i}\right]^{l}\right| \leq \frac{d-1}{2}
$$

Suppose that the above claim is not true. Then

$$
\begin{equation*}
\left|\mathcal{H}_{i}\right| \geq\binom{ k}{l-1}+1 \tag{2.9}
\end{equation*}
$$

for $i=1, \cdots, t-1$. Moreover, if $\left|\mathcal{F} \cap\left[S_{t}\right]^{l}\right| \leq \frac{d-1}{2}$, then

$$
\left|\mathcal{H}_{t}\right| \geq\binom{ k}{l-1}+1
$$

By (2.9), there exists a $(l-1)$-subset $B$ of $S_{1}$ such that

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{H}_{1}}(B) \geq 2 \tag{2.10}
\end{equation*}
$$

Assume that $A_{1}, A_{2} \in \mathcal{H}_{1}$ are chosen subject to the conditions $B \subset A_{1}$ and $B \subset A_{2}$. Since

$$
\left|\mathcal{H}_{2}\right| \geq\binom{ k}{l-1}+1>d-2
$$

we can choose $A_{3}, \ldots A_{d}$ from $\mathcal{H}_{2}$. Since $A_{1} \cap A_{2}=B \subseteq S_{1}$,

$$
A_{1} \cap \cdots \cap A_{d} \cap S_{2}=\emptyset
$$

and

$$
\left|A_{1} \cup \cdots \cup A_{d}-S_{2}\right| \leq(l+1)+(d-2)=l+d-1 \leq \frac{d l}{2}
$$

for $d \geq 4$ and $l \geq 3$. So we have reached a contradiction to Condition (2) when $d \geq 4$.

Consider the case $d=3$. Let $\left\{x_{i}\right\}=A_{i}-B$ for $i=1,2$. Since $A_{1}, A_{2} \in \mathcal{H}_{1}$, we have $x_{i} \notin S_{1}$. Let $x_{1} \in S_{i_{0}}$ for some $i_{0} \geq 2$. Choose $A_{3}$ to be either in $\mathcal{H}_{i_{0}}$ or $\mathcal{F} \cap\left[S_{i_{0}}\right]^{l}$. Since $A_{1} \cap A_{2}=B \in S_{1}$ and $S_{1} \cap S_{2}=\emptyset$, we have

$$
A_{1} \cap A_{2} \cap A_{3} \cap S_{i_{0}}=\emptyset
$$

and

$$
\left|A_{1} \cup A_{2} \cup A_{3}-S_{i_{0}}\right| \leq(l-1)+1+1=l+1 \leq \frac{d l}{2}
$$

for $l \geq 3$ and $d=3$, contradicting Condition (2) again. Thus the claim is verified.
Without loss of generality, we assume that

$$
\left|\left\{F \in \mathcal{F}:\left|F \cap S_{1}\right|=l-1\right\}\right|=\left|\mathcal{H}_{1}\right| \leq\binom{ k}{l-1} \quad \text { and } \quad\left|F \cap\left[S_{1}\right]^{l}\right| \leq \frac{d-1}{2}
$$

For any $F \in \mathcal{F}$, we may express $F$ as $F_{1} \cup F_{2}$, where $F_{1}=F \cap S_{1}$ and $F_{2}=F-F_{1}$. For a fixed $F_{1}$ of size $l-r(1 \leq r \leq l)$, let $\mathcal{F}_{r}$ be the family of all $r$-sets $F_{2} \subset S_{2} \cup S_{3} \cup \cdots \cup S_{t}$ such that $F_{1} \cup F_{2} \in \mathcal{F}$.

We claim that $\mathcal{F}_{r}$ satisfies the conditions of the lemma. For otherwise, we may assume that there exist $A_{1}, A_{2}, \ldots, A_{d} \in \mathcal{F}_{r}$ and $i \in\{2, \cdots, t\}$ such that $A_{1} \cap A_{2} \cap \cdots \cap A_{d} \cap S_{i}=\emptyset$ and

$$
\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)-S_{i}\right| \leq \frac{d}{2} r
$$

Now, let $A_{j}^{\prime}=A_{j} \cup F_{1}$ for $1 \leq j \leq d$. Clearly, $A_{1}^{\prime}, A_{2}^{\prime}, \ldots, A_{d}^{\prime} \in \mathcal{F}$ and
$A_{1}^{\prime} \cap A_{2}^{\prime} \cap \cdots \cap A_{d}^{\prime} \cap S_{i}=\emptyset$. Recalling that $l \geq r$, we find

$$
\begin{gathered}
\left|\left(A_{1}^{\prime} \cup A_{2}^{\prime} \cup \cdots \cup A_{d}^{\prime}\right)-S_{i}\right|=\left|F_{1}\right|+\left|\left(A_{1} \cup A_{2} \cup \cdots \cup A_{d}\right)-S_{i}\right| \\
\quad \leq l-r+\frac{d r}{2}=l+\frac{d-2}{2} r \leq l+\frac{d-2}{2} l=\frac{d l}{2}
\end{gathered}
$$

contradicting Condition (2). Thus we have shown that $\mathcal{F}_{r}$ satisfies the conditions of the lemma. For $r \geq 2$, by the inductive hypothesis, we see that

$$
\left|\mathcal{F}_{r}\right|<\binom{(t-1) k-1}{r-1}
$$

Since $l \geq 3$ and $d \leq k$, it is easy to check that

$$
\sum_{r=2}^{l}\binom{k}{l-r}-d \geq 0
$$

Hence $|\mathcal{F}|$ can be bounded as follows,

$$
\begin{aligned}
|\mathcal{F}| & \leq \sum_{r=2}^{l}\binom{k}{l-r}\left|\mathcal{F}_{r}\right|+\left|\left\{F \in \mathcal{F}:\left|F \cap S_{1}\right|=l-1\right\}\right|+\left|\mathcal{F} \cap\left[S_{1}\right]^{l}\right| \\
& \leq \sum_{r=1}^{l}\binom{k}{l-r}\binom{t-1) k-1}{r-1}-\sum_{r=1}^{l}\binom{k}{l-r}+\binom{k}{l-1}+\frac{d-1}{2} \\
& <\binom{t k-1}{l-1}-\sum_{r=2}^{l}\binom{k}{l-r}+d \leq\binom{ t k-1}{l-1}
\end{aligned}
$$

This completes the proof.
We are now ready to prove Theorem 2.2.
Proof of Theorem 2.2. For $d=3$, the result follows from Theorem 1.8. So we assume $d \geq 4$. Let $S_{1}, S_{2}, \ldots, S_{t}$ be a maximum subfamily of pairwise disjoint $k$-subsets from $\mathcal{F}$. We proceed by induction on $t$. If $t=1$, then $\mathcal{F}$ is intersecting and the result follows from Theorem 1.1 when $n \geq 2 k$. When $\frac{d k}{d-1} \leq n<2 k$, for any $A_{1}, \ldots, A_{d} \in \mathcal{F},\left|A_{1} \cup \cdots \cup A_{d}\right| \leq n<2 k$, it follows that their intersection is nonempty from the condition of the theorem. Hence the theorem reduces to Theorem 1.2 in this case. Now we may assume that $t \geq 2$ and the theorem holds for $t-1$. Note that $t=1$ is the only case when $\mathcal{F}$ can be a complete star. It will be shown that $|\mathcal{F}|<\binom{n-1}{k-1}$.

If $n=t k$, we set $l=k$. The condition on $\mathcal{F}$ in Theorem 2.2 implies the conditions on $\mathcal{F}$ in Lemma 2.4 with $d$ replaced by $d-1$. In fact, suppose that there exist $A_{1}, A_{2}, \ldots, A_{d-1} \in \mathcal{F}$ for which $A_{1} \cap A_{2} \cdots \cap A_{d-1} \cap S_{i}=\emptyset$. Since every $\left(d, \frac{d+1}{2}\right)$-cluster of $\mathcal{F}$ is intersecting, we see that

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d-1} \cup S_{i}\right|>\frac{d-1}{2} k
$$

hence

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d-1}-S_{i}\right|>\frac{d+1}{2} k-k=\frac{d-1}{2} k .
$$

Hence the theorem follows from Lemma 2.4 in this case.
We now assume $n>t k$ and let

$$
\begin{equation*}
Y=[n]-\bigcup_{i=1}^{t} S_{i} \tag{2.11}
\end{equation*}
$$

Given the choice of $S_{1}, S_{2}, \ldots, S_{t}, Y$ does not contain any subset $A \in \mathcal{F}$. Set

$$
\mathcal{F}^{\prime}=\{F \in \mathcal{F}:|F \cap Y|=k-1\}
$$

We claim that if $|Y|=n-t k \geq k$, then

$$
\begin{equation*}
\left|\mathcal{F}^{\prime}\right| \leq\binom{ n-t k}{k-1} \tag{2.12}
\end{equation*}
$$

If the claim is not true given the condition, then we have

$$
\left|\mathcal{F}^{\prime}\right| \geq\binom{ n-t k}{k-1}+1 \geq k+1>d
$$

Therefore, there exists a $(k-2)$-subset $B \subset Y$ such that

$$
\begin{equation*}
\operatorname{deg}_{\mathcal{F}^{\prime}}(B) \geq|Y|-k+3=(n-t k)-k+3 \tag{2.13}
\end{equation*}
$$

Otherwise, we would have

$$
\left|\mathcal{F}^{\prime}\right| \leq \frac{((n-t k)-k+2)\binom{n-t k}{k-2}}{k-1}=\binom{n-t k}{k-1}
$$

Since the number of $(k-1)$-subsets of $Y$ containing $B$ is equal to $|Y|-k+2$, there exists $=(k-1)$-subset $C$ in $Y$ containing $B$ such that $\operatorname{deg}_{\mathcal{F}^{\prime}}(C) \geq 2$. Let $A_{1}, A_{2} \in \mathcal{F}^{\prime}$ be such that $A_{1} \cap A_{2}=C \subset Y$. It is easy to see that

$$
A_{1} \cap A_{2} \cap S_{i}=\emptyset
$$

for each $1 \leq i \leq t$. Let $A_{3}, A_{4}, \ldots, A_{d-1}$ be additional subsets in $\mathcal{F}^{\prime}$ such that $B \subseteq A_{i}$ for each $i$ if $|Y|-k+3 \geq d-1$. We deduce that

$$
A_{1} \cap \cdots \cap A_{d-1} \cap S_{i}=\emptyset
$$

for each $1 \leq i \leq t$. Moreover,

$$
\left|A_{1} \cup \cdots \cup A_{d-1}\right| \leq k-2+2(d-2)+1=k+2 d-5, \quad \text { if }|Y|-k+3 \geq d-1
$$

and

$$
\left|A_{1} \cup \cdots \cup A_{d-1}\right| \leq|Y|+d-1 \leq k+2 d-6, \quad \text { if }|Y|-k+3<d-1
$$

Let $S_{h}$ be such that $S_{h} \cap A_{1} \neq \emptyset$. Since $k \geq d \geq 4$, we see that

$$
\left|\left(A_{1} \cup \cdots \cup A_{d-1}\right) \cup S_{h}\right| \leq k+2 d-5+(k-1)=2 k+2 d-6 \leq \frac{d+1}{2} k
$$

contradicting the assumption of the theorem. So the claim is justified.
Note that for any member $F$ in $\mathcal{F}$, we can write it as $F=F_{1} \cup F_{2}$, where $F_{1}=F \cap Y$ and $F_{2}=F-F_{1}$. We now consider all possible ways to construct $F$ in the above form. Let $F_{1}$ be a given subset of $Y$ size $k-l(1 \leq l \leq k)$. By the definition of $Y$ in (2.11), $F_{2}$ is a subset $\cup_{i=1}^{t} S_{i}$. Let $\mathcal{F}_{l}$ be the family of all $l$-sets $F_{2} \subset \cup_{i=1}^{t} S_{i}$ such that $F_{1} \cup F_{2} \in \mathcal{F}$. It remains to prove that $\mathcal{F}_{l}$ satisfies the conditions in Lemma 2.4 with $d$ replaced by $d-1$. For $l=k$, the assumption of the theorem implies that for every $A_{1}, A_{2}, \ldots, A_{d-1} \in \mathcal{F}_{k}$, if $A_{1} \cap A_{2} \cap \cdots \cap A_{d-1} \cap S_{i}=\emptyset$, then

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d-1} \cup S_{i}\right|>\frac{d+1}{2} k
$$

which yields that

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d-1}-S_{i}\right|>\frac{d-1}{2} k
$$

Therefore, the assertion holds when $l=k$. For $l<k$, if the assertion is not valid, then there exist $A_{1}, A_{2}, \ldots, A_{d-1} \in \mathcal{F}_{l}$ such that $A_{1} \cap A_{2} \cdots \cap A_{d-1} \cap S_{i}=\emptyset$ and

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{d-1}-S_{i}\right| \leq \frac{d-1}{2} l .
$$

Setting $A_{i}^{\prime}=A_{i} \cup F_{1}$ for $i \leq d-1$, we deduce that $A_{i}^{\prime} \in \mathcal{F}, A_{1}^{\prime} \cap A_{2}^{\prime} \cdots \cap A_{d-1}^{\prime} \cap S_{i}=$ $\emptyset$, and

$$
\begin{aligned}
& \left|\left(A_{1}^{\prime} \cup A_{2}^{\prime} \cup \cdots \cup A_{d-1}^{\prime}\right) \cup S_{i}\right|=\left|F_{1}\right|+\left|\left(A_{1} \cup=_{2} \cup \cdots \cup A_{d-1}\right)-S_{i}\right|+\left|S_{i}\right| \\
& \quad \leq k-l+\frac{d-1}{2} l+k=2 k+\frac{d-3}{2} l \leq 2 k+\frac{d-3}{2} k=\frac{d+1}{2} k,
\end{aligned}
$$

contradicting the assumption of the theorem. Up to now, we have shown that $\mathcal{F}_{l}$ satisfies the conditions in Lemma 2.4. For $l \geq 2$, by Lemma 2.4 we find that

$$
\left|\mathcal{F}_{l}\right|<\binom{t k-1}{l-1} .
$$

Evidently, for $|Y|=n-t k \leq k-2$, we have

$$
|\{F \in \mathcal{F}:|F \cap Y|=k-1\}|=0 .
$$

For the case $|Y|=k-1$, we have

$$
|\{F \in \mathcal{F}:|F \cap Y|=k-1\}|<d-1 \leq k-1
$$

Otherwise we can choose $d-1$ sets $A_{1}, \ldots, A_{d-1} \in \mathcal{F}$ together with $S_{1}$ in violation of the assumption of theorem. When $|Y| \geq k$, It follows from (2.12) that

$$
|\{F \in \mathcal{F}:|F \cap Y|=k-1\}| \leq\binom{ n-t k}{k-1}
$$

which implies

$$
|\{F \in \mathcal{F}:|F \cap Y|=k-1\}|<\sum_{l=1}^{k}\binom{n-t k}{k-l}
$$

Finally,

$$
\begin{aligned}
|\mathcal{F}| & \leq \sum_{l=2}^{k}\binom{|Y|}{k-l}\left|\mathcal{F}_{l}\right|+|\{F \in \mathcal{F}:|F \cap Y|=k-1\}| \\
& \leq \sum_{l=2}^{k}\binom{|Y|}{k-l}\left[\binom{t k-1}{l-1}-1\right]+|\{F \in \mathcal{F}:|F \cap Y|=k-1\}| \\
& =\sum_{l=1}^{k}\binom{|Y|}{k-l}\left[\binom{t k-1}{l-1}-1\right]+|\{F \in \mathcal{F}:|F \cap Y|=k-1\}| \\
& =\sum_{l=1}^{k}\binom{n-t k}{k-l}\binom{t k-1}{l-1}-\sum_{l=1}^{k}\binom{n-t k}{k-l}+|\{F \in \mathcal{F}:|F \cap Y|=k-1\}| \\
& <\binom{n-1}{k-1}
\end{aligned}
$$

as required. This completes the proof.

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