Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 08/17

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2008

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
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98693 Ilmenau
Tel.: +49 3677 69 3621
Fax: +49 3677 69 3270
http://www.tu-ilmenau.de/ifm/

ISSN xxxx-xxxx

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Edge-Injective and Edge-Surjective Vertex Labellings

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Abstract. For a graph $G = (V, E)$ we consider vertex-k-labellings $f : V \to \{1, 2, \ldots, k\}$ for which the induced edge weighting $w : E \to \{2, 3, \ldots, 2k\}$ with $w(uv) = f(u) + f(v)$ is injective or surjective or both.

We study the relation between these labellings and the number theoretic notions of an additive basis and a Sidon set, present a new construction for a so-called restricted additive basis and derive the corresponding consequences for the labellings.

We prove that a tree of order n and maximum degree Δ has a vertex-k-labelling f for which w is bijective if and only if $\Delta \leq k = n/2$. Using this result we prove a recent conjecture of Ivančo and Jendrol' concerning edge-irregular total labellings for graphs that are sparse enough.

Keywords. Labelling; weighting; additive basis; Sidon set; weak Sidon set; edge irregular total labelling

1 Introduction

For a finite, simple and undirected graph $G = (V, E)$ we consider labellings $f : V \to [1, k] := \{1, 2, \ldots, k\}$ of the vertex set V using integer labels between 1 and some $k \in \mathbb{N}$ such that the induced edge weighting $w : E \to [2, 2k]$ with $w(uv) := f(u) + f(v)$ is injective or surjective or both. If w is injective (surjective, bijective), we call f an *edge-injective (edge-surjective, edge-bijective) vertex-k-labelling*.

For a given graph $G = (V, E)$, we denote the maximal k for which G allows an edge-surjective vertex-k-labelling by $s(G)$ and the minimal k for which G allows an edge-injective vertex-k-labelling by $i(G)$. If G is of size m and maximum degree Δ and allows an edge-injective vertex-k-labelling, then m must be at most the number of possible edge weights which is $2k-1 = |[2, 2k]|$ and all Δ neighbours of a vertex of maximum degree must receive different labels, i.e.

$$i(G) \;\geq\; \max\left\{ \left\lceil \frac{m+1}{2} \right\rceil, \Delta \right\}. \tag{1}$$

Similarly, if G allows an edge-surjective vertex-k-labelling, then $m \geq 2k - 1$ and the at most k different weights on the edges incident to a vertex of maximum degree together with the at most $m-\Delta$ further weights must be at least $2k-1$, i.e.

$$s(G) \quad \leq \quad \min\left\{\left\lfloor\frac{m+1}{2}\right\rfloor, m+1-\Delta\right\}. \tag{2}$$

Clearly, for $\Delta \leq \frac{m+1}{2}$ equality in (1) is equivalent to equality in (2).

Apart from being a natural labelling concept of graphs [5] which we believe to be interesting on its own right, we have two further main motivations to study such labellings.

The first is that for complete graphs they naturally relate to well-studied number theoretic notions. In Section 2, we relate edge-surjective vertex labellings to *additive bases*. We present a new construction for a so-called restricted additive basis and derive the consequences concerning the labellings. In Section 3, we relate edge-injective vertex labellings to *Sidon sets*.

Furthermore, edge-surjective vertex labellings of graphs can be used to construct so-called *edge-irregular total labellings* which were recently introduced by Bača et al. [1]. In Section 4, we prove that (1) and (2) hold with equality for trees and, in Section 5, we use this result to establish a conjecture concerning edge-irregular total labellings due to Ivančo and Jendrol' [9] for graphs that are sparse enough.

We refer the reader to Gallian's survey [5] for a wealth of information on labelling concepts and to Graham and Sloane [6] and Pikhurko [15] who study the relation of the mentioned number theoretic concepts to harmonious graphs and edge-magic graphs.

2 Edge-surjective vertex labellings and additive bases

An *additive basis of order p* for a set A of integers is a set $\mathcal{B}$ of integers with the property that every element of A is the sum of p not necessarily distinct elements of $\mathcal{B}$. For example, Lagrange's four-squares theorem states that the squares form an additive basis of order four for the non-negative integers.

If every element of A is the sum of p distinct elements of $\mathcal{B}$, then we call the basis $\mathcal{B}$ *strong*. Furthermore, following Moser, Pounder, and Riddell [13], if $\mathcal{B} \subseteq [1,k]$ is an additive basis of order 2 for $[2,2k]$ or a strong additive basis of order 2 for $[3,2k-1]$, then we call the basis $\mathcal{B}$ *restricted*.

The systematic study of additive bases goes back to a question posed by Schur and investigated by Rohrbach [16]. Clearly, an additive basis $\mathcal{B}$ of order 2 for $[2,2k]$ satisfies $\binom{|\mathcal{B}|}{2}+|\mathcal{B}| \geq 2k-1$, i.e. $|\mathcal{B}| \geq 2\sqrt{k}+o(\sqrt{k})$. Complementing this lower bound, already Rohrbach [16] constructed bases with $|\mathcal{B}| \leq 2\sqrt{2}\sqrt{k}+o(\sqrt{k})$. Several authors improved the constants involved in the lower bounds and constructions (cf. [8, 12, 13] and the good survey in C12 of [7]). The best known lower bound which is relevant for our purposes here is due to Moser, Pounder, and Riddell [13].

Theorem 1 (Moser, Pounder, and Riddell [13]) *For every $\epsilon > 0$ there is some $k_1(\epsilon)$ such that for every integer $k \geq k_1(\epsilon)$ and every restricted additive basis $\mathcal{B} \subseteq [1,k]$ of order*

2 *for* $[2, 2k]$

$$k < \left(1 - \left(\frac{1}{1+\frac{\pi}{2}}\right)^2 + \epsilon\right) \frac{|\mathcal{B}|^2}{4} \cong (0.8487 + \epsilon)\frac{|\mathcal{B}|^2}{4}.$$

For our first result we adapt a construction of a non-restricted basis due to Hämmerer and Hofmeister [8] to our purposes.

Theorem 2 *For every $\epsilon > 0$ there is some $k_2(\epsilon)$ such that for every integer $k \geq k_2(\epsilon)$ there is a restricted strong additive basis $\mathcal{B} \subseteq [1, k]$ of order 2 for $[3, 2k-1]$ with*

$$k \geq (1-\epsilon)\frac{5}{9}\frac{|\mathcal{B}|^2}{4}.$$

Proof: For $l, h \in \mathbb{N}$ with $h \geq 3l - 2$ let $d := 10l$ and

$$\begin{aligned}
A &:= [1, 2l+1] \cup \{3l+1\} \cup [4l+1, 5l+1] \cup [6l+1, 7l] \cup \{9l+1\},\\
B_i &:= \{id + 2l + i + 1, id + 3l + i + 1, id + 5l + i + 1, id + 9l + i + 1\}\\
&\quad\ \text{for } 1 \leq i \leq l-1,\\
B &:= \bigcup_{i=1}^{l-1} B_i,\\
C &:= [(l-1)d + 7l + 1, (l-1)d + 9l],\\
D_i &:= \{(l+i)d + 4l + 1\} \text{ for } 0 \leq i \leq h - 2l + 1,\\
D &:= \bigcup_{i=0}^{h-2l+1} D_i,
\end{aligned}$$

$$\begin{aligned}
A' &:= [hd+1, hd+2l+1] \cup \{hd+3l+1\} \cup [hd+4l+1, hd+5l+1]\\
&\quad\ \cup [hd+6l+1, hd+7l+1],\\
B'_i &:= \{(h-i)d + 2l + i + 1, (h-i)d + 3l + i + 1, (h-i)d + 5l + i + 1,\\
&\quad\ (h-i)d + 9l + i + 1\} \text{ for } 1 \leq i \leq l-1,\\
B' &:= \bigcup_{i=1}^{l-1} B_i, \text{ and}\\
C' &:= [(h-l)d + 7l + 2, (h-l)d + 9l + 1].
\end{aligned}$$

See Figure 1 for an illustration of the above sets for $l = 3$ and $h = 10$ where a black circle in the y-th line from the bottom and x-th column from the left corresponds to the integer $x + (y-1)d$ and the white circles indicate the range $[1, 2k-1]$.

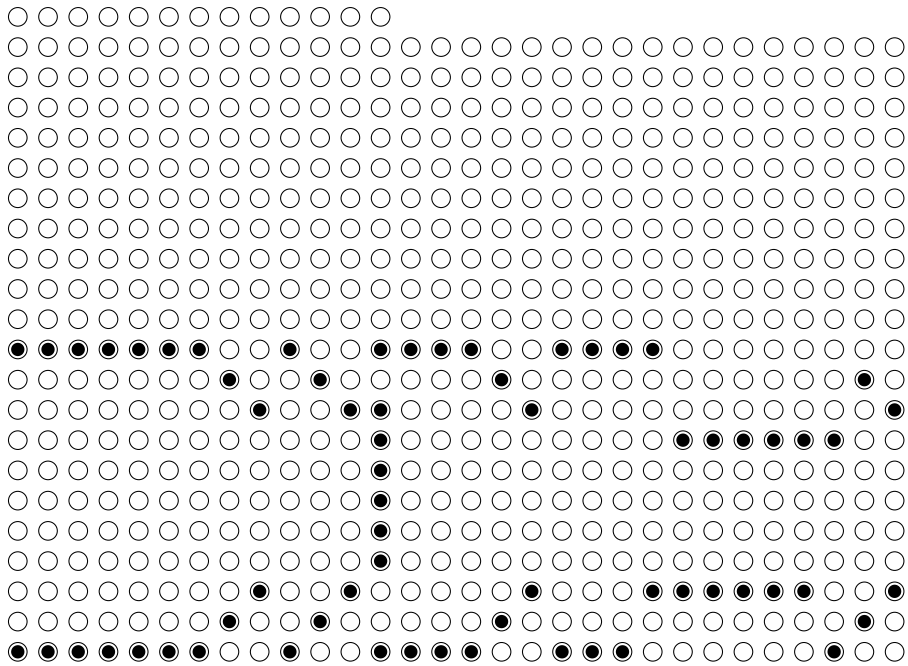

Figure 1 $\mathcal{B}$ for $l = 3$ and $h = 10$.

Note that

$$\mathcal{B} := A \cup B \cup C \cup D \cup A' \cup B' \cup C' \subseteq [1, k]$$

for $k = hd + 7l + 1$. We will show that every integer $x \in [3, 2k-1] = [3, (2h+1)d + 4l + 1]$ is the sum of two distinct elements of $\mathcal{B}$ and consider different partly overlapping intervals.

Interval $[3, d + 4l - 1]$.
All integers in this range are the sum of two distinct elements of A.

Interval $[d + 2l + 3, ld + 3l + 1]$.
For every $1 \leq i \leq l - 1$ the integers in $[id + 2l + i + 2, (i + 1)d + 2l + i + 2]$ are the sum of an element of A and an element of B_i.

Interval $[(l - 1)d + 7l + 2, ld + 6l]$.
All integers in this range are the sum of an element of A and an element of C.

Interval $[ld + 6l + 1, ld + 7l]$.
All integers in this range are the sum of an element of A and the element ld of B_{l-1}.

Interval $[ld + 6l + 2, ld + 8l + 1]$.
All integers in this range are the sum of the element $9l + 1$ of A and an element of C.

Interval $[ld + 8l + 2, (2l - 1)d + 9l + 1]$.
For $1 \leq i \leq l - 1$ all integers in $[(l + i - 1)d + 8l + 2, (l + i - 1)d + 9l + 2]$ are the sum of an element of A and the element $(l + i - 1)d + 4l + 1$ of D_{i-1}, all integers in

$[(l+i-1)d+9l+3,(l+i-1)d+9l+i+1]$ are the sum of the element $(l+i-j-1)d+4l+1$ of D_{i-j-1} and the element $jd+5l+j+1$ of B_j for $1 \leq j \leq i-1$, all integers in $[(l+i-1)d+9l+i+2,(l+i)d+4l+i+1]$ are the sum of an element of B_i and an element of C, all integers in $[(l+i)d+4l+2,(l+i)d+6l+2]$ are the sum of an element of A and an element of D_i, all integers in $[(l+i)d+6l+3,(l+i)d+6l+i+1]$ are the sum of the element $(l+i-j)d+4l+1$ of D_{i-j} and the element $jd+2l+j+1$ of B_j for $1 \leq j \leq i-1$, and all integers in $[(l+i)d+6l+i+2,(l+i)d+8l+i+2]$ are the sum of the element $\{id+9l+i+1\}$ of B_i and an element of C.

Interval $[(2l-1)d+4l+2,(h-l+2)d+4l+1]$.
Note that $A \cup B \cup C$ is a complete set of remainders modulo d. This easily implies that all integers in this range are the sum of an element of $A \cup B \cup C$ and an element of D.

Interval $[(h-l+1)d+3l+1,hd+2l+3]$.
For every $1 \leq i \leq l-1$ the integers in $[(h-i)d+2l+i+2,(h-i+1)d+2l+i+2]$ are the sum of an element of A and an element of B_i'.

Interval $[hd+2,(h+1)d+4l+2]$.
All integers in this range are the sum of an element of A and an element of A'.

Interval $[(h+1)d+2l+3,(h+l)d+3l+1]$.
For every $1 \leq i \leq l-1$ the integers in $[(h+i)d+2l+i+2,(h+i+1)d+2l+i+2]$ are the sum of an element of A' and an element of B_i.

Interval $[(h+l-1)d+7l+2,(h+l)d+6l+1]$.
All integers in this range are the sum of an element of A' and an element of C.

Interval $[(h+l)d+l+3,(2h-2l+3)d+7l]$.
Note that $A' \cup B' \cup C'$ is a complete set of remainders modulo d. This easily implies that all integers in this range are the sum of an element of $A' \cup B' \cup C'$ and an element of D.

Interval $[(2h-2l+2)d+2,(2h-l)d+9l+3]$.
For $1 \leq i \leq l-1$ all integers in $[(2h-l-i)d+9l+i+3,(2h-l-i+1)d+4l+i+2]$ are the sum of an element of B_i' and an element of C', all integers in $[(2h-l-i+1)d+4l+2,(2h-l-i+1)d+6l+2]$ are the sum of an element of A' and an element of $D_{h-2l-i+1}$, all integers in $[(2h-l-i+1)d+6l+3,(2h-l-i+1)d+6l+i+2]$ are the sum of the element $(h-l-i+j+1)d+4l+1$ of $D_{h-2l-i+j+1}$ and the element $(h-j)d+2l+j+1$ of B_j' for $1 \leq j \leq i$, all integers in $[(2h-l-i+1)d+6l+i+3,(2k-l-i+1)d+8l+i+2$ are the sum of an element of B_i' and an element of C', all integers in $[(2k-l-i+1)d+8l+2,(2k-l-i+1)d+9l+2]$ are the sum of an element of A' and an element of $D_{h-2l-i+1}$, all integers in $[(2h-l-i+1)d+9l+3,(2h-l-i+1)d+9l+i+2]$ are the sum of the element $(h-l-i+j+1)d+4l+1$ of $D_{h-2l-i+j+1}$ and the element $(h-j)d+5l+j+1$ of B_j' for $1 \leq j \leq i$.

Interval $[(2h-l)d+7l+3,(2h-l+1)d+6l+2]$.
All integers in this range are the sum of an element of A' and an element of C'.

Interval $[(2h-l+1)d+3l+1,2hd+2l+3]$.
For every $1 \leq i \leq l-1$ the integers in $[(2h-i)d+2l+i+2,(2h-i+1)d+2l+i+2]$ are the sum of an element of A' and an element of B'_i.

Interval $[2hd+3,(2h+1)d+4l+1]$.
All integers in this range are the sum of two distinct elements of A'.

Note that

$$\begin{aligned} |\mathcal{B}| &= |A \cup B \cup C \cup D \cup A' \cup B' \cup C'| \\ &= (4l+2)+(4l-4)+2l+(h-2l+2)+(4l+1)+(4l-4)+2l \\ &= h+18l-3. \end{aligned}$$

We choose $h=18l$ which asymptotically minimizes the fraction $\frac{|\mathcal{B}|^2}{k}=\frac{(h+18l-3)^2}{10hl+7l+1}$. For this choice $\frac{|\mathcal{B}|^2}{k} \to \frac{9\cdot 4}{5}$ which implies the desired result. $\Box$

In the following corollary we summarize the consequences of the last two results for edge-surjective vertex labellings of complete graphs.

Corollary 3 *For every $\epsilon > 0$ there is some $n(\epsilon)$ such that*

$$(1-\epsilon)\frac{5}{9}\frac{n^2}{4} \leq s(K_n) \leq \left(1-\left(\frac{1}{1+\frac{\pi}{2}}\right)^2+\epsilon\right)\frac{n^2}{4}$$

for all $n \in \mathbb{N}$ with $n \geq n(\epsilon)$.

Proof: Since for every edge-surjective vertex-k-labelling f of $K_n=(V,E)$ the set $\{f(v) \mid v \in V\}$ is a restricted additive basis of order 2 for $[2,2k]$, the upper bound on $s(K_n)$ is immediate from Theorem 1.

For the lower bound we consider a restricted strong additive basis $\mathcal{B} \subseteq [1,k]$ of order 2 for $[3,2k-1]$ and a corresponding vertex-k-labelling f of $K_{|\mathcal{B}|+2}=(V,E)$ with $\mathcal{B}=\{f(v) \mid v \in V\}$ in which two vertices have label 1 and two vertices have label k. Now Theorem 2 implies the desired lower bound. $\Box$

An interesting consequence of Corollary 3 is the existence of quite dense graphs which are of order n and size $(1-o(1))\frac{5}{9}\frac{n^2}{2}$ but still admit an edge-bijective vertex labelling: Starting from an edge-surjective vertex-k-labelling of K_n, we delete all but one edge of weight w for every $w \in [2,2k]$.

For $n \leq 29$, we have determined $s(K_n)$ by exhaustive computer search (cf. Table 1). It turns out that the structure of each of the corresponding optimal sets of used labels resembles the construction in the proof of Theorem 2. In each case, there is a positive integer d such that both the d smallest and the d largest elements of the set form a complete set of remainders modulo d, while the remaining elements form an arithmetic progression with common difference d. Table 2 contains lower bounds on $s(K_n)$ which have been obtained by generating complete sets of remainders modulo d for $d \leq 27$ for which the smallest element of the arithmetic progression in such a construction is largest possible. The estimate $s(K_{59}) \geq 499$ for instance is obtained using the following symmetric set $R \cup P \cup R'$ of 57 vertex labels

$$\begin{aligned} R &:= \{1,2,3,4,5,6,9,15,17,24,27,31,33,42,45,46,47\} \\ P &:= \{63+17i \mid 0 \leq i \leq 22\} \\ R' &:= \{500-i \mid i \in R\} \end{aligned}$$

on 57 vertices of K_{59} and repeating the labels 1 and $k = 499$ on the remaining two vertices.

For $30 \leq n \leq 40$ the lower bounds correspond to optimal symmetric label sets, i.e. they would be the precise value of $s(K_n)$ provided the existence of optimal labellings using symmetric label sets.

i	1	2	3	4	5	6	7	8	9	10
$s(K_i)$		1	1	2	3	4	5	7	9	12
$s(K_{i+10})$	15	18	21	25	29	33	37	42	47	52
$s(K_{i+20})$	57	63	69	76	83	90	97	106	115	

Table 1 Exact values of $s(K_n)$ for $n \leq 29$.

i	1	2	3	4	5	6	7	8	9	10
$s(K_{i+20}) \geq$										124
$s(K_{i+30}) \geq$	133	142	151	160	169	180	191	202	213	224
$s(K_{i+40}) \geq$	235	246	257	269	281	294	307	322	337	352
$s(K_{i+50}) \geq$	367	382	397	414	431	448	465	482	499	516
$s(K_{i+60}) \geq$	533	550	567	584	601	619	637	657	677	697
$s(K_{i+70}) \geq$	717	737	759	781	803	825	847	869	891	913
$s(K_{i+80}) \geq$	935	960	985	1010	1035	1060	1085	1110	1135	1160
$s(K_{i+90}) \geq$	1185	1210	1235	1262	1289	1316	1343	1370	1397	1424

Table 2 Lower bounds on $s(K_n)$ for some $n \geq 30$.

One might conjecture that a graph G admits an edge-surjective vertex-k-labelling for all values of k which are at most $s(G)$. Contrary to this, our computer search showed that K_{28} has an edge-surjective vertex-106-labelling using the labels in

$$\{1,2,3,5,7,8,13,15,18,22,31,40,49,58,67,76,85,89,92,94,99,100,102,104,105,106\}$$

but that K_{28} has no edge-surjective vertex-105-labelling.

Figure 2 shows the lower bound on $\frac{4s(K_n)}{n^2}$ corresponding to Tables 1 and 2 and indicates the difference d used for the arithmetic progression.

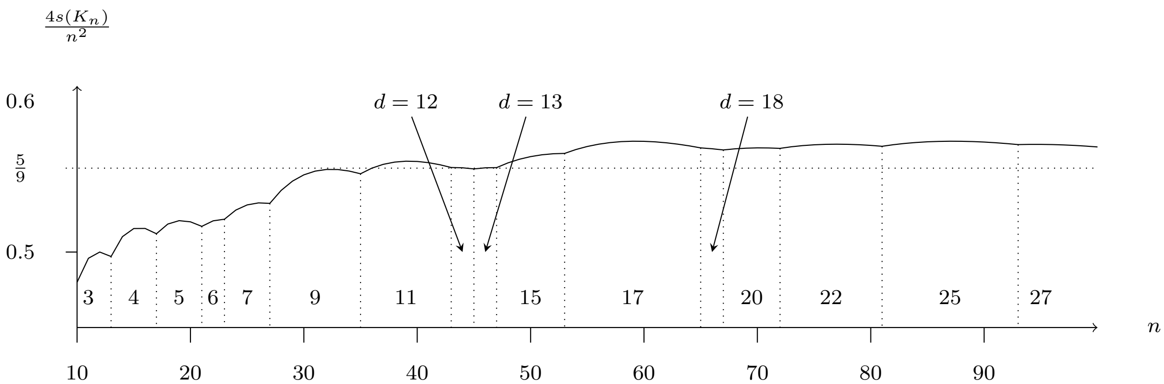

Figure 2 Lower bound on $\frac{4s(K_n)}{n^2}$.

3 Edge-injective vertex labellings and weak Sidon sets

The number theoretic notion related to edge-injective vertex labellings is that of a *weak Sidon set* or a *well-spread set* which is a set A of integers for which all $\binom{|A|}{2}$ pairwise sums $a+b$ for distinct $a, b \in A$ are different. Recall that for a usual *Sidon set* A all $\binom{|A|}{2} + |A|$ pairwise sums are required to be different. The study of Sidon sets goes back to Sidon [17] (cf. also [4, 14]).

Clearly, a weak Sidon set $A \subseteq \mathbb{N}$ gives rise to an edge-injective vertex-k-labelling of $K_{|A|}$ with $k = \max(A)$ and, conversely, an edge-injective vertex-k-labelling of K_n gives rise to a weak Sidon set $A \subseteq [1, k]$ with $|A| = n$. In view of this close correspondence it follows from known results on weak Sidon sets that

$$i(K_n)^{\frac{1}{2}} - i(K_n)^{\frac{21}{80}} + O(1) \leq n \leq i(K_n)^{\frac{1}{2}} + \sqrt{3}i(K_n)^{\frac{1}{4}} + O(1)$$

for every $n \in \mathbb{N}$ where the second inequality is due to Kayll [11] and the first inequality follows by combining a construction due to Singer [18] with prime density results (cf. [11]). Inverting these estimates, we obtain the following.

Corollary 4

$$n^2 - \left(2\sqrt{3} + o(1)\right) n^{\frac{3}{2}} \leq i(K_n) \leq n^2 + (1 + o(1))\, n^{\frac{61}{40}}$$

for every $n \in \mathbb{N}$, *i.e.* $i(K_n) = n^2 + o\left(n^2\right) = 2m + o(m)$ *for* $m = \binom{n}{2}$.

From Corollary 4 we can derive an upper bound for general graphs.

Corollary 5 *If $G=(V,E)$ is a graph of order n and size m, then*

$$i(G) \leq 4^{\frac{1}{3}} m^{\frac{4}{3}} + n + o\left(m^{\frac{4}{3}}\right).$$

Proof: We prove the existence of an edge-injective vertex-k-labelling $f: V \to [1,k]$ with $k = 4^{\frac{1}{3}} m^{\frac{4}{3}} + n + o\left(m^{\frac{4}{3}}\right)$ such that f itself is injective by an inductive argument. We may assume the "$o(.)$"-term to be positive and that the statement is true for small sizes. We consider two cases.

Case 1 *G has a vertex u of degree $d_G(u)$ less than $(4m)^{\frac{1}{3}}$.*

By induction, the graph $G' = G - u$ has an injective edge-injective vertex-k'-labelling f' with $k' \leq k$. From the possible k labels in $[1,k]$ exactly $n-1$ are forbidden for u, because we require f to be injective. Furthermore, each of the $m - d_G(u)$ edges of G' forbids at most $d_G(u)$ further labels from $[1,k]$ for u. Since

$$\begin{aligned}
& k-(n-1)-d_G(u)(m-d_G(u)) \\
\geq\; & 4^{\frac{1}{3}} m^{\frac{4}{3}} + n + o\left(m^{\frac{4}{3}}\right) - (n-1) - (4m)^{\frac{1}{3}}\left(m-(4m)^{\frac{1}{3}}\right) \\
=\; & 1 + o\left(m^{\frac{4}{3}}\right) + (4m)^{\frac{2}{3}} \\
>\; & 0
\end{aligned}$$

we can label u such that we obtain the desired labelling. (Note that, since f' is injective, all neighbours of u have distinct labels.)

Case 2 *G has no vertex of degree less than $(4m)^{\frac{1}{3}}$.*

In this case $n^2 \leq 4^{\frac{1}{3}} m^{\frac{4}{3}}$ and using Corollary 4 we obtain

$$i(G) \;\leq\; i(K_n) \leq n^2 + o(n^2) = 4^{\frac{1}{3}} m^{\frac{4}{3}} + o\left(m^{\frac{4}{3}}\right).$$

(Note that the constants implicit in the $o(\cdot)$-notation come from Corollary 4.) Since every edge-injective vertex labelling of K_n is necessarily injective, we obtain the existence of the desired labelling and the proof is complete. $\Box$

In view of the last two results, we pose the following conjecture.

Conjecture 6 *If $G=(V,E)$ is a graph of size m, then $i(G) \leq 2m$.*

For every prime power $q \geq 3$, Singer's construction [18] yields a Sidon set $A \subseteq [1, q^2+q]$ with $|A| = q+1$ which implies that $i(K_n) \leq 2\binom{n}{2}$ for $n = q+1$, i.e. Conjecture 6 is true for complete graphs of infinitely many orders.

4 Edge-bijective vertex labellings of trees

Our main result in this section states that for trees the necessary condition for the existence of an edge-bijective vertex labelling implied by (1) and (2) is already sufficient.

Theorem 7 *A tree $T = (V, E)$ of order n and maximum degree Δ admits an edge-bijective vertex-k-labelling if and only if n is even and $\Delta \leq k = \frac{n}{2}$.*

The proof of this result will rely on the following auxiliary statements. The next lemma is an easy folklore exercise.

Lemma 8 *Every tree T of order n has a vertex r such that every component of $T - r$ has order at most $\frac{n}{2}$.*

Lemma 9 *Every rooted tree T of size m contains a path P starting at the root r such that every component of $T - V(P)$ has at most $\frac{m}{2} - 1$ edges.*

Proof: Choose $P : r = v_0v_1v_2...v_p$ such that v_p is a leaf and for $1 \leq i \leq p$ the vertex v_i is a descendant of v_{i-1} for which the component of $T - v_{i-1}v_i$ containing v_i is largest possible. $\square$

Lemma 10 *Let $T = (V, E)$ be a rooted tree with m edges and root r. Let $1 \leq l \leq k$ and let $W = \{w_1, w_2, \ldots, w_m\}$ be a set of m weights satisfying*

(i) $l + 1 \leq w_1 < w_2 < \ldots < w_m \leq l + k$ *and*

(ii) $w_m - w_1 \leq \frac{k}{2}$.

There is a vertex-k-labelling $f : V \to [1, k]$ such that $f(r) = l$ and $W = \{f(u) + f(v) \mid uv \in E\}$, i.e. a partial vertex-$k$-labelling which assigns label l to the root can be extended to V such that all weights in W are realized.

Proof: We prove the statement by induction on m. For $m = 1$ the statements is clearly true. Now let $m \geq 2$. Let v be a descendant of r and let T_v and T_r be the two subtrees of $T - rv$ rooted at v and r, respectively. Let $l_1 = w_1 - l$ and $l_m = w_m - l$.

If $w_m \leq l_1 + k$, then $l_1 + 1 \leq w_1 < \ldots < w_m \leq l_1 + k$, we assign label l_1 to v and distribute the remaining weights $w_2, \ldots, w_m$ such that T_v receives $|E(T_v)|$ weights and T_r receives $|E(T_r)|$ weights. The existence of the desired labelling for T follows by induction.

If $l_m + 1 \leq w_1$, then we assign label l_m to v and proceed similarly as above by induction.

Hence, we may assume that $w_m \geq (l_1 + k) + 1 = w_1 - l + k + 1$ and $w_1 \leq (l_m + 1) - 1 = w_m - l$ which implies the contradiction $2(w_m - w_1) \geq k + 1$ and the proof is complete. $\square$

Proof of Theorem 7: Since the "only if"-part follows from (1) and (2), it only remains to prove the "if"-part.

Let the vertex r be as in Lemma 8, i.e. all components of $T - r$ have at most $\frac{n}{2}$ vertices. Let $T_1, T_2, \ldots, T_t$ be the components of $T - r$ such that the sizes $e_i = |E(T_i)|$ satisfy

$e_1 \geq e_2 \geq \ldots \geq e_t$. Root every T_i in the neighbour r_i of r in $V(T_i)$. Let $e_{\leq i} = e_1 + e_2 + \cdots + e_i$ for $1 \leq i \leq t$ and $e_{\leq 0} = 0$.

Let s be the largest possible index with $1 \leq s \leq t$ such that

$$e_{\leq s-1} \leq k - 2.$$

Note that

$$e_{\leq t} = e_1 + e_2 + \cdots + e_t = n - 1 - t = 2k - 1 - t \geq 2k - 1 - \Delta \geq k - 1$$

which implies that $s < t$ and $e_s > 0$.

We assign a label $f(v)$ to each of the roots $v \in \{r, r_1, \ldots, r_t\}$ and a weight set W_i to each of the trees $T_i \in \{T_1, \ldots, T_t\}$ using the Algorithm 1 below.

Since T has $2k - 1 = |W|$ edges, the sets $W_1, \ldots, W_t$ are well-defined by Algorithm 1.

Claim 1 *$k \in W_s$. Furthermore, if $w_s^+ > k$, then $e_s \geq 2$, $f(r_s) + 1 \leq w_s^-$ and*

$$W_s = [e_{\leq s-1} + 2, e_{\leq s} + s] \setminus [k + 1, k + (s - 1)].$$

Proof of Claim 1: The choice of s implies $e_{\leq s-1} \leq k - 2 < e_{\leq s}$. Furthermore, the definition of the sets $W_1, \ldots, W_{s-1}$ by Algorithm 1 implies

$$W_1 \cup \cdots \cup W_{s-1} = [2, e_{\leq s-1} + 1].$$

Hence, again by Algorithm 1, $k \in W_s$ and, if $w_s^+ > k$, then $e_s \geq 2$ and W_s is as stated in the claim. We obtain

$$\begin{aligned}
f(r_s) &= \left\lceil \frac{e_{\leq s} + s}{2} \right\rceil \\
&= \left\lceil \frac{(e_1 + 1) + (e_2 + 1) + \cdots + (e_{s-2} + 1) + (e_{s-1} + e_s + 2)}{2} \right\rceil \\
&= \left\lceil \frac{e_1 + 1}{2} + \frac{e_2 + 1}{2} + \cdots + \frac{e_{s-2} + 1}{2} + \frac{e_{s-1} + e_s + 2}{2} \right\rceil \\
&\leq \lceil e_1 + e_2 + \cdots + e_{s-2} + e_{s-1} + 1 \rceil \\
&= e_{\leq s-1} + 1 \\
&= w_s^- - 1 \\
&\leq k - 1
\end{aligned}$$

which completes the proof of the claim. □

Claim 2 *For $1 \leq i \leq t$ the label $f(r_i)$ assigned to r_i by Algorithm 1 is a well-defined integer between 1 and k.*

$f(r) := k$;
$W := [2, 2k]$;
for $i = 1$ **to** $s - 1$ **do**
  $W_i := [e_{\leq i-1} + 2, e_{\leq i} + 1]$;
  $f(r_i) := i$;
  $W := W \setminus (W_i \cup \{f(r_i) + k\})$;
end
$W_s := \{$ the e_s smallest elements of $W\}$;
$w_s^- := \min(W_s)$;
$w_s^+ := \max(W_s)$;
if $w_s^+ \leq k$ **then**
  $f(r_s) := s$;
else
  $f(r_s) := \left\lceil \frac{w_s^+}{2} \right\rceil$;
end
$W := W \setminus (W_s \cup \{f(r_s) + k\})$;
for $i = s + 1$ **to** t **do**
  $W_i := \{$ the e_i smallest elements of $W\}$;
  $w_i^- := \min(W_i)$;
  $w_i^+ := \max(W_i)$;
  $W := W \setminus W_i$;
  if $e_i \leq 1$ **then**
    $f(r_i) := \min(W) - k$;
  else
    $f(r_i) := \left\lceil \frac{w_i^+}{2} \right\rceil$;
  end
  $W := W \setminus \{k + f(r_i)\}$;
end

Algorithm 1 Assigning labels to $r, r_1, \ldots, r_t$ and weight sets to $T_1, \ldots, T_t$

Proof of Claim 2: This is trivially true for $1 \le i \le s-1$ and also for $i = s$ provided that $w_s^+ \le k$.

If $w_s^+ > k$, then Claim 1 implies $1 \le f(r_s) \le k-1$.

Now let $s+1 \le i \le t$. By the choice of s, we have $[2,k] \subseteq W_1 \cup \cdots \cup W_s$.

If $e_i \ge 2$, this implies $k+1 \le w_i^+ \le 2k$ and hence $1 \le f(r_i) \le k$.

If $e_i \le 1$, then at the moment of setting $f(r_i)$ in Algorithm 1, we have $|W| > 0$ and $k+1 \le \min(W) \le 2k$. Hence $1 \le f(r_i) \le k$ also in this case. $\Box$

If

$$W_i = [w_i^-, w_i^+] \setminus G_i$$

for some set

$$G_i \subseteq [w_i^- + 1, w_i^+ - 1],$$

then W_i is said to have $|G_i|$ gaps. Note that the gaps in the sets W_i are caused by the weights of the edges $rr_1, \ldots, rr_{i-1}$.

Claim 3 *If W_i has at least three gaps $g_1 < g_2 < g_3 \in G_i$ for some $s+1 \le i \le t$, then $g_3 - g_1 \ge e_i$.*

Proof of Claim 3: Clearly, $e_i \ge 2$. In view of Algorithm 1 this implies that there are three indices $s \le i_1 < i_2 < i_3 < i$ such that $g_j = f(r_{i_j}) + k = \left\lceil \frac{w_{i_j}^+}{2} \right\rceil$ for $j \in \{1,2,3\}$. (Note that if $i_1 = s$, then $w_s^+ > k$ and $f(r_s) = \left\lceil \frac{w_s^+}{2} \right\rceil$.) Since $w_{i_3}^+ - w_{i_1}^+ \ge e_{i_3-1} + e_{i_3} \ge 2e_{i_3} \ge 2e_i$, the desired result follows. $\Box$

Let $P_1, \ldots, P_t$ be paths in $T_1, \ldots, T_t$ starting at the roots $r_1, \ldots, r_t$ as described in Lemma 9 (cf. Figure 3).

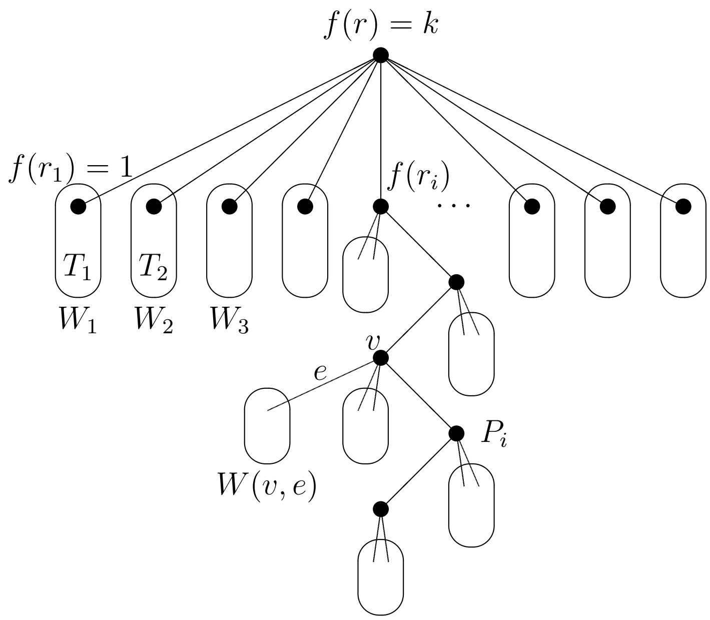

Figure 3

Claim 4 *It is possible to label the vertices in $V(P_i) \setminus \{r_i\}$ with integers between 1 and k such that if $w_i^+ \leq k$, then the edges of P_i obtain the $|E(P_i)|$ smallest weights in W_i increasingly starting from r_i, and if $w_i^+ > k$, then the edges of P_i obtain the $|E(P_i)|$ largest weights in W_i decreasingly starting from r_i.*

Proof of Claim 4: Note that $w_i^+ \leq k$ for all $i \leq s-1$ and $w_i^+ > k$ for all $i \geq s+1$.

First, we consider the case $e_i = 1$, i.e. P_i and T_i consist of exactly one edge $r_i v$. Let $W_i = \{w_i\}$. In this case the statement of the claim is trivial for $i \leq s$. If $i \geq s+1$, then by Algorithm 1, $f(r_i)+1 < k+2 \leq w_i < f(r_i)+k$ and the statement of the claim follows.

Next, we consider the case $e_i \geq 2$.

For $w_i^+ \leq k$ the set W_i contains no gaps and starting at r_i the vertices of P_i will receive the labels

$$f(r_i), w_i^- - f(r_i), f(r_i)+1, w_i^- - f(r_i)+1, f(r_i)+2, w_i^- - f(r_i)+2, \ldots.$$

Since in this case, $1 \leq f(r_i) < w_i^-$ and all weights in W_i are at most k, all assigned labels are between 1 and k and the edges of P_i obtain the desired weights.

Now let $w_i^+ > k$. If W_i contains no gaps, then starting at r_i the vertices of P_i will receive the labels

$$f(r_i) = \left\lceil \frac{w_i^+}{2} \right\rceil, \left\lfloor \frac{w_i^+}{2} \right\rfloor, \left\lceil \frac{w_i^+}{2} \right\rceil - 1, \left\lfloor \frac{w_i^+}{2} \right\rfloor - 1, \left\lceil \frac{w_i^+}{2} \right\rceil - 2, \left\lfloor \frac{w_i^+}{2} \right\rfloor - 2, \ldots,$$

i.e. the labels are non-increasing along P_i and the maximum difference of labels of adjacent vertices is 1. It is easy to see that, if W_i has $|G_i| \geq 1$ gaps, then the maximum difference of labels of adjacent vertices on P_i possibly increases to at most $|G_i|+1$. It remains to argue that all assigned labels are integers between 1 and k.

For $s+1 \leq i \leq t$ all weights in W_i are larger than $k+1$ and the first two labels are at most $f(r_i)$, which implies that all further labels lie between 1 and k as required.

For $i = s$ Claim 1 implies that W_s has exactly $s-1$ gaps and $w_s^- = e_{\leq s-1}+2$. Hence the smallest label assigned to a vertex of P_s is at least

$$\left\lceil \frac{e_{\leq s-1}+2}{2} - \frac{s}{2} \right\rceil \geq \left\lceil \frac{2(s-1)+2}{2} - \frac{s}{2} \right\rceil = \left\lceil \frac{s}{2} \right\rceil = 1$$

which completes the proof of the claim. □

Claim 5 *For every $1 \leq i \leq t$ it is possible to label the vertices in $V(T_i) \setminus V(P_i)$ with integers between 1 and k such that the edges in $E(T_i)$ obtain exactly the weights in W_i.*

Proof of Claim 5: Let $1 \leq i \leq t$. If $e_i \leq 1$, then all vertices in T_i have already been labelled and Claim 5 holds trivially. Hence we assume that $e_i \geq 2$.

For every vertex $v \in V(P_i)$ and every edge $e = vv' \in E(T_i) \setminus E(P_i)$ let $T(v,e) = (V(C) \cup \{v\}, E(C) \cup \{e\})$ where C is the component of $T_i - e$ containing v'. $T(v,e)$ can be considered a subtree of T_i rooted in the vertex v which is of degree 1 in $T(v,e)$.

To every such tree $T(v, e)$ we assign a set

$$W(v, e) \subseteq W_i \setminus \{f(u) + f(v) \mid uv \in E(P_i)\}$$

of weights such that $|W(v, e)|$ is exactly the size of $T(v, e)$ and the elements of $W(v, e)$ are consecutive elements of $W_i \setminus \{f(u) + f(v) \mid uv \in E(P_i)\}$. Note that by the choice of the path P_i according to Lemma 9, we have $|W(v, e)| \leq \frac{e_i}{2}$. It remains to argue that the trees $T(v, e)$ whose root v has already been assigned a label by Claim 4 together with the weight sets $W(v, e)$ satisfy the conditions (i) and (ii) from Lemma 10. Once this has been established the statement of Claim 5 follows immediately from that lemma.

For $w_i^+ \leq k$ the set W_i is a set of e_i consecutive integers without gaps. Since the weights of the edges of P_i are the smallest $|E(P_i)|$ elements in W_i and all weights in W_i are at most k, condition (i) from Lemma 10 trivially holds. Furthermore, since W_i has no gaps also the sets $W(v, e)$ have no gaps and

$$\max(W(v, e)) - \min(W(v, e)) \leq \frac{e_i}{2} - 1 \leq \frac{k}{2} - 1,$$

i.e. also condition (ii) from Lemma 10 holds.

If $i \geq s + 1$, then the weights of the edges of P_i are the largest $|E(P_i)|$ elements in W_i. Hence for every vertex v on P_i we obtain that all elements of $W_i \setminus \{f(u)+f(v) \mid uv \in E(P_i)\}$ are between $f(v) + 1 \leq k + 1$ and $f(v) + k$, i.e. condition (i) from Lemma 10 holds. Since $|W(v, s)| \leq \frac{e_i}{2} < e_i$, Claim 3 implies that none of the sets $W(v, s)$ contains more than 2 gaps. Since $i \geq 2$, we have $e_i \leq \frac{e_{\leq i}}{2} \leq \frac{(2k-1)-2}{2}$ and

$$\max(W(v, e)) - \min(W(v, e)) \leq \left\lfloor \frac{e_i}{2} + 2 - 1 \right\rfloor \leq \left\lfloor \frac{k}{2} + \frac{1}{4} \right\rfloor \leq \frac{k}{2},$$

i.e. also condition (ii) from Lemma 10 holds.

Finally, let $i = s$ and $w_s^+ > k$. By the choice of s, we have $e_s(s - 1) \leq k - 2$ which implies $s - 1 \leq \frac{k-2}{e_s} \leq \frac{k-2}{2}$. By Claim 4, the weights of the edges of P_i are the largest $|E(P_i)|$ elements in W_i. Since, by Claim 1, $f(r_s) + 1 \leq w_s^-$ and $w_s^+ \leq f(r_s) + k$, condition (i) from Lemma 10 holds. Furthermore, also by Claim 1,

$$w_s^+ - w_s^- = e_s + s - 2 \leq \frac{k-2}{s-1} + (s - 1) - 1.$$

The convex function $g(x) = \frac{k-2}{x-1} + (x - 1) - 1$ satisfies

$$\max_{3 \leq x \leq \frac{k-2}{2}} g(x) \leq \max\left\{g(3), g\left(\frac{k-2}{2}\right)\right\} = \frac{k}{2}.$$

Hence for $3 \leq s$, condition (ii) from Lemma 10 holds trivially. For $s = 2$ the set $W(v, e)$ has at most one gap and we obtain

$$\max(W(v, e)) - \min(W(v, e)) \leq \frac{e_i}{2} + 1 - 1 \leq \frac{k}{2}.$$

Hence also in this final case condition (ii) from Lemma 10 holds and the proof of the claim holds. $\Box$

As noted in the beginning of the proof of Claim 5, Lemma 10 finally implies the existence of an edge-bijective vertex-k-labelling of T and the proof is complete. $\Box$

Theorem 7 allows to derive best-possible result about edge-surjective and edge-injective vertex labelling of trees.

Corollary 11 *Let $T = (V, E)$ be a tree of order n and maximum degree Δ. T admits an edge-surjective vertex-k-labelling if and only if $k \leq \min\{\frac{n}{2}, n - \Delta\}$.*

Proof: Since the "only if"-part follows from (2), it only remains to prove the "if"-part. We will first prove the result in the case that n is even and $\Delta \leq \frac{n}{2}$. Then we consider the cases that n is odd and $\Delta \leq \frac{n}{2}$ and that $\Delta > \frac{n}{2}$. Let $k \leq \min\{\frac{n}{2}, n - \Delta\}$.

Claim 1 *If $\Delta = \frac{n}{2}$, then either T has a unique vertex of maximum degree which is adjacent to a leaf, or T has exactly two vertices of maximum degree which are both adjacent to a leaf.*

Proof of Claim 1: Since the statement is trivial for $n \leq 4$, we assume that $n \geq 6$.

If T has a unique vertex of maximum degree which is not adjacent to a leaf, then $1 + 2\Delta \leq n$, which is a contradiction.

If either T has exactly two vertices of maximum degree such that one is not adjacent to a leaf, or T has at least three vertices of maximum degree, then $3\Delta - 1 \leq n$. Hence $\Delta \leq \left\lfloor \frac{n+1}{3} \right\rfloor < \frac{n}{2}$, which is a contradiction. $\Box$

Claim 2 *If n is even and $\Delta \leq \frac{n}{2}$, then T admits an edge-surjective vertex-k-labelling.*

Proof of Claim 2: The proof is done by induction on $\min\{\frac{n}{2}, n - \Delta\} - k = \frac{n}{2} - k$. If $k = \frac{n}{2}$, then claim follows immediately from Theorem 7. Hence let $k < \frac{n}{2}$. By Claim 1, there are two leaves whose deletion results in a tree T' of maximum degree at most $\frac{n-2}{2}$. By induction, T' admits an edge-surjective vertex-k-labelling and hence also T. $\Box$

Claim 3 *If n is odd and $\Delta \leq \frac{n}{2}$, then T admits an edge-surjective vertex-k-labelling.*

Proof of Claim 3: Deleting a leaf results in a tree T' which admits an edge-surjective vertex-k-labelling by Claim 2. Hence so does T. $\Box$

Claim 4 *If $\Delta \geq \frac{n}{2}$, then T admits an edge-surjective vertex-k-labelling.*

Proof of Claim 4: The proof is done by induction on $\Delta - \frac{n}{2}$. If $\Delta = \frac{n}{2}$, then Claim 2 implies the result. Hence let $\Delta - \frac{n}{2} > 0$. It follows as in the proof of Claim 1 that T has a unique vertex of maximum degree which is adjacent to a leaf u. Deleting u from T results in a tree for which the difference between the maximum degree and half the order is smaller and the result follows by induction. $\Box$

The above claims clearly imply the desired result. $\Box$

Similar arguments as in the proof of Corollary 11 yield the following result.

Corollary 12 *Let $T = (V, E)$ be a tree of order n and maximum degree Δ. T admits an edge-injective vertex-k-labelling if and only if $k \geq \max\{\frac{n}{2}, \Delta\}$.*

Proof: Since the "only if"-part follows from (1), it only remains to prove the "if"-part. If $\Delta \leq \frac{n}{2}$ and n is even, then this result follows immediately from Theorem 7. For odd n we simply add a vertex and an edge joining it to a leaf and apply Theorem 7 to the resulting graph. If $\Delta > \frac{n}{2}$, then we can delete leaves incident with the unique maximum degree vertex v until we reach a tree T' with n' vertices and $k' := \Delta(T') = \frac{n'}{2}$. Applying Theorem 7 once more, we obtain an edge-bijective vertex-k'-labelling of T'. The labelling procedure described in the proof of Theorem 7 assigns label k' to v and, since $k - k' \geq n - n'$, we can label the remaining vertices of T with labels $k' + 1, \ldots, k' + n - n'$. $\Box$

The last two corollaries also imply that for trees (1) and (2) are satisfied with equality.

5 Edge-irregular total labellings of sparse graphs

An *edge-irregular total-k-labelling* of a graph $G = (V, E)$ is a labelling $f : V \cup E \to [1, k]$ of its vertices and edges with integers between 1 and k such that

$$f(u) + f(uv) + f(v) \neq f(u') + f(u'v') + f(v')$$

for all pairs of distinct edges $uv, u'v' \in E$. These labellings and the *total edge irregularity strength* $tes(G)$ which is defined as the minimum k for which G admits an edge-irregular total-k-labelling were recently introduced by Bača et al. [1]. Similar arguments as in the introduction lead to two natural lower bounds on $tes(G)$ in terms of the size m and the maximum degree Δ of G:

$$tes(G) \geq \max\left\{\left\lceil\frac{m+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}.$$

Ivančo and Jendrol' [9] posed the surprising conjecture that K_5 is the only graph which does not satisfy the last inequality with equality. Their conjecture has been verified for several special classes of graphs such as trees [9], complete and complete bipartite graphs [10] and it is known to be true if either $m \leq 3\Delta/2$ [2] or $m \geq 10^6\Delta$ [3]. Our intention in the present section is to use the results of the previous section in order to establish the conjecture for sparse enough graphs.

Let $G = (V, E)$ be a graph and let $f : V \to [1, k]$ be a vertex-k-labelling. Whether f can be extended to an edge-irregular total-k-labelling of G is essentially a matching problem in the bipartite graph — already considered in [10] —

$$BIB(G, f)$$

with partite sets E and $[3, 3k]$ where an edge $e = uv \in E$ is adjacent exactly to $[f(u) + f(v) + 1, f(u) + f(v) + k]$.

Proposition 13 *If G, f and $BIB(G, f)$ are as above, then there is an edge-irregular total-k-labeling f' such that $f' \mid_V = f$ if and only if $BIB(G, f)$ has a matching of size $|E|$.*

Proof: This equivalence follows immediate from the 1-to-1-correspondence between $f' \mid_E$ and a matching of size $|E|$ in $BIB(G, f)$. □

As shown in the next result, a natural hypothesis ensuring the Hall-condition and hence the existence of a matching as described in Proposition 13, is that the vertex-k-labelling is edge-surjective.

Corollary 14 *Let $G = (V, E)$ be a graph of order n and size m. Let $k = \lceil \frac{m+2}{3} \rceil$.*

If G admits an edge-surjective vertex-k-labelling f, then G admits an edge-irregular total-k-labelling.

Proof: We will verify the Hall-condition for the graph $BIB(G, f)$ as in Proposition 13. Therefore, let $\emptyset \neq S \subseteq E$ and let $N(S)$ denote the neighbourhood of S in $BIB(G, f)$.

If $1 \leq |S| \leq k$, then $|N(S)| \geq k \geq |S|$ by the construction of $BIB(G, f)$.

If $|S| \geq k+1$, then there are at least $(2k-1)-(m-|S|) \geq (2k-1)-((3k-2)-|S|) = |S|-k+1$ distinct partial edge weights in $\{f(u)+f(v) \mid uv \in S\}$. Hence, by the construction of $BIB(G, f)$, $|N(S)| \geq |S| - k + 1 + k - 1 = |S|$.

Therefore, the Hall-condition is satisfied, $BIB(G, f)$ has a matching of size $|E|$ and, by Proposition 13, G admits an edge-irregular total-k-labelling. □

After these preparations we proceed to our main result in this section.

Corollary 15 *Let $G = (V, E)$ be a connected graph of order n, size m and maximum degree Δ. Let $k = \lceil \frac{m+2}{3} \rceil$.*

If $m \leq \frac{3}{2}n - 1$ and $\Delta \leq k - 1$, then G admits an edge-irregular total-k-labelling.

Proof: We consider the three cases $m = 3k - 2$, $m = 3k - 3$ and $m = 3k - 4$ separately.

First let $m = 3k - 2$ which implies $2k \leq n$. Applying Theorem 7 to a subtree of G of order $2k$ yields the existence of an edge-surjective vertex-k-labelling of G and Corollary 14 implies the existence of an edge-irregular total-k-labelling of G.

Now let $m = 3k - 3$ which implies that $2k - 1 \leq n$. If $2k \leq n$, then we argue as before. Hence we may assume that $2k - 1 = n$. Applying Corollary 12 to a spanning tree T of G yields the existence of an edge-injective vertex-k-labelling f of T, i.e. all $n - 1$ edges of T have different partial edge weights induced by f.

Let $BIB(G, f)$, S and $N(S)$ be as in the proof of Corollary 14. If $|S| \leq k$, then $|N(S)| \geq |S|$. If $|S| \geq k + 1$, then at least $|S| - (m - (n - 1)) = |S| - k + 1$ many edges in S have different partial edge weights and $|N(S)| \geq (|S| - k + 1) + (k - 1) = |S|$. Hence $BIB(G, f)$ satisfies the Hall-condition and, by Proposition 13, G admits an edge-irregular total-k-labelling.

Finally, let $m = 3k - 4$ which implies $2k - 2 \leq n$. Applying Corollary 12 to a subtree T' of G of order $2(k - 1)$ yields the existence of an edge-injective vertex-$(k - 1)$-labelling f of T'. As before, $m - (2(k - 1) - 1) = k - 1$ and we can argue as above which completes the proof. $\square$

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