# Note on bipartite graph tilings

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**Abstract**

Let $s < t$ be two fixed positive integers. We study sufficient minimum degree conditions for a bipartite graph $G$, with both color classes of size $n = k(s+t)$, which ensure that $G$ has a $K_{s,t}$-factor. Our result extends the work of Zhao, who determined the minimum degree threshold which guarantees that a bipartite graph has a $K_{s,s}$-factor.

## 1 Introduction

For two (finite, loopless, simple) graphs $H$ and $G$, we say that $G$ contains an *$H$-factor* if there exist $v(G)/v(H)$ vertex-disjoint copies of $H$ in $G$. As a synonym, we say that there exists an *$H$-tiling of $G$*. Obviously, if $G$ contains an $H$-factor, then $v(G)$ is a multiple of $v(H)$. For a fixed graph $H$, necessary and sufficient conditions on the minimum-degree of $G$ which guarantee that $G$ contains an $H$-factor were studied extensively. Results in this spirit include the Tutte 1-factor Theorem (see [7]), the Hajnal-Szemerédi Theorem [4], and series of improving results by Alon and Yuster [1, 2], Komlós [5], Zhao and Shokoufandeh [8], and by Kühn and Osthus [6]. In [6] the answer to the problem is settled (up to a constant) for any $H$. It was shown that the relevant parameters are the chromatic number and the critical chromatic number of $H$.

The additional information that $G$ is $r$-partite might help to decrease the minimum-degree threshold for $G$ containing an $H$-factor. The conjectured $r$-partite version of the Hajnal-Szemerédi Theorem [3] is such an example. (Recently a proof of the approximate version of the $r$-partite Hajnal-Szemerédi Theorem was announced by Csaba.) In this paper we determine the threshold for the minimum-degree of a balanced bipartite graph $G$ which guarantees that $G$ contains a $K_{s,t}$-factor, for arbitrary integers $s < t$. If the cardinalities of both color classes of $G$ are $n$, a necessary condition for $G$ having a $K_{s,t}$-factor is that $n$ is a multiple of $s+t$. The sufficient minimum-degree condition is given in Theorem 2, and a matching lower bound is provided in Theorem 3. Our work can be seen as an extension of the work of Zhao [9], who investigated the case $s = t$.

**Theorem 1** (Zhao, [9])**.** *For each $s \geq 2$ there exists a number $k_0$ such that if $G = (A, B; E)$ is a bipartite graph, $|A| = |B| = n = ks$, where $k > k_0$, and*

$$\delta(G) \geq \begin{cases} \frac{n}{2} + s - 1 & \text{if } k \text{ is even,} \\ \frac{n+3s}{2} - 2 & \text{if } k \text{ is odd,} \end{cases}$$

*then $G$ has a $K_{s,s}$-factor.*

Moreover, Zhao showed that the bounds in Theorem 1 are tight. We extend those results to $K_{s,t}$-factors with $s < t$.

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**Theorem 2.** *Let $1 \le s < t$ be fixed integers. There exists a number $k_0 \in \mathbb{N}$ such that if $G = (A, B; E)$ is a bipartite graph, $|A| = |B| = n = k(s+t)$, with $k > k_0$, and*

$$\delta(G) \ge \begin{cases} \frac{n}{2} + s - 1 & \text{if } k \text{ is even,} \\ \frac{n+t+s}{2} - 1 & \text{if } k \text{ is odd,} \end{cases}$$

*then $G$ has a $K_{s,t}$-factor.*

On the other hand, we show that these bounds are best possible.

**Theorem 3.** *Let $1 \le s < t$ be fixed integers. There exists a number $k_0 \in \mathbb{N}$ such that for every $k > k_0$ there exists a bipartite graph $G = (A, B; E)$, $|A| = |B| = k(s+t) = n$, such that*

$$\delta(G) = \begin{cases} \frac{n}{2} + s - 2 & \text{if } k \text{ is even,} \\ \frac{n+t+s}{2} - 2 & \text{if } k \text{ is odd and } t \le 2s+1, \end{cases}$$

*and $G$ does not have a $K_{s,t}$-factor.*

The bounds in Theorem 2 and 3 exhibit a somewhat surprising phenomenon: for the case when $k$ is even the bound is independent of the value $t$, while for the case $k$ is odd, the minimum-degree condition depends on $t$. Moreover, we note that our results are not tight for the case $t > 2s + 1$ and $k$ odd. We are very grateful to Andrzej Czygrinow and Louis DeBiasio for drawing our attention to an oversight in Theorem 3 in an earlier version of this note.

# 2 Lower bound

In this section we prove Theorem 3. We treat three cases (based on the parity of $k$ and on the relation between $s$ and $t$) separately. The proof of Theorem 3 is constructive, i.e., we will construct a graph $G$ with the demanded minimum-degree and then argue that $G$ does not contain a $K_{s,t}$-factor.

The building blocks of our constructions are the graphs $P(m, p)$, where $m, p \in \mathbb{N}$. The graphs $P(m, p)$ were introduced in [9]. We just state their properties, which will be used throughout this section.

**Lemma 4.** *For any $p \in \mathbb{N}$ there exists a number $m_0$ such that for any $m \in \mathbb{N}, m > m_0$ there exists a bipartite graph $P(m, p) = (P_1, P_2; E_P)$ satisfying*

- *$|P_1| = |P_2| = m$,*
- *$P(m, p)$ is $p$-regular, and*
- *$P(m, p)$ does not contain a copy of $K_{2,2}$.*

In all constructions we assume that $n$ is large enough.

## 2.1 Case $k$ is even

For two integers $m$ and $q$ we write $Q(m, q)$ to denote (any of possibly many) bipartite graph $Q(m, q) = (Q_1, Q_2; E_Q)$ with the following properties:

- $|Q_1| = m, |Q_2| = m - 2$,
- $Q(m, q)$ does not contain any $K_{2,2}$,
- $\deg(x) \in \{q - 1, q\}$ for any vertex $x \in Q_1$, and
- $\deg(y) = q$ for any vertex $y \in Q_2$.

Such graphs $Q(m, q)$ do exist for fixed $q$ and large $m$. One way to construct them is by taking the graph $P(m, q) = (P_1, P_2; E_P)$ from Lemma 4, selecting two vertices $w_1, w_2 \in P_2$ such that they do not share a common neighbor in $P_1$, and then take $Q(m, q)$ to be the subgraph of $P(m, q)$ induced by the vertex sets $P_1$, $P_2 \setminus \{w_1, w_2\}$. In particular, the graph $Q(m, 0)$ is the empty graph.

Now we describe the construction of the graph $G$. Partition $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_1| = \frac{n}{2} + 1$, $|A_2| = |B_2| = \frac{n}{2} - 1$. The graph $G$ is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$, and
- $G[A_1, B_2] \cong G[B_1, A_2] \cong Q(n/2 + 1, s - 1)$.

We have $\delta(G) = \frac{n}{2} + s - 2$. The fact that there exists no $K_{s,t}$-factor is implied immediately by the fact that there is no subgraph isomorphic to $K_{s,t}$ whose vertices would touch both $A_1$ and $B_2$, or $A_2$ and $B_1$.

### 2.2 Case $k$ is odd, $2s + 1 \geq t > s + 1$

Let $k = 2l + 1$, $n = k(s + t)$. Note that $\frac{n-t+s+2}{2}$ is an integer. Partition $A = A_1 + A_2 + A_*$, $B = B_1 + B_2 + B_*$, $|A_1| = |A_2| = |B_1| = |B_2| = \frac{n-t+s+2}{2}$, $|A_*| = |B_*| = t - s - 2$. The graph $G$ is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$,
- $G[A_*, B_i]$ and $G[B_*, A_i]$ are complete bipartite graphs for $i = 1, 2$,
- $G[A_1, B_2] \cong G[A_2, B_1] \cong P(\frac{n-t+s+2}{2}, s - 1)$,
- the graph $G[A_*, B_*]$ is empty.

We have $\delta(G) = \frac{n+t+s}{2} - 2$. To see that $G$ does not have a $K_{s,t}$-factor, we argue as follows. Suppose for contradiction that $G$ has a $K_{s,t}$-factor. Fix a $K_{s,t}$-factor of $G$. First, observe that there cannot be a copy isomorphic to $K_{s,t}$ intersecting both $A_1 \cup B_1$ and $A_2 \cup B_2$. Let $r_1$ and $r_2$ be the number of copies of $K_{s,t}$ in the tiling whose color class of size $t$ touches $A_1$ and $B_1$, respectively. Let $A_c$ and $B_c$ be vertices covered by these $r_1 + r_2$ copies. It holds

$$A_1 \subset A_c \subset A_1 \cup A_* \quad \text{and} \quad B_1 \subset B_c \subset B_1 \cup B_* \,. \tag{1}$$

If $r_1 \neq r_2$ then $||A_c| - |B_c|| \geq t - s$, which contradicts (1). Thus, $r_1 = r_2$. We conclude that

$$\frac{l(s+t) + s + 1}{s + t} \leq r_1 \leq \frac{l(s+t) + t - 1}{s + t},$$

a contradiction to the integrality of $r_1$.

### 2.3 Case $k$ is odd, $t = s + 1$

By $R(m, q)$ we denote (any of possibly many) bipartite graph $R(m, q) = (R_1, R_2; E_R)$ with the following properties:

- $|R_1| = m, |R_2| = m - 1$,
- $R(m, q)$ does not contain any $K_{2,2}$,
- for any vertex $x$ in $R_1$, it holds $\deg(x) \in \{q - 1, q\}$, and
- for any vertex $y$ in $R_2$, it holds $\deg(y) = q$.

For fixed $q$ and large $m$ the existence of such a graph $R(m, q)$ follows by a construction analogous to the construction of the graph $Q(m, q)$.

Let $k = 2l + 1$. Partition $A = A_1 + A_2$, $B = B_1 + B_2$, $|A_1| = |B_1| = l(s + t) + s$, $|A_2| = |B_2| = l(s + t) + s + 1$. The graph $G$ is described by

- $G[A_i, B_i]$ is a complete bipartite graph for $i = 1, 2$,
- $G[B_2, A_1] \cong G[A_2, B_1] \cong R((n + 1)/2, s - 1)$.

One immediately sees that $\delta(G) = \frac{n+t+s}{2} - 2$ and no $K_{s,t}$-tiling of $G$ exists.

## 3 Upper bound

We prove Theorem 2 in this section. The proof of Theorem 2 utilizes the previous work of Zhao [9]. We will need the following lemma, which allows us to find many vertex disjoint copies of certain stars. For $h \in \mathbb{N}$, an $h$-star is a graph $K_{1,h}$, its *center* is the unique vertex in the part of size one. Moreover, for a graph $G$ and two disjoint sets $A, B \subset V(G)$ we define

$$\delta(A,B) = \min\{\deg(v,B) \, : \, v \in A\} \, , \quad \Delta(A,B) = \max\{\deg(v,B) \, : \, v \in A\}$$

and

$$d(A,B) = \frac{e(A,B)}{|A||B|} \, .$$

**Lemma 5** (Zhao, [9])**.** *Let $1 \le h \le \delta \le M$ and $0 < c < 1/(6h+7)$. Suppose that $H = (U_1, U_2; E_H)$ is a bipartite graph such that $||U_i| - M| \le cM$ for $i = 1, 2$. If $\delta = \delta(U_1, U_2) \le cM$ and $\Delta = \Delta(V_2, V_1) \le cM$, then we can find a family of vertex-disjoint $h$-stars, $2(\delta - h + 1)$ of which have centers in $U_1$ and $2(\delta - h + 1)$ of which have centers in $U_2$.*

As in [9] we distinguish between an extremal and a non-extremal case. If we find a $K_{s+t,s+t}$-factor in $G$ we are done, as each copy of $K_{s+t,s+t}$ can be split into two copies of $K_{s,t}$ and hence we have a $K_{s,t}$-factor. Thus the theorem stated next is just a corollary of [9, Theorem 4.1].

**Theorem 6** (Zhao, [9])**.** *For every $\alpha > 0$ and positive integers $s < t$, there exist $\beta > 0$ and a positive integer $k_0$ such that the following holds for all $n = k(s+t)$ with $k > k_0$. Given a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$, if $\delta(G) > (\frac{1}{2} - \beta)n$, then either $G$ contains a $K_{s,t}$-factor, or there exist*

$$A_1 \subset A, \quad B_1 \subset B \quad \textit{such that} \quad |A_1| = |B_1| = \lfloor n/2 \rfloor, \quad d(A_1, B_1) < \alpha.$$

Therefore, we reduce the problem to the extremal case. Let $\alpha = \alpha(t) > 0$ be small. As in the proof of Theorem 11 in [9], define

$$\begin{aligned} A_1' &= \left\{ x \in A : \deg(x, B_1) < \alpha^{\frac{1}{3}} \tfrac{n}{2} \right\}, & B_1' &= \left\{ x \in B : \deg(x, A_1) < \alpha^{\frac{1}{3}} \tfrac{n}{2} \right\}, \\ A_2' &= \left\{ x \in A : \deg(x, B_1) > (1 - \alpha^{\frac{1}{3}}) \tfrac{n}{2} \right\}, & B_2' &= \left\{ x \in B : \deg(x, A_1) > (1 - \alpha^{\frac{1}{3}}) \tfrac{n}{2} \right\}, \\ A_0 &= A - A_1' - A_2', & B_0 &= B - B_1' - B_2', \\ G_1 &= G[A_1', B_1'], & G_2 &= G[A_2', B_2']. \end{aligned}$$

Similarly as in the proof of Theorem 11 in [9], we assume that removing any edge from $G$ would violate the minimum-degree condition and then change $A_i'$ and $B_i'$ a little so that $\Delta(G_1), \Delta(G_2) < \alpha^{\frac{1}{9}} n$. Vertices in $A_0 \cup B_0$ are called *special*.

### 3.1 $k$ is even

To exhibit the existence of a tiling in this case, it is sufficient to translate carefully the proof of Case I of Theorem 11 from [9]. We give a sketch of the proof below and refer the reader to the corresponding places in [9] for more details.

Set $\mathcal{V} = (A_1', B_1', A_2', B_2')$. First assume, that no member of $\mathcal{V}$ contains more than $n/2$ vertices. We add vertices from $A_0$ and $B_0$ into sets of $\mathcal{V}$ in such a way, that every set has size exactly $n/2$. Then, we may apply arguments used in [9], based on Hall's Marriage Theorem, to find a $K_{s+t,s+t}$ tiling.

Next, assume that there is only one set in $\mathcal{V}$ which has more than $n/2$ elements. Without loss of generality, assume that it is $A_2'$, i.e., $|A_2'| = c > n/2$. Lemma 5 applied to the graph $G[A_2', B_2']$ yields the existence of $c - n/2$ disjoint $s$-stars with centers in $A_2'$. We move the centers of the stars into $A_1'$ and extend each of the stars into a copy of $K_{s,t}$ (each of these copies lies entirely in $A_1' \cup B_2'$, with the color class of size $s$ being contained in $B_2'$). We distribute vertices of $B_0$ into $B_1'$ and $B_2'$ so, that $|B_1'| = |B_2'| = n/2$. Then, it is easy to finish the entire tiling. This is done in three steps. In the first step, we find in an arbitrary manner $c - n/2$ copies of $K_{s,t}$ (disjoint with the previous ones) in $G[A_1', B_2']$ placed in such a way, that the color-class of size $s$ lies in $A_1'$. This step ensures us, that the cardinalities of untiled (i.e., those vertices which are not covered by the partial $K_{s,t}$-factor) vertices in the both color-classes of $G[A_1', B_2']$ are equal and divisible by $s + t$. In the second step, all yet untiled vertices of $G[A'_1, B'_2]$ which were originally special vertices are tiled. In the third step, the tiling is in an analogous manner defined for $G[A'_2, B'_1]$.

Now, assume that two diagonal sets of $\mathcal{V}$, say $A'_2$ and $B'_1$ have sizes more than $n/2$. Then we apply separately Lemma 5 to $G[A'_2, B'_2]$ and $G[A'_1, B'_1]$ to obtain families $\mathcal{S}_A$ and $\mathcal{S}_B$ of disjoint $s$-stars with centers in $A'_2$ and $B'_1$, such that $|A'_2| - |\mathcal{S}_A| = |B'_1| - |\mathcal{S}_B| = n/2$. We move the centers of the stars to $A'_1$ and $B'_2$ and proceed as in the previous case.

The remaining case is when two non-diagonal sets from $\mathcal{V}$ have size more than $n/2$. Assume these are $A'_2$ and $B'_1$. We apply Lemma 5 to the graph $G[A'_2, B'_2]$ to obtain families $\mathcal{S}_A, \mathcal{S}_B$ of disjoint $s$-stars with centers in $A'_2$ and $B'_2$, such that $|A'_2| - |\mathcal{S}_A| = |B'_2| - |\mathcal{S}_B| = n/2$. We proceed as in the previous cases.

## 3.2 $k$ is odd

Let $k = 2l + 1$. We say that a set of special vertices ($A_0$ and/or $B_0$) is *small* if its size is less than $t - s$. Otherwise, it is called *big*.

We distinguish four cases.

- *Both $A_0$ and $B_0$ are small.* Then there exist $i, j \in \{1, 2\}$, such that $|A'_i|, |B'_j| \geq l(s + t) + s + 1$. If $i = j$, then we apply Lemma 5 to the graph $G_i$ and find families $\mathcal{S}_A$, $\mathcal{S}_B$ of pairwise disjoint $s$-stars with centers in $A'_i$ and $B'_i$ respectively, so that $|A'_i| - |\mathcal{S}_A| = |B'_i| - |\mathcal{S}_B| = l(s + t) + s$. Move the centers of the stars in $A'_{3-i}$ and $B'_{3-i}$. After the changes we shall tile two graphs: $G[A'_1, B'_2]$ and $G[A'_2, B'_1]$. Note, that both those graphs are not balanced. The tiling procedure is analogous to the previous cases (when $k$ is even); the only difference is that one copy of $K_{s,t}$ has to be found in the graphs first to make each of them balanced.

  If $i \neq j$, we can assume that $|A'_j|, |B'_i| \leq l(s + t) + s$. Since if this does not hold, then we could change one index and continue as in the case when $i = j$. We will show that one can add vertices to $A'_j$ and to $B'_i$ so that $|A'_j| = l(s + t) + s$ and $|B'_i| = l(s + t) + t$. Then, the existence of the tiling will follow by standard arguments. We apply Lemma 5 to the graph $G_j$ to obtain a family of $|B'_j| - (l(s + t) + s)$ vertex disjoint $s$-stars with centers in $B'_j$ and end-vertices in $A'_j$. If we moved all the centers to $B'_i$ and all the vertices of $B_0$, the cardinality of $B'_i$ would be

  $$|B'_i| + (|B'_j| - (l(s + t) + s)) + |B_0| = l(s + t) + t \; .$$

  The same applies for $A'_j$. Therefore, by removing some of the vertices, we may attain $|A'_j| = l(s + t) + s$ and $|B'_i| = l(s + t) + t$. Then, the existence of a tiling follows.

- *$A_0$ is small and $B_0$ is big.* Then at least one $B'_i$ (say $B'_2$) has at most $l(s+t)+s$ vertices. Lemma 5 asserts that we can find a family $\mathcal{S}_B$ of disjoint $s$-stars with centers in $B'_1$ and end-vertices in $A'_1$, such that $|B'_1| - |\mathcal{S}_B| \leq l(s + t) + s$. This implies, that we can find vertices (in $B_0$ or centers of the stars of $\mathcal{S}_B$) which can be moved to $B'_2$ so that $|B'_2| = l(s + t) + t$.

  As $A_0$ is small, one of $A'_1$ and $A'_2$ must have at least $l(s+t)+s+1$ vertices. The tiling can be found by standard arguments if we achieve to have $|A'_1| = l(s + t) + s$. If $|A'_1| > l(s + t) + s$, Lemma 5 yields existence of a family $\mathcal{S}_A$ of disjoint $s$-stars with centers in $A'_1$ and end-vertices in $B'_1$ such that $|A'_1| - |\mathcal{S}_A| = l(s+t)+s$. Moving the centers to $A'_2$, we achieve $|A'_1| = l(s+t)+s$. Assume that $|A'_1| \leq l(s+t)+s$. The size of $A'_2$ is $k(s+t) - |A'_1| - |A_0| > l(s+t)+s$. Lemma 5 yields existence of a family $\mathcal{S}_A$ of disjoint $s$-stars in $G_2$ centered in $A'_2$ with the property that $|A'_1| + |\mathcal{S}_A| = l(s+t)+s$. Moving the centers to $A'_1$ yields demanded $A'_1 = l(s + t) + s$.

- *$A_0$ is big and $B_0$ is small.* The analysis of this case is analogous to the previous one.

- *Both $A_0$ and $B_0$ are big.* We shall show in the next paragraph, that we can achieve $A'_1$ to be of size $l(s + t) + s$ and of size $l(s + t) + t$. An analogous procedure can be used to show the same property for the set $B'_1$. Then, the existence of the tiling follows immediately; one prescribes the cardinalities of $A'_1$ and $B'_1$ to be equal to the same number $l(s + t) + s$.

  If $|A'_i \cup A_0| < l(s + t) + t$ for some $i \in \{1, 2\}$, then we have $|A'_{3-i}| > l(s + t) + s$. Appealing to Lemma 5 we can remove centers of $s$-stars from $A'_{3-i}$ in such a way that $|A'_{3-i}| = l(s + t) + s$.

Also, by moving $t - s$ vertices from the big set $A_0$ to $A'_{3-i}$ arrive at $|A'_{3-i}| = l(s+t)+t$. Then, the partial $K_{s,t}$-tiling can be extended to a $K_{s,t}$-factor.

Finally, if both $|A'_1| \le l(s+t)+s$ and $|A'_2| \le l(s+t)+s$ then we redistribute some vertices (again, appealing to Lemma 5, and using the set $A_0$) to arrive at the situation when $|A'_1| = l(s+t)+s$, $|A'_2| = l(s+t)+t$. Then the tiling can be extended as before.

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