# DISCONTINUOUS GALERKIN FINITE ELEMENT APPROXIMATION OF THE CAHN-HILLIARD EQUATION WITH CONVECTION 

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#### Abstract

The paper is concerned with the construction and convergence analysis of a discontinuous Galerkin finite element method for the Cahn-Hilliard equation with convection. Using discontinuous piecewise polynomials of degree $p \geq 1$ and backward Euler discretization in time, we show that the order-parameter $c$ is approximated in the broken $\mathrm{L}^{\infty}\left(\mathrm{H}^{1}\right)$ norm with optimal order, $\mathcal{O}\left(h^{p}+\tau\right)$; the associated chemical potential $w=\Phi^{\prime}(c)-\gamma^{2} \Delta c$ is shown to be approximated with optimal order, $\mathcal{O}\left(h^{p}+\tau\right)$, in the broken $\mathrm{L}^{2}\left(\mathrm{H}^{1}\right)$ norm. Here $\Phi(c)=\frac{1}{4}\left(1-c^{2}\right)^{2}$ is a quartic free-energy function and $\gamma>0$ is an interface parameter. Numerical results are presented with polynomials of degree $p=1,2,3$.


Key words. Cahn-Hilliard equation, discontinuous Galerkin finite element method, convergence, error analysis.

1. Introduction. This paper is devoted to the discontinuous Galerkin finite element approximation of an initial-boundary-value problem for the Cahn-Hilliard equation with a convection term, stated as follows:
(R) Find real-valued functions $c$ and $w$ defined on $\Omega \times[0, T]$, where $T>0$, such that

$$
\begin{align*}
\partial_{t} c-\frac{1}{\mathrm{Pe}} \Delta w+\nabla \cdot(\mathbf{u} c)=0 & \text { in } \Omega_{T}:=\Omega \times(0, T],  \tag{1.1a}\\
w=\Phi^{\prime}(c)-\gamma^{2} \Delta c & \text { in } \Omega_{T},  \tag{1.1b}\\
c(\cdot, 0)=c_{0}(\cdot) & \text { in } \Omega,  \tag{1.1c}\\
\partial_{\mathbf{n}} c=\partial_{\mathbf{n}} w=0 & \text { on } \partial \Omega_{T}:=\partial \Omega \times(0, T] . \tag{1.1d}
\end{align*}
$$

Here $\Omega$ is a bounded convex polygonal domain in $\mathbb{R}^{2}$, with boundary $\partial \Omega$ that has an outward-pointing unit normal $\mathbf{n}$. The order-parameter $c$ is such that $c(x, t) \approx 1$ (respectively $c(x, t) \approx-1$ ) if, and only if, at time $t \in[0, T]$ fluid 1 (respectively fluid 2 ) is present at the point $x \in \Omega$. Finally, $\mathbf{u} \in \mathrm{H}(\operatorname{div}, \Omega) \cap[\mathrm{C}(\bar{\Omega})]^{2}$ is a given function such that $\nabla \cdot \mathbf{u}=0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n}=0$ on $\partial \Omega$. Here $\mathrm{H}(\operatorname{div}, \Omega):=\left\{\mathbf{v} \in\left[\mathrm{L}^{2}(\Omega)\right]^{2}\right.$ : $\left.\nabla \cdot \mathbf{v} \in \mathrm{L}^{2}(\Omega)\right\}$.

The interface parameter $\gamma>0$ is a given constant that is assumed to be small, typically in the range $10^{-3}-10^{-2}$. We take the free-energy $\Phi(\cdot)$ in (1.1b) to be

$$
\begin{equation*}
\Phi(c):=\frac{1}{4}\left(1-c^{2}\right)^{2} . \tag{1.2}
\end{equation*}
$$

Finally, Pe is the Péclet number which, for ease of presentation, we will assume to be 1 in the analysis. By $\partial_{t} \eta$ we mean $\frac{\partial \eta}{\partial t}$ and $\partial_{\mathbf{n}} \eta:=\nabla \eta \cdot \mathbf{n}$.

The Cahn-Hilliard equation [17,18] was originally introduced as a phenomenological model of phase separation in a binary alloy. More recently it has been used to study phase transitions and interface dynamics, related free-boundary problems, multiphase fluids and polymer solutions; see [11, 40, 45] and the references therein. For the derivation and analysis of the equation we refer to [28] and the references therein. Results for continuous finite element approximations of the Cahn-Hilliard equation include optimal order error estimates for a semidiscrete splitting method obtained by Elliott, French and Milner in [30], optimal order error estimates for a fully-discrete splitting method in one space dimension with weaker regularity assumptions by Du and Nicolaides in $[27]$ and convergence of a fully-discrete splitting method with a nonsmooth logarithmic free-energy proved by Copetti and Elliott in [25]. More recently results for continuous finite element approximations of Cahn-Hilliard systems, which model phase separation of multi-component alloys, have also been established; see for example $[9,7,8]$ and the references therein. In [36] near-optimal error estimates are shown for a fully discrete mixed finite element approximation of the Cahn-Hilliard equation, with emphasis on the dependence on the interface parameter $\gamma$.

[^0]Explicit numerical discretizations of the Cahn-Hilliard equation require severe time-step restrictions of the form $\tau \sim h^{4}$ (where $\tau$ is the time-step and $h$ is the spatial mesh-size), and therefore implicit methods should be used. Additionally, in order to fully capture the interface dynamics, high spatial resolution is required: typically at least $8-10$ elements are needed across the interfacial region (see [31] where it is shown that if there are an insufficient number of elements across the interfacial region spurious numerical solutions can be introduced).

In the neighbourhood of the interface, the leading-order term in the asymptotic expansion of $c$ for $0<\gamma \ll 1$ is (see, for example, [29])

$$
\tanh \left(\frac{1}{\gamma \sqrt{2}} \mathrm{~d}\right)
$$

where $d$ is the signed distance to the centre of the interface. Thus if we consider the interfacial region to be located where the order-parameter $c$ varies between -0.99 and 0.99 a simple calculation yields that the width of the interface is $\approx 7.5 \gamma$.

Introducing a flow into the Cahn-Hilliard system leads to models of the Cahn-Hilliard-NavierStokes type (cf. [5, 10, 12, 38, 41, 42]), which include a convection term in the Cahn-Hilliard equation. The Péclet number in such problems is usually taken to be large, leading to a convection-dominated problem. If no numerical smoothing (such as streamline-diffusion or least-squares stabilization) is present in the discretization of the equation, computational modelling with continuous finite elements may lead to poor approximations. In contrast with this behaviour, due to their built-in numerical dissipation, no such added numerical diffusion is needed with a discontinuous Galerkin finite element method; see [23, 24]. Discontinuous Galerkin methods have a number of other attractive features: as has been noted by the authors of [35] "the trial and test spaces are very easy to construct; they can naturally handle inhomogeneous boundary conditions and curved boundaries; they also allow the use of highly nonuniform and unstructured meshes, and have built-in parallelism which permits coarse-grain parallelization. In addition, the fact that the mass matrices are block diagonal is an attractive feature in the context of time-dependent problems, especially if explicit time discretizations are used." One could add to this list the relative ease of designing $h p$-adaptive discontinuous Galerkin discretizations, even on meshes with hanging nodes; and the fact that methods of this kind are locally conservative, which is a particularly relevant feature in the realm of numerical approximation of nonlinear hyperbolic conservation laws.

Pioneering research on discontinuous Galerkin methods was pursued in [53, 44, 2, 26, 57, 52, 39]. We refer to the survey papers $[22,21]$ for a detailed historical overview. For more recent developments, see, for example, $[54,37,3,15]$ and references therein. The papers of Babuška and Zlámal [4] and Baker [6] are the earliest contributions to the theory of discontinuous Galerkin finite element methods for fourth-order elliptic problems; for more recent results, including historical notes, see [47, 34, 49, 50, 48]. The application of discontinuous Galerkin methods to the Cahn-Hilliard equation is discussed in $[20,56,58,35]$. In particular, in the article of Feng and Karakashian [35] a fully-discrete discontinuous Galerkin method is analyzed for the Cahn-Hilliard equation written as a fourth-order PDE, and an optimal-order error bound is derived for the order-parameter $c$ in the broken $\mathrm{L}^{2}\left(\mathrm{H}^{2}\right)$ norm with discontinuous piecewise polynomials of degree $p \geq 2$. Error bounds in the $\mathrm{L}^{\infty}\left(\mathrm{L}^{2}\right)$ and broken $\mathrm{L}^{2}\left(\mathrm{H}^{1}\right)$ norms have also been established by the authors of [35]; these are fully optimal when $p \geq 3$, while in the case of $p=2$ the latter estimates are suboptimal by one complete order with respect to the spatial discretization parameter $h$ (cf. (4.10)-(4.12) in Theorem 4.1 on p. 1107 of [35]).

The objective of the present paper is to derive optimal-order error bounds, with the inclusion of convection, for the order-parameter $c$ and the chemical potential $w$ in the $\mathrm{L}^{\infty}\left(\mathrm{H}^{1}\right)$ and $\mathrm{L}^{2}\left(\mathrm{H}^{1}\right)$ norm, respectively, with discontinuous piecewise polynomials of degree $\geq 1$. We will also present numerical results for discontinuous piecewise polynomial approximations of degree 1,2 and 3 . The main difference between the work of Feng and Karakashian [35] and our own results here is that we discretize the CahnHilliard equation as a system of two coupled second-order elliptic equations while in [35] the Cahn-Hilliard equation was approximated as a single fourth-order PDE. This choice crucially influences the magnitudes of the various penalty parameters in the respective discontinuous Galerkin discretizations: in particular, the penalty parameters in the method proposed herein are bounded by $\mathcal{O}(1 / h)$ as $h \rightarrow 0$, whereas in [35] (cf. in particular equation (3.4) in [35]) the largest of the two penalty parameters is $\mathcal{O}\left(1 / h^{3}\right)$ as the mesh-size $h \rightarrow 0$.

The paper is organized in the following manner. In Section 2 we introduce the basic notation and some fundamental properties of the discontinuous Galerkin method. In Section 3 we obtain bounds on various norms of the sequence of discontinuous Galerkin approximations to $c$ and $w$, independent of the spatial and temporal discretization parameters, as well as optimal-order error estimates for the sequence of approximations. In [51] the authors consider the model (R) and show that the problem possesses two dominant length scales, associated with bubbles and filaments. In particular it is shown that the convective term in the model can, for some parameter regimes, arrest the coarsening dynamics of the Cahn-Hilliard equation in a way that leads to the formation of bubbles, and that, for sufficiently strong velocity fields $\mathbf{u}$, these bubbles are replaced by filament structures. In Section 4 we shall present numerical simulations that display these features; we also present other computational results, including numerical simulation of spinodal decomposition. We close with some concluding remarks.
2. Notation and auxiliary results. Suppose that $q \in[1, \infty]$. For a bounded open set $\omega \subset \mathbb{R}^{2}$, let $\mathrm{L}^{q}(\omega)$ denote the space of $q$-integrable functions (with the usual modification for $q=\infty$ ) with norm denoted by $\|\cdot\|_{0, q, \omega}$, where, for simplicity of notation, in the case of $q=2$ we shall write $\|\cdot\|:=\|\cdot\|_{0,2, \Omega}$. Furthermore, for $m \in \mathbb{N}_{\geq 0}$, let $\mathrm{W}^{m, q}(\omega)$ and $\mathrm{H}^{m}(\omega)$ be the usual Sobolev spaces with norms $\|\cdot\|_{m, q, \omega}$ and $\|\cdot\|_{m, \omega}:=\|\cdot\|_{m, 2, \omega}$, respectively. Let $\left(\mathrm{H}^{1}(\omega)\right)^{\prime}$ denote the dual space of $\mathrm{H}^{1}(\omega)$ with respect to the pivot space $\mathrm{L}^{2}(\omega)$, and let $|\cdot|_{m, \omega}$ denote the usual Sobolev seminorm on $\mathrm{H}^{m}(\omega)$. When $\omega=\Omega$, we shall omit the index $\Omega$ from the subscript for norms and seminorms. Finally, we let $(\cdot, \cdot)_{\omega}$ and $\langle\cdot, \cdot\rangle_{\omega}$, respectively, denote the $\mathrm{L}^{2}(\omega)$ inner product and the duality pairing between $\left(\mathrm{H}^{1}(\omega)\right)^{\prime}$ and $\mathrm{H}^{1}(\omega)$ with respect to the pivot space $\mathrm{L}^{2}(\omega)$ where, for simplicity of notation, we shall write $(\cdot, \cdot):=(\cdot, \cdot)_{\Omega}$ and $\langle\cdot, \cdot\rangle:=\langle\cdot, \cdot\rangle_{\Omega}$. Throughout the paper $C$ will denote a generic positive constant, independent of the discretization parameters, whose value may change from line to line; $C_{1}$ will denote a generic positive constant, independent of the discretization parameters, whose value may change from line to line, and which can be taken to be arbitrarily small.

We consider the following function spaces:

$$
\begin{aligned}
\mathrm{V} & :=\left\{v \in \mathrm{H}^{1}(\Omega):(v, 1)=0\right\}, \\
\mathcal{F} & :=\left\{v \in\left(\mathrm{H}^{1}(\Omega)\right)^{\prime}:\langle v, 1\rangle=0\right\}, \\
\mathrm{H}_{\mathrm{N}}^{2}(\Omega) & :=\left\{v \in \mathrm{H}^{2}(\Omega): \partial_{\mathbf{n}} v=0 \text { on } \partial \Omega\right\} .
\end{aligned}
$$

Here, 1 denotes the function that is identically equal to 1 on $\Omega$. It is also convenient to introduce the linear operator $\mathcal{G}: \mathcal{F} \rightarrow \mathrm{V}$, referred to as Green's operator, by

$$
(\nabla(\mathcal{G} z), \nabla \eta)=\langle z, \eta\rangle \quad \forall \eta \in \mathrm{H}^{1}(\Omega)
$$

The existence of a unique element $\mathcal{G} z \in \mathrm{~V}$ for any $z \in \mathcal{F}$ follows by the Lax-Milgram theorem on V , on noting that $\mathrm{H}^{1}(\Omega)=\mathrm{V} \oplus \operatorname{span}\{1\}$ and $\langle z, 1\rangle=0$ for all $z \in \mathcal{F}$.
2.1. Weak formulation of the problem. We begin by stating the weak formulation of our initial-boundary-value problem.
(P) Given $\mathbf{u} \in \mathrm{H}(\operatorname{div} ; \Omega) \cap[\mathrm{C}(\bar{\Omega})]^{2}$, with $\nabla \cdot \mathbf{u}=0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n}=0$ on $\partial \Omega$, find $\{c(\cdot, t), w(\cdot, t)\}$ in $\mathrm{V} \times \mathrm{H}^{1}(\Omega), t \in[0, T]$, such that

$$
\begin{align*}
\left(\partial_{t} c, \eta\right)+(\nabla w, \nabla \eta)=b(\mathbf{u} ; c, \eta) & \forall \eta \in \mathrm{H}^{1}(\Omega),  \tag{2.1a}\\
(w, \eta)=\left(\Phi^{\prime}(c), \eta\right)+\gamma^{2}(\nabla c, \nabla \eta) & \forall \eta \in \mathrm{H}^{1}(\Omega),  \tag{2.1b}\\
c(\cdot, 0)=c_{0}(\cdot) \in \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V}, & \tag{2.1c}
\end{align*}
$$

where

$$
b(\mathbf{u} ; c, \eta):=\int_{\Omega} c \mathbf{u} \cdot \nabla \eta \mathrm{~d} x
$$

2.2. Discontinuous Galerkin finite element approximation. Let $\left\{\mathcal{T}_{h}\right\}_{h>0}$ be a shape-regular family of partitions of $\Omega$ into disjoint open triangles/quadrilaterals $\kappa$, such that $\bar{\Omega}=\cup_{\kappa \in \mathcal{T}_{h}} \bar{\kappa}$; here $h:=\max _{\kappa \in \mathcal{T}_{h}} h_{\kappa}$ is the spatial discretization parameter and $h_{\kappa}:=\operatorname{diam}(\kappa)$. Our stability and convergence analysis in Sections 3.1 and 3.2 require the use of inverse inequalities. We shall therefore assume in what follows that the family of discretisations $\left\{\mathcal{T}_{h}\right\}_{h>0}$ is quasiuniform.

Suppose that $p \geq 1$. Associated with $\mathcal{T}_{h}$ and $p$ is the finite element space

$$
\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right):=\left\{v \in \mathrm{~L}^{2}(\Omega):\left.v\right|_{\kappa} \text { is a polynomial of degree } \leq p \text { on each } \kappa \in \mathcal{T}_{h}\right\}
$$

We also define the broken Sobolev spaces

$$
\mathrm{H}^{\mathbf{s}}\left(\Omega, \mathcal{T}_{h}\right):=\left\{v \in \mathrm{~L}^{2}(\Omega):\left.v\right|_{\kappa} \in \mathrm{H}^{s_{\kappa}}(\kappa) \quad \forall \kappa \in \mathcal{T}_{h}\right\}
$$

and

$$
\mathrm{V}\left(\Omega, \mathcal{T}_{h}\right):=\left\{v \in \mathrm{H}^{\mathbf{1}}\left(\Omega, \mathcal{T}_{h}\right):(v, 1)=0\right\}
$$

where $\mathbf{s}=\left\{s_{\kappa}\right\}_{\kappa \in \mathcal{T}_{h}}$ is a set of positive integers. These spaces are equipped with the norms

$$
\|u\|_{\mathbf{s}, \mathcal{T}_{h}}:=\left(\sum_{\kappa \in \mathcal{T}_{h}}\|u\|_{\mathbf{s}_{\kappa}, \kappa}^{2}\right)^{\frac{1}{2}}
$$

and $\|\cdot\|_{\mathbf{1}, \mathcal{T}_{h}}$, respectively. It will be assumed throughout the paper that $s_{\kappa}=s$ for all $\kappa \in \mathcal{T}_{h}$, where $s$ is a positive integer; we shall then write $\mathrm{H}^{s}\left(\Omega, \mathcal{T}_{h}\right)$ instead of $\mathrm{H}^{\mathrm{s}}\left(\Omega, \mathcal{T}_{h}\right)$.

In analogy with V , we define the finite element space

$$
\mathrm{V}_{h}:=\left\{v \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right):(v, 1)=0\right\}
$$

For $v \in \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)$, we define the piecewise gradient $\nabla_{h} v$ of $v$ by $\left.\left(\nabla_{h} v\right)\right|_{\kappa}:=\nabla\left(\left.v\right|_{\kappa}\right), \kappa \in \mathcal{T}_{h}$, where $\nabla$ is the weak gradient of $v$ on $\kappa$. Next, for any interior (open) edge $e$ shared by the (open) elements $\kappa^{+}$ and $\kappa^{-}$, we define the edge-jump and edge-average of $v \in \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)$ by

$$
\llbracket v \rrbracket_{e}:=\left(\left.v^{+}\right|_{e}\right) \mathbf{n}^{+}+\left(\left.v^{-}\right|_{e}\right) \mathbf{n}^{-} \quad \text { and } \quad\left\{\{v\}_{e}:=\frac{1}{2}\left(\left.v^{+}\right|_{e}+\left.v^{-}\right|_{e}\right)\right.
$$

where, for $i=+,-, v^{i}=\left.v\right|_{\overline{\kappa^{i}}}$ and $\mathbf{n}^{i}$ is the unit normal vector on $e$ pointing outward of $\kappa^{i}$. By interior $e d g e$ we mean that $e \subset \Omega$ (i.e. $e$ has empty intersection with $\partial \Omega$ ). Similarly, for a vector-valued function $\mathbf{v} \in\left[\mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)\right]^{2}$, with an analogous definition of $\mathbf{v}^{i}$ to $v^{i}$ above, and an interior (open) edge $e$ shared by the (open) elements $\kappa^{+}$and $\kappa^{-}$, we define

$$
\llbracket \mathbf{v} \rrbracket_{e}:=\left(\left.\mathbf{v}^{+}\right|_{e}\right) \cdot \mathbf{n}^{+}+\left(\left.\mathbf{v}^{-}\right|_{e}\right) \cdot \mathbf{n}^{-} \quad \text { and } \quad\{\mathbf{v}\}_{e}:=\frac{1}{2}\left(\left.\mathbf{v}^{+}\right|_{e}+\left.\mathbf{v}^{-}\right|_{e}\right)
$$

In what follows, for ease of writing we shall suppress the subscript $e$ in our notations $\llbracket v \rrbracket_{e},\left\{\{v\}_{e}, \llbracket \mathbf{v} \rrbracket_{e}\right.$ and $\{\{\mathbf{v}\}\}_{e}$ and will simply write $\llbracket v \rrbracket,\{\{v\}, \llbracket \mathbf{v} \rrbracket$ and $\{\{\mathbf{v}\}$; the particular choice of the edge $e$ will be clear from the context.

Finally, we define

$$
B_{\mathcal{T}_{h}}(v, w):=\sum_{\kappa \in \mathcal{T}_{h}}(\nabla v, \nabla w)_{\kappa}-\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}}\left[\left(\llbracket v \rrbracket,\{[\nabla w\})_{e}+(\llbracket w \rrbracket,\{\llbracket v v\})_{e}-(\sigma \llbracket v \rrbracket, \llbracket w \rrbracket)_{e}\right]\right.
$$

and

$$
b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w):=\sum_{\kappa \in \mathcal{T}_{h}} \int_{\kappa} v \mathbf{u} \cdot \nabla w \mathrm{~d} x-\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}} \int_{e}\{\{\mathbf{u} v\}\} \cdot \llbracket w \rrbracket \mathrm{~d} s-\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}} c_{e} \int_{e} \llbracket v \rrbracket \cdot \llbracket w \rrbracket \mathrm{~d} s
$$

Here $\mathcal{E}_{\mathcal{T}_{h}}$ is the set of all interior edges of all elements $\kappa \in \mathcal{T}_{h},\left.\sigma\right|_{e}:=\sigma_{e}=\frac{\alpha}{h_{e}}$, where $\alpha$ is a sufficiently large positive constant, $c_{e} \geq \theta_{0}|\mathbf{u} \cdot \mathbf{n}|$, with $\theta_{0}$ a positive constant, independent of $e$ and $h_{e}, h_{e}$ is the edge length, and $(u, v)_{\kappa}:=\int_{\kappa} u v \mathrm{~d} x$, with a similar definition of $(u, v)_{e}$.

Let us define ||| •||| by

$$
\||w|\|^{2}:=\left\|\nabla_{h} w\right\|^{2}+\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}}\left(2 \sigma_{e}\|\llbracket w \rrbracket\|_{e}^{2}+\frac{1}{\sigma_{e}}\left\|\left\{\left\{\nabla_{h} w\right\}\right\}\right\|_{e}^{2}\right) .
$$

We observe that $\|\|\cdot\|\|$ is a seminorm on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ and a norm on $\mathrm{V}_{h}$. Whenever $B_{\mathcal{T}_{h}}(w, w) \geq 0$ (see Remark 2.1, Item 3, for a sufficient condition), we also define the broken energy (semi)norm $\|\|\cdot\|\|_{B}$ by

$$
\||w|\|_{B}^{2}:=B_{\mathcal{T}_{h}}(w, w)
$$

It follows from Items 2 and 3 of Remark 2.1 below that $\||\cdot|\|_{B}$ is equivalent to $\|\|\cdot\|\|$, uniformly in $h$, on $\mathrm{V}_{h}$ as a norm and on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ as a seminorm.

REMARK 2.1. The bilinear form $B_{\mathcal{T}_{h}}(\cdot, \cdot)$ and the (semi)norm \|\| $\cdot \| \mid$ have the following properties:

1. Consistency: let $v \in \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V}$; then,

$$
B_{\mathcal{T}_{h}}(v, w)=(-\Delta v, w) \quad \forall w \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)
$$

The identity follows from equation (2.1) in [2]; see also line 2 on p. 746 in [2].
2. Continuity: There exists a positive constant $C$, independent of the dicretization parameter $h$, such that

$$
\begin{equation*}
\left|B_{\mathcal{T}_{h}}(v, w)\right| \leq C\left|\left\|v \left|\left\|\left|\|w \mid\| \quad \forall v, w \in \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right)\right.\right.\right.\right.\right. \tag{2.2}
\end{equation*}
$$

This follows directly from the definition of $B_{\mathcal{T}_{h}}(\cdot, \cdot)$ by applying the Cauchy-Schwarz inequality.
3. Coercivity: There exists a positive constant $\alpha_{0}>0$, and for each $\alpha \geq \alpha_{0}$ there exists a constant $C_{0}=C_{0}(\alpha)$, independent of the discretization parameter $h$, such that

$$
\begin{equation*}
C_{0}\| \| w \mid \|^{2} \leq B_{\mathcal{T}_{h}}(w, w) \quad \forall w \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.3}
\end{equation*}
$$

Henceforth, we shall assume that $\alpha=\alpha_{0}$ in the definition of the penalty parameter $\sigma$ featuring in the definition of $B_{\mathcal{T}_{h}}(\cdot, \cdot)$. For a proof of (2.3) see the argument leading to inequality (3.1) in the paper of Arnold [2], with the minor alteration that Arnold assumes a Dirichlet boundary condition on $\partial \Omega$, whereas we have a homogeneous Neumann boundary condition here; therefore contributions from boundary edges to the norm ||| ||| in [2] can be omitted.
An explicit expression for the penalty parameter $\alpha_{0}$ in the interior-penalty discontinuous Galerkin finite element approximation of a second-order elliptic problem was proposed by Shabhazi [55] for meshes consisting of simplicial elements. The explicit dependence of the coercivity constant $C_{0}$ on the polynomial degree and the angles of the triangular/quadrilateral mesh elements was derived by Epshteyn and Rivière [33]; Mozolevski and Bösing [46] derived explicit expressions for penalty parameters in symmetric interior-penalty discontinuous Galerkin approximations of fourth-order elliptic problems on meshes consisting of parallelepipeds.
4. Broken Friedrichs' inequality: Let $r \in[2, \infty)$; there exists a positive constant $C=C(r)$, independent of the discretization parameter $h$, such that

$$
\begin{equation*}
\|w\|_{0, r} \leq C\left(\left\|\nabla_{h} w\right\|^{2}+\sum_{e \in \mathcal{E}_{\tau_{h}}} 2 \sigma_{e}\|\llbracket w \rrbracket\|_{e}^{2}\right)^{\frac{1}{2}} \quad \forall w \in \mathrm{~V}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.4}
\end{equation*}
$$

Trivially, $\|w\|_{0, r} \leq C\||w|\|$ for all $w \in \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right) \cap \mathrm{V}\left(\Omega, \mathcal{T}_{h}\right)$; in particular, both inequalities hold for all $w$. For a proof, see [13, 43]; see also [16].
Furthermore, for any $\mathbf{u} \in \mathrm{H}(\operatorname{div}, \Omega) \cap[\mathrm{C}(\bar{\Omega})]^{2}$, with $\nabla \cdot \mathbf{u}=0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n}=0$ on $\partial \Omega$, the bilinear form $b_{\mathcal{T}_{h}}(\mathbf{u} ; \cdot, \cdot)$ satisfies the following identities and bounds.
5. Consistency: Suppose that $(v, w) \in \mathrm{H}^{1}(\Omega) \times \mathrm{H}^{1}(\Omega)$; then,

$$
b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w)=b(\mathbf{u} ; v, w) .
$$

For a proof, see [37, 15], or Section 3 in [14].
6. Continuity: For $\alpha_{0}$ as in Item 3 above there exists a positive constant $C$ such that, for any $C_{1}>0$,

$$
\begin{align*}
& b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w) \leq C\|\mathbf{u}\|_{0, \infty}\left[\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}}\left(\frac{C_{1}}{2 \sigma_{e}}\|\{v v\}\|_{e}^{2}+\frac{C_{1}}{2 \sigma_{e}} c_{e}^{2}\|\llbracket v \rrbracket\|_{e}^{2}+\frac{1}{C_{1}} \sigma_{e}\|\llbracket w \rrbracket\|_{e}^{2}\right)\right. \\
& \left.\quad+\sum_{\kappa \in \mathcal{T}_{h}}\left(\frac{C_{1}}{2}\|v\|_{\kappa}^{2}+\frac{1}{2 C_{1}}\|\nabla w\|_{\kappa}^{2}\right)\right] \quad \forall(v, w) \in \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right) \times \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.5}
\end{align*}
$$

and thereby,

$$
\begin{equation*}
b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w) \leq C\|\mathbf{u}\|_{0, \infty}\left(\left.C_{1}\|v\|^{2}+\frac{1}{C_{1}} \right\rvert\,\|w\| \|^{2}\right) \quad \forall(v, w) \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.6}
\end{equation*}
$$

These inequalities follow directly from the definition of $b_{\mathcal{T}_{h}}(\mathbf{u} ; \cdot, \cdot)$ by applying the Cauchy-Schwarz inequality. Furthermore, it follows from (2.6) by (2.4) (with $r=2$ ) and the equivalence of $\|\|\cdot\|\|$ and $\|\|\cdot\|\|_{B}$ that, for $\alpha_{0}$ as in Item 3 above, there exists a positive constant $C$ such that, for any $C_{1}>0$,

$$
\begin{equation*}
b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w) \leq C\|\mathbf{u}\|_{0, \infty}\left(C_{1}\||v|\|^{2}+\frac{1}{C_{1}}\||w|\|_{B}^{2}\right) \quad \forall(v, w) \in \mathrm{V}_{h} \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.7}
\end{equation*}
$$

Integrating the first term of $b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w)$ by parts and combining the element-edge integrals with the second term we obtain

$$
\left.\left.b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w)=-\sum_{\kappa \in \mathcal{T}_{h}} \int_{\kappa} w \mathbf{u} \cdot \nabla v \mathrm{~d} x+\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}} \int_{e} \llbracket \mathbf{u} w\right\}\right\} \cdot \llbracket v \rrbracket \mathrm{~d} s-\sum_{e \in \mathcal{E}_{\tau_{h}}} c_{e} \int_{e} \llbracket v \rrbracket \cdot \llbracket w \rrbracket \mathrm{~d} s
$$

Hence, similarly to (2.6), we deduce that for $\alpha_{0}$ as in Item 3 above, there exists a positive constant $C$ such that, for any $C_{1}>0$, we have

$$
\begin{equation*}
b_{\mathcal{T}_{h}}(\mathbf{u} ; v, w) \leq C\|\mathbf{u}\|_{0, \infty}\left(C_{1}\|w\|^{2}+\frac{1}{C_{1}}\|\mid v\|^{2}\right) \quad \forall(v, w) \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.8}
\end{equation*}
$$

We note that when a solution $(c(\cdot, t), w(\cdot, t)), t \in[0, T]$, to problem $(\mathbf{P})$ belongs to $\mathrm{H}^{2}(\Omega) \times \mathrm{H}^{2}(\Omega)$, $t \in(0, T]$ - which we shall henceforth assume to be the case - then $\{c(\cdot, t), w(\cdot, t)\} \in\left(\mathrm{H}^{2}(\Omega) \cap \mathrm{V}\right) \times \mathrm{H}^{2}(\Omega)$, $t \in(0, T]$, and

$$
\begin{align*}
\left(\partial_{t} c, \eta\right)+B_{\mathcal{T}_{h}}(w, \eta)=b_{\mathcal{T}_{h}}(\mathbf{u} ; c, \eta) & \forall \eta \in \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right)  \tag{2.9a}\\
(w, \eta)=\left(\Phi^{\prime}(c), \eta\right)+\gamma^{2} B_{\mathcal{T}_{h}}(c, \eta) & \forall \eta \in \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right),  \tag{2.9b}\\
c(\cdot, 0)=c_{0}(\cdot) \in \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V} . & \tag{2.9c}
\end{align*}
$$

Let us define the discrete Laplacian $\Delta_{h}$ as follows: given $w \in \mathrm{~S}\left(\Omega, \mathcal{I}_{h}\right)$, find $\Delta_{h} w \in \mathrm{~V}_{h}$ such that

$$
\begin{equation*}
\left(-\Delta_{h} w, v\right)=B_{\mathcal{T}_{h}}(w, v) \quad \forall v \in \mathrm{~V}_{h} \tag{2.10}
\end{equation*}
$$

The existence and uniqueness of $\Delta_{h} w$ in $\mathrm{V}_{h}$ follows by the Riesz representation theorem: we equip $\mathrm{V}_{h}$ with the $\mathrm{L}^{2}(\Omega)$ inner product to obtain a Hilbert space, and we then note that, for $w \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$ fixed, $v \in \mathrm{~V}_{h} \mapsto B_{\mathcal{T}_{h}}(w, v) \in \mathbb{R}$ is a bounded linear functional over this Hilbert space by (2.2) and the property of norm-equivalence in finite-dimensional vector spaces. Since $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)=\mathrm{V}_{h} \oplus \operatorname{span}\{1\}$, and the equality in (2.10) holds trivially for $v=1$ as $B_{\mathcal{T}_{h}}(w, 1)=0$ for all $w \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$, the test space $\mathrm{V}_{h}$ in (2.10) can be replaced by $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ to deduce that

$$
\begin{equation*}
\left(-\Delta_{h} w, v\right)=B_{\mathcal{T}_{h}}(w, v) \quad \forall v \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.11}
\end{equation*}
$$

Let us also consider the discrete Green's function $\mathcal{G}_{h}: \mathcal{F} \cap \mathrm{L}^{2}(\Omega) \rightarrow \mathrm{V}_{h}$, defined by

$$
\begin{equation*}
B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} z, v\right)=(z, v) \quad \forall v \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.12}
\end{equation*}
$$

We note that since, for $z \in \mathcal{F} \cap \mathrm{~L}^{2}(\Omega)$ we have $(z, 1)=0$ and $B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} z, 1\right)=0$, the equality in (2.12) holds trivially for any constant function $v$. Thus, equivalently, we can define, for $z \in \mathcal{F} \cap \mathrm{~L}^{2}(\Omega)$, the function $\mathcal{G}_{h} z \in \mathrm{~V}_{h}$ by

$$
\begin{equation*}
B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} z, v\right)=(z, v) \quad \forall v \in \mathrm{~V}_{h} \tag{2.13}
\end{equation*}
$$

Since $\|\|\cdot\|\|$ is a norm on $\mathrm{V}_{h}$ and the bilinear form $B_{\mathcal{T}_{h}}(\cdot, \cdot)$ is coercive on $\mathrm{V}_{h} \times \mathrm{V}_{h}$ with respect to $\|\|\cdot\|\|$, the existence of a unique $\mathcal{G}_{h} z$, for any $z \in \mathcal{F} \cap \mathrm{~L}^{2}(\Omega)$, follows from the Lax-Milgram theorem and the fact that $\mathrm{V}_{h}$ is a finite-dimensional vector space. Furthermore, by (2.11),

$$
\begin{equation*}
B_{\mathcal{T}_{h}}(w, w)=\left(-\Delta_{h} w, w\right) \leq\left\|\Delta_{h} w\right\|\|w\| \quad \forall w \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.14}
\end{equation*}
$$

Now, from (2.13) and (2.2) we have that, for any $w \in \mathrm{~V}_{h} \subset \mathcal{F} \cap \mathrm{~L}^{2}(\Omega)$,

$$
\|w\|^{2}=B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} w, w\right) \leq\left\|\left|\mathcal{G}_{h} w\right|\right\|\| \| w \mid \| \quad \forall w \in \mathrm{~V}_{h}
$$

which yields

$$
\begin{equation*}
\left.\|w\| \leq\left\|\left|\mathcal{G}_{h} w\right|\right\|^{\frac{1}{2}} \right\rvert\,\|w\| \|^{\frac{1}{2}} \quad \forall w \in \mathrm{~V}_{h} \tag{2.15}
\end{equation*}
$$

Substituting (2.15) into (2.14), and noting that $\mathrm{V}_{h} \subset \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$, we obtain

$$
\begin{equation*}
B_{\mathcal{T}_{h}}(w, w) \leq\left\|\Delta_{h} w\right\|\left\|\left|\mathcal{G}_{h} w\| \|^{\frac{1}{2}}\||w|\|^{\frac{1}{2}} \quad \forall w \in \mathrm{~V}_{h}\right.\right. \tag{2.16}
\end{equation*}
$$

Using (2.3) in (2.16), we then deduce that

$$
C_{0}\||w|\|^{2} \leq\left\|\Delta_{h} w\right\|\| \| \mathcal{G}_{h} w\left\|^{\frac{1}{2}}\right\||w| \|^{\frac{1}{2}} \quad \forall w \in \mathrm{~V}_{h}
$$

and hence

$$
\begin{equation*}
\left\|\| w \| \left|\leq C_{0}^{-\frac{2}{3}}\left\|\left|\mathcal{G}_{h} w\right|\right\|^{\frac{1}{3}}\left\|\Delta_{h} w\right\|^{\frac{2}{3}} \quad \forall w \in \mathrm{~V}_{h}\right.\right. \tag{2.17}
\end{equation*}
$$

Now, (2.16) and (2.17) imply that

$$
B_{\mathcal{T}_{h}}(w, w) \leq C_{0}^{-\frac{1}{3}}\left\|\mid \mathcal{G}_{h} w\right\|^{\frac{2}{3}}\left\|\Delta_{h} w\right\|^{\frac{4}{3}} \quad \forall w \in \mathrm{~V}_{h}
$$

Let us note that for any $z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$ we have $\Delta_{h} z \in \mathrm{~V}_{h} \subset \mathcal{F} \cap \mathrm{~L}^{2}(\Omega)$, and therefore, by (2.12) and (2.11),

$$
B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} \Delta_{h} z, v\right)=\left(\Delta_{h} z, v\right)=-B_{\mathcal{T}_{h}}(z, v) \quad \forall(z, v) \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)
$$

This implies that

$$
B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h} \Delta_{h} z+z, v\right)=0 \quad \forall(z, v) \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)
$$

On selecting $v=\mathcal{G}_{h} \Delta_{h} z+z$ and using (2.3) we deduce that $\mathcal{G}_{h} \Delta_{h} z+z=c$, where $c$ is a constant function on $\Omega$. Since, by the definition of $\mathcal{G}_{h},\left(\mathcal{G}_{h} \Delta_{h} z, 1\right)=0$, it follows that $(z, 1)=(c, 1)=c$ meas $(\Omega)$, and therefore

$$
c=\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} z \mathrm{~d} x=: f_{\Omega} z \mathrm{~d} x
$$

Thus, we deduce that

$$
\begin{equation*}
z-f_{\Omega} z \mathrm{~d} x=-\mathcal{G}_{h} \Delta_{h} z \quad \forall z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.18}
\end{equation*}
$$

We introduce the (broken elliptic) projection operator $P_{h}: \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right) \rightarrow \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ defined, for $v \in$ $\mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right)$, by

$$
\begin{equation*}
B_{\mathcal{T}_{h}}\left(P_{h} v, \chi\right)=B_{\mathcal{T}_{h}}(v, \chi) \quad \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \quad \text { and } \quad\left(P_{h} v, 1\right)=(v, 1) \tag{2.19}
\end{equation*}
$$

We note that $P_{h}: \mathrm{H}^{2}\left(\Omega, \mathcal{T}_{h}\right) \cap \mathrm{V} \rightarrow \mathrm{V}_{h}$ and that this operator satisfies the bounds

$$
\begin{equation*}
\left\|P_{h} v-v\right\| \leq h\left\|\mid P_{h} v-v\right\| \| \quad \text { and } \quad\left\|\mid P_{h} v-v\right\|\left\|\leq C h^{s}\right\| v \|_{s+1} \quad \forall v \in \mathrm{H}^{s+1}(\Omega), \quad 1 \leq s \leq p \tag{2.20}
\end{equation*}
$$

Finally we note that the orthogonal projector $\Pi_{h}: \mathrm{L}^{2}(\Omega) \rightarrow \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ defined by

$$
\left(v-\Pi_{h} v, \chi\right)=0 \quad \forall \chi \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)
$$

satisfies the error bound

$$
\begin{equation*}
\left\|\Pi_{h} v-v\right\| \leq C h^{s}\|v\|_{s} \quad \forall v \in \mathrm{H}^{s}(\Omega), \quad 0 \leq s \leq p+1 \tag{2.21}
\end{equation*}
$$

Observe in particular that if $(v, 1)=0$ then $\left(P_{h} v, 1\right)=0$; thus, if $v \in \mathrm{~V}\left(\Omega, \mathcal{T}_{h}\right)$ then $P_{h} v \in \mathrm{~V}_{h}$.
The following broken version of Agmon's inequality will be required in our arguments below.
Lemma 2.2 .

$$
\begin{equation*}
\|z\|_{0, \infty} \leq C\|z\|^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}} \quad \forall z \in \mathrm{~V}_{h} \tag{2.22}
\end{equation*}
$$

Proof. Let as assume for the moment that $z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$. On writing $\xi=-\Delta_{h} z$ and noting that $\xi \in \mathrm{V}_{h}\left(\subset \mathrm{~L}^{2}(\Omega)\right)$ and, by elliptic regularity, $\mathcal{G} \xi \in \mathrm{H}^{2}(\Omega)$, the identity (2.18) implies, by the triangle inequality, Agmon's inequality (cf. Theorem 3 in the paper of Adams and Fournier [1]), and an inverse inequality on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ (recall that quasiuniformity of the family of partitions $\left\{\mathcal{T}_{h}\right\}_{h>0}$ has been assumed), that

$$
\begin{align*}
\left\|z-f_{\Omega} z \mathrm{~d} x\right\|_{0, \infty} & =\left\|\mathcal{G}_{h} \Delta_{h} z\right\|_{0, \infty} \\
& \leq\|\mathcal{G} \xi\|_{0, \infty}+\left\|\left(I-\pi_{h}\right) \mathcal{G} \xi\right\|_{0, \infty}+\left\|\left(\pi_{h} \mathcal{G}-\mathcal{G}_{h}\right) \xi\right\|_{0, \infty}  \tag{2.23}\\
& \leq C\|\mathcal{G} \xi\|^{\frac{1}{2}}\|\mathcal{G} \xi\|_{2}^{\frac{1}{2}}+C h\|\mathcal{G} \xi\|_{2}+C h^{-1}\left\|\left(\pi_{h} \mathcal{G}-\mathcal{G}_{h}\right) \xi\right\|
\end{align*}
$$

where $\pi_{h}: \mathrm{C}(\bar{\Omega}) \rightarrow \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ is the nodal interpolation operator. However,

$$
\begin{align*}
\|\mathcal{G} \xi\| & \leq\left\|\mathcal{G}_{h} \xi\right\|+\left\|\left(\mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& =\left\|z-f_{\Omega} z \mathrm{~d} x\right\|+\left\|\left(\mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& \leq\left(1+|\Omega|^{-\frac{1}{2}}\right)\|z\|+C h^{2}\|\mathcal{G} \xi\|_{2} \tag{2.24}
\end{align*}
$$

The bound appearing in the last term of inequality (2.24) comes from the error analysis of the symmetric version of the discontinuous Galerkin finite element method in the $\mathrm{L}^{2}(\Omega)$-norm; see [54].

On substituting (2.24) into (2.23) and using the approximation properties of the interpolant $\pi_{h}$, a standard error bound for the symmetric interior penalty discontinuous Galerkin method to estimate the closeness of $\mathcal{G}_{h}$ to $\mathcal{G}$, the elliptic regularity estimate $\|\mathcal{G} \xi\|_{2} \leq C\|\xi\|$, and recalling the definition of $\xi$, we deduce that

$$
\begin{align*}
\left\|z-f_{\Omega} z \mathrm{~d} x\right\|_{0, \infty} & \leq C\|z\|^{\frac{1}{2}}\|\mathcal{G} \xi\|_{2}^{\frac{1}{2}}+C h\|\mathcal{G} \xi\|_{2}+C h^{-1}\left\|\left(\pi_{h} \mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& \leq C\|z\|^{\frac{1}{2}}\|\mathcal{G} \xi\|_{2}^{\frac{1}{2}}+C h\|\mathcal{G} \xi\|_{2}+C h^{-1}\left\|\left(\pi_{h}-I\right) \mathcal{G} \xi\right\|+C h^{-1}\left\|\left(\mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& \leq C\|z\|^{\frac{1}{2}}\|\mathcal{G} \xi\|_{2}^{\frac{1}{2}}+C h\|\mathcal{G} \xi\|_{2} \\
& \leq C\|z\|^{\frac{1}{2}}\|\xi\|^{\frac{1}{2}}+C h\|\xi\| \\
& =C\|z\|^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}}+C h\left\|\Delta_{h} z\right\| \\
& =C\|z\|^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}}+C\left(h^{2}\left\|\Delta_{h} z\right\|\right)^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}} \quad \forall z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) . \tag{2.25}
\end{align*}
$$

Noting (2.10) and (2.2), and using inverse inequalities, we have that

$$
\left\|\Delta_{h} z\right\|^{2}=B_{\mathcal{T}_{h}}\left(z,-\Delta_{h} z\right) \leq C\left\|\left|\|z\|\| \| \Delta_{h} z\| \| \leq h^{-1}\right|\right\| z\| \|\left\|\Delta_{h} z\right\| \leq C h^{-2}\||z|\|^{2} \leq C h^{-4}\|z\|^{2} .
$$

Therefore,

$$
\begin{equation*}
h^{2}\left\|\Delta_{h} z\right\| \leq C\|z\|_{8} \quad \forall z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.26}
\end{equation*}
$$

Substituting (2.26) into (2.25) we obtain

$$
\left\|z-f_{\Omega} z \mathrm{~d} x\right\|_{0, \infty} \leq C\|z\|^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}} \quad \forall z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)
$$

In particular,

$$
\begin{equation*}
\|z\|_{0, \infty} \leq C\|z\|^{\frac{1}{2}}\left\|\Delta_{h} z\right\|^{\frac{1}{2}} \quad \forall z \in \mathrm{~V}_{h} \tag{2.27}
\end{equation*}
$$

That completes the proof.
We shall also require the following broken Gagliardo-Nirenberg inequality.
Lemma 2.3. Let $\|\cdot\|_{0,3}$ denote the $\mathrm{L}^{3}(\Omega)$ norm on $\Omega$. There exists a positive constant $C$, independent of $h$, such that

$$
\begin{equation*}
\left\|\nabla_{h} z\right\|_{0,3} \leq C\|z\|^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}} \quad \forall z \in \mathrm{~V}_{h} \tag{2.28}
\end{equation*}
$$

Proof. With the same definitions of $z, \pi_{h}, \mathcal{G}_{h}$ and $\mathcal{G}$ as in the proof of (2.27), and proceeding in a very similar manner, on noting that, by (2.18), $\nabla_{h} z=-\nabla_{h} \mathcal{G}_{h} \Delta_{h} z$ for all $z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right)$, letting, as before, $\xi=\Delta_{h} z$, and using the Gagliardo-Nirenberg inequality stated in Theorem 3 in the paper of Adams and Fournier [1], we have that

$$
\begin{align*}
\left\|\nabla_{h} z\right\|_{0,3} & =\left\|\nabla_{h} \mathcal{G}_{h} \xi\right\|_{0,3} \\
& \leq\|\nabla \mathcal{G} \xi\|_{0,3}+\left\|\nabla_{h}\left(I-\pi_{h}\right) \mathcal{G} \xi\right\|_{0,3}+\left\|\nabla_{h}\left(\pi_{h} \mathcal{G}-\mathcal{G}_{h}\right) \xi\right\|_{0,3} \\
& \leq C\|\mathcal{G} \xi\|^{\frac{1}{3}}\|\mathcal{G} \xi\|_{2}^{\frac{2}{3}}+C h^{\frac{2}{3}}\|\mathcal{G} \xi\|_{2}+C h^{-\frac{4}{3}}\left\|\left(\pi_{h} \mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& \leq C\|\mathcal{G} \xi\|^{\frac{1}{3}}\|\mathcal{G} \xi\|_{2}^{\frac{2}{3}}+C h^{\frac{2}{3}}\|\mathcal{G} \xi\|_{2}+C h^{-\frac{4}{3}}\left\|\left(\pi_{h}-I\right) \mathcal{G} \xi\right\|+C h^{-\frac{4}{3}}\left\|\left(\mathcal{G}-\mathcal{G}_{h}\right) \xi\right\| \\
& \leq C\|\mathcal{G} \xi\|^{\frac{1}{3}}\|\mathcal{G} \xi\|_{2}^{\frac{2}{3}}+C h^{\frac{2}{3}}\|\mathcal{G} \xi\|_{2} \\
& \leq C\left\|\mathcal{G}_{h} \xi\right\|^{\frac{1}{3}}\|\mathcal{G} \xi\|_{2}^{\frac{2}{3}}+C h^{\frac{2}{3}}\|\mathcal{G} \xi\|_{2} \\
& \leq C\left\|z-f_{\Omega} z \mathrm{~d} x\right\|^{\frac{1}{3}}\|\xi\|^{\frac{2}{3}}+C h^{\frac{2}{3}}\|\xi\| \\
& =C\left\|z-f_{\Omega} z \mathrm{~d} x\right\|^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}}+C h^{\frac{2}{3}}\left\|\Delta_{h} z\right\| \\
& =C\left\|z-f_{\Omega} z \mathrm{~d} x\right\|^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}}+C\left(h^{2}\left\|\Delta_{h} z\right\|\right)^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}} \\
& \leq C\left\|z-f_{\Omega} z \mathrm{~d} x\right\|^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}}+C\|z\|^{\frac{1}{3}}\left\|\Delta_{h} z\right\|^{\frac{2}{3}} \quad \forall z \in \mathrm{~S}\left(\Omega, \mathcal{T}_{h}\right) \tag{2.29}
\end{align*}
$$

The stated result follows by noting that $f_{\Omega} z \mathrm{~d} x=0$ for $z \in \mathrm{~V}_{h}$.
3. Finite element discretization. In this section we study a finite element approximation $\left(\mathbf{P}_{h, \tau}\right)$ of $(\mathbf{P})$. Here $\mathcal{T}_{h}$ is chosen as in Section 2.2, and in addition we let $0=t_{0}<t_{1}<\cdots<t_{N}=T$ be a partition of $[0, T]$ with $\tau:=t_{n}-t_{n-1}, n=1 \rightarrow N$.

We now define our discontinuous Galerkin finite element approximation of $(\mathbf{P})$ as follows:
$\left(\mathbf{P}_{h, \tau}\right)$ Given $\mathbf{u} \in \mathrm{H}(\operatorname{div} ; \Omega) \cap[\mathrm{C}(\bar{\Omega})]^{2}$, with $\nabla \cdot \mathbf{u}=0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n}=0$ on $\partial \Omega$, for $n=1 \rightarrow N$ find $\left\{c_{h}^{n}, w_{h}^{n}\right\} \in \mathrm{V}_{h} \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ such that

$$
\begin{array}{cc}
\left(\delta_{\tau} c_{h}^{n}, \chi\right)+B_{\mathcal{T}_{h}}\left(w_{h}^{n}, \chi\right)=b_{\mathcal{T}_{h}}\left(\mathbf{u} ; c_{h}^{n}, \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \\
\left(w_{h}^{n}, \chi\right)=\gamma^{2} B_{\mathcal{T}_{h}}\left(c_{h}^{n}, \chi\right)+\left(\Phi^{\prime}\left(c_{h}^{n}\right), \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \\
c_{h}^{0}:=\Pi_{h} c_{0} \in \mathrm{~V}_{h} & \tag{3.1c}
\end{array}
$$

where

$$
\delta_{\tau} c_{h}^{n}:=\frac{c_{h}^{n}-c_{h}^{n-1}}{\tau}, \quad n=1 \rightarrow N
$$

REmark 3.1. It follows from (2.9c), (3.1c) and the definition of $\||\cdot|| |$ that there exists a positive constant $C$, independent of $h$ and $\tau$, such that

$$
\begin{equation*}
\left(\Phi\left(c_{h}^{0}\right), 1\right)+\| \| c_{h}^{0} \| \mid \leq C \tag{3.2}
\end{equation*}
$$

We also note that the choice of the $\mathrm{L}^{2}$ projection operator $\Pi_{h}$ in (3.1c) is not mandatory: $\Pi_{h}$ can be replaced by any projector that is stable in the broken $\mathrm{H}^{1}$ norm.
3.1. Uniform bounds on the sequence of numerical solutions. We begin by establishing the following bounds, independent of $h$ and $\tau$, on the sequence of numerical solutions.

Lemma 3.1. For any $h>0$ and $\tau \leq C_{\star} \gamma^{2}$, where $C_{\star}$ is a sufficiently small but fixed positive constant, there exists a unique solution $\left\{c_{h}^{n}, w_{h}^{n}\right\} \in \mathrm{V}_{h} \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ to the $n$-th step of $\left(\mathbf{P}_{h, \tau}\right), n \in\{1,2, \ldots, N\}$; in addition, there exists a positive constant $C=C\left(C_{\star}, \gamma, \alpha_{0}, \theta_{0}, T\right)$, independent of $h$ and $\tau$, such that

$$
\begin{gather*}
\max _{n=1 \rightarrow N}\left[\gamma^{2}\left\|\left|c_{h}^{n}\right|\right\|^{2}+\left\|c_{h}^{n}\right\|_{0, \infty}+\left(\Phi\left(c_{h}^{n}\right), 1\right)+\left\|w_{h}^{n}\right\|^{2}\right]+\sum_{n=1}^{N} \tau\left\|\mid w_{h}^{n}\right\|^{2}+\gamma^{2} \sum_{n=1}^{N} \tau\left\|\delta_{\tau} c_{h}^{n}\right\|^{2} \leq C  \tag{3.3}\\
\max _{n=1 \rightarrow N}\left\|\Delta_{h} c_{h}^{n}\right\| \leq C \tag{3.4}
\end{gather*}
$$

and

$$
\begin{equation*}
\sum_{n=1}^{N} \tau\left\|c_{h}^{n}\right\|_{0, \infty}^{4} \leq C \tag{3.5}
\end{equation*}
$$

Proof. The existence of a unique solution $\left\{c_{h}^{n}, w_{h}^{n}\right\} \in \mathrm{V}_{h} \times \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ to (3.1a,b) follows similarly as in [32]; for the sake of brevity the details are omitted. Taking $\chi=w_{h}^{n}$ in (3.1a) and $\chi=c_{h}^{n}-c_{h}^{n-1}$ in (3.1b), noting (2.7) and the following inequality, arising from an application of Taylor's remainder theorem on observing that $\Phi^{\prime \prime}(s) \geq-1$ for all $s \in \mathbb{R}$,

$$
\Phi^{\prime}(r)(r-s) \geq \Phi(r)-\Phi(s)-\frac{1}{2}(r-s)^{2} \quad \forall s, r \in \mathbb{R}
$$

we obtain

$$
\begin{aligned}
\gamma^{2} B_{\mathcal{T}_{h}}\left(c_{h}^{n}, c_{h}^{n}-c_{h}^{n-1}\right) & +\tau B_{\mathcal{T}_{h}}\left(w_{h}^{n}, w_{h}^{n}\right)=\tau b_{\mathcal{T}_{h}}\left(\mathbf{u} ; c_{h}^{n}, w_{h}^{n}\right)-\left(\Phi^{\prime}\left(c_{h}^{n}\right), c_{h}^{n}-c_{h}^{n-1}\right) \\
& \leq C \tau\| \| c_{h}^{n}\| \|^{2}+\frac{\tau}{4}\left\|\left|w_{h}^{n}\right|\right\|_{B}^{2}-\left(\Phi\left(c_{h}^{n}\right), 1\right)+\left(\Phi\left(c_{h}^{n-1}\right), 1\right)+\frac{1}{2}\left\|c_{h}^{n}-c_{h}^{n-1}\right\|^{2}
\end{aligned}
$$

Setting $\chi=c_{h}^{n}-c_{h}^{n-1}$ in (3.1a) gives, using (2.2), (2.7) and the equivalence of $\|\|\cdot\|\|$ and $\|\mid \cdot\| \|_{B}$ as norms on $\mathrm{V}_{h}$ and seminorms on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$,

$$
\begin{aligned}
\left\|c_{h}^{n}-c_{h}^{n-1}\right\|^{2} & =\tau b_{\mathcal{T}_{h}}\left(\mathbf{u} ; c_{h}^{n}, c_{h}^{n}-c_{h}^{n-1}\right)-\tau B_{\mathcal{T}_{h}}\left(w_{h}^{n}, c_{h}^{n}-c_{h}^{n-1}\right) \\
& \leq \frac{\tau}{2}\left|\left\|w _ { h } ^ { n } \left|\left\|_{B}^{2}+C \tau\left|\left\|c_{h}^{n}-c_{h}^{n-1} \mid\right\|^{2}+C \tau\| \| c_{h}^{n}\| \|^{2}\right.\right.\right.\right.\right.
\end{aligned}
$$

and hence we deduce that

$$
\begin{aligned}
\gamma^{2} B_{\mathcal{T}_{h}}\left(c_{h}^{n}, c_{h}^{n}-c_{h}^{n-1}\right)+ & \tau B_{\mathcal{T}_{h}}\left(w_{h}^{n}, w_{h}^{n}\right)+\left(\Phi\left(c_{h}^{n}\right), 1\right) \\
& \leq C \tau\left\|c_{h}^{n}\left|\left\|^{2}+\frac{\tau}{2}\left|\left\|w_{h}^{n}\right\|\right|_{B}^{2}+C \tau\right\|\right| c_{h}^{n}-c_{h}^{n-1}\right\| \|^{2}+\left(\Phi\left(c_{h}^{n-1}\right), 1\right)
\end{aligned}
$$

Noting that

$$
B_{\mathcal{T}_{h}}\left(c_{h}^{n}, c_{h}^{n}-c_{h}^{n-1}\right)=\frac{1}{2}\left(\left\|\left|c_{h}^{n}\| \|_{B}^{2}-\left\|\left|c_{h}^{n-1}\left\|_{B}^{2}+\right\|\right| c_{h}^{n}-c_{h}^{n-1} \mid\right\|_{B}^{2}\right)\right.\right.
$$

and the equivalence of $\||\cdot|\|$ and $\||\cdot|\|_{B}$ as norms on $\mathrm{V}_{h}$, using (3.2) and a discrete Grönwall inequality we deduce that

$$
\begin{equation*}
\max _{n=1 \rightarrow N}\left[\gamma^{2} \mid\left\|c_{h}^{n}\right\| \|^{2}+\left(\Phi\left(c_{h}^{n}\right), 1\right)\right]+\sum_{n=1}^{N} \tau\left\|\left|w_{h}^{n}\right|\right\|^{2} \leq C \tag{3.6}
\end{equation*}
$$

Thus we have proved the first, third and fifth bound in (3.3). Next we prove (3.5). Taking $\chi=\Delta_{h} c_{h}^{n}$ in (3.1b) and using (2.10) we have

$$
\begin{align*}
\gamma^{2}\left\|\Delta_{h} c_{h}^{n}\right\|^{2} & =-\gamma^{2} B_{\mathcal{T}_{h}}\left(c_{h}^{n}, \Delta_{h} c_{h}^{n}\right) \\
& =-\left(w_{h}^{n}, \Delta_{h} c_{h}^{n}\right)+\left(\Phi^{\prime}\left(c_{h}^{n}\right), \Delta_{h} c_{h}^{n}\right) \tag{3.7}
\end{align*}
$$

Using (2.10), the symmetry of $B_{\mathcal{T}_{h}}(\cdot, \cdot)$, the definition of $\|\mid \cdot\| \|_{B}$, the definition of $\Phi$, the equivalence of $\|\|\cdot\|\|$ and $\|\|\cdot\|\|_{B}$ as norms on $\mathrm{V}_{h}$ and seminorms on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$ and (2.4) (with $r=6,4$ and 2) we obtain from (3.7) that

$$
\begin{align*}
\gamma^{2}\left\|\Delta_{h} c_{h}^{n}\right\|^{2} & \leq B_{\mathcal{T}_{h}}\left(w_{h}^{n}, c_{h}^{n}\right)+C\left\|\Phi^{\prime}\left(c_{h}^{n}\right)\right\|^{2}+\frac{\gamma^{2}}{2}\left\|\Delta_{h} c_{h}^{n}\right\|^{2} \\
& \leq \frac{1}{2}\left|\left\|w_{h}^{n}\right\|\left\|_{B}^{2}+\frac{1}{2}\left|\left\|c_{h}^{n}\right\|\right|_{B}^{2}+C\left(\left\|c_{h}^{n}\right\|_{0,6}^{6}+\left\|c_{h}^{n}\right\|_{0,4}^{4}+\left\|c_{h}^{n}\right\|^{2}\right)+\frac{\gamma^{2}}{2}\right\| \Delta_{h} c_{h}^{n} \|^{2}\right. \\
& \leq C\left|\| w _ { h } ^ { n } \| \left\|^{2}+C\left|\left\|c_{h}^{n}\right\|\left\|^{2}+C\right\|\left\|c_{h}^{n} \mid\right\|^{6}+\frac{\gamma^{2}}{2}\left\|\Delta_{h} c_{h}^{n}\right\|^{2}\right.\right.\right. \tag{3.8}
\end{align*}
$$

Summing (3.8) from $n=1 \rightarrow N$ and using (2.22), (2.4) (with $r=2$ ) and (3.6) we obtain (3.5).
Next we prove the remaining bounds in (3.3). To this end, we subtract (3.1b) with $n$ replaced by $n-1$ from (3.1b) to obtain

$$
\begin{equation*}
\left(w_{h}^{n}-w_{h}^{n-1}, \chi\right)=\tau \gamma^{2} B_{\mathcal{T}_{h}}\left(\delta_{\tau} c_{h}^{n}, \chi\right)+\left(\Phi^{\prime}\left(c_{h}^{n}\right)-\Phi^{\prime}\left(c_{h}^{n-1}\right), \chi\right) \tag{3.9}
\end{equation*}
$$

Setting $\chi=\tau \gamma^{2} \delta_{\tau} c_{h}^{n}$ in (3.1a) and $\chi=w_{h}^{n}$ in (3.9), combining the resulting equations and noting the symmetry of $B_{\mathcal{T}_{h}}(\cdot, \cdot),(2.8)$ and (1.2) gives

$$
\begin{align*}
\tau \gamma^{2}\left\|\delta_{\tau} c_{h}^{n}\right\|^{2}+\left(w_{h}^{n}-w_{h}^{n-1}, w_{h}^{n}\right) & =\tau\left(\left[\left(c_{h}^{n}\right)^{2}+\left(c_{h}^{n}+c_{h}^{n-1}\right) c_{h}^{n-1}-1\right] \delta_{\tau} c_{h}^{n}, w_{h}^{n}\right)+\tau \gamma^{2} b_{\mathcal{T}_{h}}\left(\mathbf{u} ; c_{h}^{n}, \delta_{\tau} c_{h}^{n}\right) \\
& \leq \frac{\tau \gamma^{2}}{2}\left\|\delta_{\tau} c_{h}^{n}\right\|^{2}+\tau \tilde{C}_{h, \tau}\left\|w_{h}^{n}\right\|^{2}+C \tau\left\|c_{h}^{n}\right\| \|^{2} \tag{3.10}
\end{align*}
$$

where $\tilde{C}_{h, \tau}^{n}:=C\left(1+\left\|c_{h}^{n}\right\|_{0, \infty}^{4}+\left\|c_{h}^{n-1}\right\|_{0, \infty}^{4}\right)$. Note that, by (3.5), $\tau \sum_{n=1}^{N} \tilde{C}_{h, \tau}^{n}$ is bounded by a constant independent of $h$ and $\tau$. Combining (3.6) with (3.10) and using a discrete Grönwall inequality gives the fourth and sixth bound of (3.3).

Using (3.7) and the previously obtained bounds we deduce (3.4) as follows:

$$
\begin{align*}
\frac{\gamma^{2}}{2}\left\|\Delta_{h} c_{h}^{n}\right\|^{2} & \leq C\left\|w_{h}^{n}\right\|^{2}+C\left\|\Phi^{\prime}\left(c_{h}^{n}\right)\right\|^{2} \\
& \leq C\left\|w_{h}^{n}\right\|^{2}+C\left(\left\|c_{h}^{n}\right\|_{0,6}^{6}+\left\|c_{h}^{n}\right\|_{0,4}^{4}+\left\|c_{h}^{n}\right\|^{2}\right) \\
& \leq C \tag{3.11}
\end{align*}
$$

Applying (3.6), (2.4) (with $r=2$ ), (3.11) and (2.22) gives the final bound

$$
\max _{n=1 \rightarrow N}\left\|c_{h}^{n}\right\|_{0, \infty} \leq C
$$

3.2. Error estimate. Before we prove the main result of this section we introduce some notation. We define the continuous-in-time functions

$$
c_{h, \tau}(\cdot, t):=\frac{\left(t-t_{n-1}\right)}{\tau} c_{h}^{n}(\cdot)+\frac{\left(t_{n}-t\right)}{\tau} c_{h}^{n-1}(\cdot), \quad t \in\left(t_{n-1}, t_{n}\right], \quad n=1 \rightarrow N
$$

and

$$
w_{h, \tau}(\cdot, t):=\frac{\left(t-t_{n-1}\right)}{\tau} w_{h}^{n}(\cdot)+\frac{\left(t_{n}-t\right)}{\tau} w_{h}^{n-1}(\cdot), \quad t \in\left(t_{n-1}, t_{n}\right], \quad n=1 \rightarrow N .
$$

We also define the piecewise-constant-in-time functions

$$
\widehat{c}_{h, \tau}(\cdot, t):=c_{h}^{n} \quad \text { and } \quad \widehat{w}_{h, \tau}(\cdot, t):=w_{h}^{n}, \quad t \in\left(t_{n-1}, t_{n}\right], \quad n=1 \rightarrow N
$$

When there is no danger of ambiguity, for ease of writing we shall omit $(\cdot, t)$ from our notation. Using the above notation problem $\left(\mathbf{P}_{\mathbf{h}, \tau}\right)$ can be restated as follows.

Given $\mathbf{u} \in \mathrm{H}(\operatorname{div} ; \Omega) \cap[\mathrm{C}(\bar{\Omega})]^{2}$, $\operatorname{div} \mathbf{u}=0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n}=0$ on $\partial \Omega$, find $\left\{c_{h, \tau}, w_{h, \tau}\right\} \in \mathrm{H}^{1}\left(0, T ; \mathrm{V}_{h}\right) \times$ $\mathrm{L}^{2}\left(0, T ; \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)\right)$ such that

$$
\begin{array}{ll}
\left(\delta_{\tau} \widehat{c}_{h, \tau}, \chi\right)+B_{\mathcal{T}_{h}}\left(\widehat{w}_{h, \tau}, \chi\right)=b_{\mathcal{T}_{h}}\left(\mathbf{u} ; \widehat{c}_{h, \tau}, \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right), \\
\left(\widehat{w}_{h, \tau}, \chi\right)=\gamma^{2} B_{\mathcal{T}_{h}}\left(\widehat{c}_{h, \tau}, \chi\right)+\left(\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{3.12b}
\end{array}
$$

Next we define

$$
\widehat{S}^{c}(\cdot, t):=\delta_{\tau} P_{h} c(\cdot, t)-\partial_{t} P_{h} c(\cdot, t), \quad t \in\left(t_{n-1}, t_{n}\right]
$$

and we note that for $c \in \mathrm{~W}^{2, \infty}\left((0, T) ; \mathrm{H}^{2}(\Omega) \cap \mathrm{V}\right)$ using (2.4) with $r=2$, the equivalence of the norms $\|\|\cdot \mid\|$ and $\|\|\cdot\| \|_{B}$ on $\mathrm{V}_{h}$, and the stability of $P_{h}$ in the norm $\|\|\cdot\|\|_{B}$, we have

$$
\begin{equation*}
\left\|\widehat{S}^{c}(\cdot, t)\right\| \leq C \tau \tag{3.13}
\end{equation*}
$$

for a.e. $t \in[0, T]$; here and below $C$ will denote a positive constant, independent of $h, \tau$ and $t$, whose actual value may vary from line to line. Finally, for $t \in\left(t_{n-1}, t_{n}\right], n=1 \rightarrow N$, we define

$$
E^{c}(\cdot, t)=c(\cdot, t)-\widehat{c}_{h, \tau}(\cdot, t), \quad E_{h}^{c}(\cdot, t):=P_{h} c(\cdot, t)-\widehat{c}_{h, \tau}(\cdot, t), \quad E_{A}^{c}(\cdot, t):=c(\cdot, t)-P_{h} c(\cdot, t)
$$

and $\widehat{c}_{h, \tau}(\cdot, 0)=c_{h}^{0}=P_{h} c_{0}=P_{h} c(\cdot, 0)$ (whereby $E_{h}^{c}(\cdot, 0)=0$ ), with analogous error functions for $w$ and $\partial_{t} c$.

For $1 \leq s \leq p$, assuming that $c \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}\right), w \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega)\right),(2.20)$ yields

$$
\begin{equation*}
\left\|E_{A}^{c}(\cdot, t)\right\|+h\left|\left\|E_{A}^{c}(\cdot, t)\right\|\right| \leq C h^{s+1}, \quad\left\|E_{A}^{w}(\cdot, t)\right\|+h \mid\left\|E_{A}^{w}(\cdot, t)\right\| \| \leq C h^{s+1} \tag{3.14}
\end{equation*}
$$

for a.e. $t \in[0, T]$. Using the definition of $\|\|\cdot\|\|_{B},(2.4)$ with $r=2,(2.13)$, the equivalence of this norm with $\|\|\cdot\|\|$ on $\mathrm{V}_{h}$ and $(2.21)$, for $c \in \mathrm{~W}^{1, \infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}\right), 1 \leq s \leq p$, we obtain

$$
\begin{equation*}
\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{A}^{c}\right)(\cdot, t)\right\|\|\leq C\| \delta_{\tau} E_{A}^{c}(\cdot, t) \| \leq C h^{s+1} \quad \text { for a.e. } t \in[0, T] \tag{3.15}
\end{equation*}
$$

Recalling the definition of $E^{c}$ and noting (2.4) with $r=2$ and (3.14) gives

$$
\begin{equation*}
\left\|E^{c}(\cdot, t)\right\| \leq\left\|E_{h}^{c}(\cdot, t)\right\|+\left\|E_{A}^{c}(\cdot, t)\right\| \leq C \mid\left\|E_{h}^{c}(\cdot, t)\right\| \|+C h^{s+1} \tag{3.16}
\end{equation*}
$$

for a.e. $t \in[0, T]$ and all $s$ with $1 \leq s \leq p$.
Using the above notation and (2.19) we deduce from (2.9a,b) that:

$$
\begin{align*}
\left(\delta_{\tau} P_{h} c, \chi\right)+B_{\mathcal{T}_{h}}\left(P_{h} w, \chi\right)=b_{\mathcal{T}_{h}}(\mathbf{u} ; c, \chi)+\left(\widehat{S}^{c}, \chi\right)-\left(\partial_{t} E_{A}^{c}, \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)  \tag{3.17a}\\
\left(P_{h} w, \chi\right)=\gamma^{2} B_{\mathcal{T}_{h}}\left(P_{h} c, \chi\right)+\left(\Phi^{\prime}(c), \chi\right)-\left(E_{A}^{w}, \chi\right) & \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{3.17b}
\end{align*}
$$

We shall also need the following lemma for our subsequent bounds.
Lemma 3.2. For $c \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{2}(\Omega)\right)$,

$$
\begin{equation*}
\left\|\left\|\Phi^{\prime}(c(\cdot, t))-\Phi^{\prime}\left(\widehat{c}_{h, \tau}(\cdot, t)\right)\right\|\right\| \leq C\left\|\mid E^{c}(\cdot, t)\right\| \| \quad \text { for a.e. } t \in[0, T] \tag{3.18}
\end{equation*}
$$

Proof. For ease of writing we shall suppress the dependence of $c, \widehat{c}_{h, \tau}$ and $E^{c}$ on $t$. Let us define

$$
Q^{c}:=c^{2}+c \widehat{c}_{h, \tau}+\widehat{c}_{h, \tau}^{2} .
$$

Clearly,

$$
\begin{equation*}
\left\|\left\|\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right) \mid\right\| \leq\right\|\left\|E^{c} Q^{c}\right\|\|+\|\left\|E^{c}\right\| \| \tag{3.19}
\end{equation*}
$$

We must now bound the first term in (3.19). Using the definition of $|\| \cdot||\mid$ we have

$$
\begin{align*}
\left\|\left\|E^{c} Q^{c}\right\|\right\|^{2} & =\left\|\nabla_{h}\left(E^{c} Q^{c}\right)\right\|^{2}+\sum_{e \in \mathcal{E}_{\mathcal{T}_{h}}}\left(2 \sigma_{e}\left\|\llbracket E^{c} Q^{c} \rrbracket\right\|_{e}^{2}+\frac{1}{\sigma_{e}} \|\left\{\left\{E^{c} Q^{c}\right\} \|_{e}^{2}\right)\right. \\
& :=\mathrm{T}_{1}+\mathrm{T}_{2}+\mathrm{T}_{3} . \tag{3.20}
\end{align*}
$$

We begin by bounding the term $\mathrm{T}_{1}$. For any element $\kappa \in \mathcal{T}_{h}$ we have

$$
\begin{aligned}
\left\|\nabla\left(E^{c} Q^{c}\right)\right\|_{\kappa} \leq & \left\|Q^{c} \nabla E^{c}\right\|_{\kappa}+\left\|E^{c} \nabla Q^{c}\right\|_{\kappa} \\
\leq & \left\|Q^{c}\right\|_{0, \infty}\left\|\nabla E^{c}\right\|_{\kappa}+\left\|E^{c} \nabla Q^{c}\right\|_{\kappa} \\
\leq & \frac{3}{2}\left(\|c\|_{0, \infty}^{2}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}^{2}\right)\left\|\nabla E^{c}\right\|_{\kappa} \\
& \quad+\left\|E^{c}\left(2 c \nabla c+c \nabla \widehat{c}_{h, \tau}+\widehat{c}_{h, \tau} \nabla c+2 \widehat{c}_{h, \tau} \nabla \widehat{c}_{h, \tau}\right)\right\|_{\kappa} \\
\leq & \frac{3}{2}\left(\|c\|_{0, \infty}^{2}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}^{2}\right)\left\|\nabla E^{c}\right\|_{\kappa} \\
& \quad+2\left\|E^{c}\right\|_{0,6, \kappa}\left(\|c\|_{0, \infty}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}\right)\left(\|\nabla c\|_{0,3, \kappa}+\left\|\nabla \widehat{c}_{h, \tau}\right\|_{0,3, \kappa}\right) .
\end{aligned}
$$

After squaring, summing over $\kappa \in \mathcal{T}_{h}$, using inequality (2.4) with $r=6$, taking square roots, applying Hölder's inequality for finite sums and the interpolation inequality (2.28) with $z=\widehat{c}_{h, \tau}$, we obtain

$$
\begin{aligned}
\left\|\nabla_{h}\left(E^{c} Q^{c}\right)\right\| & \leq C\left(c,\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty},\left\|\Delta_{h} \widehat{c}_{h, \tau}\right\|\right) \mid\left\|E^{c}\right\| \| \\
& \leq C\left|\left\|E^{c} \mid\right\|\right.
\end{aligned}
$$

The last bound follows from the second inequality in (3.3) and from (3.4). Hence our bound on term $\mathrm{T}_{1}$.
Next we bound the term $\mathrm{T}_{2}$. For any $e \in \mathcal{E}_{\mathcal{T}_{h}}$ we have

$$
\begin{align*}
\sigma_{e}\left\|\llbracket E^{c} Q^{c} \rrbracket\right\|_{e}^{2} & =\sigma_{e}\left\|\llbracket E^{c} \rrbracket\left(Q^{c}\right)^{+}+\llbracket Q^{c} \rrbracket\left(E^{c}\right)^{-}\right\|_{e}^{2} \\
& \leq 2 \sigma_{e} \int_{e}\left(\left|\llbracket E^{c} \rrbracket\right|^{2}\left(\left(Q^{c}\right)^{+}\right)^{2}+\left|\llbracket Q^{c} \rrbracket\right|^{2}\left(\left(E^{c}\right)^{-}\right)^{2}\right) \mathrm{d} s \tag{3.21}
\end{align*}
$$

Since $c$ is continuous, we have

$$
\llbracket Q^{c} \rrbracket=c \llbracket \widehat{c}_{h, \tau} \rrbracket+\llbracket \widehat{c}_{h, \tau}^{2} \rrbracket .
$$

Squaring yields

$$
\begin{align*}
\left|\llbracket Q^{c} \rrbracket\right|^{2} & \leq 2\left(c^{2}\left|\llbracket \widehat{c}_{h, \tau} \rrbracket\right|^{2}+\left|\llbracket \widehat{c}_{h, \tau}^{2} \rrbracket\right|^{2}\right) \\
& =2\left(c^{2}\left|\llbracket \widehat{c}_{h, \tau} \rrbracket\right|^{2}+\left\{\left\{\widehat{c}_{h, \tau}\right\}\right\}^{2}\left|\llbracket \widehat{c}_{h, \tau} \rrbracket\right|^{2}\right) \\
& =2\left(c^{2}+\left\{\left\{\widehat{c}_{h, \tau}\right\}\right\}^{2}\right)\left|\llbracket E^{c} \rrbracket\right|^{2} . \tag{3.22}
\end{align*}
$$

Combining (3.21) and (3.22) we obtain

$$
\begin{aligned}
\sigma_{e}\left\|\llbracket E^{c} Q^{c} \rrbracket\right\|_{e}^{2} & \leq 2 \sigma_{e} \int_{e}\left(\left|\llbracket E^{c} \rrbracket\right|^{2}\left(\left(Q^{c}\right)^{+}\right)^{2}+2\left(c^{2}+\left\{\widehat{c}_{h, \tau}\right\}^{2}\right)\left|\llbracket E^{c} \rrbracket\right|^{2}\left(\left(E^{c}\right)^{-}\right)^{2}\right) \mathrm{d} s \\
& \leq C\left(\|c\|_{0, \infty}^{4}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}^{4}\right) \sigma_{e} \int_{e}\left|\llbracket E^{c} \rrbracket\right|^{2} \mathrm{~d} s \\
& \leq C \sigma_{e} \int_{e}\left|\llbracket E^{c} \rrbracket\right|^{2}
\end{aligned}
$$

where the last line follows by the second inequality in (3.3). Summing over element edges $e \in \mathcal{E}_{\mathcal{T}_{h}}$ yields the desired bound on term $\mathrm{T}_{2}$.

Finally, we bound the term $\mathrm{T}_{3}$. For any $e \in \mathcal{E}_{\mathcal{T}_{h}}$ and any $\kappa \in \mathcal{T}_{h}$ such that $e \subset \partial \kappa$, we have

$$
\begin{aligned}
\left.\frac{1}{\sigma_{e}} \|\left\{E^{c} Q^{c}\right\}\right\} \|_{e}^{2} & =\frac{1}{4 \sigma_{e}}\left\|\left(E^{c}\right)^{+}\left(Q^{c}\right)^{+}+\left(Q^{c}\right)^{-}\left(E^{c}\right)^{-}\right\|_{e}^{2} \\
& =\frac{1}{4 \sigma_{e}} \int_{e}\left[\left(\left(E^{c}\right)^{+}+\left(E^{c}\right)^{-}\right)\left(Q^{c}\right)^{+}-\left(\left(Q^{c}\right)^{+}-\left(Q^{c}\right)^{-}\right)\left(E^{c}\right)^{-}\right]^{2} \mathrm{~d} s \\
& \leq \frac{2}{\sigma_{e}} \int_{e}\left\{\left\{E^{c}\right\}^{2}\left(\left(Q^{c}\right)^{+}\right)^{2} \mathrm{~d} s+\frac{1}{2 \sigma_{e}} \int_{e}\left|\llbracket Q^{c} \rrbracket\right|^{2}\left(\left(E^{c}\right)^{-}\right)^{2} \mathrm{~d} s\right. \\
& \leq C\left(\|c\|_{0, \infty}^{4}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}^{4}\right)\left(\frac{1}{\sigma_{e}} \int_{e}\left\{\left\{E^{c}\right\}^{2} \mathrm{~d} s+\frac{1}{\sigma_{e}} \int_{e}\left(\left(E^{c}\right)^{-}\right)^{2} \mathrm{~d} s\right)\right. \\
& \leq C\left(\|c\|_{0, \infty}^{4}+\left\|\widehat{c}_{h, \tau}\right\|_{0, \infty}^{4}\right)\left(\frac{1}{\sigma_{e}} \int_{e}\left\{\left\{E^{c}\right\}\right\}^{2} \mathrm{~d} s+\int_{\kappa}\left|\nabla E^{c}\right|^{2} \mathrm{~d} x\right) .
\end{aligned}
$$

We sum over all element edges $e \in \mathcal{E}_{\mathcal{T}_{h}}$ to obtain our bound on $\mathrm{T}_{3}$. We then substitute our bounds on $\mathrm{T}_{1}, \mathrm{~T}_{2}$ and $\mathrm{T}_{3}$ into (3.20), insert the resulting bound on $\left\|\left|E^{c} Q^{c}\right|\right\|$ into (3.19), and we note that $c(\cdot, t) \in \mathrm{H}^{2}(\Omega) \cap \mathrm{V} \subset \mathrm{L}^{\infty}(\Omega) \cap \mathrm{W}^{1,3}(\Omega)$ to complete the proof.

We require two further preparatory lemmas, which are stated and proved below.
Lemma 3.3. Suppose that $1 \leq s \leq p$, and assume that $c_{0} \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V}, c \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega) \cap\right.$ $\mathrm{V}) \cap \mathrm{W}^{2, \infty}\left((0, T) ; \mathrm{L}^{2}(\Omega)\right)$ and $w \in \mathrm{~L}^{\infty}\left(0, T ; \mathrm{H}^{s+1}(\Omega)\right)$. Then, for almost every $t \in[0, T]$, we have

$$
\begin{equation*}
\left|\left(\Phi^{\prime}(c(\cdot, t))-\Phi^{\prime}\left(\widehat{c}_{h, \tau}(\cdot, t)\right), \delta_{\tau} E_{h}^{c}(\cdot, t)\right)\right| \leq C\left(h^{2 s+2}+\tau^{2}+\left\|\left|E^{c}(\cdot, t)\right|\right\|^{2}+\| \| E_{h}^{c}(\cdot, t)\| \|^{2}\right)+\frac{1}{8}\left|\left\|E_{h}^{w}(\cdot, t) \mid\right\|_{B}^{2}\right. \tag{3.23}
\end{equation*}
$$

Proof. Noting (2.13), (2.2), (2.20) and using standard inverse estimates it follows that

$$
\begin{align*}
&\left|\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)\right| \leq\left\|P_{h}\left[\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right)\right]-\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right)\right)\right\|\left\|\delta_{\tau} E_{h}^{c}\right\| \\
&+C \mid\left\|P_{h}\left[\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right)\right]\right\|\| \|\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\right\| \| \\
& \leq C h\left|\left\|\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right) \mid\right\|\| \| \delta_{\tau} E_{h}^{c} \|\right. \\
&+C \mid\left\|P_{h}\left[\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right)\right]\right\|\| \|\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\right\| \| \\
& \leq C\left\|\left|\left\|\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right)\left|\left\|^{2}+C_{1}\right\|\right| \mid \mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\right\| \|^{2} .\right.\right. \tag{3.24}
\end{align*}
$$

We shall use (3.18) to bound the first term on the right-hand side of (3.24). In order to handle the second term on the right-hand side of (3.24), we now bound $\left\|\mid \mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\right\| \|$. To this end, subtracting (3.12a) from (3.17a) and noting (2.13), (2.2), (2.6) and (2.4) with $r=2$, we have, for any $\chi \in \mathrm{V}_{h}$ and a.e. $t \in[0, T]$,

$$
\begin{align*}
& B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right), \chi\right)=\left(\delta_{\tau} E_{h}^{c}, \chi\right)=-B_{\mathcal{T}_{h}}\left(E_{h}^{w}, \chi\right)+b_{\mathcal{T}_{h}}\left(\mathbf{u} ; E^{c}, \chi\right)+\left(\widehat{S}^{c}, \chi\right)-\left(\delta_{\tau} E_{A}^{c}, \chi\right)  \tag{3.25}\\
& \leq C\left|\left\|E _ { h } ^ { w } \left|\left\|\left|\| \chi \| \| + | b _ { \mathcal { T } _ { h } } ( \mathbf { u } ; E _ { h } ^ { c } , \chi ) | + | b _ { \mathcal { T } _ { h } } ( \mathbf { u } ; E _ { A } ^ { c } , \chi ) | + \| \widehat { S } ^ { c } \| \| \chi \left\|+C\left|\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{A}^{c}\right) \mid\right\|\| \| \chi\| \|\right.\right.\right.\right.\right.\right.\right. \\
& \leq C\left|\left\|E _ { h } ^ { w } \left|\left\|\left|\| \chi \| \| + C \| E _ { h } ^ { c } \| | \| \chi \| | + | b _ { \mathcal { T } _ { h } } ( \mathbf { u } ; E _ { A } ^ { c } , \chi ) | + C \| \widehat { S } ^ { c } \| \| \| \| \left\|+C\left|\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{A}^{c}\right)|\||\|\chi \mid\| .\right.\right.\right.\right.\right.\right.\right.\right. \tag{3.26}
\end{align*}
$$

Now, by (2.5), the multiplicative trace inequality stated in (3.22) in [37], and (2.20), we have that

$$
\begin{equation*}
\left|b_{\mathcal{T}_{h}}\left(\mathbf{u} ; E_{A}^{c}, \chi\right)\right| \leq C h^{s+1} \mid\|\chi\| \| \quad \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right) \tag{3.27}
\end{equation*}
$$

On substituting (3.27), (3.13) and (3.15) into (3.26) and recalling the equivalence of the seminorms \|\|•\|\| and $\|\|\cdot\|\|_{B}$ on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$, we have that

$$
\begin{equation*}
B_{\mathcal{T}_{h}}\left(\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right), \chi\right) \leq C\left(h^{s+1}+\tau+\| \| E_{h}^{w}\| \|_{B}+\left\|E_{h}^{c}\right\|\right)\|\chi\| \| \tag{3.28}
\end{equation*}
$$

where $1 \leq s \leq p$. On taking $\chi=\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\left(\in \mathrm{V}_{h}\right)$ in (3.28), applying (2.3) to the left-hand side of (3.28) and (2.4) with $r=2$ to the last term in the brackets on the right-hand side of (3.28), it follows that

$$
\begin{equation*}
\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)(\cdot, t)\right\| \|^{2} \leq C_{2}\left(h^{2 s+2}+\tau^{2}+\left\|\left|E_{h}^{w}(\cdot, t)\| \|_{B}^{2}+\left\|\mid E_{h}^{c}(\cdot, t)\right\| \|^{2}\right) \quad \text { for a.e. } t \in[0, T]\right.\right. \tag{3.29}
\end{equation*}
$$

where $1 \leq s \leq p$. Choosing $C_{1}$ such that $8 C_{1} C_{2}<1$ and using (3.24), (3.18) and (3.29) we obtain the desired result.

Lemma 3.4. Suppose that $1 \leq s \leq p$, and assume that $c_{0} \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V}, c \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega) \cap\right.$ $\mathrm{V}) \cap \mathrm{W}^{2, \infty}\left((0, T) ; \mathrm{L}^{2}(\Omega)\right)$ and $w \in \mathrm{~L}^{2}\left(0, T ; \mathrm{H}^{s+1}(\Omega)\right)$. Then, for almost every $t \in[0, T]$, we have that

$$
\begin{equation*}
\gamma^{2} B_{\mathcal{T}_{h}}\left(E_{h}^{c}(\cdot, t), \delta_{\tau} E_{h}^{c}(\cdot, t)\right)+B_{\mathcal{T}_{h}}\left(E_{h}^{w}(\cdot, t), E_{h}^{w}(\cdot, t)\right) \leq C\left(h^{2 s}+\tau^{2}\right)+C\left|\left\|E_{h}^{c}(\cdot, t)\right\|\right|^{2}+\frac{1}{2}\left|\left\|E_{h}^{w}(\cdot, t) \mid\right\|_{B}^{2}\right. \tag{3.30}
\end{equation*}
$$

Proof. Subtracting (3.12b) from (3.17b) and choosing $\chi=\delta_{\tau} E_{h}^{c}$ in the resulting equation we obtain

$$
\begin{equation*}
\left(E_{h}^{w}, \delta_{\tau} E_{h}^{c}\right)=\gamma^{2} B_{\mathcal{T}_{h}}\left(E_{h}^{c}, \delta_{\tau} E_{h}^{c}\right)+\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)-\left(E_{A}^{w}, \delta_{\tau} E_{h}^{c}\right) \tag{3.31}
\end{equation*}
$$

Next, setting $\chi=E_{h}^{w}$ in (3.25), combining the resulting equation with (3.31) we have, for a.e. $t \in[0, T]$ and any real number $\beta$, that

$$
\begin{aligned}
\gamma^{2} B_{\mathcal{T}_{h}}\left(E_{h}^{c}, \delta_{\tau} E_{h}^{c}\right)+B_{\mathcal{T}_{h}}\left(E_{h}^{w}, E_{h}^{w}\right)= & b_{\mathcal{T}_{h}}\left(\mathbf{u} ; E^{c}, E_{h}^{w}\right)+\left(\widehat{S}^{c}-\delta_{\tau} E_{A}^{c}, E_{h}^{w}\right) \\
& -\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)+\left(E_{A}^{w}, \delta_{\tau} E_{h}^{c}\right) \\
= & b_{\mathcal{T}_{h}}\left(\mathbf{u} ; E_{h}^{c}, E_{h}^{w}\right)+b_{\mathcal{T}_{h}}\left(\mathbf{u} ; E_{A}^{c}, E_{h}^{w}\right)+\left(\widehat{S}^{c}-\delta_{\tau} E_{A}^{c}, E_{h}^{w}-\beta\right) \\
& -\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)+\left(E_{A}^{w}, \delta_{\tau} E_{h}^{c}\right),
\end{aligned}
$$

since $\left(\widehat{S}^{c}, 1\right)=0$ and $\left(\delta_{\tau} E_{A}^{c}, 1\right)=0$. Noting (2.6), (3.27), (3.15), (3.14) and the broken Poincaré-Friedrichs inequality (1.4) in the paper of Brenner [13], which implies that

$$
\inf _{\beta \in \mathbb{R}}\left\|E_{h}^{w}-\beta\right\| \leq C \mid\left\|E_{h}^{w}\right\| \|
$$

we deduce, for $1 \leq s \leq p$ and a.e. $t \in[0, T]$, that

$$
\begin{align*}
\gamma^{2} B_{\mathcal{T}_{h}}\left(E_{h}^{c}, \delta_{\tau} E_{h}^{c}\right)+B_{\mathcal{T}_{h}}\left(E_{h}^{w}, E_{h}^{w}\right) \leq & C\left\|E_{h}^{c}\right\|\left|\left\|E _ { h } ^ { w } \left|\left\|\left|+C h^{s+1}\left\|\left|E_{h}^{w}\right|\right\|+\left(\left\|\widehat{S}^{c}\right\|+C h^{s+1}\right) \inf _{\beta \in \mathbb{R}}\left\|E_{h}^{w}-\beta\right\|\right.\right.\right.\right.\right. \\
& +\left|\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)\right|+C h^{s+1}\left\|\delta_{\tau} E_{h}^{c}\right\| \\
\leq & C\left\|\widehat{S}^{c}\right\|^{2}+C_{1}\left|\left\|\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\left|\| \|^{2}+C\right|\right\| E_{h}^{c} \|\right|^{2} \\
& +\frac{1}{4}\left|\left\|E _ { h } ^ { w } \left|\|_{B}^{2}+\left|\left(\Phi^{\prime}(c)-\Phi^{\prime}\left(\widehat{c}_{h, \tau}\right), \delta_{\tau} E_{h}^{c}\right)\right|+C h^{2 s}\right.\right.\right. \tag{3.32}
\end{align*}
$$

In the transition to the first line of the second inequality in (3.32) we used the inverse inequality $h\left\|\delta_{\tau} E_{h}^{c}\right\| \leq$ $C\left|\left|\left|\mathcal{G}_{h}\left(\delta_{\tau} E_{h}^{c}\right)\right| \|\right.\right.$.

Noting (3.13), (3.29), (3.23), the triangle inequality $\left\|\left\|E^{c}\right\|\left|\leq\left|\left\|E_{h}^{c}|\|+\||\left|E_{A}^{c} \|\right|\right.\right.\right.\right.$ in conjunction with (3.14), and choosing $C_{1}$ as in the proof of Lemma 3.3, the inequality (3.32) yields the required bound.

We now state and prove the main result of the paper. We shall write $\|\cdot\|_{\mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)}:=|\||\cdot|| \mid$.
TheOrem 3.5. Suppose that $p \geq 1$ and $1 \leq s \leq p$. Assume further that $c_{0} \in \mathrm{H}^{s+1}(\Omega) \cap \mathrm{H}_{\mathrm{N}}^{2}(\Omega) \cap \mathrm{V}$, $c \in \mathrm{~L}^{\infty}\left((0, T) ; \mathrm{H}^{s+1}(\Omega) \cap \mathrm{V}\right) \cap \mathrm{W}^{1, \infty}\left(0, T ; \mathrm{H}^{2}(\Omega)\right) \cap \mathrm{W}^{2, \infty}\left((0, T) ; \mathrm{L}^{2}(\Omega)\right)$ and $w \in \mathrm{~L}^{\infty}\left(0, T ; \mathrm{H}^{s+1}(\Omega)\right) \cap$ $\mathrm{W}^{1, \infty}\left(0, T ; \mathrm{H}^{2}(\Omega)\right)$. Then,

$$
\left\|c_{0}-c_{h}^{0}\right\|_{\mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)} \leq C h^{s}
$$

and

$$
\left\|c-c_{h, \tau}\right\|_{\mathrm{L}^{\infty}\left(0, T ; \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)\right)}+\left\|w-w_{h, \tau}\right\|_{\mathrm{L}^{2}\left(0, T ; \mathrm{H}^{1}\left(\Omega, \mathcal{T}_{h}\right)\right)} \leq C\left(h^{s}+\tau\right)
$$

Proof. The bound on the error between $c_{0}$ and $c_{h}^{0}:=\Pi_{h} c_{0}$ is a simple consequence of (2.21).
Setting $t=t_{n}$ in (3.30) and applying a discrete Grönwall inequality, we deduce that for all $\tau \in\left(0, \tau_{0}\right)$, where $\tau_{0}=\tau_{0}(\gamma)$ is a sufficiently small (depending on $\gamma$ ) but fixed real number and $1 \leq s \leq p$,

$$
\left\|\mid E_{h}^{c}\left(\cdot, t_{n}\right)\right\|\left\|_{B}^{2}+\tau \sum_{m=1}^{n}\right\|\left\|E_{h}^{w}\left(\cdot, t_{m}\right)\right\| \|_{B}^{2} \leq C\left(\| \| E_{h}^{c}(\cdot, 0)\| \|_{B}^{2}+h^{2 s}+\tau^{2}\right) \leq C\left(h^{2 s}+\tau^{2}\right), \quad n=1 \rightarrow N
$$

For $t \in\left(t_{n-1}, t_{n}\right], n=1 \rightarrow N$, we have that

$$
\begin{aligned}
\left\|E_{h}^{c}(\cdot, t)\right\| \|_{B} & =\| \| P_{h} c(\cdot, t)-c_{h}^{n}\| \|_{B} \leq\left\|\left|\left\|P_{h}\left(c(\cdot, t)-c\left(\cdot, t_{n}\right)\right)\right\|_{B}+\|\mid\| P_{h} c\left(\cdot, t_{n}\right)-c_{h}^{n}\| \|_{B}\right.\right. \\
& \leq\left\|c(\cdot, t)-c\left(\cdot, t_{n}\right)\right\|\left\|_{B}+\right\|\left\|P_{h} c\left(\cdot, t_{n}\right)-c_{h}^{n} \mid\right\|_{B} \\
& \leq \tau \cdot \operatorname{ess}^{2} \sup _{t \in[0, T]} \mid\left\|\partial_{t} c(\cdot, t)\right\|\left\|_{B}+\right\|\left\|E_{h}^{c}\left(\cdot, t_{n}\right)\right\| \|_{B} \\
& \leq C \tau+\left\|E_{h}^{c}\left(\cdot, t_{n}\right)\right\| \|_{B} .
\end{aligned}
$$

Therefore,

$$
\left\|E_{h}^{c}(\cdot, t)\right\| \|_{B}^{2} \leq C\left(h^{2 s}+\tau^{2}\right), \quad \text { for a.e. } t \in[0, T], \quad 1 \leq s \leq p
$$

Similarly,

$$
\int_{0}^{T}\| \| E_{h}^{w}(\cdot, t)\| \|_{B}^{2} \mathrm{~d} t=\sum_{m=1}^{N} \int_{t_{m-1}}^{t_{m}}\left\|E_{h}^{w}(\cdot, t)\right\| \|_{B}^{2} \mathrm{~d} t \leq C\left(h^{2 s}+\tau^{2}\right), \quad 1 \leq s \leq p
$$

Thus we deduce, on noting the definitions of $E_{h}^{c}$ and $E_{h}^{w}$, that

$$
\operatorname{ess} \cdot \sup _{t \in[0, T]}\left\|\left|P_{h} c(\cdot, t)-\widehat{c}_{h \tau}(\cdot, t)\| \|_{B}^{2}+\int_{0}^{T}\left\|\mid P_{h} w(\cdot, t)-\widehat{w}_{h \tau}(\cdot, t)\right\| \|_{B}^{2} \mathrm{~d} t \leq C\left(h^{2 s}+\tau^{2}\right), \quad 1 \leq s \leq p\right.\right.
$$

By recalling the equivalence of $\|\|\cdot\|\|_{B}$ and $\|\|\cdot\|\|$ as norms on $\mathrm{V}_{h}$ and seminorms on $\mathrm{S}\left(\Omega, \mathcal{T}_{h}\right)$, the desired bounds follow on applying the triangle inequality and the approximation results (3.14).
4. Numerical results. For the numerical solution of the nonlinear system (3.1a), (3.1b) we use a Newton iteration: Given $c_{h}^{0}$, at each time-step, $n$, we perform an inner iteration for $k=1 \rightarrow K$ to obtain $\left\{c^{n, k}, w^{n, k}\right\}$ satisfying

$$
\begin{array}{r}
\left(\frac{c_{h}^{n, k}-c_{h}^{n-1}}{\tau}, \chi\right)+\frac{1}{\operatorname{Pe}} B_{\mathcal{T}_{h}}\left(w_{h}^{n, k}, \chi\right)=b_{\mathcal{T}_{h}}\left(\mathbf{u} ; c_{h}^{n, k}, \chi\right) \quad \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right), \\
\left(w_{h}^{n, k}, \chi\right)=\gamma^{2} B_{\mathcal{T}_{h}}\left(c_{h}^{n, k}, \chi\right)+\left(3 c_{h}^{n, k}\left(c_{h}^{n, k}\right)^{2}-2\left(c_{h}^{n, k-1}\right)^{3}-c_{h}^{n, k-1}, \chi\right) \quad \forall \chi \in \mathrm{S}\left(\Omega, \mathcal{T}_{h}\right), \tag{4.2}
\end{array}
$$

and we define $c_{h}^{n}:=c_{h}^{n, K}$. Throughout Sections 4.2-4.4, $\Omega=(0,1)^{2}$.
REmARK 4.1. In practice $K=2$ or 3 inner iterations were seen to provide a sufficiently accurate approximation.
4.1. Model problem with known solution. In our first numerical experiment we consider the Cahn-Hilliard equation with known solution

$$
c=t \cos \left(\frac{\pi x}{3}\right) \cos \left(\frac{\pi y}{3}\right),
$$

and apply an appropriate non zero term to the right-hand side of (1.1a). Clearly this problem will not produce interfacial layers, but it will provide insight into the discontinuous Galerkin approximation developed in this paper.

The domain $\Omega$ is taken to be $\Omega_{3}:=(-3,3) \times(-3,3)$ and $T=0.1$. The convective velocity is

$$
\mathbf{u}(x, y):=f(r)(y,-x)^{\mathrm{T}}, \quad(x, y) \in \Omega_{3}
$$

where

$$
f(r):=\frac{1}{2}(1+\tanh (\beta(1-r))), \quad r^{2}:=x^{2}+y^{2}
$$

with $\beta=10$. Clearly $\nabla \cdot \mathbf{u}=0$, and $\mathbf{u} \cdot \mathbf{n}=0$ to machine precision on $\partial \Omega_{3}$. We will investigate rates of convergence for the values $\gamma=\frac{1}{10}, \frac{1}{20}, \frac{1}{40}$ and $\mathrm{Pe}=50,100,200$.

We consider three uniform refinements for both linear and quadratic elements. Finally, we choose $\tau=\frac{\gamma^{2}}{10}$.

Table 4.1 documents the results of our experiments: We have observed first-order, respectively secondorder, convergence of $c_{h, \tau}$ to $c$ in the $\mathrm{L}^{\infty}\left((0, T) ; \mathrm{H}^{1}(\Omega) \cap \mathrm{V}\right)$ seminorm, with $p=1$ and $p=2$, for all three values of $\gamma$ considered.

|  |  | $\begin{gathered} \mathrm{Pe}=50 \\ p=1\left(10^{-2}\right) \quad p=2\left(10^{-3}\right) \end{gathered}$ |  | $\begin{array}{r} \mathrm{Pe} \\ p=1\left(10^{-2}\right) \end{array}$ | $\begin{aligned} & 100 \\ & p=2\left(10^{-3}\right) \end{aligned}$ | $\begin{array}{r} \mathrm{Pe}= \\ p=1\left(10^{-2}\right) \end{array}$ | $\begin{aligned} & 200 \\ & p=2\left(10^{-3}\right) \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\gamma=\frac{1}{10} \quad$Level 1 <br> Level 2 <br>  <br> Level 3 |  | 3.30 | 4.80 | 3.30 | 4.80 | 3.31 | 4.80 |
|  |  | 1.58 | 1.20 | 1.59 | 1.20 | 1.60 | 1.20 |
|  |  | 0.78 | 0.31 | 0.78 | 0.30 | 0.78 | 0.30 |
| $\gamma=\frac{1}{20}$ | Level 1 | 3.32 | 4.90 | 3.32 | 4.90 | 3.31 | 4.90 |
|  | Level 2 | 1.60 | 1.20 | 1.61 | 1.20 | 1.61 | 1.20 |
|  | Level 3 | 0.78 | 0.32 | 0.79 | 0.31 | 0.79 | 0.31 |
| $\gamma=\frac{1}{40}$ | Level 1 | 3.30 | 4.90 | 3.32 | 4.90 | 3.32 | 4.90 |
|  | Level 2 | 1.61 | 1.30 | 1.61 | 1.30 | 1.61 | 1.30 |
|  | Level 3 | 0.80 | 0.35 | 0.80 | 0.32 | 0.81 | 0.32 |

The error $\left(\int_{0}^{T}\left\|\nabla\left(c-c_{h, \tau}\right)\right\|^{2} \mathrm{~d} t\right)^{\frac{1}{2}}$ for $c=t \cos \left(\frac{\pi x}{3}\right) \cos \left(\frac{\pi y}{3}\right)$ on $\Omega_{3}$.
4.2. The evolution of an ellipse without convection: linear elements. In our second example we apply the discontinuous Galerkin method using linear elements on a 32 by 32 uniform quadrilateral mesh. We shall consider the Cahn-Hilliard equation without convection (i.e. $\mathbf{u}=\mathbf{0}$ ) and choose $\gamma=$ $1 / 100$. The initial datum $c_{0}$ is a piecewise constant function whose jump-set is an ellipse:

$$
c_{0}(x, y):=\left\{\begin{aligned}
0.95 & \text { if } 9(x-0.5)^{2}+(y-0.5)^{2}<1 / 9 \\
-0.95 & \text { otherwise }
\end{aligned}\right.
$$



Fig. 4.1. The evolution of an ellipse without convection
As expected the initial datum $c_{0}$ with the ellipse-shaped jump-set evolves to a steady state exhibiting a circular interface; see Figure 4.1. Thereafter no motion will occur as the interface has constant curvature. Furthermore, as is expected from a Cahn-Hilliard system, mass is conserved.

Remark 4.2. This rather coarse mesh does not have the desired $8-10$ elements in the interface; see [31]; in fact, the number of elements in the interface is, on average, 2-3. For this model problem and subsequent problems this did not seem to cause any difficulties when using discontinuous elements.
4.3. The evolution of a cross without convection: quadratic elements. In this example we use the same parameters as in Section 4.1. The initial datum $c_{0}$ is a piecewise constant function whose jump-set has the shape of a cross; see Figure 4.2 . We use a 32 by 32 uniform quadrilateral mesh and quadratic elements.

As was the case in the previous example, the initial datum $c_{0}$ with a cross-shaped jump-set is seen to evolve to a steady state exhibiting a circular interface; see Figure 4.2.
4.4. Spinodal decomposition: quadratic elements. Spinodal decomposition is the separation of a mixture of two, or more, components to bulk regions of each. Such a phenomenon occurs when a high-temperature mixture of two, or more, alloys is rapidly cooled. To model this separation the initial datum $c_{0}$ is chosen to be a small uniformly distributed random perturbation about zero; see Figure


Fig. 4.2. The evolution of a cross
4.3. We consider this phenomenon with $\gamma=1 / 100$ using quadratic elements on a 32 by 32 uniform quadrilateral mesh.


Fig. 4.3. Early stages of spinodal decomposition


Fig. 4.4. Later stages of spinodal decomposition

The separation of the two components into bulk regions can quite clearly be seen in Figure 4.3. This initial separation happens over a very small time-scale relative to the motion thereafter. In Figure 4.4 the bulk regions begin to move more slowly and separation will continue until the interface(s) develop a constant curvature.
4.5. Convection-dominated problems: quadratic elements. In all of the following examples we will take $\mathrm{Pe}=200$,

$$
\mathbf{u}(x, y):=f(r)(2 y-1,1-2 x)^{\mathrm{T}}, \quad(x, y) \in \Omega:=(0,1)^{2},
$$

where

$$
f(r):=\frac{1}{2}\left(1+\tanh \left(\beta\left(\frac{1}{2}-\varepsilon-r\right)\right)\right), \quad r^{2}:=\left(x-\frac{1}{2}\right)^{2}+\left(y-\frac{1}{2}\right)^{2},
$$

with $\beta=200, \varepsilon=0.1$. Clearly $\nabla \cdot \mathbf{u}=0$, and $\mathbf{u} \cdot \mathbf{n}=0$ to machine precision on $\partial \Omega$.
4.5.1. Evolution of a cross. In this example we start from the same cross-shaped initial datum as that in Section 4.2. We use a quadratic discontinuous Galerkin method on a 32 by 32 uniform quadrilateral mesh and apply the above velocity field, taking $\gamma=1 / 100$.


Fig. 4.5. Evolution of a cross with circular convection at $t=0.02,0.04,0.3,1.0$
We quite clearly see the effects of both convection and interfacial motion on the resulting evolution; see Figure 4.5. The convection term is rotating the two components anti-clockwise while the interface is reducing to a circle. Note that in the final frame of Figure 4.5 both bulk regions are still rotating under the velocity field.
4.5.2. Evolution of spinodal decomposition under convection: cubic elements. In this final example we show the effects of a velocity field on spinodal decomposition. We use a cubic discontinuous Galerkin approximation and take $\gamma=1 / 200$. Note that to model the resulting thinner interface we apply a cubic polynomial approximation on a 64 by 64 uniform quadrilateral mesh. The initial datum $c_{0}$ within the circular domain is a small uniformly distributed random perturbation about zero; see the initial figure in Figure 4.6.


Fig. 4.6. Formation of bulk regions: spinodal decomposition under circular convection: $\gamma=1 / 200, \mathrm{Pe}=200$, $t=0,0.05,0.1,0.15$

As in spinodal decomposition in the absence of a velocity field we see that initially the two components are driven into bulk regions; see Figure 4.6. As before this initial motion occurs over a relatively short time-scale.

Due to the convection term, these bulk regions form concentric circles exhibiting a filament type structure as seen in [51]; see Figure 4.7. This convection-dominated motion occurs on a relatively short time-scale. Note that, when the order-parameter is in the form of a set of concentric circular regions $\nabla \cdot(\mathbf{u c})=0$, leading to the standard Cahn-Hilliard system that will drive any interface to one with constant curvature.


Fig. 4.7. Convection of bulk regions into circular regions: spinodal decomposition under circular convection: $\gamma=$ $1 / 200, \mathrm{Pe}=200, t=0.3,0.35,0.4,10$


FIG. 4.8. Spinodal decomposition under circular convection: $\gamma=1 / 200, \mathrm{Pe}=200, t=80,140,200,300$

Finally, motion of the phases continues due to the fact that we have considered a constant mobility function of $B(c)=1$ in our model; see Figure 4.8. This motion, due to the diffusion coefficient, occurs over a very large time-scale. In general, such a function restricts diffusion away from interfaces by degenerating to zero when $c= \pm 1$ and is introduced into (1.1a) as follows:

$$
\partial_{t} c-\frac{1}{\mathrm{Pe}} \nabla \cdot(B(c) \nabla w)+\nabla \cdot(\mathbf{u} c)=0
$$

see [19]. Hence, this model will allow diffusion away from the interface, ultimately leading to only two bulk regions.
5. Conclusions. We introduced a discontinuous Galerkin finite element method for the numerical approximation of the Cahn-Hilliard equation with a convection term. The model can be used to describe the competing processes of stirring and separation in a two-phase flow. Unlike a standard continuous finite element method, the discontinuous Galerkin method does not require any additional numerical stabilization in the presence of a convection term in the equation. We derived bounds, uniform in the discretization parameters, on the sequence of numerical solutions delivered by the method. We established an optimal-order bound in the broken $\mathrm{L}^{\infty}\left(\mathrm{H}^{1}\right)$ norm on the error between the order-parameter $c$ and its discontinuous Galerkin approximation; in addition, an optimal-order error bound was derived for the discontinuous Galerkin finite element approximation of the chemical potential $w$ in the broken $\mathrm{L}^{2}\left(\mathrm{H}^{1}\right)$ norm. The analytical results were illustrated by numerical simulations that compare solutions of the Cahn-Hilliard equation with and without a convection term.

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