# CONVEXITY IN SEMI-ALGEBRAIC GEOMETRY AND POLYNOMIAL OPTIMIZATION 

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#### Abstract

We review several (and provide new) results on the theory of moments, sums of squares and basic semi-algebraic sets when convexity is present. In particular, we show that under convexity, the hierarchy of semidefinite relaxations for polynomial optimization simplifies and has finite convergence, a highly desirable feature as convex problems are in principle easier to solve. In addition, if a basic semi-algebraic set $\mathbf{K}$ is convex but its defining polynomials are not, we provide two algebraic certificate of convexity which can be checked numerically. The second is simpler and holds if a sufficient (and almost necessary) condition is satisfied, it also provides a new condition for $\mathbf{K}$ to have semidefinite representation. For this we use (and extend) some of recent results from the author and Helton and Nie 6. Finally, we show that when restricting to a certain class of convex polynomials, the celebrated Jensen's inequality in convex analysis can be extended to linear functionals that are not necessarily probability measures.


## 1. Introduction

Motivation. This paper is a contribution to the new emerging field of convex semi-algebraic geometry, and its purpose is threefold: First we show that the moment approach for global polynomial optimization proposed in 13, and based on semidefinite programming (SDP), is consistent as it simplifies and/or has better convergence properties when solving convex problems. In other words, the SDP moment approach somehow "recognizes" convexity, a highly desirable feature for a general purpose method because, in principle, convex problems should be easier to solve.

We next review some recent results (and provide a new one) on the representation of convex basic semi-algebraic sets by linear matrix inequalities which show how convexity permits to derive relatively simple and explicit semidefinite representations. In doing so we also provide a certificate of convexity for $\mathbf{K}$ when its defining polynomials are not convex.

Finally, we consider the important Jensen's inequality in convex analysis. When restricting its application to a class of convex polynomials, we provide an extension to a class of linear functionals that are not necessarily probability measures.

To do so, we use (and sometimes extend) some recent results of the author [16, 17] and Helton and Nie [6]. We hope to convince the reader that convex semi-algebraic geometry is indeed a very specific subarea of real algebraic geometry which should

[^0]deserve more attention from both the optimization and real algebraic geometry research communities.

Background. I. Relatively recent results in the theory of moments and its dual theory of positive polynomials have been proved useful in polynomial optimization as they provide the basis of a specific convergent numerical approximation scheme. Namely, one can define a hierarchy of semidefinite relaxations (in short SDP-relaxations) of the original optimization problem whose associated monotone sequence of optimal values converges to the global optimum. For a more detail account of this approach, the interested reader is referred to e.g. Lasserre [13, 14, Parrilo [21], Schweighofer [29], and the many references therein.

Remarkably, practice seems to reveal that convergence is often fast and even finite. However, the size of the SDP-relaxations grows rapidly with the rank in the hierarchy; typically the $r$-th SDP-relaxation in the hierarchy has $O\left(n^{2 r}\right)$ variables and semidefinite matrices of $O\left(n^{r}\right)$ sizes (where $n$ is the number of variables in the original problem). On the other hand, it is well-known that a large class of convex optimization problems can be solved efficiently; see e.g. Ben Tal and Nemirovski [1]. Therefore, as the SDP-based moment approach is dedicated to solving difficult non convex (most of the time NP-hard) problems, it should have the highly desirable feature to somehow recognize "easy" problems like convex ones. That is, when applied to such easy problems it should show some significant improvement or a particular nice behavior not necessarily valid in the general case. Notice that this is not the case of the LP-based moment-approach described in [14, 15] for which only asymptotic (and not finite) convergence occurs in general (and especially for convex problems), a rather annoying feature. However, for SDP-relaxations, some results of [17] already show that indeed convexity helps as one provides specialized representation results for convex polynomials that are nonnegative on a basic semialgebraic set.
II. Next, in view of the potential of semidefinite programming techniques, an important issue is the characterization of convex sets that are semidefinite representable (in short called SDr sets). A SDr set $\mathbf{K} \subset \mathbb{R}^{n}$ is the projection of a set defined by linear matrix inequalities (LMIs). That is,

$$
\mathbf{K}:=\left\{x \in \mathbb{R}^{n}: \exists y \in \mathbb{R}^{s} \text { s.t. } \quad A_{0}+\sum_{i=1}^{n} x_{i} A_{i}+\sum_{j=1}^{s} y_{j} B_{j} \succeq 0\right\}
$$

for some real symmetric matrices $\left(A_{i}, B_{j}\right)$ (and where $A \succeq 0$ stands for $A$ is positive semidefinite). For more details, the interested reader is referred to Ben Tal and Nemirovski [1], Lewis et al. [19], Parrilo [22], and more recently, Chua and Tuncel [2], Helton and Nie [6, 7], Henrion [8] and Lasserre [16]. For compact basic semialgebraic sets

$$
\begin{equation*}
\mathbf{K}:=\left\{x \in \mathbb{R}^{n}: g_{j}(x) \geq 0, \quad j=1, \ldots, m\right\} \tag{1.1}
\end{equation*}
$$

recent results of Helton and Nie [6, 7] and the author [16] provide sufficient conditions on the defining polynomials $\left(g_{j}\right) \subset \mathbb{R}[X]$ for the convex hull co $(\mathbf{K})(\equiv \mathbf{K}$ if $\mathbf{K}$ is convex) to be SDr. Again, an interesting issue is to analyze whether convexity of $\mathbf{K}$ (with or without concavity of the defining polynomials $\left(g_{j}\right)$ ) provides some additional insights and/or simplifications. Another interesting issue is how to detect whether a basic semi-algebraic set $\mathbf{K}$ is convex, or equivalently, how to obtain an algebraic certificate of convexity of $\mathbf{K}$ from its defining polynomials $\left(g_{j}\right)$. By
certificate we mean a mathematical statement that obviously implies convexity of $\mathbf{K}$, can be checked numerically and does not require infinitely many tests. So far, and to the best of our knowledge, such a certificate does not exist.
III. The celebrated Jensen's inequality is an important result in convex analysis which states that $E_{\mu}(f(x)) \geq f\left(E_{\mu}(x)\right)$ for a convex function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and a probability measure $\mu$ with $E_{\mu}(x)<\infty$. A third goal of this paper is to analyze whether when restricted to a certain class of convex polynomials, Jensen's inequality can be extended to a class of linear functionals larger than the class of probability measures.

Contribution. Concerning issue I: We first recall two previous results proved in [17: (a) the cone of convex SOS is dense (for the $l_{1}$-norm of coefficients) in the cone of nonnegative convex polynomials, and (b) a convex Positivstellensatz for convex polynomials nonnegative on $\mathbf{K}$ (a specialization of Putinar's Positivstellensatz). We then analyze the role of convexity for the polynomial optimization problem

$$
\begin{equation*}
\mathbf{P}: \quad f^{*}=\min _{x}\{f(x): x \in \mathbf{K}\} \tag{1.2}
\end{equation*}
$$

with $\mathbf{K}$ as in (1.1), and show that indeed convexity helps and makes the SDPrelaxations more efficient. In particular, when $\mathbf{K}$ is convex and Slater's condition 1 holds, by using some recent results of Helton and Nie [6], we show that
(i) If the polynomials $f,\left(-g_{j}\right)$ are all convex and $\nabla^{2} f$ is positive definite (and so $f$ is strictly convex) on $\mathbf{K}$, then the hierarchy of SDP-relaxations has finite convergence.
(ii) If $f$ and $\left(-g_{j}\right)$ are all SOS-convex (i.e. their Hessian is a SOS matrix polynomial), then $\mathbf{P}$ reduces to solving a single SDP whose index in the hierarchy is readily available.

Concerning II: Under certain sufficient conditions on the $\left(g_{j}\right)$ (typically some second order positive curvature conditions) Helton and Nie [6, 7] have proved that co (K) (or $\mathbf{K}$ if convex) has a semidefinite representation that uses Schmüdgen or Putinar SOS representation of polynomials positive on $\mathbf{K}$; see [6, 17. Yet, in general its dimension depends on an unknown degree parameter in Schmüdgen (or Putinar) SOS representation. Our contribution is to provide a new sufficient condition for existence of a SDr when $\mathbf{K}$ is compact with nonempty interior and its boundary satisfies some nondegeneracy assumption. It translates the geometric property of convexity of $\mathbf{K}$ into a SOS Putinar representation of some appropriate polynomial obtained from each $g_{j}$. When satisfied, this representation provides an algebraic certificate of convexity for $\mathbf{K}$ and it is almost necessary in the sense that it always holds true when relaxed by an arbitrary $\epsilon>0$. It also contains as special cases Helton and Nie 6] sufficient conditions of SOS-convexity or strict convexity on $\partial \mathbf{K}$ of the $-g_{j}$ 's, and leads to an explicit semidefinite representation of $\mathbf{K}$. We also provide a more general algebraic certificate based on Stengle's Positivstellensatz, but more complex and heavy to implement and so not very practical. In practice both certificates are obtained by solving a semidefinite program. Therefore, because of unavoidable numerical inaccuracies, the certificate is valid only up to machine precision.

[^1]Concerning III, we prove that when restricting its application to the subclass of SOS-convex polynomials, Jensen's inequality can be extended to all linear functionals $L_{\mathbf{y}}$ (with $L_{\mathbf{y}}(1)=1$ ) in the dual cone of SOS polynomials, hence not necessarily probability measures.

Some of the results already obtained in [6, 16] and in the present paper strongly suggest that the class of SOS-convex polynomials introduced in Helton and Nie 6] is particularly nice and should deserve more attention.

## 2. Notation, definitions and preliminary Results

Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables $X=\left(X_{1}, \ldots, X_{n}\right)$, and let $\Sigma^{2}[X] \subset \mathbb{R}[X]$ be the subset of sums of squares (SOS) polynomials. Denote $\mathbb{R}[X]_{d} \subset \mathbb{R}[X]$ be the set of polynomials of degree at most $d$, which forms a vector space of dimension $s(d)=\binom{n+d}{d}$. If $f \in \mathbb{R}[X]_{d}$, write $f(X)=\sum_{\alpha \in \mathbb{N}^{n}} f_{\alpha} X^{\alpha}$ in the usual canonical basis $\left(X^{\alpha}\right)$, and denote by $\mathbf{f}=\left(f_{\alpha}\right) \in \mathbb{R}^{s(d)}$ its vector of coefficients. Also write $\|f\|_{1}\left(=\|\mathbf{f}\|_{1}:=\sum_{\alpha}\left|f_{\alpha}\right|\right)$ the $l_{1}$-norm of $f$. Finally, denote by $\Sigma^{2}[X]_{d} \subset \Sigma^{2}[X]$ the subset of SOS polynomials of degree at most $2 d$.

We use the notation $X$ for the variable of a polynomial $X \mapsto f(X)$ and $x$ when $x$ is a point of $\mathbb{R}^{n}$, as for instance in $\left\{x \in \mathbb{R}^{n}: f(x) \geq 0\right\}$.

Moment matrix. With $\mathbf{y}=\left(y_{\alpha}\right)$ being a sequence indexed in the canonical basis $\left(X^{\alpha}\right)$ of $\mathbb{R}[X]$, let $L_{\mathbf{y}}: \mathbb{R}[X] \rightarrow \mathbb{R}$ be the linear functional

$$
f \quad\left(=\sum_{\alpha} f_{\alpha} X^{\alpha}\right) \quad \mapsto \quad L_{\mathbf{y}}(f)=\sum_{\alpha} f_{\alpha} y_{\alpha}
$$

and let $M_{d}(\mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical basis $\left(X^{\alpha}\right)$, and defined by:

$$
M_{d}(\mathbf{y})(\alpha, \beta):=L_{\mathbf{y}}\left(X^{\alpha+\beta}\right)=y_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{N}_{d}^{n}
$$

with $\mathbb{N}_{d}^{n}:=\left\{\alpha \in \mathbb{N}^{n}:|\alpha|\left(=\sum_{i} \alpha_{i}\right) \leq d\right\}$.
Localizing matrix. Similarly, with $\mathbf{y}=\left(y_{\alpha}\right)$ and $g \in \mathbb{R}[X]$ written

$$
X \mapsto g(X)=\sum_{\gamma \in \mathbb{N}^{n}} g_{\gamma} X^{\gamma}
$$

let $M_{d}(g \mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical basis $\left(X^{\alpha}\right)$, and defined by:

$$
M_{d}(g \mathbf{y})(\alpha, \beta):=L_{\mathbf{y}}\left(g(X) X^{\alpha+\beta}\right)=\sum_{\gamma} g_{\gamma} y_{\alpha+\beta+\gamma}
$$

for every $\alpha, \beta \in \mathbb{N}_{d}^{n}$.
Putinar Positivstellensatz. Let $Q(g) \subset \mathbb{R}[X]$ be the quadratic module generated by the polynomials $\left(g_{j}\right) \subset \mathbb{R}[X]$, that is,

$$
\begin{equation*}
Q(g):=\left\{\sigma_{0}+\sum_{j=1}^{m} \sigma_{j} g_{j}: \quad\left(\sigma_{j}\right) \subset \Sigma^{2}[X]\right\} . \tag{2.1}
\end{equation*}
$$

Assumption 2.1. $K \subset \mathbb{R}^{n}$ is a compact basic semi-algebraic set defined as in (1.1) and the quadratic polynomial $X \mapsto M-\|X\|^{2}$ belongs to $Q(g)$.

Assumption 2.1 is not very restrictive. For instance, it holds if every $g_{j}$ is affine (i.e., $\mathbf{K}$ is a convex polytope) or if the level set $\left\{x: g_{j}(x) \geq 0\right\}$ is compact for some $j \in\{1, \ldots, m\}$. In addition, if $M-\|x\| \geq 0$ for all $x \in \mathbf{K}$, then it suffices to add the redundant quadratic constraint $M^{2}-\|x\|^{2} \geq 0$ to the definition (1.1) of $\mathbf{K}$ and Assumption 2.1 will hold true.

Theorem 2.2 (Putinar's Positivstellensatz [24]). Let Assumption 2.1 hold. If $f \in \mathbb{R}[X]$ is (strictly) positive on $\mathbf{K}$, then $f \in Q(g)$. That is:

$$
\begin{equation*}
f=\sigma_{0}+\sum_{j=1}^{m} \sigma_{j} g_{j} \tag{2.2}
\end{equation*}
$$

for some SOS polynomials $\left(\sigma_{j}\right) \subset \Sigma^{2}[X]$.
2.1. A hierarchy of semidefinite relaxations (SDP-relaxations). Let $\mathbf{P}$ be the optimization problem (1.2) with $\mathbf{K}$ as in (1.1) and let $r_{j}=\left\lceil\left(\operatorname{deg} g_{j}\right) / 2\right\rceil, j=$ $1, \ldots, m$. With $f \in \mathbb{R}[X]$ and $2 r \geq \max \left[\operatorname{deg} f, \max _{j} 2 r_{j}\right]$, consider the hierarchy of semidefinite relaxations $\left(\mathbf{Q}_{r}\right)$ defined by:

$$
\mathbf{Q}_{r}:\left\{\begin{array}{cl}
\begin{array}{cl}
\inf & L_{\mathbf{y}}(f) \\
\text { s.t. } & M_{r}(\mathbf{y}) \succeq 0 \\
& M_{r-r_{j}}\left(g_{j} \mathbf{y}\right) \succeq 0, \quad j=1, \ldots, m \\
& y_{0}=1
\end{array}, \tag{2.3}
\end{array}\right.
$$

with optimal value denoted by $\inf \mathbf{Q}_{r}$. One says that $\mathbf{Q}_{r}$ is solvable if it has an optimal solution (in which case one writes $\inf \mathbf{Q}_{r}=\min \mathbf{Q}_{r}$ ). The dual of $\mathbf{Q}_{r}$ reads

$$
\mathbf{Q}_{r}^{*}:\left\{\begin{array}{ll}
\sup & \lambda  \tag{2.4}\\
\text { s.t. } & f-\lambda=\sigma_{0}+\sum_{j=1}^{m} \sigma_{j} g_{j} \\
& \sigma_{j} \in \Sigma^{2}[X], \quad j=0,1, \ldots, m \\
& \operatorname{deg} \sigma_{0}, \operatorname{deg} \sigma_{j}+\operatorname{deg} g_{j} \leq 2 r, \quad j=1, \ldots, m
\end{array},\right.
$$

with optimal value denoted by $\sup \mathbf{Q}_{r}^{*}\left(\right.$ or $\max \mathbf{Q}_{r}^{*}$ if the sup is attained).
By weak duality $\sup \mathbf{Q}_{r}^{*} \leq \inf \mathbf{Q}_{r}$ for every $r \in \mathbb{N}$ and under Assumption 2.1, $\inf \mathbf{Q}_{r} \uparrow f^{*}$ as $r \rightarrow \infty$. For a more detailed account see e.g. [13].
2.2. Convexity and SOS-convexity. We first briefly recall basic facts on a multivariate convex function. If $C \subseteq \mathbb{R}^{n}$ is a nonempty convex set, a function $f: C \rightarrow \mathbb{R}$ is convex on $C$ if and only if

$$
f(\lambda x+(1-\lambda) y) \leq \lambda f(x)+(1-\lambda) f(y), \quad \forall \lambda \in(0,1), x, y \in C
$$

Similarly, $f$ is strictly convex on $C$ if and only if the above inequality is strict for every $x, y \in C, x \neq y$, and all $\lambda \in(0,1)$.

If $C \subseteq \mathbb{R}^{n}$ is an open convex set and $f$ is twice differentiable on $C$, then $f$ is convex on $C$ if and only if its Hessian $\nabla^{2} f$ is positive semidefinite on $C$ (denoted $\nabla^{2} f \succeq 0$ on $C$ ). Finally, if $\nabla^{2} f$ is positive definite on $C$ (denoted $\nabla^{2} f \succ 0$ on $C$ ) then $f$ is strictly convex on $C$.

SOS-convexity. Helton and Nie [6] have introduced the following interesting subclass of convex polynomials, called SOS-convex polynomials.

Definition 2.3 (Helton and Nie [6]). A polynomial $f \in \mathbb{R}[X]_{2 d}$ is said to be SOSconvex if $\nabla^{2} f$ is SOS, that is, $\nabla^{2} f=L L^{T}$ for some real matrix polynomial $L \in$ $\mathbb{R}[X]^{n \times s}$ (for some $s \in \mathbb{N}$ ).

As noted in [6, an important feature of SOS-convexity is that it can be can be checked numerically by solving a SDP. They have also proved the following important property:

Lemma 2.4 (Helton and Nie [6, Lemma 7]). If a symmetric matrix polynomial $P \in \mathbb{R}[X]^{r \times r}$ is SOS then for any $u \in \mathbb{R}^{n}$, the double integral

$$
X \mapsto \quad F(X, u):=\int_{0}^{1} \int_{0}^{t} P(u+s(X-u)) d s d t
$$

is also a symmetric SOS matrix polynomial in $\mathbb{R}[X]^{r \times r}$.
And also:
Lemma 2.5 (Helton and Nie [6, Lemma 8]). For a polynomial $f \in \mathbb{R}[X]$ and every $x, u \in \mathbb{R}^{n}$ :

$$
\begin{aligned}
f(x)= & f(u)+\nabla f(u)^{T}(x-u) \\
& +(x-u)^{T} \underbrace{\int_{0}^{1} \int_{0}^{t} \nabla^{2} f(u+s(x-u)) d s d t}_{F(x, u)}(x-u) .
\end{aligned}
$$

And so if $f$ is SOS-convex and $f(u)=0, \nabla f(u)=0$, then $f$ is a SOS polynomial.
2.3. An extension of Jensen's inequality. Recall that if $\mu$ is a probability measure on $\mathbb{R}^{n}$ with $E_{\mu}(x)<\infty$, Jensen's inequality states that if $f \in L_{1}(\mu)$ and $f$ is convex, then

$$
E_{\mu}(f(x)) \geq f\left(E_{\mu}(x)\right)
$$

a very useful property in many applications.
We now provide an extension of Jensen's inequality when one restricts its application to the class of SOS-convex polynomials. Namely, we may consider the linear functionals $L_{\mathbf{y}}: \mathbb{R}[X]_{2 d} \rightarrow \mathbb{R}$ in the dual cone of $\Sigma^{2}[X]_{d}$, that is, vectors $\mathbf{y}=\left(y_{\alpha}\right)$ such that $M_{d}(\mathbf{y}) \succeq 0$ and $y_{0}=L_{\mathbf{y}}(1)=1$; hence $\mathbf{y}$ is not necessarily the (truncated) moment sequence of some probability measure $\mu$. Crucial in the proof is Lemma 2.4 of Helton and Nie.

Theorem 2.6. Let $f \in \mathbb{R}[X]_{2 d}$ be $S O S$-convex, and let $\mathbf{y}=\left(y_{\alpha}\right)_{\alpha \in \mathbb{N}_{2 d}^{n}}$ satisfy $y_{0}=1$ and $M_{d}(\mathbf{y}) \succeq 0$. Then:

$$
\begin{equation*}
L_{\mathbf{y}}(f(X)) \geq f\left(L_{\mathbf{y}}(X)\right) \tag{2.5}
\end{equation*}
$$

where $L_{\mathbf{y}}(X)=\left(L_{\mathbf{y}}\left(X_{1}\right), \ldots, L_{\mathbf{y}}\left(X_{n}\right)\right)$.

Proof. Let $z \in \mathbb{R}^{n}$ be fixed, arbitrary, and consider the polynomial $X \mapsto f(X)-$ $f(z)$. Then,

$$
\begin{equation*}
f(X)-f(z)=\langle\nabla f(z), X-z\rangle+\langle(X-z), F(X)(X-z)\rangle, \tag{2.6}
\end{equation*}
$$

with $F: \mathbb{R}^{n} \rightarrow \mathbb{R}[X]^{n \times n}$ being the matrix polynomial

$$
X \mapsto \quad F(X):=\int_{0}^{1} \int_{0}^{t} \nabla^{2} f(z+s(X-z)) d s d t
$$

As $f$ is SOS-convex, by Lemma [2.4, $F$ is a SOS matrix polynomial and so the polynomial $X \mapsto \Delta(X):=\left\langle(X-z), F(X)(X-z)\right.$ is SOS, i.e., $\Delta \in \Sigma^{2}[X]$. Then applying $L_{\mathbf{y}}$ to the polynomial $X \mapsto f(X)-f(z)$ and using (2.6) yields (recall that $y_{0}=1$ )

$$
\begin{aligned}
L_{\mathbf{y}}(f(X))-f(z) & =\left\langle\nabla f(z), L_{\mathbf{y}}(X)-z\right\rangle+L_{\mathbf{y}}(\Delta(X)) \\
& \geq\left\langle\nabla f(z), L_{\mathbf{y}}(X)-z\right\rangle \quad\left[\text { because } L_{\mathbf{y}}(\Delta(X)) \geq 0\right]
\end{aligned}
$$

As $z \in \mathbb{R}^{n}$ was arbitrary, taking $z:=L_{\mathbf{y}}(X)\left(=\left(L_{\mathbf{y}}\left(X_{1}\right), \ldots, L_{\mathbf{y}}\left(X_{n}\right)\right)\right.$ yields the desired result.

As a consequence we also get:
Corollary 2.7. Let $f$ be a convex univariate polynomial, $g \in \mathbb{R}[X]$ (and so $f \circ g \in$ $\mathbb{R}[X])$. Let $d:=\lceil(\operatorname{deg} f \circ g) / 2\rceil$, and let $\mathbf{y}=\left(y_{\alpha}\right)_{\alpha \in \mathbb{N}_{2 d}^{n}}$ be such that $y_{0}=1$ and $M_{d}(\mathbf{y}) \succeq 0$. Then:

$$
\begin{equation*}
L_{\mathbf{y}}[f(g(X))] \geq f\left(L_{\mathbf{y}}[g(X)]\right) \tag{2.7}
\end{equation*}
$$

Proof. Again let $z \in \mathbb{R}^{n}$ be fixed, arbitrary, and consider the univariate polynomial $Y \mapsto f(Y)-f(z)$ so that (2.6) holds. That is,

$$
f(Y)-f(z)=f^{\prime}(z)(Y-z)+F(Y)(Y-z)^{2}
$$

with $F: \mathbb{R} \rightarrow \mathbb{R}[Y]$ being the univariate polynomial

$$
Y \mapsto \quad F(Y):=\int_{0}^{1} \int_{0}^{t} f^{\prime \prime}(z+s(Y-z)) d s d t
$$

As $f$ is convex, $f^{\prime \prime} \geq 0$, and so the univariate polynomial $Y \mapsto F(Y)(Y-z)^{2}$ is nonnegative, and being univariate, is SOS. Therefore, with $Y:=g(X)$,

$$
f(g(X))-f(z)=f^{\prime}(z)(g(X)-z)+F(g(X))(g(X)-z)^{2}
$$

and so

$$
\begin{aligned}
L_{\mathbf{y}}[f(g(X))]-f(z) & =f^{\prime}(z)\left(L_{\mathbf{y}}[g(X)]-z\right)+L_{\mathbf{y}}\left[F(g(X))(g(X)-z)^{2}\right] \\
& \geq f^{\prime}(z)\left(L_{\mathbf{y}}[g(X)]-z\right)
\end{aligned}
$$

and taking $z:=L_{\mathbf{y}}[g(X)]$ yields the desired result.

Hence the class of SOS-convex polynomials has the very interesting property to extend Jensen's inequality to some linear functionals that are not necessarily coming from a probability measure.

## 3. Semidefinite relaxations in the convex case

3.1. A convex Positivstellensatz. Let $\mathbf{K}$ be as in (1.1) and define $Q_{c}(g) \subset \mathbb{R}[X]$ to be the set:

$$
\begin{equation*}
Q_{c}(g):=\left\{\sigma_{0}+\sum_{j=1}^{m} \lambda_{j} g_{j}: \quad \lambda \in \mathbb{R}_{+}^{m} ; \sigma_{0} \in \Sigma^{2}[X], \sigma_{0} \text { convex }\right\} \subset Q(g) \tag{3.1}
\end{equation*}
$$

The set $Q_{c}(g)$ is a specialization of $Q(g)$ in (2.1) to the convex case, in that the weights asociated with the $g_{j}$ 's are nonnegative scalars, i.e., SOS polynomials of degree 0 , and the SOS polynomial $\sigma_{0}$ is convex. In particular, every $f \in Q_{c}(g)$ is nonnegative on $\mathbf{K}$. Let $\mathcal{F}_{\mathbf{K}} \subset \mathbb{R}[X]$ be the convex cone of convex polynomials nonnegative on $\mathbf{K}$.

Theorem 3.1 (Lasserre [17]). Let $\mathbf{K}$ be as in (1.1), Slater's condition hold and $g_{j}$ be concave for every $j=1, \ldots, m$.

Then with $Q_{c}(g)$ as in (3.1), the set $Q_{c}(g) \cap \mathcal{F}_{\mathbf{K}}$ is dense in $\mathcal{F}_{\mathbf{K}}$ for the $l_{1}$-norm $\|\cdot\|_{1}$. In particular, if $\mathbf{K}=\mathbb{R}^{n}$ (so that $\mathcal{F}_{\mathbb{R}^{n}}=: \mathcal{F}$ is now the set of nonnegative convex polynomials), then $\Sigma^{2}[X] \cap \mathcal{F}$ is dense in $\mathcal{F}$.

Theorem 3.1 states that if $f$ is convex and nonnegative on $\mathbf{K}$ (including the case $\mathbf{K} \equiv \mathbb{R}^{n}$ ) then one may approximate $f$ by a sequence $\left\{f_{\epsilon r}\right\} \subset Q_{c}(g) \cap \mathcal{F}_{\mathbf{K}}$ with $\left\|f-f_{\epsilon r}\right\|_{1} \rightarrow 0$ as $\epsilon \rightarrow 0$ (and $r \rightarrow \infty$ ). For instance, with $r_{0}:=\lfloor(\operatorname{deg} f) / 2\rfloor+1$,

$$
\begin{align*}
X \mapsto f_{\epsilon r}(X) & :=f+\epsilon\left(\theta_{r_{0}}(X)+\theta_{r}(X)\right), \quad \text { with } \\
X \mapsto \theta_{r}(X) & :=1+\sum_{k=1}^{r} \sum_{i=1}^{n} \frac{X_{i}^{2 k}}{k!} \quad r \geq r_{\epsilon} \tag{3.2}
\end{align*}
$$

for some $r_{\epsilon}$; see Lasserre [17] for details. Observe that Theorem 3.1] provides $f$ with a certificate of nonnegativity on $\mathbf{K}$. Indeed, let $x \in \mathbf{K}$ be fixed arbitrary. Then as $f_{\epsilon r} \in Q_{c}(g)$ one has $f_{\epsilon r}(x) \geq 0$. Letting $\epsilon \downarrow 0$ yields $0 \leq \lim _{\epsilon \rightarrow 0} f_{\epsilon r}(x)=f(x)$. And as $x \in \mathbf{K}$ was arbitray, $f \geq 0$ on $\mathbf{K}$.

Theorem 3.1 is a convex (weak) version of Theorem[2.2 (Putinar's Positivstellensatz) where one replaces the quadratic module $Q(g)$ with its subset $Q_{c}(g)$. We call it a weak version of Theorem 2.2 because it invokes a density result (i.e. $f_{\epsilon r} \in Q_{c}(g)$ whereas $f$ might not be an element of $\left.Q_{c}(g)\right)$. Notice that $f$ is allowed to be nonnegative (instead of strictly positive) on $\mathbf{K}$ and $\mathbf{K}$ need not be compact; recall that extending Theorem 2.2 to non compact basic semi-algebraic sets $\mathbf{K}$ and to polynomials $f$ nonnegative on $\mathbf{K}$ is hopeless in general; see Scheiderer [26].

Corollary 3.2. Let $\mathbf{K}$ be as in (1.1), $f \in \mathbb{R}[X]$ with $f^{*}:=\min _{x}\{f(x): x \in \mathbf{K}\}$ and let $d:=\max \left[\lceil(\operatorname{deg} f) / 2\rceil, \max _{j}\left\lceil\left(\operatorname{deg} g_{j}\right) / 2\right\rceil\right]$. Consider the simplified SDPrelaxation

$$
\widehat{\mathbf{Q}}:\left\{\begin{array}{cl}
\underset{\mathbf{i n f}}{ }: & L_{\mathbf{y}}(f)  \tag{3.3}\\
\text { s.t. } & M_{d}(\mathbf{y}) \succeq 0 \\
& L_{\mathbf{y}}\left(g_{j}\right) \geq 0, \\
& y_{0}=1
\end{array} \quad j=1, \ldots, m\right.
$$

and its dual

$$
\widehat{\mathbf{Q}}^{*}: \begin{cases}\sup _{\gamma, \sigma_{0}, \lambda} & \gamma  \tag{3.4}\\ \text { s.t. } & f-\gamma=\sigma_{0}+\sum_{j=1}^{m} \lambda_{j} g_{j} \\ & \sigma_{0} \in \Sigma^{2}[X]_{d} ; \lambda_{j} \geq 0, \quad j=1, \ldots, m\end{cases}
$$

(a) If $f-f^{*} \in Q_{c}(g)$ then the SDP-relaxation $\widehat{\mathbf{Q}}$ and its dual $\widehat{\mathbf{Q}}^{*}$ are exact.
(b) If $f,-g_{j} \in \mathbb{R}[X]$ are convex, $j=1, \ldots, m$, and if $\mathbf{y}$ is an optimal solution of $\widehat{\mathbf{Q}}$ which satisfies

$$
\begin{equation*}
\operatorname{rank} M_{d}(\mathbf{y})=\operatorname{rank} M_{d-1}(\mathbf{y}) \tag{3.5}
\end{equation*}
$$

then $\widehat{\mathbf{Q}}$ is exact and $x^{*}:=\left(L_{\mathbf{y}}\left(X_{i}\right)\right) \in \mathbf{K}$ is a (global) minimizer of $f$ on $\mathbf{K}$.
Proof. (a) If $f-f^{*} \in Q_{c}(g)$, i.e., if $f-f^{*}=\sigma_{0}+\sum_{j=1}^{m} \lambda_{j} g_{j}$, with $\sigma_{0} \in \Sigma^{2}[X]_{d}$ and $\lambda \in \mathbb{R}_{+}^{m}$, the triplet $\left(f^{*}, \sigma_{0}, \lambda\right)$ is a feasible solution of $\widehat{\mathbf{Q}}^{*}$ with value $f^{*}$. Therefore, as $\sup \widehat{\mathbf{Q}}^{*} \leq \inf \widehat{\mathbf{Q}} \leq f^{*}$, the SDP-relaxation $\widehat{\mathbf{Q}}$ and its dual $\widehat{\mathbf{Q}}^{*}$ are exact. In fact, $\left(f^{*}, \sigma_{0}, \lambda\right)$ is an optimal solution of $\widehat{\mathbf{Q}}^{*}$.
(b) If $\mathbf{y}$ satisfies the rank condition (3.5) then by the flat extension theorem of Curto and Fialkow [4], y is the (truncated) moment sequence of an atomic probability measure $\mu$ on $\mathbb{R}^{n}$, say $\mu=\sum_{k=1}^{s} \lambda_{k} \delta_{x(k)}$ with $s=\operatorname{rank} M_{d}(\mathbf{y}), 0<$ $\lambda_{k} \leq 1, \sum_{k} \lambda_{k}=1$, and $\delta_{x(k)}$ being the Dirac measure at $x(k) \in \mathbb{R}^{n}, k=1, \ldots, s$. Let $x^{*}:=\sum_{k} \lambda_{k} x(k)=\left(L_{\mathbf{y}}\left(X_{i}\right)\right) \in \mathbb{R}^{n}$. Then $f^{*} \geq L_{\mathbf{y}}(f)$ and by convexity of $f$, $L_{\mathbf{y}}(f)=\sum_{k} \lambda_{k} f(x(k)) \geq f\left(\sum_{k} \lambda_{k} x(k)\right)=f\left(x^{*}\right)$. Similarly, by convexity of $-g_{j}$, $0 \leq L_{\mathbf{y}}\left(g_{j}\right)=\sum_{k} \lambda_{k} g_{j}(x(k)) \leq g_{j}\left(\sum_{k} \lambda_{k} x(k)\right)=g_{j}\left(x^{*}\right), j=1, \ldots, m$. Therefore, $x^{*} \in \mathbf{K}$ and as $f\left(x^{*}\right) \leq f^{*}, x^{*}$ is a global minimizer of $f$ on $\mathbf{K}$.

Notice that K in Corollary 3.2 need not be compact. Also, Corollary 3.2 (b) has practical value because in general one does not know whether $f-f^{*} \in Q_{c}(g)$ (despite that in the convex case, $f-f^{*} \in \mathcal{F}_{\mathbf{K}}$ and $Q_{c}(g) \cap \mathcal{F}_{\mathbf{K}}$ is dense in $\mathcal{F}_{\mathbf{K}}$ ). However, one may still solve $\widehat{\mathbf{Q}}$ and check whether the rank condition (3.5) is satisfied. If in solving $\widehat{\mathbf{Q}}_{r}$, the rank condition (3.5) is not satisfied, then other sufficient conditions can be exploited as we next see.
3.2. The SOS-convex case. Part (a) of the following result is already contained in Lasserre [17, Cor. 2.5].

Theorem 3.3. Let $\mathbf{K}$ be as in (1.1) and Slater's condition hold. Let $f \in \mathbb{R}[X]$ be such that $f^{*}:=\inf _{x}\{f(x): x \in \mathbf{K}\}=f\left(x^{*}\right)$ for some $x^{*} \in \mathbf{K}$. If $f$ is SOS-convex and $-g_{j}$ is SOS-convex for every $j=1, \ldots, m$, then:
(a) $f-f^{*} \in Q_{c}(g)$.
(b) The simplified SDP-relaxation $\widehat{\mathbf{Q}}$ in (3.3) and its dual (3.4) are exact and solvable. If $\mathbf{y}$ is an optimal solution of $\widehat{\mathbf{Q}}$ then $x^{*}:=\left(L_{\mathbf{y}}\left(X_{i}\right)\right) \in \mathbf{K}$ is a global minimizer of $f$ on $\mathbf{K}$.
Proof. (a) is proved in [17, Cor. 2.5]. (b) That $\widehat{\mathbf{Q}}$ is exact follows from (a) and Corollary 3.2(a). Hence it is solvable (e.g. take $\mathbf{y}$ to be the moment sequence associated with the Dirac measure at a global minimizer $\left.x^{*} \in \mathbf{K}\right)$. So let $\mathbf{y}$ be an
optimal solution of $\widehat{\mathbf{Q}}$, hence with $f^{*}=L_{\mathbf{y}}(f)$. As $-g_{j}$ is SOS-convex for every $j$, then by Theorem 2.6, $0 \leq L_{\mathbf{y}}\left(g_{j}\right) \leq g_{j}\left(x^{*}\right)$ with $x^{*}:=\left(L_{\mathbf{y}}\left(X_{i}\right)\right)$ and so $x^{*} \in \mathbf{K}$. Similarly, as $f$ is SOS-convex, we also have $f^{*}=L_{\mathbf{y}}(f) \geq f\left(x^{*}\right)$ which proves that $f\left(x^{*}\right)=f^{*}$ and $x^{*}$ is a global minimizer of $f$ on $\mathbf{K}$. Finally, as by (a) $f-f^{*} \in Q_{c}(g)$ then $\widehat{\mathbf{Q}}^{*}$ is exact and solvable.
(Again notice that $\mathbf{K}$ in Theorem 3.3 need not be compact.) So the class of SOS-convex polynomials is particularly interesting. Not only Jensen's inequality can be extended to some linear functionals that are not coming from a probability measure, but one may also solve SOS-convex optimization problems $\mathbf{P}$ in (1.2) (i.e. with $f$ and $\mathbf{K}$ defined with SOS-convex polynomials) by solving the single semidefinite program (3.3).

Notice that a self-concordant ${ }^{2}$ logarithmic barrier function exists for (3.3) whereas the logarithmic barrier function with barrier parameter $\mu$ :

$$
\begin{equation*}
x \mapsto \phi_{\mu}(x):=\mu f(x)-\sum_{j=1}^{m} \ln \left(-g_{j}(x)\right), \tag{3.6}
\end{equation*}
$$

associated with $\mathbf{P}$, is not self-concordant in general. Therefore, despite (3.3) involves additional variables (a lifting), solving (3.3) via an interior point method might be more efficient than solving $\mathbf{P}$ by using the logarithmic barrier function (3.6) with no lifting. In addition, all SOS-convex polynomials nonnegative on $\mathbf{K}$ and which attain their minimum on $\mathbf{K}$, belong to $Q_{c}(g)$, a very specific version of Putinar Positivstellensatz (as $f$ is only nonnegative and $\mathbf{K}$ need not be compact).
3.3. The strictly convex case. If $f$ or some of the $-g_{j}$ 's is not SOS-convex but $\nabla^{2} f \succ 0$ (so that $f$ is strictly convex) and $-g_{j}$ is convex for every $j=1, \ldots, m$, then inspired by a nice argument from Helton and Nie 6] for existence of a semidefinite representation of convex sets, one obtains the following result.

Theorem 3.4. Let $\mathbf{K}$ be as in (1.1) and let Assumption 2.1 and Slater's condition hold. Assume that $f,-g_{j} \in \mathbb{R}[X]$ are convex, $j=1, \ldots, m$, with $\nabla^{2} f \succ 0$ on $\mathbf{K}$.

Then the hierarchy of SDP-relaxations defined in (2.3) has finite convergence. That is, $f^{*}=\sup \mathbf{Q}_{r}^{*}=\inf \mathbf{Q}_{r}$ for some index $r$. In addition, $\mathbf{Q}_{r}$ and $\mathbf{Q}_{r}^{*}$ are solvable so that $f^{*}=\max \mathbf{Q}^{*}=\min \mathbf{Q}_{r}$.

Proof. Let $x^{*} \in \mathbf{K}$ be a global minimizer (i.e. $f^{*}=f\left(x^{*}\right)$ ). As Slater's condition holds, there exists a vector of Karush-Kuhn-Tucker (KKT) multipliers $\lambda \in \mathbb{R}_{+}^{m}$ such that the (convex) Lagrangian $L_{f} \in \mathbb{R}[X]$ defined by

$$
\begin{equation*}
X \mapsto L_{f}(X):=f(X)-f^{*}-\sum_{j=1}^{m} \lambda_{j} g_{j}(X) \tag{3.7}
\end{equation*}
$$

has a global minimum at $x^{*} \in \mathbf{K}$, i.e., $\nabla L_{f}\left(x^{*}\right)=0$. In addition, $\lambda_{j} g_{j}\left(x^{*}\right)=0$ for every $j=1, \ldots, m$ and $L_{f}\left(x^{*}\right)=0$. Then, by Lemma 2.5.

$$
L_{f}(X)=\left\langle\left(X-x^{*}\right), F\left(X, x^{*}\right)\left(X-x^{*}\right)\right\rangle
$$

[^2]with
$$
F\left(X, x^{*}\right):=\left(\int_{0}^{1} \int_{0}^{t} \nabla^{2} L_{f}\left(x^{*}+s\left(X-x^{*}\right)\right) d s d t\right)
$$

Next, let $I_{n}$ be the $n \times n$ identity matrix. As $\nabla^{2} f \succ 0$ on $\mathbf{K}$, continuity of the (strictly positive) smallest eigenvalue of $\nabla^{2} f$ and compactness of $\mathbf{K}$ yield that $\nabla^{2} f \succeq \delta I_{n}$ on $\mathbf{K}$, for some $\delta>0$. Next, as $-g_{j}$ is convex for every $j$, and in view of the definition (3.7) of $L_{f}, \nabla^{2} L_{f} \succeq \nabla^{2} f \succeq \delta I_{n}$ on $\mathbf{K}$. Hence for every $\xi \in \mathbb{R}^{n}$, $\xi^{T} F\left(x, x^{*}\right) \xi \geq \delta \int_{0}^{1} \int_{0}^{t} \xi^{T} \xi d s d t=\frac{\delta}{2} \xi^{T} \xi$, and so $F\left(x, x^{*}\right) \succeq \frac{\delta}{2} I_{n}$ for every $x \in \mathbf{K}$. Therefore, by the matrix polynomial version of Putinar Positivstellensatz,

$$
F\left(X, x^{*}\right)=F_{0}(X)+\sum_{j=1}^{m} F_{j}(X) g_{j}(X)
$$

for some real SOS matrix polynomials $X \mapsto F_{j}(X)=L_{j}(X) L_{j}(X)^{T}$ (for some apppropriate $\left.L_{j} \in \mathbb{R}[X]^{n \times p_{j}}\right), j=0, \ldots, m$. See Helton and Nie [6, Kojima and Maramatsu [10, Hol and Scherer [11]. But then

$$
X \mapsto\left\langle\left(X-x^{*}\right), F_{j}\left(X, x^{*}\right)\left(X-x^{*}\right)\right\rangle=\sigma_{j}(X) \in \Sigma^{2}[X], \quad j=0, \ldots, m
$$

and so

$$
\begin{aligned}
f(X)-f^{*} & =L_{f}(X)+\sum_{j=1}^{m} \lambda_{j} g_{j}(X) \\
& =\sigma_{0}(X)+\sum_{j=1}^{m}\left(\lambda_{j}+\sigma_{j}(X)\right) g_{j}(X)
\end{aligned}
$$

Let $2 s$ be the maximum degree of the SOS polynomials $\left(\sigma_{j}\right)$. Then $\left(f^{*},\left\{\sigma_{j}+\lambda_{j}\right\}\right)$ is a feasible solution of the SDP-relaxation $\mathbf{Q}_{r}^{*}$ in (2.4) with $r:=s+\max _{j} r_{j}$. Therefore, as $\sup \mathbf{Q}_{r}^{*} \leq \inf \mathbf{Q}_{r} \leq f^{*}$, the $\operatorname{SDP}$-relaxations $\mathbf{Q}_{r}$ and $\mathbf{Q}_{r}^{*}$ are exact, finite convergence occurs and $\mathbf{Q}_{r}^{*}$ is solvable. But this also implies that $\mathbf{Q}_{r}$ is solvable (take $\mathbf{y}$ to be the moment sequence of the Dirac measure $\delta_{x^{*}}$ at any global minimizer $\left.x^{*} \in \mathbf{K}\right)$.

When compared to Theorem 3.3 for the SOS-convex case, in the strictly convex case the simplified SDP-relaxation $\widehat{\mathbf{Q}}$ in (3.3) is not guaranteed to be exact. However, finite convergence still occurs for the SDP-relaxations $\left(\mathbf{Q}_{r}\right)$ in (2.3).

Remark 3.5. It is worth emphasizing that in general, the hierarchy of LP-relaxations (as opposed to SDP-relaxations) defined in [15] and based on Krivine's representation [12, 30] for polynomials positive on $\mathbf{K}$, cannot have finite convergence, especially in the convex case! For more details, the interested reader is referred to [14, 15]. Therefore, and despite LP software packages can solve LP problems of very large size, using LP-relaxations does not seem a good idea even for solving a convex polynomial optimization problem.

## 4. Convexity and semidefinite Representation of convex sets

We now consider the semidefinite representation of convex sets. First recall the following result.

Theorem 4.1 (Lasserre [16). Let $\mathbf{K}$ in (1.1) be compact with $g_{j}$ concave, $j=$ $1, \ldots, m$, and assume that Slater's condition holds. If the Lagrangian polynomial $L_{f}$ in (3.7) associated with every linear polynomial $f \in \mathbb{R}[X]$ is SOS, then with $d:=\max _{j}\left\lceil\left(\operatorname{deg} g_{j}\right) / 2\right\rceil$, the set

$$
\Omega:=\left\{(x, \mathbf{y}) \in \mathbb{R}^{n} \times \mathbb{R}^{s(2 d)}: \begin{cases}M_{d}(\mathbf{y}) & \succeq 0  \tag{4.1}\\ L_{\mathbf{y}}\left(g_{j}\right) & \geq 0, \quad j=1, \ldots, m \\ L_{\mathbf{y}}\left(X_{i}\right) & =x_{i}, \\ y_{0} & =1\end{cases}\right.
$$

is a semidefinite representation of $\mathbf{K}$.
Next, Helton and Nie [6, 7] have provided several interesting second-order positive curvature (sufficient and necessary) conditions on the defining polynomials $\left(g_{j}\right)$ for $\mathbf{K}$ (or its convex hull $\mathrm{co}(\mathbf{K})$ ) to have a SDr. In particular (recall that $r_{j}=\left\lceil\left(\operatorname{deg} g_{j}\right) / 2\right\rceil$ for every $\left.j=1, \ldots, m\right)$ :

Theorem 4.2 (Helton and Nie [6]). Let $\mathbf{K}$ in (1.1) be convex, Asssumption 2.1 hold, and assume that Slater's condition holds and $g_{j}$ is concave on $\mathbf{K}, j=1, \ldots, m$.
(a) If $-g_{j}$ is SOS-convex for every $j=1, \ldots, m$, then for every linear $f \in \mathbb{R}[X]$, the associated Lagrangian $L_{f}$ (3.7) is SOS and the set $\Omega$ in (4.1) is a semidefinite representation of $\mathbf{K}$.
(b) If every $-g_{j}$ is either SOS-convex or satisfies $-\nabla^{2} g_{i} \succ 0$ on $\mathbf{K} \cap\left\{x: g_{j}(x)=\right.$ $0\}$, then there exists $r \in \mathbb{N}$ such that the set

$$
\Omega:=\left\{(x, \mathbf{y}) \in \mathbb{R}^{n} \times \mathbb{R}^{s(2 r)}:\left\{\begin{array}{ll}
M_{r}(\mathbf{y}) & \succeq 0  \tag{4.2}\\
M_{r-r_{j}}\left(g_{j} \mathbf{y}\right) & \succeq 0, \quad j=1, \ldots, m \\
L_{\mathbf{y}}\left(X_{i}\right) & =x_{i}, \quad i=1, \ldots, n \\
y_{0} & =1
\end{array}\right\}\right.
$$

is a semidefinite representation of $\mathbf{K}$.
See [6, Theor. 6, and 9]. This follows from the fact that the Hessian $\nabla^{2} L_{f}$ associated with a linear $f \in \mathbb{R}[X]$ has a Putinar representation in terms of SOS matrix polynomials, and with degree of the weights bounded uniformly in $f$. In principle, the degree parameter $d$ in Theorem4.2(b) may be computed by solving a hierarchy of semidefinite programs. Some other (more technical) weaker secondorder positive curvature sufficient conditions (merely for existence of a SDr ) are also provided in [6, 7] but the semidefinite representation is not explicit any more in terms of the defining polynomials $\left(g_{j}\right)$. Notice that if $\mathbf{K}$ is compact but Assumption 2.1 does not hold, then one still obtains a semidefinite representation for $\mathbf{K}$ but more complicated as it is now based on Schmüdgen's representation [27] instead of Putinar's representation; see [6, Theor. 5].

We next provide a sufficient condition in the case where $\mathbf{K}$ is convex but its defining polynomials $\left(-g_{j}\right)$ are not necessarily convex. Among its distinguishing features, it is checkable numerically, contains Theorem 4.2 as a special case and leads to the explicit semidefinite representation (4.2) of $\mathbf{K}$.
4.1. Algebraic certificate of convexity. We first present the following characterization of convexity when $\mathbf{K}$ is closed, satisfies a nondegeneracy assumption on its boundary, and Slater's condition holds.

Lemma 4.3. Let $\mathbf{K}$ be as in (1.1) (hence closed), Slater's condition hold and assume that for every $j=1, \ldots, m, \nabla g_{j}(y) \neq 0$ if $y \in \mathbf{K}$ and $g_{j}(y)=0$. Then $\mathbf{K}$ is convex if and only if for every $j=1, \ldots, m$,

$$
\begin{equation*}
\left\langle\nabla g_{j}(y), x-y\right\rangle \geq 0, \quad \forall x \in \mathbf{K} \text { and } \forall y \in \mathbf{K} \text { with } g_{j}(y)=0 \tag{4.3}
\end{equation*}
$$

Proof. The only if part is obvious. Indeed if $\left\langle\nabla g_{j}(y), x-y\right\rangle<0$ for some $x \in \mathbf{K}$ and $y \in \mathbf{K}$ with $g_{j}(y)=0$, then there is some $\bar{t}>0$ such that $g_{j}(y+t(x-y))<0$ for all $t \in(0, \bar{t})$ and so the point $x^{\prime}:=t x+(1-t) y$ does not belong to $\mathbf{K}$, which in turn implies that $\mathbf{K}$ is not convex.

For the if part, (4.3) implies that at every point of the boundary, there exists a supporting hyperplane for $\mathbf{K}$. As $\mathbf{K}$ is closed with nonempty interior, the result follows from [28, Theor. 1.3.3] ${ }^{3}$.

The nondegeneracy assumption is crucial as demonstrated in the following simple example kindly provided by an anonymous referee:

Example 1. Consider the non convex set $\mathbf{K} \subset \mathbb{R}^{2}$ defined by:

$$
\mathbf{K}:=\left\{x \in \mathbb{R}^{2}:\left(1-x_{1}^{2}+x_{2}^{2}\right)^{3} \geq 0,10-x_{1}^{2}-x_{2}^{2} \geq 0\right\}
$$

Then it is straightforward to see that (4.3) is satisfied. This is because $\nabla g_{1}$ vanishes on the piece of boundary determined by $g_{1}(x)=0$.

Next, using the above characterization (4.3), we provide an algebraic certificate of convexity.

Corollary 4.4 (Algebraic certificate of convexity). Let K be as in (1.1), Slater's condition hold and assume that for every $j=1, \ldots, m, \nabla g_{j}(y) \neq 0$ if $y \in \mathbf{K}$ and $g_{j}(y)=0$. Then $\mathbf{K}$ is convex if and only if for every $j=1, \ldots, m$,

$$
\begin{equation*}
h_{j}(X, Y)\left\langle\nabla g_{j}(Y), X-Y\right\rangle=\left\langle\nabla g_{j}(Y), X-Y\right\rangle^{2 l}+\theta_{j}(X, Y)+\varphi_{j}(X, Y) g_{j}(Y) \tag{4.4}
\end{equation*}
$$

for some integer $l \in \mathbb{N}$, some polynomial $\varphi_{j} \in \mathbb{R}[X, Y]$ and some polynomials $h_{j}, \theta_{j}$ in the preordering ${ }^{4}$ of $\mathbb{R}[X, Y]$ generated by the family of polynomials $\left(g_{k}(X), g_{p}(Y)\right)$, $k, p \in\{1, \ldots, m\}, p \neq j$.

Proof. By Lemma 4.3, $\mathbf{K}$ is convex if and only if for every $j=1, \ldots, m$, the polynomial $(X, Y) \mapsto\left\langle\nabla g_{j}(Y), X-Y\right\rangle$ is nonnegative on the set $\Omega_{j}$ defined by:

$$
\begin{equation*}
\Omega_{j}:=\left\{(x, y) \in \mathbf{K} \times \mathbf{K}: g_{j}(y)=0\right\} \tag{4.5}
\end{equation*}
$$

Equivalently, $\mathbf{K}$ is convex if and only if for every $j=1, \ldots, m$ :

$$
\begin{array}{ll}
\emptyset=\left\{(x, y) \in \mathbb{R}^{n}: \quad\right. & (x, y) \in \mathbf{K} \times \mathbf{K} ; \quad g_{j}(y)=0 \\
& \left.\left\langle\nabla g_{j}(y), x-y\right\rangle \leq 0 ;\left\langle\nabla g_{j}(y), x-y\right\rangle \neq 0\right\}
\end{array}
$$

Then (4.4) follows from Stengle's Positivstellensatz [25, Theor. 4.4.2, p. 92].

[^3]Observe that Corollary 4.4 provides an algebraic certificate of convexity when $\mathbf{K}$ is closed with nonempty interior and a nondegeneracy assumption holds on its boundary. If one fixes an a priory bound $s$ on $l \in \mathbb{N}$ and on the degree of $h_{j}, \theta_{j}$ and $\varphi_{j}$, then checking whether (4.4) holds reduces to solving a semidefinite program. If $\mathbf{K}$ is convex, by increasing $s$, eventually one would obtain such a certificate if one could solve semidefinite programs exactly. In practice, and because of unavoidable numerical inaccuracies, one only obtains a numerical approximation of the optimal value and so, a certificate valid up to machine precision only.

However, implementing such a procedure is extremely costly because one has potentially $2 \times 2^{m}$ unknown SOS polynomials to define $h_{j}$ and $\theta_{j}$ in (4.4)! Therefore, it is highly desirable to provide a less costly certificate but with no guarantee to hold for every $\mathbf{K}$ as in Corollary 4.4.

In particular one only considers compact sets $\mathbf{K}$. Indeed, if $\mathbf{K}$ is compact, one has the following result (recall that $g_{0} \equiv 1$ ).

Lemma 4.5. Let K be convex, Assumption 2.1 and Slater's condition hold. Assume that for every $j=1, \ldots, m, \nabla g_{j}(y) \neq 0$ if $y \in \mathbf{K}$ and $g_{j}(y)=0$. Then for every $\epsilon>0$ and every $j=1, \ldots, m$ :

$$
\begin{align*}
\left\langle\nabla g_{j}(Y), X-Y\right\rangle+\epsilon= & \sum_{k=0}^{m} \sigma_{j k}(X, Y) g_{k}(X)+\sum_{k=0, k \neq j}^{m} \psi_{j k}(X, Y) g_{k}(Y) \\
& +\psi_{j}(X, Y) g_{j}(Y) \tag{4.6}
\end{align*}
$$

for some SOS polynomials $\left(\sigma_{j k}\right)$ and $\left(\psi_{j k}\right)_{k \neq j} \subset \Sigma^{2}[X, Y]$, and some polynomial $\psi_{j} \in \mathbb{R}[X, Y]$.

Proof. By Lemma 4.3, for every $j=1, \ldots, m$, and every $x, y \in \mathbf{K}$ such that $g_{j}(y)=$ 0 , (4.3) holds and therefore, for every $j=1, \ldots, m$,

$$
\begin{equation*}
\left\langle\nabla g_{j}(y), x-y\right\rangle+\epsilon>0 \quad \forall(x, y) \in \Omega_{j} \tag{4.7}
\end{equation*}
$$

where $\Omega_{j}$ has been defined in (4.5). As $\mathbf{K}$ satisfies Assumption 2.1 then so does $\Omega_{j}$ for every $j=1, \ldots, m$. Hence (4.6) follows from (4.7) and Theorem [2.2.

Therefore, inspired by Lemma 4.5, introduce the following condition:
Assumption 4.6 (Certificate of convexity). For every $j=1, \ldots, m$, 4.6) holds with $\epsilon=0$. Then let $d_{j} \in \mathbb{N}$ be such that $2 d_{j}$ is larger than the maximum degree of the polynomials $\sigma_{j k} g_{k}, \psi_{j k} g_{k}, \psi_{j} g_{j} \in \mathbb{R}[X, Y]$ in 4.6), $j=1, \ldots, m$.

When $\mathbf{K}$ is closed (and not necessarily compact), Slater's condition holds and the nondegeneracy assumption on the boundary holds (i.e., $\nabla g_{j}(y) \neq 0$ if $y \in$ $\mathbf{K}$ and $g_{j}(y)=0$ ) Assumption 4.6 is indeed a certificate of convexity because then (4.3) holds for every $x, y \in \mathbf{K}$ with $g_{j}(y)=0$, and by Lemma 4.3. $\mathbf{K}$ is convex. It translates the geometric property of convexity of $\mathbf{K}$ into an algebraic SOS Putinar representation of the polynomial $(X, Y) \mapsto\left\langle\nabla g_{j}(Y), X-Y\right\rangle$ nonnegative on $\Omega_{j}, j=1, \ldots, m$. On the other hand, if $\mathbf{K}$ is convex and Assumption 2.1 Slater's condition and the nondegeneracy assumption all hold, then Assumption 4.6 is almost necessary as, by Lemma 4.5, (4.6) holds with $\epsilon>0$ arbitrary.

With $d_{j}$ fixed a priori, checking whether (4.6) hold with $\epsilon=0$ can be done numerically. (However, again it provides a certificate of convexity valid up to machine precision only.) For instance, for every $j=1, \ldots, m$, it suffices to solve the semidefinite program (recall that $\left.r_{k}=\left\lceil\left(\operatorname{deg} g_{k}\right) / 2\right\rceil, k=1 \ldots, m\right)$

$$
\begin{cases}\rho_{j}:= & \min _{\mathbf{z}} L_{\mathbf{z}}\left(\left\langle\nabla g_{j}(Y), X-Y\right\rangle\right)  \tag{4.8}\\ \text { s.t. } & M_{d_{j}}(\mathbf{z}) \succeq 0 \\ & M_{d_{j}-r_{k}}\left(g_{k}(X) \mathbf{z}\right) \succeq 0, \quad k=1, \ldots, m \\ & M_{d_{j}-r_{k}}\left(g_{k}(Y) \mathbf{z}\right) \succeq 0, \quad k=1, \ldots, m ; k \neq j \\ & M_{d_{j}-r_{j}}\left(g_{j}(Y) \mathbf{z}\right)=0 \\ & y_{0}=1\end{cases}
$$

If $\rho_{j}=0$ for every $j=1, \ldots, m$, then Assumption 4.6 holds. This is in contrast to the PP-BDR property in [17] that cannot be checked numerically as it involves infinitely many linear polynomials $f$.

Remark 4.7. Observe that the usual rank condition (3.5) used as a stopping criterion to detect whether (4.8) is exact (i.e. $\rho_{1}=0$ ), cannot be satisfied in solving (4.8) with primal dual interior point methods (as in the SDP-solvers used by GloptiPoly) because one tries to find an optimal solution $\mathbf{z}^{*}$ in the relative interior of the feasible set of (4.8) and this gives maximum rank to the moment matrix $M_{d_{j}}\left(\mathbf{z}^{*}\right)$. Therefore, in the context of (4.8), if indeed $\rho_{j}=0$ then $\mathbf{z}^{*}$ corresponds to the moment vector of some probability measure $\mu$ supported on the set of points $(x, x) \in \mathbf{K} \times \mathbf{K}$ that satisfy $g_{j}(x)=0$ (as indeed $\left.\left.L_{\mathbf{z}^{*}}\left(\left\langle\nabla g_{j}(Y), X-Y\right)\right\rangle\right)=0=\rho_{j}\right)$. Therefore $\rho_{j}=0$ as $d_{j}$ increases but the rank of $M_{d_{j}}\left(\mathbf{z}^{*}\right)$ does not stabilize because $\mu$ is not finitely supported. In particular, a good candidate $\mathbf{z}^{*}$ for optimal solution is the moment vector of the probability measure uniformly distributed on the set $\left\{(x, x) \in \mathbf{K} \times \mathbf{K}: g_{j}(x)=0\right\}$.

Alternatively, if $\rho_{j} \approx 0$ and the dual of (4.8) has an optimal solution $\left(\sigma_{j k}, \psi_{j k}, \psi_{j}\right)$, then in some cases one may check if (4.6) holds exactly after appropriate rounding of coefficients of the solution. But in general, obtaining an exact certificate (i.e., $\rho_{j}=0$ in the primal or (4.6) with $\epsilon=0$ in the dual) numerically is hopeless.

Example 2. Consider the following simple illustrative example in $\mathbb{R}^{2}$ :

$$
\begin{equation*}
\mathbf{K}:=\left\{x \in \mathbb{R}^{2}: x_{1} x_{2}-1 / 4 \geq 0 ; 0.5-\left(x_{1}-0.5\right)^{2}-\left(x_{2}-0.5\right)^{2} \geq 0\right\} \tag{4.9}
\end{equation*}
$$

Obviously $\mathbf{K}$ is convex but its defining polynomial $x \mapsto g_{1}(x):=x_{1} x_{2}-1 / 4$ is not concave whereas $x \mapsto g_{2}(x):=0.5-\left(x_{1}-0.5\right)^{2}-\left(x_{2}-0.5\right)^{2}$ is.

With $d_{1}=3$, solving (4.8) using GloptiPoly 3 司 yields the optimal value $\rho_{1} \approx-4.58 .10^{-11}$ which, in view of the machine precision for the SDP solvers used in GloptiPoly, could be considered to be zero, but of course with no guarantee. However, and according to Remark 4.7. we could check that (again up to machine precision) for every $\alpha \in \mathbb{N}^{n}$ with $|\alpha| \leq 2 d_{j}$, $z_{\alpha, \alpha}^{*}=z_{2 \alpha, 0}^{*}$ and $z_{\alpha, 0}^{*}=z_{0, \alpha}^{*}$. In addition, because of symmetry, $z_{\alpha, \beta}=z_{\alpha^{\prime}, \beta^{\prime}}$ whenever $\alpha_{1}^{\prime}=\alpha_{2}$ and $\alpha_{2}^{\prime}=\alpha_{1}$ (and similarly for $\beta$ and $\beta^{\prime}$ ). Indeed for moments of order 1 we have $z_{\alpha, \beta}^{*}=(0.5707,0.5707,0.5707,0.5707)$ and for moments of order 2 ,

$$
z_{\alpha, \beta}^{*}=(0.4090,0.25,0.4090,0.25,0.4090,0.25,0.4090,0.4090,0.25,0.4090)
$$

[^4]For $j=2$ there is no test to perform because $-g_{2}$ being quadratic and convex yields

$$
\begin{equation*}
\left\langle\nabla g_{2}(Y), X-Y\right\rangle=g_{2}(X)-g_{2}(Y)+\underbrace{(X-Y)^{T}\left(-\nabla^{2} g_{2}(Y)\right)(X-Y)}_{\text {SOS }} \tag{4.10}
\end{equation*}
$$

which is in the form (4.6) with $d_{2}=1$.
We next show the role of Assumption 4.6 in obtaining a semidefinite representation of $\mathbf{K}$.

Theorem 4.8. Let Assumption 2.1 and Slater's condition hold. Moreover, assume that for every $j=1, \ldots, m, \nabla g_{j}(y) \neq 0$ whenever $y \in \mathbf{K}$ and $g_{j}(y)=0$. If Assumption 4.6 holds then $\mathbf{K}$ is convex and $\Omega$ in (4.2) with $d:=\max _{j} d_{j}$, is a semidefinite representation of $\mathbf{K}$.

Proof. That K is convex follows from Lemma 4.3. We next prove that the PPBDR property defined in Lasserre [16] holds for $\mathbf{K}$. Let $f \in \mathbb{R}[X]$ be a linear polynomial with coefficient vector $\mathbf{f} \in \mathbb{R}^{n}$ (i.e., $X \mapsto f(X)=\mathbf{f}^{T} X$ ) and consider the optimization problem $\mathbf{P}: \min \left\{\mathbf{f}^{T} x: x \in \mathbf{K}\right\}$. As $\mathbf{K}$ is compact, let $x^{*} \in \mathbf{K}$ be a global minimizer of $f$. The Fritz-John optimality conditions state that there exists $0 \neq \lambda \in \mathbb{R}_{+}^{m+1}$ such that

$$
\begin{equation*}
\lambda_{0} \mathbf{f}=\sum_{j=1}^{m} \lambda_{j} \nabla g_{j}\left(x^{*}\right) ; \quad \lambda_{j} g_{j}\left(x^{*}\right)=0 \quad \forall j=1, \ldots, m \tag{4.11}
\end{equation*}
$$

(See e.g. 33.) We first prove by contradiction that if Slater's condition and the nondegeneracy assumption hold then $\lambda_{0}>0$. Suppose that $\lambda_{0}=0$ and let $J:=$ $\left\{j \in\{1, \ldots, m\}: \lambda_{j}>0\right\}$; hence $J$ is nonempty as $\lambda \neq 0$. With $x_{0} \in \mathbf{K}$ such that $g_{j}\left(x_{0}\right)>0$ (as Slater's condition holds, one such $x_{0}$ exists), let $B\left(x_{0}, \rho\right):=$ $\left\{z:\left\|z-x_{0}\right\| \leq \rho\right\}$. For $\rho$ sufficiently small, $B\left(x_{0}, \rho\right) \subset \mathbf{K}$ and $g_{j}(z)>0$ for all $z \in B\left(x_{0}, \rho\right)$ and every $j=1, \ldots, m$. Then by (4.11) and $\lambda_{0}=0$,

$$
0=\sum_{j=1}^{m} \lambda_{j}\left\langle\nabla g_{j}\left(x^{*}\right), z-x^{*}\right\rangle, \quad \forall z \in B\left(x_{0}, \rho\right)
$$

which in turn implies (by nonnegativity of each term in the above sum)

$$
\left\langle\nabla g_{j}\left(x^{*}\right), z-x^{*}\right\rangle=0, \quad \forall z \in B\left(x_{0}, \rho\right), j \in J
$$

But this clearly implies $\nabla g_{j}\left(x^{*}\right)=0$ for every $j \in J$, in contradiction with the nondegeneracy assumption. Hence $\lambda_{0}>0$ and by homogeneity, we may and will take $\lambda_{0}=1$.

Therefore, letting $Y:=x^{*}$ in (4.6), the polynomial $X \mapsto f(X)-f^{*}$ can be written

$$
\begin{aligned}
\mathbf{f}^{T} X-f^{*}= & \sum_{j=1}^{m} \lambda_{j}\left[\left\langle\nabla g_{j}\left(x^{*}\right), X-x^{*}\right\rangle\right] \\
= & \sum_{j=1}^{m} \lambda_{j}\left[\sum_{k=0}^{m} \sigma_{j k}\left(X, x^{*}\right) g_{k}(X)+\sum_{k=0, k \neq j}^{m} \psi_{j k}\left(X, x^{*}\right) g_{k}\left(x^{*}\right)\right. \\
& \left.+\psi_{j}\left(X, x^{*}\right) g_{j}\left(x^{*}\right)\right]
\end{aligned}
$$

where we have used (4.6) with $Y=x^{*}$ and $\epsilon=0$. Next, observe that :

$$
\begin{array}{rll}
X \mapsto \sigma_{j k}\left(X, x^{*}\right) & \in \Sigma^{2}[X] \quad\left[\text { as } \sigma_{j k} \in \Sigma^{2}[X, Y]\right] \\
X \mapsto \psi_{j k}\left(X, x^{*}\right) g_{k}\left(x^{*}\right) & \in \Sigma^{2}[X] \quad\left[\text { as } \psi_{j k} \in \Sigma^{2}[X, Y] \text { and } g_{j}\left(x^{*}\right) \geq 0\right] \\
\lambda_{j} g_{j}\left(x^{*}\right) & =0 \quad j=1, \ldots, m
\end{array}
$$

And so, as $\lambda \in \mathbb{R}_{+}^{m}$,

$$
\begin{equation*}
X \mapsto \mathbf{f}^{T} X-f^{*}=\Delta_{0}(X)+\sum_{j=1}^{m} \Delta_{j}(X) g_{j}(X), \tag{4.12}
\end{equation*}
$$

for SOS polynomials $\left(\Delta_{j}\right)_{j=0}^{m} \subset \Sigma^{2}[X]$ defined by

$$
\begin{aligned}
& X \mapsto \Delta_{0}(X)=\sum_{j=1}^{m} \lambda_{j}\left(\sum_{k=0, k \neq j}^{m} \psi_{j k}\left(X, x^{*}\right) g_{k}\left(x^{*}\right)\right) \\
& X \mapsto \Delta_{j}(X)=\sum_{l=1}^{m} \lambda_{l} \sigma_{l j}\left(X, x^{*}\right), \quad j=1, \ldots, m
\end{aligned}
$$

Write every affine polynomial $f \in \mathbb{R}[X]$ as $\mathbf{f}^{T} X+f_{0}$ for some $\mathbf{f} \in \mathbb{R}^{n}$ and $f_{0}=f(0)$. If $f$ is nonnegative on $\mathbf{K}$ then from (4.12),

$$
\begin{aligned}
f(X)=\mathbf{f}^{T} X-f^{*}+f^{*}+f_{0} & =f^{*}+f_{0}+\Delta_{0}(X)+\sum_{j=1}^{m} \Delta_{j}(X) g_{j}(X) \\
& =\widehat{\Delta}_{0}(X)+\sum_{j=1}^{m} \Delta_{j}(X) g_{j}(X) \quad \forall X
\end{aligned}
$$

with $\widehat{\Delta}_{0} \in \Sigma^{2}[X]$ (because $f^{*}+f_{0} \geq 0$ ) and so, the PP-BDR property holds for $\mathbf{K}$ with order $d$. By [16, Theor. 2], $\mathbf{K}$ is SDr with the semidefinite representation (4.2).

We next show that the two sufficient conditions of strict convexity and SOSconvexity of Helton and Nie [6] in Theorem 4.2 both imply that Assumption 4.6 holds and so Theorem 4.8 contains Theorem 4.2 as a special case.

Corollary 4.9. Let $\mathbf{K}$ in (1.1) be convex and both Assumption 2.1 and Slater's condition hold. Assume that either $-g_{j}$ is SOS-convex or $-g_{j}$ is convex on $\mathbf{K}$ and $-\nabla^{2} g_{j} \succ 0$ on $\mathbf{K} \cap\left\{x: g_{j}(x)=0\right\}$, for every $j=1, \ldots, m$. Then Assumption 4.6 holds and so Theorem 4.8 applies.

Proof. By Lemma 2.5, for every $j=1, \ldots, m$, write

$$
\begin{gathered}
(X, Y) \quad \mapsto \quad g_{j}(X)-g(Y)-\left\langle\nabla g_{j}(Y), X-Y\right\rangle= \\
\langle(X-Y), \underbrace{\left(\int_{0}^{1} \int_{0}^{t} \nabla^{2} g_{j}(Y+s(X-Y)) d s d t\right)}_{F_{j}(X, Y)}(X-Y)\rangle
\end{gathered}
$$

If $-\nabla^{2} g_{j} \succ 0$ on $y \in \mathbf{K}$ with $g_{j}(y)=0$, then from the proof of [6] Lemma 19], $-F_{j}(x, y) \succ 0$ for all $x, y \in \mathbf{K}$ with $g_{j}(y)=0$. In other words, $-F_{j}(x, y) \succeq \delta I_{n}$ on
$\Omega_{j}$ (defined in (4.5)) for some $\delta>0$. Therefore, by the matrix polynomial version of Putinar Positivstellensatz in [6, Theor. 29],

$$
\begin{equation*}
-F_{j}(X, Y)=\sum_{k=0}^{m} \widehat{\sigma}_{j k}(X, Y) g_{k}(X)+\sum_{k=0, k \neq j}^{m} \widehat{\psi}_{j k}(X, Y) g_{k}(Y)+\widehat{\psi}_{j}(X, Y) g_{j}(Y) \tag{4.13}
\end{equation*}
$$

for some SOS matrix polynomials $\left(\widehat{\sigma}_{j k}(X, Y)\right),\left(\widehat{\psi}_{j k}(X, Y)\right)$ and some matrix polynomial $\widehat{\psi}_{j}(X, Y)$.

On the other hand, if $-g_{j}$ is SOS-convex then by Lemma 2.4 $-F_{j}(X, Y)$ is SOS and therefore (4.13) also holds (take $\widehat{\sigma}_{j k} \equiv 0$ for all $k \neq 0, \widehat{\psi}_{j k} \equiv 0$ for all $k$ and $\left.\widehat{\psi}_{j} \equiv 0\right)$. But then

$$
\begin{aligned}
g_{j}(X)-g(Y)-\left\langle\nabla g_{j}(Y), X-Y\right\rangle & =\left\langle(X-Y), F_{j}(X, Y)(X-Y)\right\rangle \\
& =-\sum_{k=0}^{m}\left\langle(X-Y), \widehat{\sigma}_{j k}(X, Y)(X-Y)\right\rangle g_{k}(X) \\
& -\sum_{k=0, k \neq j}^{m}\left\langle(X-Y), \widehat{\psi}_{j k}(X, Y)(X-Y)\right\rangle g_{k}(Y) \\
& -\left\langle(X-Y), \widehat{\psi}_{j}(X, Y)(X-Y)\right\rangle g_{j}(Y) \\
= & -\sum_{k=0}^{m} \sigma_{j k}(X, Y) g_{k}(X)- \\
& \sum_{k=0, k \neq j}^{m} \psi_{j k}(X, Y) g_{k}(Y)-\psi_{j}(X, Y) g_{j}(Y)
\end{aligned}
$$

for all $X, Y$ and for some SOS polynomials $\sigma_{j k}, \psi_{j k} \in \mathbb{R}[X, Y]$ and some polynomial $\psi_{j} \in \mathbb{R}[X, Y]$. Equivalently,

$$
\begin{aligned}
\left\langle\nabla g_{j}(Y), X-Y\right)= & g_{j}(X)-g_{j}(Y)+\sum_{k=0}^{m} \sigma_{j k}(X, Y) g_{k}(X) \\
& +\sum_{k=0, k \neq j}^{m} \psi_{j k}(X, Y) g_{k}(Y)+\psi_{j}(X, Y) g_{j}(Y) \\
= & \sum_{k=0}^{m} \sigma_{j k}^{\prime}(X, Y) g_{k}(X)+\sum_{k=0, k \neq j}^{m} \psi_{j k}(X, Y) g_{k}(Y) \\
& +\psi_{j}^{\prime}(X, Y) g_{j}(Y)
\end{aligned}
$$

for some SOS polynomials $\sigma_{j k}^{\prime}, \psi_{j k} \in \Sigma^{2}[X, Y]$ and some polynomial $\psi_{j}^{\prime} \in \mathbb{R}[X, Y]$. In other words, Assumption 4.6 holds, which concludes the proof.

Hence if each $-g_{j}$ is SOS-convex or convex on $\mathbf{K}$ with $-\nabla^{2} g_{j} \succ 0$ on $\mathbf{K} \cap\{x$ : $\left.g_{j}(x)=0\right\}$, one obtains a numerical scheme to obtain the parameter $d$ in Theorem 4.8 as well as the semidefinite representation (4.2) of $\mathbf{K}$. Solve the semidefinite programs (4.8) with degree parameter $d_{j}$. Eventually, $\rho_{j}=0$ for every $j=1, \ldots, m$.

Example 3. Consider the convex set $\mathbf{K}$ in (4.9) of Example 2 for which the defining polynomial $g_{1}$ of $\mathbf{K}$ is not concave. We have seen that Assumption 4.6 holds (up to $\rho_{1} \approx 10^{-11}$, close to machine precision) and $\max \left[d_{1}, d_{2}\right]=3$. By Theorem 4.8 if $\rho_{1}$ would be exactly 0 , the set

$$
\Omega:=\left\{(x, \mathbf{y}) \in \mathbb{R}^{n} \times \mathbb{R}^{s(6)}: \begin{cases}M_{3}(\mathbf{y}) & \succeq 0  \tag{4.14}\\ M_{2}\left(g_{j} \mathbf{y}\right) & \geq 0, \quad j=1,2 \\ L_{\mathbf{y}}\left(X_{i}\right) & =x_{i}, \quad i=1,2 \\ y_{0} & =1\end{cases}\right.
$$

would be a semidefinite representation of $\mathbf{K}$.
At least in practice, for every linear polynomial $f \in \mathbb{R}[X]$, minimizing $L_{\mathbf{y}}(f)$ over $\Omega$ yields the desired optimal value $f^{*}:=\min _{x \in \mathbf{K}} f(x)$, up to $\rho_{1} \approx-10^{-11}$.

Indeed, let $f \in \mathbb{R}[X]$ be $\mathbf{f}^{T} X$ for some vector $\mathbf{f} \in \mathbb{R}^{n}$. In minimizing $f$ over $\mathbf{K}$, one has $\mathbf{f}=\lambda_{1} \nabla g_{1}\left(x^{*}\right)+\lambda_{2} \nabla g_{2}\left(x^{*}\right)$ for some $\lambda \in \mathbb{R}_{+}^{2}$, some $x^{*} \in \mathbf{K}$ with $\lambda_{i} g_{i}\left(x^{*}\right)=0, i=1,2$, and $f^{*}=\lambda_{1}\left\langle\nabla g_{1}\left(x^{*}\right), x^{*}\right\rangle+\lambda_{2}\left\langle\nabla g_{2}\left(x^{*}\right), x^{*}\right\rangle=\min _{x \in \mathbf{K}} \mathbf{f}^{T} x$. Let $x$ be as in (4.14), arbitrary. Then

$$
\mathbf{f}^{T} x-f^{*}=L_{\mathbf{y}}\left(f(X)-f^{*}\right)=\sum_{i=1}^{2} \lambda_{i} L_{\mathbf{y}}\left(\left\langle\nabla g_{i}\left(x^{*}\right), X-x^{*}\right\rangle\right) .
$$

If $\lambda_{1}>0$ so that $g_{1}\left(x^{*}\right)=0$, use (4.12) to obtain

$$
L_{\mathbf{y}}\left(\left\langle\nabla g_{1}\left(x^{*}\right), X-x^{*}\right\rangle\right)=L_{\mathbf{y}}\left(\rho_{1}+\Delta_{0}(X)+\sum_{j=1}^{2} \Delta_{j}(X) g_{j}(X)\right) \geq \rho_{1}
$$

because $L_{\mathbf{y}}\left(\Delta_{0}\right) \geq 0$ follows from $M_{3}(\mathbf{y}) \succeq 0$, and $L_{\mathbf{y}}\left(\Delta_{j} g_{j}\right) \geq 0, j=1,2$, follows from $M_{2}\left(g_{1} \mathbf{y}\right), M_{2}\left(g_{2} \mathbf{y}\right) \succeq 0$. If $\lambda_{2}>0$ so that $g_{2}\left(x^{*}\right)=0$, then from (4.10)

$$
L_{\mathbf{y}}\left(\left\langle\nabla g_{2}\left(x^{*}\right), X-x^{*}\right\rangle\right)=L_{\mathbf{y}}\left(g_{2}(X)-\left\langle\left(X-x^{*}\right), \nabla^{2} g_{2}\left(x^{*}\right)\left(X-x^{*}\right)\right\rangle\right) \geq 0
$$

because $L_{\mathbf{y}}\left(g_{2}\right) \geq 0$ follows from $M_{2}\left(g_{2} \mathbf{y}\right) \succeq 0$ whereas the second term is nonnegative as $\left\langle\left(X-x^{*}\right),-\nabla^{2} g_{2}\left(x^{*}\right)\left(X-x^{*}\right)\right\rangle$ is SOS and $M_{3}(\mathbf{y}) \succeq 0$. Hence $\mathbf{f}^{T} x-f^{*} \geq \rho_{1}$. On the other hand, from $\mathbf{K} \subseteq\{x:(x, y) \in \Omega\}$, one finally obtains the desired result

$$
f^{*}+\rho_{1} \leq \min \left\{\mathbf{f}^{T} x:(x, y) \in \Omega\right\} \leq f^{*}
$$

## 5. Conclusion

As well-known, convexity is a highly desirable property in optimization. We have shown that it also has important specific consequences in polynomial optimization. For instance, for polynomial optimization problems with SOS-convex or strictly convex polynomial data, the basic SDP-relaxations of the moment approach [13] recognizes convexity and finite convergence occurs. Similarly, the set $\mathbf{K}$ has a semidefinite representation, explicit in terms of the defining polynomials $\left(g_{j}\right)$.

The class of SOS-convex polynomials introduced in Helton and Nie [6] is particularly interesting because the semidefinite constraint to handle in the semidefinite relaxation only involves the Hankel-like moment matrix which does not depend on the problem data! Hence one might envision a dedicated SDP solver that would take into account this peculiarity as Hankel-like or Toeplitz-like matrices enjoy very specific properties. Moreover, if restricted to this class of polynomials, Jensen's inequality can be extended to linear functionals in the dual cone of SOS polynomials (hence not necessarily probability measures).

Therefore, a topic of further research is to evaluate how large is the subclass of SOS-convex polynomials in the class of convex polynomials, and if possible, to also provide simple sufficient conditions for SOS-convexity.

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[^1]:    ${ }^{1}$ Slater's condition holds for $\mathbf{K}$ in (1.1) if for some $x_{0} \in \mathbf{K}, g_{j}\left(x_{0}\right)>0, j=1, \ldots, m$.

[^2]:    ${ }^{2}$ The self-concordance property introduced in [20] is fundamental in the design and efficiency of interior point methods for convex programming.

[^3]:    ${ }^{3}$ The author is grateful to L . Tuncel for providing us with the reference 28 .
    ${ }^{4}$ The preordering of $\mathbb{R}[X]$ generated by a family $\left(g_{1}, \ldots, g_{m}\right) \subset \mathbb{R}[X]$ is the set of polynomials $\left\{p: p=\sum_{J \subseteq\{1, \ldots, m\}} \sigma_{J}\left(\prod_{j \in J} g_{j}\right)\right.$, with $\left.\sigma_{J} \in \Sigma^{2}[X]\right\}$.

[^4]:    ${ }^{5}$ GloptiPoly 3 (a Matlab based public software) is an extension of GloptiPoly 9] to solve the generalized problem of moments described in 18 . For more details see www.laas.fr/~henrion/software/.

