

Stability for time-dependent inverse transport

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Abstract

This paper concerns the reconstruction of the absorption and scattering parameters in a time-dependent linear transport equation from full knowledge of the albedo operator at the boundary of a bounded domain of interest. We present optimal stability results on the reconstruction of the absorption and scattering parameters for a given error in the measured albedo operator.

1 Introduction

Inverse transport theory has many applications in e.g. medical and geophysical imaging. It consists of reconstructing optical parameters in a domain of interest from measurements of the transport solution at the boundary of that domain. The optical parameters are the total absorption (extinction) parameter $\sigma(x)$ and the scattering parameter $k(x, v', v)$, which measures the probability of a particle at position x to scatter from direction v' to direction v .

The domain of interest is probed as follows. A known flux of particles enters the domain and the flux of outgoing particles is measured at the domain's boundary. Several inverse theories may then be envisioned based on available data. The least favorable situation is when the density of outgoing particles is angularly averaged, which means that only the spatial density of particles may be estimated and not the phase space (space and direction) density. Angular averaging may be necessitated by equipment cost, time of acquisition of the measurements, or low particle counts. For uniqueness and stability results in this setting, we refer the reader e.g. to Bal and Jollivet [BJ2], Bal et al. [BLM], and Langmore [L].

A much more favorable situation is when the density of outgoing particles is angularly resolved. We may then be able to sample the outgoing distribution of particles as a function of time if sufficiently accurate equipment is available. In many setting however, only time independent measurements are feasible.

The uniqueness of the reconstruction of the optical parameters from knowledge of angularly resolved measurements both in the time-dependent and time-independent settings was proved in Choulli and Stefanov [CS1, CS2]. We also refer the reader to Stefanov [S] for a review of uniqueness results in inverse transport theory. Stability in the time-independent case has been analyzed in dimension $d = 2, 3$ under smallness assumptions for both optical parameters by Romanov [R1, R2] and

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in dimension $d = 2$ under smallness assumption for the scattering parameter by Stefanov and Uhlmann [SU]. Partial results on the stability of the reconstruction in the time-independent setting in dimension $d = 3$ were obtained in Wang [W] without smallness assumptions. Complete stability results in the time-independent case in dimension $d \geq 3$ were obtained by the authors in [BJ1]. The present paper proves stability results for the time-dependent inverse transport problem. We restrict ourselves to the case of elastic scattering, where the velocity space may be modeled by the unit sphere \mathbb{S}^{d-1} . Optimal results on the stability of the optical parameters are obtained in all dimensions $d \geq 2$.

The rest of the paper is structured as follows. Section 2 recalls useful results on the time-dependent linear transport equation. The main stability results of this paper are stated in section 3. They are based on a decomposition of the albedo operator used in [CS1] and recalled in section 3.2. Useful regularity results on the decomposition are stated in Proposition 3.2 and proved in section 4. Our first stability result is stated in Theorem 3.1. It shows how the Radon transform of the absorption parameter and a weighted L^1 norm of the scattering coefficient may be stably reconstructed from knowledge of the albedo operator. Under additional regularity assumptions, Theorem 3.2 shows the stability of the reconstruction of both optical parameters. Both stability results are proved in section 5.

2 The forward problem

In this section we introduce some notation and recall known facts about the well-posedness of the forward transport problem.

2.1 The linear Boltzmann transport equation

Let X be a bounded open subset of \mathbb{R}^d , $d \geq 2$, with a C^1 boundary ∂X . We denote the diameter of X by $\text{diam}(X)$ ($\text{diam}(X) := \sup_{(x,y) \in X^2} |x - y|$). Let $\nu(x)$ denote the outward normal unit vector to ∂X at $x \in \partial X$. Let $\Gamma_{\pm} = \{(x, v) \in \partial X \times \mathbb{S}^{d-1} \mid \pm \nu(x)v > 0\}$. For $(x, v) \in \bar{X} \times \mathbb{S}^{d-1}$ we define $\tau_{\pm}(x, v)$ and $\tau(x, v)$ by $\tau_{\pm}(x, v) := \inf\{s \in (0, +\infty) \mid x \pm sv \notin X\}$ and $\tau(x, v) := \tau_{-}(x, v) + \tau_{+}(x, v)$.

Consider $\sigma : X \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ and $k : X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ two nonnegative measurable functions. We assume that (σ, k) is admissible when

$$\begin{aligned} 0 &\leq \sigma \in L^{\infty}(X \times \mathbb{S}^{d-1}), \\ 0 &\leq k(x, v', \cdot) \in L^1(\mathbb{S}^{d-1}) \text{ a.e. } (x, v') \in X \times \mathbb{S}^{d-1} \\ \sigma_p(x, v') &= \int_{\mathbb{S}^{d-1}} k(x, v', v) dv \text{ belongs to } L^{\infty}(X \times \mathbb{S}^{d-1}). \end{aligned} \tag{2.1}$$

Let $T > \eta > 0$. We consider the following linear Boltzmann transport equation with boundary conditions

$$\frac{\partial u}{\partial t}(t, x, v) + v \nabla_x u(t, x, v) + \sigma(x, v)u(t, x, v) = \int_{\mathbb{S}^{d-1}} k(x, v', v)u(t, x, v')dv', \quad (t, x, v) \in (0, T) \times X \times \mathbb{S}^{d-1}, \tag{2.2}$$

$$\begin{aligned} u|_{(0,T) \times \Gamma_-}(t, x, v) &= \phi(t, x, v), \\ u(0, x, v) &= 0, \quad (x, v) \in X \times \mathbb{S}^{d-1}, \end{aligned}$$

where $\phi \in L^1((0, T), L^1(\Gamma_-, d\xi))$ and $\text{supp}\phi \subseteq [0, \eta]$.

We assume here that scattering is elastic, which implies that the speed of the particles is preserved by scattering while only the direction of propagation may change. Elastic scattering is a good approximation in many applications in medical and geophysical imaging. Our results are stated for a (normalized) velocity space equal to the unit sphere \mathbb{S}^{d-1} . Generalizations to other velocity spaces may be obtained as in e.g. [BJ2] and [CS1, CS2].

2.2 Semigroups and unbounded operators

We introduce the following space

$$\mathcal{Z} := \{f \in L^1(X \times \mathbb{S}^{d-1}) \mid v\nabla_x f \in L^1(X \times \mathbb{S}^{d-1})\}, \quad (2.3)$$

$$\|f\|_{\mathcal{Z}} := \|f\|_{L^1(X \times \mathbb{S}^{d-1})} + \|v\nabla_x f\|_{L^1(X \times \mathbb{S}^{d-1})}; \quad (2.4)$$

where $v\nabla_x$ is understood in the distributional sense.

It is known [C1, C2] that the trace map β_- from $C^1(\bar{X} \times \mathbb{S}^{d-1})$ to $C(\Gamma_-)$ defined by

$$\beta_-(f) = f|_{\Gamma_-} \quad (2.5)$$

extends to a continuous operator from \mathcal{Z} onto $L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v))$ and admits a continuous lifting. Note that $L^1(\Gamma_-, d\xi)$ is a subset of the space $L^1(\Gamma_-, \tau_+(x, v)d\xi(x, v))$.

We introduce the following notation

$$A_1 f = -\sigma f, \quad A_2 f = \int_{\mathbb{S}^{d-1}} k(x, v', v) f(x, v') dv'. \quad (2.6)$$

As (σ, k) is admissible, the operators A_1 and A_2 are bounded operators in $L^1(X \times \mathbb{S}^{d-1})$.

Consider the following unbounded operators

$$T_1 f = -v\nabla_x f + A_1 f, \quad D(T_1) = \{f \in \mathcal{Z} \mid f|_{\Gamma_-} = 0\}, \quad (2.7)$$

$$T f = T_1 f + A_2 f, \quad D(T) = D(T_1). \quad (2.8)$$

The unbounded operators T_1 and T are generators of strongly continuous semigroups $U_1(t)$ and $U(t)$, respectively, in $L^1(X \times \mathbb{S}^{d-1})$ (see e.g. [DL, Proposition 2 p.226]). In addition, $U_1(t)$ and $U(t)$ preserve the cone of positive functions and $U_1(t)$ is given explicitly by the following formula

$$U_1(t)f = e^{-\int_0^t \sigma(x-sv, v) ds} f(x-tv, v) \theta(x-tv, x), \quad \text{for a.e. } (x, v) \in X \times \mathbb{S}^{d-1}, \quad (2.9)$$

for $f \in L^1(X \times \mathbb{S}^{d-1})$, where

$$\theta(x, y) = \begin{cases} 1 & \text{if } x + p(y-x) \in X \text{ for all } p \in [0, 1], \\ 0 & \text{otherwise,} \end{cases} \quad (2.10)$$

for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

We will use the Duhamel formula

$$U(r') = U_1(r') + \int_0^{r'} U_1(r' - s') A_2 U(s') ds', \quad \text{for } r' \geq 0. \quad (2.11)$$

2.3 Trace results

We introduce the following space

$$\mathcal{W} := \left\{ u \in L^1((0, T) \times X \times \mathbb{S}^{d-1}) \mid \left(\frac{\partial}{\partial t} + v \nabla_x \right) u \in L^1((0, T) \times X \times \mathbb{S}^{d-1}) \right\}, \quad (2.12)$$

$$\|u\|_{\mathcal{W}} := \|u\|_{L^1((0, T) \times X \times \mathbb{S}^{d-1})} + \left\| \left(\frac{\partial}{\partial t} + v \nabla_x \right) u \right\|_{L^1((0, T) \times X \times \mathbb{S}^{d-1})}; \quad (2.13)$$

where $\frac{\partial}{\partial t}$ and $v \nabla_x$ are understood in the distributional sense.

It is known [C1, C2] that the trace map γ_- (respectively γ_+) from $C^1([0, T] \times \bar{X} \times \mathbb{S}^{d-1})$ to $C(X \times \mathbb{S}^{d-1}) \times C((0, T) \times \Gamma_-)$ (respectively $C(X \times \mathbb{S}^{d-1}) \times C((0, T) \times \Gamma_+)$) defined by

$$\gamma_-(\psi) = (\psi(0, \cdot), \psi|_{(0, T) \times \Gamma_-}) \quad (\text{respectively } \gamma_+(\psi) = (\psi(T, \cdot), \psi|_{(0, T) \times \Gamma_+})) \quad (2.14)$$

extends to a continuous operator from \mathcal{W} onto $L^1(X \times \mathbb{S}^{d-1}, \tau_+(x, v) dx dv) \times L^1((0, T) \times \Gamma_-, \min(T-t, \tau_+(x, v)) dt d\xi(x, v))$ (respectively $L^1(X \times \mathbb{S}^{d-1}, \tau_-(x, v) dx dv) \times L^1((0, T) \times \Gamma_+, \min(t, \tau_-(x, v)) dt d\xi(x, v))$). In addition γ_{\pm} admits a continuous lifting. Note that $L^1(X \times \mathbb{S}^{d-1})$ is a subset of $L^1(X \times \mathbb{S}^{d-1}, \tau_+(x, v) dx dv)$. Note also that $L^1((0, T) \times \Gamma_-, dt d\xi)$ (respectively $L^1((0, T) \times \Gamma_+, dt d\xi)$) is a subset of $L^1((0, T) \times \Gamma_-, \min(T-t, \tau_+(x, v)) dt d\xi(x, v))$ (respectively $L^1((0, T) \times \Gamma_+, \min(t, \tau_-(x, v)) dt d\xi(x, v))$).

We now introduce the space

$$W := \{u \in \mathcal{W} \mid \gamma_-(u) \in L^1(X \times \mathbb{S}^{d-1}) \times L^1((0, T) \times \Gamma_-, dt d\xi)\}. \quad (2.15)$$

We recall the following trace results (see [C1, C2] in a more general setting).

Lemma 2.1. *The following equality is valid*

$$W = \{u \in \mathcal{W} \mid \gamma_+(u) \in L^1(X \times \mathbb{S}^{d-1}) \times L^1((0, T) \times \Gamma_+, dt d\xi)\}. \quad (2.16)$$

In addition the trace maps

$$\begin{aligned} \gamma_{\pm} : W &\rightarrow L^1(X \times \mathbb{S}^{d-1}) \times L^1((0, T) \times \Gamma_{\pm}, dt d\xi) \\ &\text{are continuous, onto, and admit continuous liftings.} \end{aligned} \quad (2.17)$$

2.4 Solution to equation (2.2)

For any $r > 0$, we identify the space $L^1((0, r), L^1(\Gamma_{\pm}, d\xi))$ with the space $L^1((0, r) \times \Gamma_{\pm}, dt d\xi)$, and we extend any function $\phi \in L^1((0, r), L^1(\Gamma_-, d\xi))$ by 0 on $\mathbb{R} \setminus (0, r)$ (the extension is still denoted by ϕ).

Let $\phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi))$. We extend ϕ by 0 outside $(0, \eta)$. Then we consider the lifting $G_-(t)\phi \in W$ of $(0, \phi)$ defined by

$$G_-(t)\phi(x, v) := e^{-\int_0^{\tau_-(x, v)} \sigma(x-sv, v) ds} \phi_-(t - \tau_-(x, v), x - \tau_-(x, v)v, v), \quad (2.18)$$

for $(t, x, v) \in (0, T) \times X \times \mathbb{S}^{d-1}$. Note that $G_-(.)\phi$ is a solution in the distributional sense of the equation $(\frac{\partial}{\partial t} + v\nabla_x)u + \sigma u = 0$ in $(0, T) \times X \times \mathbb{S}^{d-1}$ and

$$\|G_-(.)\phi\|_W \leq (1 + \|\sigma\|_\infty) \|G_-(.)\phi\|_{L^1((0, T) \times X \times \mathbb{S}^{d-1})} \leq (1 + \|\sigma\|_\infty) T \|\phi_-\|_{L^1((0, \eta) \times \Gamma_-, d\xi)}. \quad (2.19)$$

To prove the latter statements, one can use the change of variables given by Lemma 4.1. From (2.19) we obtain that the map $i : L^1((0, \eta), L^1(\Gamma_-, d\xi)) \rightarrow W$ defined by

$$i(\phi) = G_-(.)\phi, \quad \phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi)), \quad (2.20)$$

is continuous.

The following result holds (see [DL, Theorem 3 p. 229]).

Lemma 2.2. *The equation (2.2) admits a unique solution u in W which is given by*

$$u(t) = G_-(t)\phi + \int_0^t U(t-s)A_2G_-(s)\phi ds. \quad (2.21)$$

where $U(t)$ is the strongly continuous semigroup in $L^1(X \times \mathbb{S}^{d-1})$ introduced in section 2.2.

From (2.20), Lemma 2.2 and (2.17), we obtain the existence of the albedo operator.

Lemma 2.3. *The albedo operator A given by the formula*

$$A\phi = u|_{(0, T) \times \Gamma_+}, \quad \text{for } \phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi)) \text{ where } u \text{ is given by (2.21),} \quad (2.22)$$

is well-defined and is a bounded operator from $L^1((0, \eta), L^1(\Gamma_-, d\xi))$ to $L^1((0, T), L^1(\Gamma_+, d\xi))$.

3 Stability results for the inverse problem

3.1 Recall of uniqueness results

Choulli-Stefanov [CS1] studied the uniqueness of the reconstruction of (σ, k) from the albedo operator by analyzing the distributional kernel of that operator. They considered the following problem

$$\frac{\partial u}{\partial t}(t, x, v) + v\nabla_x u(t, x, v) + \sigma(x, v)u(t, x, v) = \int_{\mathbb{S}^{d-1}} k(x, v', v)u(t, x, v')dv', \quad (t, x, v) \in \mathbb{R} \times X \times V, \quad (3.1)$$

$$\begin{aligned} u|_{\mathbb{R} \times \Gamma_-}(t, x, v) &= \phi(t, x, v), \\ u|_{t \leq 0} &= 0, \end{aligned}$$

for $\phi \in L^1_{\text{comp}}(\mathbb{R}, L^1(\Gamma_-, d\xi))$, where V is an open subset of \mathbb{R}^d , $d \geq 2$. The albedo operator is defined as an operator from $L^1_{\text{comp}}(\mathbb{R}, L^1(\Gamma_-, d\xi))$ to $L^1_{\text{loc}}(\mathbb{R}, L^1(\Gamma_+, d\xi))$. They proved, in particular, that the albedo operator uniquely determines the absorption and scattering coefficient (σ, k) provided that σ is a function of x and $|v|$ only. It is straightforward from the proof of this result (see [CS1, Theorem 5.1, Propositions 5.1 and 5.2]) that the following result holds.

Proposition 3.1. *Assume that (σ, k) are admissible and $\sigma(x, v) = f(x, |v|)$ for some real function f . Let $T > \eta > 0$. Then the following statements are valid:*

i if $T > \text{diam}(X)$ then the albedo operator

$A : L^1((0, \eta), L^1(\Gamma_-, d\xi)) \rightarrow L^1((0, T), L^1(\Gamma_+, d\xi))$ uniquely determines σ ,

ii if $T > 2\text{diam}(X)$ then the albedo operator

$A : L^1((0, \eta), L^1(\Gamma_-, d\xi)) \rightarrow L^1((0, T), L^1(\Gamma_+, d\xi))$ uniquely determines (σ, k) ,

In this paper we analyze the stability of the reconstruction of (σ, k) from the albedo operator. Our study is also based on the distributional kernel of the albedo operator. In a first stage, we do not assume that $\sigma(x, v) = f(x, |v|)$ for some real function f .

3.2 Decomposition of the albedo operator

Consider the distributional kernels

$$\alpha_1(\tau, x, v, x', v') = e^{-\int_0^{\tau_-(x,v)} \sigma(x-sv, v) ds} \delta_v(v') \delta_{x-\tau_-(x,v)v}(x') \delta(\tau - \tau_-(x, v)), \quad (3.2)$$

$$\begin{aligned} \alpha_2(\tau, x, v, x', v') &= \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma(x-pv, v) ds - \int_0^{\tau_-(x-sv, v')} \sigma(x-tv-pv', v') dp} \\ &\quad \times k(x-sv, v', v) \delta_{x-sv-\tau_-(x-sv, v')v'}(x') \delta(\tau - s - \tau_-(x-sv, v')) ds, \end{aligned} \quad (3.3)$$

for a.e. $(\tau, x, v, x', v') \in \mathbb{R} \times \Gamma_+ \times \Gamma_-$ and where we have defined $\int_{\mathbb{S}^{d-1}} f_1(v') \delta_v(v') dv' = f_1(v)$, $\int_{\partial X} \delta_{x'_0}(x') f_2(x') d\mu(x') = f_2(x'_0)$ and $\int_{\mathbb{R}} \delta(\tau - s) f_3(\tau) d\tau = f_3(s)$ for $(v, x'_0, s) \in \mathbb{S}^{d-1} \times \partial X \times \mathbb{R}$ and for $(f_1, f_2, f_3) \in C(\mathbb{S}^{d-1}) \times C(\partial X) \times C(\mathbb{R})$.

We consider the usual decomposition of the albedo operator as a sum of three terms: the ballistic part (whose distributional kernel is given by α_1), the single scattering part (whose distributional kernel is given by α_2) and the multiple scattering (whose distributional kernel is denoted by α_3). Using [CS1, Theorem 5.1], we know that $|\nu(x')v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-, L^1_{\text{loc}}(\mathbb{R}, L^1(\Gamma_+, d\xi)))$. The following Proposition 3.2 improves on the latter statement provided that $k \in L^\infty(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$. The result will be used in the proof of Theorem 3.1.

Proposition 3.2. *Assume $d \geq 2$ and (σ, k) admissible. Assume that $k \in L^\infty(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$. Then*

$$A(\phi)(t, x, v) := \int_{(0, \eta) \times \Gamma_-} (\alpha_1 + \alpha_2 + \alpha_3)(t - t', x, v, x', v') \phi(t', x', v') dt' d\mu(x') dv', \quad (3.4)$$

for a.e. $(t, x, v) \in (0, T) \times \Gamma_+$ and for any continuous and compactly supported function ϕ on $(0, \eta) \times \Gamma_-$, where

$$|\nu(x')v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-, L^p((-\eta, T), L^p(\Gamma_+, d\xi))), \quad \text{for any } 1 \leq p < \frac{d+1}{d}. \quad (3.5)$$

Proposition 3.2 is proved in section 4.

3.3 First stability result

Now we assume that X is a bounded open convex subset of \mathbb{R}^d , $d \geq 2$, with C^1 boundary and that

$$\begin{aligned} &\text{the function } 0 \leq \sigma \text{ is continuous and bounded on } X \times \mathbb{S}^{d-1}, \\ &\text{the function } 0 \leq k \text{ is continuous and bounded on } X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}. \end{aligned} \quad (3.6)$$

Let $(\tilde{\sigma}, \tilde{k})$ be a pair of absorption and scattering coefficients that also satisfy (3.6). Let \tilde{A} be the albedo operator from $L^1((0, \eta), L^1(\Gamma_-, d\xi))$ to $L^1((0, T), L^1(\Gamma_+, d\xi))$ related to $(\tilde{\sigma}, \tilde{k})$.

For $(x, v, s, w) \in \Gamma_{\pm} \times \mathbb{R} \times \mathbb{S}^{d-1}$, $0 < s < \tau_{\mp}(x, v)$, let $E_{\mp}(x, v, s, w) \geq 0$ be defined by

$$E_{\mp}(x, v, s, w) = \exp \left(- \int_0^s \sigma(x \mp pv, v) ds - \int_0^{\tau_{\mp}(x \mp sv, w)} \sigma(x \mp sv \mp pw, w) dp \right). \quad (3.7)$$

Replacing σ by $\tilde{\sigma}$ in (3.7) we define $\tilde{E}_{\mp}(x, v, s, w)$ similarly for $(x, v, s, w) \in \Gamma_{\pm} \times \mathbb{R} \times \mathbb{S}^{d-1}$, $0 < s < \tau_{\mp}(x, v)$.

Let $(x'_0, v'_0) \in \Gamma_-$. For $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, let $f_{\varepsilon_1} \in C^1(\Gamma_-)$ and $g_{\varepsilon_2} \in C^\infty(\mathbb{R})$ be such that

$$f_{\varepsilon_1} \geq 0, \quad \text{supp } f_{\varepsilon_1} \subseteq \{(x', v') \in \Gamma_- \mid |x - x'_0| + |v' - v'_0| \leq \varepsilon_1\}, \quad (3.8)$$

$$\int_{\Gamma_-} f_{\varepsilon_1}(x', v') d\xi(x', v') = 1,$$

$$g_{\varepsilon_2} \geq 0, \quad \text{supp } g_{\varepsilon_2} \subseteq (0, \min(\eta, \varepsilon_2)), \quad \int_0^{+\infty} g_{\varepsilon_2}(t) dt = 1. \quad (3.9)$$

Consider the function $\phi_{\varepsilon_1, \varepsilon_2} \in C^1(\mathbb{R} \times \Gamma_-)$ defined by

$$\phi_{\varepsilon_1, \varepsilon_2}(t', x', v') = g_{\varepsilon_2}(t') f_{\varepsilon_1}(x', v'), \quad (3.10)$$

for $t' \in (0, +\infty)$ and $(x', v') \in \Gamma_-$. Note that $\text{supp } \phi_{\varepsilon_1, \varepsilon_2} \subseteq (0, \eta) \times \Gamma_-$ (see (3.9)). From (3.8) and (3.9) it follows that $|\nu(x')v'| \phi_{\varepsilon_1, \varepsilon_2}$ is a smooth approximation of the delta function on $\mathbb{R} \times \Gamma_-$ at $(0, x'_0, v'_0)$ as $\varepsilon_1 \rightarrow 0^+$ and $\varepsilon_2 \rightarrow 0^+$.

Let ψ be any compactly supported continuous function on $(0, T) \times \Gamma_+$ such that $\|\psi\|_{\infty} \leq 1$. First we note that upon using the estimate $\|\psi\|_{\infty} \leq 1$ and the equality $\int_{(0, \eta) \times \Gamma_-} \phi_{\varepsilon_1, \varepsilon_2}(t, x, v) dt d\xi(x, v) = 1$ we obtain that

$$\begin{aligned} \left| \int_{(0, T) \times \Gamma_+} \psi(t, x, v) (A - \tilde{A}) \phi_{\varepsilon_1, \varepsilon_2}(t, x, v) dt d\xi(x, v) \right| &\leq \| (A - \tilde{A}) \phi_{\varepsilon_1, \varepsilon_1} \|_{L^1((0, T), L^1(\Gamma_+, d\xi))} \\ &\leq \| A - \tilde{A} \|_{\eta, T}, \end{aligned} \quad (3.11)$$

where $\|\cdot\|_{\eta, T} := \|\cdot\|_{\mathcal{L}(L^1((0, \eta), L^1(\Gamma_-, d\xi)), L^1((0, T), L^1(\Gamma_+, d\xi)))}$.

In addition, it follows from (3.4)–(3.5) that for any compactly supported continuous function ψ on $(0, T) \times \Gamma_+$, we have

$$\int_{(0, T) \times \Gamma_+} \psi(t, x, v) (A - \tilde{A}) \phi_{\varepsilon_1, \varepsilon_2}(t, x, v) dt d\xi(x, v) = I_1(\psi, \varepsilon_1, \varepsilon_2) + I_2(\psi, \varepsilon_1, \varepsilon_2) + I_3(\psi, \varepsilon_1, \varepsilon_2), \quad (3.12)$$

where

$$I_1(\psi, \varepsilon_1, \varepsilon_2) = \int_0^T \int_{\Gamma_+} \left(e^{-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds} - e^{-\int_0^{\tau_-(x,v)} \tilde{\sigma}(x-sv,v)ds} \right) \times \psi(t, x, v) \phi_{\varepsilon_1, \varepsilon_2}(t - \tau_-(x, v), x - \tau_-(x, v)v, v) d\xi(x, v) dt \quad (3.13)$$

$$I_2(\psi, \varepsilon_1, \varepsilon_2) = \int_0^T \int_{\Gamma_+} \psi(t, x, v) \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(x,v)} \left(E_-(x, v, s, v') k(x - sv, v', v) - \tilde{E}_-(x, v, s, v') \tilde{k}(x - sv, v', v) \right) \psi_{\varepsilon_1, \varepsilon_2}(t - s - \tau_-(x - sv, v'), x - sv - \tau_-(x - sv, v')v', v') ds dv' d\xi(x, v) dt \quad (3.14)$$

$$|I_3(\psi, \varepsilon_1, \varepsilon_2)| \leq C \left(\int_0^T \int_{\Gamma_-} |\psi(t, x, v)|^{p'} d\xi(x, v) dt \right)^{\frac{1}{p'}}, \quad (3.15)$$

where C is a constant that does not depend on $\varepsilon_1, \varepsilon_2$ (for (3.15) we also used Hölder inequality and the equality $\|\phi_{\varepsilon_1, \varepsilon_2}\|_{L^1((0, \eta), L^1(\Gamma_-, d\xi))} = 1$).

Using (3.8)–(3.10), (3.6), we obtain the following preparatory Lemma 3.1.

Lemma 3.1. *Assume that X is convex and that (σ, k) and $(\tilde{\sigma}, \tilde{k})$ both satisfy (3.6). Then the following statements are valid:*

i. *if $T > \text{diam}(X)$ then*

$$\lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_1(\psi, \varepsilon_1, \varepsilon_2) = \psi(\tau_+(x'_0, v'_0), x'_0 + \tau_+(x'_0, v'_0)v'_0, v'_0) \times \left(e^{-\int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 + sv'_0, v'_0)ds} - e^{-\int_0^{\tau_+(x'_0, v'_0)} \tilde{\sigma}(x'_0 + sv'_0, v'_0)ds} \right), \quad (3.16)$$

for any compactly supported and continuous function ψ on $(0, T) \times \Gamma_+$;

ii. *if $T > 2\text{diam}(X)$ then*

$$\lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_2(\psi, \varepsilon_1, \varepsilon_2) = I_2^1(\psi) + I_2^2(\psi), \quad (3.17)$$

for any compactly supported and continuous function ψ on $(0, T) \times \Gamma_+$, where

$$I_2^1(\psi) = \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \psi(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) (k - \tilde{k})(x'_0 + sv'_0, v'_0, v) E_+(x'_0, v'_0, s, v) ds dv, \quad (3.18)$$

$$I_2^2(\psi) = \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \psi(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \tilde{k}(x'_0 + sv'_0, v'_0, v) (E_+ - \tilde{E}_+)(x'_0, v'_0, s, v) ds dv, \quad (3.19)$$

where E_+ and \tilde{E}_+ are defined by (3.7).

Lemma 3.1 is proved in section 5.

Taking account of Lemma 3.1 and (3.11), and choosing an appropriate sequence of functions “ ψ ”, we obtain the main result of this paper:

Theorem 3.1. *Let $T > \eta > 0$. Assume that $d \geq 2$ and X is convex and (σ, k) and $(\tilde{\sigma}, \tilde{k})$ both satisfy condition (3.6). Then the following statements are valid:*

i. if $T > \text{diam}(X)$, then

$$\left| \exp \left(- \int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 + sv'_0, v'_0) ds \right) - \exp \left(- \int_0^{\tau_+(x'_0, v'_0)} \tilde{\sigma}(x'_0 + sv'_0, v'_0) ds \right) \right| \leq \|A - \tilde{A}\|_{\eta, T}; \quad (3.20)$$

ii. if $T > 2\text{diam}(X)$, then

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \left| k - \tilde{k} \right| (x'_0 + sv'_0, v'_0, v) E_+(x'_0, v'_0, s, v) ds dv \\ & \leq \tau_+(x'_0, v'_0) \sup_{s \in (0, \tau_+(x'_0, v'_0))} \tilde{\sigma}_p(x'_0 + sv'_0, v'_0) \sup_{\substack{s \in (0, \tau_+(x'_0, v'_0)) \\ v \in \mathbb{S}^{d-1}}} \left| E_+ - \tilde{E}_+ \right| (x'_0, v'_0, s, v) \\ & + \|A - \tilde{A}\|_{\eta, T}, \end{aligned} \quad (3.21)$$

where $\|\cdot\|_{\eta, T} := \|\cdot\|_{\mathcal{L}(L^1((0, \eta), L^1(\Gamma_-, d\xi)), L^1((0, T), L^1(\Gamma_+, d\xi)))}$ and where E_+ and \tilde{E}_+ are defined by (3.7).

The proof of Theorem 3.1 is given in section 5.

Remark 3.1. One can prove that estimate (3.20) still holds a.e. $(x'_0, v'_0) \in \Gamma_-$ provided that $T > \text{diam}(X)$ and $k \in L^\infty(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ and without assuming (3.6) nor that X is convex.

3.4 Second stability result

We now impose that the absorption coefficient σ does not depend on the velocity variable, i.e. $\sigma(x, v) = \sigma(x)$, $x \in X$. Then let

$$\begin{aligned} \mathcal{M} &:= \{(\sigma(x), k(x, v', v)) \in L^\infty(X) \times L^\infty(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}) \mid (\sigma, k) \text{ satisfies (3.6),} \\ & \text{and } \sigma \in H^{\frac{d}{2} + \tilde{r}}(X), \|\sigma\|_{H^{\frac{d}{2} + \tilde{r}}(X)} \leq M, \|\sigma_p\|_\infty \leq M\}, \end{aligned} \quad (3.22)$$

for some $\tilde{r} > 0$ and $M > 0$. Using Theorem 3.1 for any $(x'_0, v'_0) \in \Gamma_-$ we obtain the following Theorem 3.2.

Theorem 3.2. Assume that $d \geq 2$ and X is convex. Let $T > \eta > 0$. For any $(\sigma, k) \in \mathcal{M}$ and $(\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$ the following stability estimates are valid:

i. if $T > \text{diam}(X)$ then

$$\|\sigma - \tilde{\sigma}\|_{H^s(X)} \leq C_1 \|A - \tilde{A}\|_{\eta, T}^\kappa, \quad (3.23)$$

where $-\frac{1}{2} \leq s < \frac{d}{2} + \tilde{r}$, $\kappa = \frac{d+2(\tilde{r}-s)}{d+1+2\tilde{r}}$, and $C_1 = C_1(X, M, s, \tilde{r})$;

ii. if $T > 2\text{diam}(X)$ then

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \left| k(x'_0 + s'v'_0, v'_0, v) - \tilde{k}(x'_0 + s'v'_0, v'_0, v) \right| ds' dv \\ & \leq C_2 \|A - \tilde{A}\|_{\eta, T}^\kappa \left(1 + \|A - \tilde{A}\|_{\eta, T}^{1-\kappa} \right), \end{aligned} \quad (3.24)$$

for $(x'_0, v'_0) \in \Gamma_-$, and where $\kappa = \frac{2(\tilde{r}-r)}{d+1+2\tilde{r}}$, $0 < r < \tilde{r}$, and $C_2 = C_2(X, M, r, \tilde{r})$;

iii. in addition, if $T > 2\text{diam}(X)$ then

$$\|k - \tilde{k}\|_{L^1(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1})} \leq C_3 \|A - \tilde{A}\|_{\eta, T}^{\kappa} \left(1 + \|A - \tilde{A}\|_{\eta, T}^{1-\kappa}\right), \quad (3.25)$$

where $\kappa = \frac{2(\tilde{r}-r)}{d+1+2\tilde{r}}$, $0 < r < \tilde{r}$, and $C_3 = C_3(X, M, r, \tilde{r})$.

Theorem 3.2 is proved in section 5.

Remark 3.2. Stability estimates similar to (3.23) were given by Cipolatti-Motta-Roberty [CMR, Theorem 1.1]. They proved (3.23) for $s = -\frac{1}{2}$ under the assumptions $k, \tilde{k} \in L^\infty(X, L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}))$, $\max(\|\sigma\|_\infty, \|\tilde{\sigma}\|_\infty) \leq M$ (and $\max(\|\sigma_p\|_\infty, \|\tilde{\sigma}_p\|_\infty) < \infty$). They also proved (3.23) for $0 < s < \tilde{r}$ under the assumptions $k, \tilde{k} \in L^\infty(X, L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}))$, $\sigma, \tilde{\sigma} \in H^{\frac{d}{2}+\tilde{r}}(X)$ and $\max(\|\sigma\|_{H^{\frac{d}{2}+\tilde{r}}}, \|\tilde{\sigma}\|_{H^{\frac{d}{2}+\tilde{r}}}) \leq M$.

4 Proof of Proposition 3.2

Before giving the proof of Proposition 3.2, we need Lemmas 4.1, 4.2, 4.3.

Lemma 4.1. For $f \in L^1(X \times \mathbb{S}^{d-1})$ we have

$$\int_{X \times \mathbb{S}^{d-1}} f(x, v) dx dv = \int_{\Gamma_\pm} \int_0^{\tau_\mp(x, v)} f(x \mp sv, v) ds d\xi(x, v). \quad (4.1)$$

For the proof of Lemma 4.1, see [CS2, Lemma 2.1].

Let $m \geq 1$. For U a subset of \mathbb{R}^m , we denote by χ_U the function from \mathbb{R}^m to \mathbb{R} defined by

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Lemma 4.2. Let $T > 0$ and let $1 < p < \frac{d+1}{d}$. Consider the nonnegative measurable function $\beta : X \rightarrow \mathbb{R}$ defined by

$$\beta(x') = \int_{\Gamma_+} \int_{|x-x'|}^{T+\text{diam}(X)} \frac{(s - (x - x')v)^{p(d-3)}}{|x - x' - sv|^{p(2d-4)}} ds d\xi(x, v) \quad (4.3)$$

for a.e. $x' \in X$. Then

$$\beta \in L^\infty(X). \quad (4.4)$$

Proof of Lemma 4.2. We first consider the case $d = 2$.

We have

$$\begin{aligned} \beta(x') &\leq \int_{\partial X} \int_{|x-x'|}^{+\infty} \int_{\mathbb{S}^1} \frac{1}{(s - (x - x')v)^p} dv ds d\mu(x) \\ &\leq \int_{\partial X} \int_0^{2\pi} \int_{|x-x'|}^{+\infty} \frac{-1}{p-1} \frac{d}{ds} \frac{1}{(s - |x - x'| \cos \omega)^{p-1}} d\omega ds d\mu(x) \\ &= \frac{1}{p-1} \int_{\partial X} \frac{1}{|x - x'|^{p-1}} dx \int_0^{2\pi} \frac{1}{(1 - \cos \omega)^{p-1}} d\omega, \end{aligned}$$

for a.e. $x' \in X$. Hence using the estimate $p < 1 + \frac{1}{2}$, we obtain

$$\|\beta\|_{L^\infty(X)} \leq \frac{1}{p-1} \int_0^{2\pi} \frac{1}{(1-\cos\omega)^{p-1}} d\omega \sup_{z \in X} \int_{\partial X} \frac{1}{|x-z|^{p-1}} d\mu(x) < \infty. \quad (4.5)$$

Now assume $d = 3$. Using (4.3), spherical coordinates and performing the change of variables “ s ” = $|x - x'|s$, we obtain

$$\begin{aligned} \beta(x') &\leq \int_{\partial X} \int_{\mathbb{S}^2} \int_{|x-x'|}^{+\infty} \frac{1}{|sv - (x - x')|^{2p}} ds dv d\mu(x) \\ &= 2\pi \int_{\partial X} \frac{1}{|x - x'|^{2p-1}} d\mu(x) \int_1^\infty \frac{1}{2s(p-1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d}{d\omega} \frac{1}{(s^2 + 1 - 2s \sin \omega)^{p-1}} d\omega ds \\ &\leq 2\pi \int_{\partial X} \frac{1}{|x - x'|^{2p-1}} d\mu(x) \int_1^\infty \frac{1}{2(p-1)s(s-1)^{2p-2}} ds. \end{aligned}$$

Therefore using the estimate $1 < p < 1 + \frac{1}{3}$, we obtain

$$\|\beta\|_{L^\infty(X)} \leq 2\pi \sup_{z \in X} \int_{\partial X} \frac{1}{|x - z|^{2p-1}} d\mu(x) \int_1^\infty \frac{1}{2(p-1)s(s-1)^{2p-2}} ds < \infty. \quad (4.6)$$

Finally assume $d \geq 4$. Note that $|s - v(x - x')| = |(sv - (x - x'))v| \leq |sv - (x - x')|$ for $s \in \mathbb{R}$ and $x, x' \in \mathbb{R}^d$. Using in particular the latter estimate and (4.3), we obtain

$$\begin{aligned} \beta(x') &\leq \int_{\partial X} \int_{\mathbb{S}^{d-1}} \int_{|x-x'|}^{+\infty} \frac{(s - v(x - x'))}{|sv - (x - x')|^{p(d-1)+1}} ds dv d\mu(x) \\ &= \frac{1}{(p(d-1)-1)} \int_{\partial X} \int_{\mathbb{S}^{d-1}} \frac{1}{(|x - x'|v - (x - x'))^{p(d-1)-1}} dv d\mu(x) \\ &= \text{Vol}(\mathbb{S}^{d-2}) \int_{\partial X} \frac{1}{(p(d-1)-1)(|x - x'|\sqrt{2})^{p(d-1)-1}} d\mu(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \omega^{d-2}}{(1 - \sin \omega)^{\frac{p(d-1)-1}{2}}} d\omega, \end{aligned}$$

for a.e. $x' \in X$. Hence using the estimate $p < 1 + \frac{1}{d}$, we obtain

$$\begin{aligned} \|\beta\|_{L^\infty(X)} &\leq \text{Vol}(\mathbb{S}^{d-2}) \frac{1}{(p(d-1)-1)2^{\frac{p(d-1)-1}{2}}} \sup_{z \in X} \int_{\partial X} \frac{1}{|x - z|^{p(d-1)-1}} d\mu(x) \\ &\quad \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \omega^{d-2}}{(1 - \sin \omega)^{\frac{p(d-1)-1}{2}}} d\omega < \infty. \end{aligned} \quad (4.7)$$

□

Finally, we need the following Lemma 4.3.

Lemma 4.3. *Consider the nonnegative measurable function $\gamma : (0, T) \times X \times \mathbb{S}^{d-1} \times X \times \mathbb{S}^{d-1} \rightarrow \mathbb{R}$ defined by*

$$\begin{aligned} \gamma(t, x, v, x', v') &= 2^{d-2} \chi_{(0,t)}(|x - x'|) \left[e^{-\int_0^{s_1} \sigma(x-sv, v) ds - \int_0^{t-s_1} \sigma(x-s_1v-pv_1, v_1) dp} \theta(x, x - s_1v) \right. \\ &\quad \times \theta(x - s_1v, x') k(x - s_1v, v_1) k(x', v', v_1) \Big]_{s_1 = \frac{t^2 - (x-x')^2}{2(t-(x-x')v)}}^{\frac{(t - (x-x')v)^{d-3}}{|tv - x - x'|^{2d-4}}}, \end{aligned} \quad (4.8)$$

where θ is defined by (2.10). Then

$$\left(\int_0^t U_1(t-s_1) A_2 U_1(s_1) A_2 f ds_1 \right) (x, v) = \int_{X \times \mathbb{S}^{d-1}} \gamma(t, x, v, x', v') f(x', v') dx' dv' \quad (4.9)$$

for $t \in (0, T)$ and for a.e. $(x, v) \in X \times \mathbb{S}^{d-1}$ and for $f \in L^1(X \times \mathbb{S}^{d-1})$.

Proof of Lemma 4.3. Let $t \in (0, T)$ and let $f \in L^1(X \times \mathbb{S}^{d-1})$. From (2.9) and (2.6), it follows that

$$\begin{aligned} \int_0^t U_1(t-s_1)A_2U_1(s_1)A_2f ds_1 &= \int_0^t \int_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} e^{-\int_0^{t-s_1} \sigma(x-pv, v) dp - \int_0^{s_1} \sigma(x-(t-s_1)v-pv_1, v_1) dp} \\ &\quad \times k(x-(t-s_1)v, v_1, v) k(x-(t-s_1)v-s_1v_1, v', v_1) \\ &\quad \times \theta(x-(t-s_1)v, x) \theta(x-(t-s_1)v-s_1v_1, x-(t-s_1)v) \\ &\quad \times f(x-(t-s_1)v-s_1v_1, v') dv' dv_1 ds_1. \end{aligned} \quad (4.10)$$

Performing the change of variables “ $s_1 = t - s_1$ ” and then performing the change of variables “ $x' = x - (t - s_1)v_1 - s_1v$ ” ($2^{d-2} \frac{(tv-(x-x')v)^{d-3}}{|x-x'-tv|^{2d-4}} dx' = dv_1 ds_1$), we obtain (4.9). \square

Proof of Proposition 3.2. Let $\phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi))$. Let u be the solution of (2.2). Using twice Duhamel’s formula (2.11) and using (2.21) we obtain

$$u(t) = R_1(t) + R_2(t) + R_3^1(t) + R_3^2(t), \quad (4.11)$$

for $t \in (0, T)$ where

$$R_1(t) = G_-(t)\phi, \quad (4.12)$$

$$R_2(t) = \int_0^t U_1(t-t')A_2G_-(t')\phi dt', \quad (4.13)$$

$$R_3^1(t) = \int_0^t \int_0^{t-t'} U_1(t-t'-s_1)A_2U_1(s_1)A_2G_-(t')\phi ds_1 dt', \quad (4.14)$$

$$\begin{aligned} R_3^2(t) &= \int_0^t \int_0^{t-t'} \int_0^{t-t'-s_2} U_1(t-t'-s_2-s_1)A_2U_1(s_1)A_2 \\ &\quad U(s_2)A_2G_-(t')\phi ds_1 ds_2 dt'. \end{aligned} \quad (4.15)$$

From (4.12) and (2.18), it follows that

$$\begin{aligned} R_1|_{(0,T) \times \Gamma_+}(t, x, v) &= e^{-\int_0^{\tau_-(x,v)} \sigma(x-sv, v) ds} \phi(t - \tau_-(x, v), x - \tau_-(x, v)v, v) \\ &= \int_{(0,\eta) \times \Gamma_-} \alpha_1(t-t', x, v, x', v') \phi(t', x', v') dt' d\xi(x', v'), \end{aligned} \quad (4.16)$$

where α_1 is defined by (3.2).

From (2.9), (2.18) and (4.13), it follows that

$$\begin{aligned} R_2(t, x, v) &= \int_0^t \theta(x-t'v, x) \int_{\mathbb{S}^{d-1}} k(x-t'v, v', v) e^{-\int_0^{t'} \sigma(x-pv, v) dp - \int_0^{\tau_-(x-t'v, v')} \sigma(x-t'v-pv', v') dp} \\ &\quad \times \phi(t-t'-\tau_-(x-t'v, v'), x-t'v-\tau_-(x-t'v, v')v', v') dv' dt'. \end{aligned}$$

Hence

$$R_2|_{(0,T) \times \Gamma_-}(t, x, v) = \int_{(0,\eta) \times \Gamma_-} \alpha_2(t-t', x, v, x', v') \phi(t', x', v') dt' d\xi(x', v'), \quad (4.17)$$

where α_2 is defined by (3.3).

From (4.14) and (4.9) (with “ t ” = $t - t'$), it follows that

$$R_3^1(t) = \int_{(0,+\infty) \times X \times \mathbb{S}^{d-1}} \chi_{(0,+\infty)}(t - t') \gamma(t - t', x, v, x', v') G_-(t') \phi(x', v') dt' dx' dv', \quad (4.18)$$

where γ is defined by (4.8). Using (4.18) and (4.8) we obtain

$$R_{3|(0,T) \times \Gamma_+}^1(t, x, v) = \int_{(0,T) \times X \times \mathbb{S}^{d-1}} \tilde{\alpha}_3^1(t - t', x, v, x', v') (G_-(t') \phi)(x', v') dt' dx' dv', \quad (4.19)$$

where

$$\begin{aligned} \tilde{\alpha}_3^1(\tau, x, v, x', v') &= \chi_{(0,+\infty)}(\tau - |x - x'|) \left[e^{-\int_0^{s_1} \sigma(x - pv, v) dp - \int_0^{\tau - s_1} \sigma(x - s_1 v - pv_1, v_1) dp} \right. \\ &\quad \times k(x - s_1 v, v_1, v) k(x', v', v_1) \theta(x, x - s_1 v) \theta(x - s_1 v, x') \Big]_{s_1 = \frac{\tau^2 - |x - x'|^2}{2(\tau - (x - x')v)}} \\ &\quad \times 2^{d-2} \frac{(\tau - (x - x')v)^{d-3}}{|x - x' - \tau v|^{2d-4}} \end{aligned} \quad (4.20)$$

for a.e. $(\tau, x, v, x', v') \in \mathbb{R} \times \Gamma_+ \times X \times \mathbb{S}^{d-1}$.

From (4.15) and (4.9) it follows that

$$R_3^2(t) = \int_0^t \int_0^{t-t'} \int_{X \times \mathbb{S}^{d-1}} \gamma(t - t' - s_2, x, v, x', v') (U(s_2) A_2 G_-(t') \phi)(x', v') dx' dv' ds_2 dt', \quad (4.21)$$

where γ is defined by (4.8). Hence

$$R_{3|(0,T) \times \Gamma_+}^2(t, x, v) = \int_0^t \int_0^{t-t'} \int_{X \times \mathbb{S}^{d-1}} \tilde{\gamma}(t - t' - s_2, x, v, x', v') (U(s_2) A_2 G_-(t') \phi)(x', v') dx' dv' ds_2 dt', \quad (4.22)$$

for a.e. $(t, x, v) \in (0, T) \times \Gamma_+$ where

$$\begin{aligned} \tilde{\gamma}(r, x, v, x', v') &= 2^{d-2} \chi_{(0,r)}(|x - x'|) \left[e^{-\int_0^{s_1} \sigma(x - sv, v) ds - \int_0^{r - s_1} \sigma(x - s_1 v - pv_1, v_1) dp} \theta(x, x - s_1 v) \right. \\ &\quad \times \theta(x - s_1 v, x') k(x - s_1 v, v_1, v) k(x', v', v_1) \Big]_{s_1 = \frac{r^2 - (x - x')^2}{2(r - (x - x')v)}} \frac{(r - (x - x')v)^{d-3}}{|rv - x - x'|^{2d-4}}, \end{aligned} \quad (4.23)$$

for a.e. $(r, x, v, x', v') \in (0, T) \times \Gamma_+ \times X \times \mathbb{S}^{d-1}$.

Let $\psi \in L^\infty((0, T) \times \Gamma_+)$. Assume that $k \in L^\infty(X \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$. From (4.19) it follows that

$$\begin{aligned} &\left| \int_0^T \int_{\Gamma_+} \psi(t, x, v) R_{3|(0,T) \times \Gamma_+}^1(t, x, v) dt d\xi(x, v) \right| \\ &= \left| \int_{(0,T) \times X \times \mathbb{S}^{d-1}} (G_-(t') \phi)(x', v') \int_0^T \int_{\Gamma_+} \tilde{\alpha}_3^1(t - t', x, v, x', v') \psi(t, x, v) d\xi(x, v) dt dt' dx' dv' \right| \\ &\leq \|G_-(\cdot) \phi\|_{L^1((0,T) \times X \times \mathbb{S}^{d-1})} \left\| \int_0^T \int_{\Gamma_+} \tilde{\alpha}_3^1(t - t', x, v, x', v') \psi(t, x, v) d\xi(x, v) dt dt' \right\|_{L^\infty(\mathbb{R}_{t'} \times X_{x'} \times \mathbb{S}_{v'}^{d-1})}. \end{aligned} \quad (4.24)$$

From Lemma 4.1 and (2.18), it follows that

$$\|G_-(t')\phi\|_{L^1(X \times \mathbb{S}^{d-1})} \leq \|\phi\|_{L^1((0,\eta), L^1(\Gamma_-, d\xi))}, \text{ for } t' \in (0, T). \quad (4.25)$$

From (4.20), Hölder's inequality and Lemma 4.2, it follows that

$$\begin{aligned} & \left| \int_0^T \int_{\Gamma_+} \tilde{\alpha}_3^{-1}(t-t', x, v, x', v') \psi(t, x, v) d\xi(x, v) dt \right| \\ & \leq \|k\|_\infty^2 \int_0^T \int_{\Gamma_+} \frac{\chi_{(0,+\infty)}(t-t'-|x-x'|)(t-t'-(x-x')v)^{d-3}}{2^{2-d}|x-x'-(t-t')v|^{2d-4}} |\psi|(t, x, v) d\xi(x, v) dt \\ & \leq 2^{d-2} \|k\|_\infty^2 \left(\int_0^T \int_{\Gamma_+} |\psi(t, x, v)|^{p'} d\xi(x, v) dt \right)^{\frac{1}{p'}} \beta(x')^{\frac{1}{p}} \\ & \leq 2^{d-2} \|k\|_\infty^2 \|\beta\|_{L^\infty(X)}^{\frac{1}{p}} \left(\int_0^T \int_{\Gamma_+} |\psi(t, x, v)|^{p'} d\xi(x, v) dt \right)^{\frac{1}{p'}}, \end{aligned} \quad (4.26)$$

for a.e. $(t', x', v') \in (0, T) \times X \times \mathbb{S}^{d-1}$.

Using (4.24)–(4.26) we obtain

$$\begin{aligned} & \left| \int_0^T \int_{\Gamma_+} \psi(t, x, v) R_{3|(0,T) \times \Gamma_+}^1(t, x, v) dt d\xi(x, v) \right| \\ & \leq 2^{d-2} T \|k\|_\infty^2 \|\beta\|_{L^\infty(X)}^{\frac{1}{p}} \|\phi\|_{L^1((0,\eta), L^1(\Gamma_-, d\xi))} \left(\int_0^T \int_{\Gamma_+} |\psi(t, x, v)|^{p'} d\xi(x, v) dt \right)^{\frac{1}{p'}}. \end{aligned} \quad (4.27)$$

In addition from (4.22), Hölder inequality, (4.23) and Lemma 4.2 it follows that

$$\begin{aligned} & \left| \int_0^T \int_{\Gamma_+} \psi(t, x, v) R_{3|(0,T) \times \Gamma_+}^2(t, x, v) d\xi(x, v) dt \right| \\ & = \left| \int_0^T \int_{\Gamma_+} \psi(t, x, v) \int_0^t \int_0^{t-t'} \int_{X \times \mathbb{S}^{d-1}} \tilde{\gamma}(t-t'-s_2, x, v, x', v') \right. \\ & \quad \times (U(s_2) A_2 G_-(t') \phi)(x', v') dx' dv' ds_2 dt' d\xi(x, v) dt \Big| \\ & = \left| \int_0^T \int_0^{T-t'} \int_{X \times \mathbb{S}^{d-1}} (U(s_2) A_2 G_-(t') \phi)(x', v') \int_0^T \int_{\Gamma_+} \chi_{(t'+s_2, T)}(t) \psi(t, x, v) \right. \\ & \quad \times \tilde{\gamma}(t-t'-s_2, x, v, x', v') dt d\xi(x, v) dx' dv' ds_2 dt' \Big| \\ & \leq \int_0^T \int_0^{T-t'} \int_{X \times \mathbb{S}^{d-1}} |U(s_2) A_2 G_-(t') \phi|(x', v') \\ & \quad \times \left(\int_0^T \int_{\Gamma_+} \chi_{(t'+s_2, T)}(t) |\tilde{\gamma}|^p(t-t'-s_2, x, v, x', v') dt d\xi(x, v) \right)^{\frac{1}{p}} ds_2 dt' dx' dv' \\ & \quad \times \left(\int_{(0,T) \times \Gamma_+} |\psi(t, x, v)|^{p'} dt d\xi(x, v) \right)^{\frac{1}{p'}} \\ & \leq 2^{d-2} \|k\|_\infty^2 \|\beta\|_\infty^{\frac{1}{p}} \int_0^T \int_0^{T-t'} \int_{X \times \mathbb{S}^{d-1}} |U(s_2) A_2 G_-(t') \phi|(x', v') dx' dv' ds_2 dt \\ & \quad \left(\int_{(0,T) \times \Gamma_+} |\psi(t, x, v)|^{p'} dt d\xi(x, v) \right)^{\frac{1}{p'}}. \end{aligned} \quad (4.28)$$

Moreover using (4.25), the equality $\|A_2\| = \|\sigma_p\|_\infty$ and the estimate $\|U(s_2)\| \leq e^{s_2\|\sigma_p\|_\infty}$ for $s_2 \geq 0$ (see Trotter's formula [T] $U(s_2) = s - \lim_{n \rightarrow \infty} \left(U_1(\frac{s_2}{n}) e^{\frac{s_2}{n} A_2} \right)^n$ where U_1 and A_2 are defined by (2.9) and (2.6) respectively), we obtain

$$\begin{aligned} & \int_0^T \int_0^{T-t'} \int_{X \times \mathbb{S}^{d-1}} |U(s_2) A_2 G_-(t') \phi|(x', v') dx' dv' ds_2 dt \\ & \leq \int_0^T \int_0^{T-t'} \frac{d}{ds_2} e^{s_2\|\sigma_p\|_\infty} ds_2 dt' \|\phi\|_{L^1((0, \eta) \times \Gamma_-, dt d\xi)} \\ & \leq T(e^{T\|\sigma_p\|_\infty} - 1) \|\phi\|_{L^1((0, \eta), L^1(\Gamma_-, d\xi))}. \end{aligned} \quad (4.29)$$

Combining (4.27)–(4.29), we finally obtain

$$\begin{aligned} & \left| \int_0^T \int_{\Gamma_+} \psi(t, x, v) (R_3^1 + R_3^2)|_{(0, T) \times \Gamma_+}(t, x, v) d\xi(x, v) dt \right| \\ & \leq 2^{d-2} \|k\|_\infty^2 \|\beta\|_{L^\infty(X)}^{\frac{1}{p}} T e^{T\|\sigma_p\|_\infty} \|\phi\|_{L^1((0, \eta), L^1(\Gamma_-, d\xi))} \\ & \quad \times \left(\int_0^T \int_{\Gamma_+} |\psi(t, x, v)|^{p'} d\xi(x, v) dt \right)^{\frac{1}{p'}}. \end{aligned} \quad (4.30)$$

Proposition 3.2 follows from (4.16), (4.17) and (4.30). \square

5 Proof of Lemma 3.1, Theorems 3.1, 3.2

Proof of Lemma 3.1. First note that using twice Lemma 4.1 we obtain

$$\int_{\Gamma_+} \int_0^{\tau_-(x, v)} f(x - wv, v) dw d\xi(x, v) = \int_{X \times \mathbb{S}^{d-1}} f(x, v) dx dv = \int_{\Gamma_-} \int_0^{\tau_+(x', v')} f(x' + sv', v') ds d\xi(x', v'), \quad (5.1)$$

for $f \in L^1(X \times \mathbb{S}^{d-1})$.

We first prove (3.16). Let $T > \text{diam}(X)$. We have, in particular, $T > \tau_-(x, v)$ for any $(x, v) \in \bar{X} \times \mathbb{S}^{d-1}$. From (3.13) and (3.10) it follows that

$$I_1(\psi, \varepsilon_1, \varepsilon_2) = \int_{\Gamma_+} \Phi_{\varepsilon_2}(x, v) f_{\varepsilon_1}(x - \tau_-(x, v)v, v) d\xi(x, v), \quad (5.2)$$

where Φ_{ε_2} is the continuous function on Γ_+ given by

$$\Phi_{\varepsilon_2}(x, v) = \int_{\tau_-(x, v)}^T \psi(t, x, v) g_{\varepsilon_2}(t - \tau_-(x, v)) dt \left(e^{-\int_0^{\tau_-(x, v)} \sigma(x - sv, v) ds} - e^{-\int_0^{\tau_-(x, v)} \bar{\sigma}(x - sv, v) ds} \right), \quad (5.3)$$

for $(x, v) \in \Gamma_+$ (we used also $\text{supp } g_{\varepsilon_2} \subseteq (0, +\infty)$). The continuity of Φ_{ε_2} follows from the assumptions (3.6), the continuity of ψ and g_{ε_2} and the continuity of τ_- on Γ_+ (X is convex with C^1 boundary). From (5.1) (“ $f(x, v) = \frac{1}{\tau(x, v)} \Phi_{\varepsilon_2}(x + \tau_+(x, v)v, v) f_{\varepsilon_2}(x - \tau_-(x, v)v, v)$ ”) and (3.8), we obtain

$$\begin{aligned} & \int_{\Gamma_+} \Phi_{\varepsilon_2}(x, v) f_{\varepsilon_1}(x - \tau_-(x, v)v, v) d\xi(x, v) = \int_{\Gamma_-} \Phi_{\varepsilon_2}(x' + \tau_+(x', v')v', v') f_{\varepsilon_1}(x', v') d\xi(x', v') \\ & \xrightarrow{\varepsilon_1 \rightarrow 0^+} \int_{\tau_+(x'_0, v'_0)}^T \psi(t, x'_0 + \tau_+(x'_0, v'_0)v'_0, v'_0) \left(e^{-\int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 + sv'_0) ds} - e^{-\int_0^{\tau_+(x'_0, v'_0)} \bar{\sigma}(x'_0 + sv'_0) ds} \right) \\ & \quad \times g_{\varepsilon_2}(t - \tau_+(x'_0, v'_0)) dt. \end{aligned} \quad (5.4)$$

The limit (3.16) follows from (5.2), (5.4), (3.9) and the continuity of ψ and (3.6).

We prove (3.17). Let $T > 2\text{diam}(X)$. We have, in particular, $T > s + \tau_+(x' + sv', v)$ for $s \in (0, \tau_+(x', v'))$ and $(x', v') \in \Gamma_-$. From (3.14), it follows that

$$\begin{aligned} I_2(\psi, \varepsilon_1, \varepsilon_2) &= \int_{\Gamma_+} \int_0^{\tau_-(x, v)} \int_0^T \psi(t, x, v) \\ &\times \int_{\mathbb{S}^{d-1}} (k(x - wv, v', v)E_-(x, v, w, v') - \tilde{k}(x - wv, v', v)\tilde{E}_-(x, v, w, v')) \\ &\times g_{\varepsilon_2}(t - w - \tau_-(x - wv, v'))f_{\varepsilon_1}(x - wv - \tau_-(x - wv, v')v', v')dv'dtdwd\xi(x, v). \end{aligned} \quad (5.5)$$

Using (5.5) and (5.1), we obtain

$$I_2(\psi, \varepsilon_1, \varepsilon_2) = \int_{\Gamma_-} \Psi_{\varepsilon_2}(x', v')f_{\varepsilon_1}(x', v')d\xi(x', v'), \quad (5.6)$$

where

$$\begin{aligned} \Psi_{\varepsilon_2}(x', v') &= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x', v')} \int_{s+\tau_+(x'+sv', v)}^T \psi(t, x' + sv' + \tau_+(x' + sv', v)v, v) \\ &\times \left(k(x' + sv', v', v)E_+(x', v', s, v) - \tilde{k}(x' + sv', v', v)\tilde{E}_+(x', v', s, v) \right) \\ &\times g_{\varepsilon_2}(t - s - \tau_+(x' + sv', v))dtds dv, \end{aligned} \quad (5.7)$$

for $(x', v') \in \Gamma_-$. From (3.6), (3.7) and the continuity of ψ and g_{ε_2} , it follows that Ψ_{ε_2} is continuous on Γ_- . From (3.8) and (5.7) it follows that

$$\begin{aligned} &\int_{\Gamma_-} \Psi_{\varepsilon_2}(x', v')f_{\varepsilon_1}(x', v')d\xi(x', v') \\ &\xrightarrow{\varepsilon_1 \rightarrow 0^+} \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \left(\int_{s+\tau_+(x'_0+sv'_0, v)}^T \psi(t, x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \right. \\ &\quad \left(E_+(x'_0, v'_0, s, v)k(x'_0 + sv'_0, v'_0, v) - \tilde{E}_+(x'_0, v'_0, s, v)\tilde{k}(x'_0 + sv'_0, v'_0, v) \right) \\ &\quad \times g_{\varepsilon_2}(t - s - \tau_+(x'_0 + sv'_0, v))dt) ds dv. \end{aligned} \quad (5.8)$$

The limit (3.17) follows from (5.6), (5.8), (3.9), (3.6), the continuity of ψ and the Lebesgue dominated convergence theorem. \square

Proof of Theorem 3.1. We first prove (3.20). Let $T > \text{diam}(X)$. Let $\varepsilon_3 > 0$ and let ψ_{ε_3} be a continuous and compactly supported function on $(0, T) \times \Gamma_+$ that satisfies

$$\begin{aligned} 0 &\leq \psi_{\varepsilon_3} \leq 1 \text{ and } \text{supp} \psi_{\varepsilon_3} \subseteq \{(t, x, v) \in (0, T) \times \Gamma_+ \mid |v - v'_0| < \varepsilon_3\}, \\ \psi_{\varepsilon_3}(t, x, v) &= 1 \text{ for } (t, x, v) \in (0, T) \times \Gamma_+ \text{ such that} \\ |v - v'_0| &\leq \frac{\varepsilon_3}{2}, |t - \tau_+(x'_0, v'_0)| \leq \frac{T - \tau_+(x'_0, v'_0)}{2}. \end{aligned} \quad (5.9)$$

From (3.16) and (5.9), it follows that

$$I_1(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2) = e^{-\int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 - sv'_0, v'_0)ds} - e^{-\int_0^{\tau_+(x'_0, v'_0)} \tilde{\sigma}(x'_0 - sv'_0, v'_0)ds}. \quad (5.10)$$

Using (3.17), (5.9) and the estimate $\sigma \geq 0$, we obtain

$$\left| \lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_2(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2) \right| \leq \text{diam}(X) \left(\|k\|_\infty + \|\tilde{k}\|_\infty \right) \int_{\substack{v \in \mathbb{S}^{d-1} \\ |v-v'_0| < \varepsilon_3}} dv.$$

Hence

$$\lim_{\varepsilon_3 \rightarrow 0^+} \lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_2(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2) = 0. \quad (5.11)$$

From (3.15) and (5.9), it follows that

$$\begin{aligned} |I_3(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2)| &\leq C \left(\int_0^T \int_{\partial X} \int_{\substack{v \in \mathbb{S}^{d-1} \\ |v-v'_0| < \varepsilon_3}} dv d\mu(x) dt \right)^{\frac{1}{p'}} \\ &\leq C (\text{Vol}(\partial X) T)^{\frac{1}{p'}} \left(\int_{\substack{v \in \mathbb{S}^{d-1} \\ |v-v'_0| < \varepsilon_3}} dv \right)^{\frac{1}{p'}}, \end{aligned}$$

for $\varepsilon_i > 0$, $i = 1 \dots 3$. Therefore

$$\lim_{\varepsilon_3 \rightarrow 0^+} \limsup_{\varepsilon_2 \rightarrow 0^+} \limsup_{\varepsilon_1 \rightarrow 0^+} I_3(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2) = 0. \quad (5.12)$$

In addition, from (3.11)–(3.12) it follows that

$$|I_1(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2)| \leq \|A - \tilde{A}\|_{\eta, T} + |I_2(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2) + I_3(\psi_{\varepsilon_3}, \varepsilon_1, \varepsilon_2)|, \quad (5.13)$$

for $\varepsilon_i > 0$, $i = 1 \dots 3$.

Combining (5.10)–(5.13) we obtain (3.20).

Now we prove (3.21). Let $T > 2\text{diam}(X)$. Let $U := \{(t', v) \in (0, \tau_+(x'_0, v'_0)) \times \mathbb{S}^{d-1} \mid (k - \tilde{k})(x'_0 + t'v'_0, v'_0, v) > 0\}$. From (3.6) it follows that U is an open subset of $\mathbb{R} \times \mathbb{S}^{d-1}$. Let (K_m) be a sequence of compact sets such that $\bigcup_{m \in \mathbb{N}} K_m = U$ and $K_m \subseteq K_{m+1}$ for $m \in \mathbb{N}$. For $m \in \mathbb{N}$ let $\chi_m \in C^\infty(\mathbb{R} \times \mathbb{S}^{d-1}, \mathbb{R})$ such that $\chi_{K_m} \leq \chi_m \leq \chi_U$ (where χ_{K_m} and χ_U are defined in (4.2)), and let

$$\rho_m = 2\chi_m - 1. \quad (5.14)$$

Thus we obtain

$$\lim_{m \rightarrow +\infty} (k - \tilde{k})(x'_0 + t'v'_0, v'_0, v) \rho_m(t', v) = |k - \tilde{k}|(x'_0 + t'v'_0, v'_0, v), \quad (5.15)$$

for $v \in \mathbb{S}^{d-1}$ and $t' \in (0, \tau_+(x'_0, v'_0))$.

Consider

$$\mathcal{V}_\delta := \{(t, x, v) \in (0, T) \times \Gamma_+ \mid |v - (vv'_0)v'_0| > \delta, \frac{\delta}{2} < t < T - \frac{\delta}{2}\}, \quad (5.16)$$

$$\mathcal{V}_{\delta, l} := \{(t, x, v) \in (0, T) \times \Gamma_+ \mid |v - (vv'_0)v'_0| \geq \delta + \frac{1}{l}, \delta \leq t \leq T - \delta\}, \quad (5.17)$$

for $0 < \delta < \frac{\min(1, T)}{2}$ and $l \in \mathbb{N}$, $l \geq 2$. For $0 < \delta < \frac{\min(1, T)}{2}$ and $l \in \mathbb{N}$, $l \geq 2$, let $\chi_{\delta, l}$ be a continuous and compactly supported function on $(0, T) \times \Gamma_+$ such that

$$\chi_{\mathcal{V}_{\delta, l}} \leq \chi_{\delta, l} \leq \chi_{\mathcal{V}_\delta} \quad (5.18)$$

(where $\chi_{\mathcal{V}_{\delta,l}}$ and $\chi_{\mathcal{V}_\delta}$ are defined in (4.2)). Finally, for $0 < \delta < \frac{\min(1,T)}{2}$ and $m, l \in \mathbb{N}$, $l \geq 2$, let $\psi_{\delta,m,l,\varepsilon_3}$ be the continuous compactly supported function on $(0, T) \times \Gamma_+$ defined by

$$\psi_{\delta,m,l,\varepsilon_3}(t, x, v) := \chi_{\delta,l}(t, x, v) (\zeta_{\varepsilon_3}(t - s - s') \rho_m(s, v))_{\substack{s = \frac{(x-x'_0)(v'_0 - (vv'_0)v)}{1-(vv'_0)^2} \\ s' = \frac{(x-x'_0)(v - (vv'_0)v'_0)}{1-(vv'_0)^2}}}, \quad (5.19)$$

where $\zeta_{\varepsilon_3} \in C^\infty(\mathbb{R})$, $\zeta_{\varepsilon_3}(s'') = 1$ for $s'' \in [-\varepsilon_3, \varepsilon_3]$, $0 \leq \zeta_{\varepsilon_3} \leq 1$ and $\zeta_{\varepsilon_3}(s'') = 0$ for $|s''| \geq 2\varepsilon_3$.

From (3.16), (5.19) and the equality $\chi_{\delta,l}(t, x'_0 + \tau_+(x'_0, v'_0)v'_0, v'_0) = 0$ for $t \in (0, T)$ (see (5.16)–(5.18)), it follows that

$$\lim_{\varepsilon_2 \rightarrow 0^+} \lim_{\varepsilon_1 \rightarrow 0^+} I_1(\psi_{\delta,m,l,\varepsilon_3}, \varepsilon_1, \varepsilon_2) = 0 \quad (5.20)$$

for $0 < \delta < \frac{\min(1,T)}{2}$, $m, l \geq 2$, $\varepsilon_3 > 0$.

From (3.18)–(3.19) and (5.19), it follows that

$$\begin{aligned} I_2^1(\psi_{\delta,m,l,\varepsilon_3}) &:= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \zeta_{\varepsilon_3}(0) \chi_{\delta,l}(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \\ &\quad \times \rho_m(s, v) (k - \tilde{k})(x'_0 + sv'_0, v'_0, v) E_+(x'_0, v'_0, s, v) ds dv. \end{aligned} \quad (5.21)$$

$$\begin{aligned} I_2^2(\psi_{\delta,m,l,\varepsilon_3}) &:= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \zeta_{\varepsilon_3}(0) \chi_{\delta,l}(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \\ &\quad \times \rho_m(s, v) \tilde{k}(x'_0 + sv'_0, v'_0, v) (E_+ - \tilde{E}_+)(x'_0, v'_0, s, v) ds dv, \end{aligned} \quad (5.22)$$

for $0 < \delta < \frac{\min(1,T)}{2}$, $m, l \geq 2$, $\varepsilon_3 > 0$.

Note that using (5.16)–(5.18) we obtain

$$\lim_{l \rightarrow \infty} \chi_{\mathcal{V}_{\delta,l}}(t, x, v) = \chi_{\mathcal{V}_\delta}(t, x, v), \quad (5.23)$$

for $(t, x, v) \in (0, T) \times \Gamma_+$ and $0 < \delta < \frac{\min(1,T)}{2}$.

From equality $\zeta_{\varepsilon_3}(0) = 1$, (5.21), (5.23) and the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned} \lim_{l \rightarrow +\infty} I_2^1(\psi_{\delta,m,l,\varepsilon_3}) &= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \chi_{\mathcal{V}_\delta}(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \\ &\quad \times \rho_m(s, v) (k - \tilde{k})(x'_0 + sv'_0, v'_0, v) E_+(x'_0, v'_0, s, v) ds dv, \end{aligned}$$

for $0 < \delta < \frac{\min(1,T)}{2}$, $m \in \mathbb{N}$, $l \geq 2$, $\varepsilon_3 > 0$. Therefore, using (5.15) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{m \rightarrow +\infty} \lim_{l \rightarrow +\infty} I_2^1(\psi_{\delta,m,l,\varepsilon_3}) &= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \chi_{\mathcal{V}_\delta}(s + \tau_+(x'_0 + sv'_0, v), x'_0 + sv'_0 + \tau_+(x'_0 + sv'_0, v)v, v) \\ &\quad \times E_+(x'_0, v'_0, s, v) |k - \tilde{k}|(x'_0 + sv'_0, v'_0, v) ds dv, \end{aligned}$$

for $0 < \delta < \frac{\min(1,T)}{2}$, $\varepsilon_3 > 0$. Using this latter equality and (5.16), we obtain

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon_3 \rightarrow 0^+} \lim_{m \rightarrow +\infty} \lim_{l \rightarrow +\infty} I_2^1(\psi_{\delta,m,l,\varepsilon_3}) = \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} |k - \tilde{k}|(x'_0 + sv'_0, v'_0, v) E_+(x'_0, v'_0, s, v) ds dv. \quad (5.24)$$

From equality $\zeta_{\varepsilon_3}(0) = 1$ and (5.22) it follows that

$$|I_2^2(\psi_{\delta,m,l,\varepsilon_3})| \leq \text{diam}(X) \sup_{s \in (0, \tau_+(x'_0, v'_0))} \tilde{\sigma}_p(x'_0 + sv'_0, v'_0) \sup_{\substack{v \in \mathbb{S}^{d-1} \\ 0 < s < \tau_+(x'_0, v'_0)}} \left| (E_+ - \tilde{E}_+)(x'_0, v'_0, s, v) \right|, \quad (5.25)$$

for $0 < \delta < \frac{\min(1, T)}{2}$, $m \in \mathbb{N}$, $l \geq 2$, $\varepsilon_3 > 0$.

Note that using (5.19) and the estimate $0 \leq \rho_m \leq 1$ for all m , we obtain

$$\begin{aligned} & \int_0^T \int_{\Gamma_+} |\psi_{\delta,m,l,\varepsilon_3}(t, x, v)|^{p'} d\xi(x, v) dt \\ & \leq \int_0^T \int_{\Gamma_+} \chi_{\delta,l}(t, x, v)^{p'} \zeta_{\varepsilon_3}(t - s - s')^{p'} d\xi(x, v) dt, \end{aligned} \quad (5.26)$$

$s = \frac{(x-x'_0)(v'_0 - (vv'_0)v)}{1 - (vv'_0)^2}$
 $s' = \frac{(x-x'_0)(v - (vv'_0)v'_0)}{1 - (vv'_0)^2}$

for $0 < \delta < \frac{\min(1, T)}{2}$, $m \in \mathbb{N}$, $l \geq 2$, $\varepsilon_3 > 0$. Therefore using the definition of ζ_{ε_3} we obtain

$$\int_0^T \int_{\Gamma_+} |\psi_{\delta,m,l,\varepsilon_3}(t, x, v)|^{p'} d\xi(x, v) dt \leq \int_0^T \int_{\substack{\Gamma_+ \\ |v - (vv'_0)v'_0| > \delta}} \chi_{[-2\varepsilon_3, 2\varepsilon_3]}(t - s - s')^{p'} d\xi(x, v) dt, \quad (5.27)$$

$s = \frac{(x-x'_0)(v'_0 - (vv'_0)v)}{1 - (vv'_0)^2}$
 $s' = \frac{(x-x'_0)(v - (vv'_0)v'_0)}{1 - (vv'_0)^2}$

for $0 < \delta < \frac{\min(1, T)}{2}$, $m \in \mathbb{N}$, $l \geq 2$, $\varepsilon_3 > 0$. Using (5.26)–(5.27) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon_3 \rightarrow 0^+} \limsup_{m \rightarrow +\infty} \limsup_{l \rightarrow +\infty} \int_0^T \int_{\Gamma_+} |\psi_{\delta,m,l,\varepsilon_3}(t, x, v)|^{p'} d\xi(x, v) dt = 0,$$

for $0 < \delta < \frac{\min(1, T)}{2}$. Using this latter equality and (3.15), we obtain

$$\lim_{\delta \rightarrow 0^+} \lim_{\varepsilon_3 \rightarrow 0^+} \limsup_{m \rightarrow +\infty} \limsup_{l \rightarrow +\infty} \lim_{\varepsilon_2 \rightarrow 0^+} \sup_{\varepsilon_1 \rightarrow 0^+} \limsup |I_3(\psi_{\delta,m,l,\varepsilon_3}, \varepsilon_1, \varepsilon_2)| = 0. \quad (5.28)$$

In addition, from (3.11)–(3.12), it follows that

$$|I_2(\psi_{\delta,m,l,\varepsilon_3}, \varepsilon_1, \varepsilon_2)| \leq \|A - \tilde{A}\|_{\eta, T} + |I_1(\psi_{\delta,m,l,\varepsilon_3}, \varepsilon_1, \varepsilon_2) + I_3(\psi_{\delta,m,l,\varepsilon_3}, \varepsilon_1, \varepsilon_2)|, \quad (5.29)$$

for $0 < \delta < \frac{\min(1, T)}{2}$, $m \in \mathbb{N}$, $l \geq 2$, $\varepsilon_i > 0$, $i = 1 \dots 3$. Combining (5.29), (5.24), (5.25) and (5.28) we obtain (3.21). \square

Proof of Theorem 3.2. The method used to prove (3.23) is the same as in [W] and [BJ1]. For the reader's convenience, we adapt the proof given in [BJ1] with minor modification.

Let $(\sigma, k), (\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$. We extend σ and $\tilde{\sigma}$ outside X by 0. Let $f = \sigma - \tilde{\sigma}$ and consider Pf the X-ray transform of $f = \sigma - \tilde{\sigma}$ defined by $Pf(x, \varphi) := \int_{-\infty}^{+\infty} f(t\varphi + x) dt$ for $(x, \varphi) \in T\mathbb{S}^{d-1} := \{(z, v) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \mid vz = 0\}$.

From $f|_X \in H^{\frac{d}{2} + \tilde{r}}(X)$, it follows that

$$\|f\|_{H^{-\frac{1}{2}}(X)} \leq D_1(d, X) \|Pf\|_*, \quad (5.30)$$

where

$$\|Pf\|_* := \left(\int_{\mathbb{S}^{d-1}} \int_{\Pi_\varphi} |Pf(x, \varphi)|^2 dx d\varphi \right)^{\frac{1}{2}}$$

and $D_1(d, X)$ is a real constant which does not depend on f and $\Pi_\varphi := \{x \in \mathbb{R}^d \mid x\varphi = 0\}$ for $\varphi \in \mathbb{S}^{d-1}$. Note that $Pf(x, \varphi) = 0$ for $(x, \varphi) \in T\mathbb{S}^{d-1}$ and $|x| \geq \sup_{z \in X} |z|$. Therefore using also (5.30) we obtain

$$\|f\|_{H^{-\frac{1}{2}}(X)} \leq D_2(d, X) \|Pf\|_{L^\infty(T\mathbb{S}^{d-1})}, \quad (5.31)$$

where $D_2(d, X)$ is a real constant which does not depend on $\sigma, \tilde{\sigma}$.

We also use the following interpolation inequality:

$$\|f\|_{H^s(X)} \leq \|f\|_{H^{\frac{d}{2}+\tilde{r}}}^{\frac{2s+1}{d+1+2\tilde{r}}} \|f\|_{H^{-\frac{1}{2}}}^{\frac{d+2\tilde{r}}{d+1+2\tilde{r}}}, \quad (5.32)$$

for $-\frac{1}{2} \leq s \leq \frac{d}{2} + \tilde{r}$. As $(\sigma, k) \in \mathcal{M}$, it follows that

$$\|\sigma\|_\infty \leq D_3(d, \tilde{r}) \|\sigma\|_{H^{\frac{d}{2}+\tilde{r}}} \leq D_3(d, \tilde{r}) M. \quad (5.33)$$

Therefore,

$$\int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 + sv'_0) ds \leq \text{diam}(X) D_3(d, \tilde{r}) M, \quad (5.34)$$

for $(x'_0, v'_0) \in \Gamma_-$. From (5.34), it follows that

$$\left| e^{-\int_0^{\tau_+(x'_0, v'_0)} \sigma(x'_0 + sv'_0, v'_0) ds} - e^{-\int_0^{\tau_+(x'_0, v'_0)} \tilde{\sigma}(x'_0 + sv'_0, v'_0) ds} \right| \geq e^{-\text{diam}(X) D_3(d, \tilde{r}) M} |P(\sigma - \tilde{\sigma})(x'_0, v'_0)|, \quad (5.35)$$

for $(x'_0, v'_0) \in \Gamma_-$ (we used the equality $e^{t_1} - e^{t_2} = e^c(t_2 - t_1)$ for $t_1 < t_2 \in \mathbb{R}$ and for some $c \in [t_1, t_2]$, which depends on t_1 and t_2).

Combining (5.35), (5.31) and (3.20), we obtain

$$\frac{e^{-\text{diam}(X) D_3(d, \tilde{r}) M}}{D_2(d, X)} \|\sigma - \tilde{\sigma}\|_{H^{-\frac{1}{2}}(X)} \leq \|A - \tilde{A}\|_{\eta, T}. \quad (5.36)$$

Combining (5.36) and (5.32), we obtain (3.23).

We now prove (3.24). Using (3.7) and (5.33), we obtain that

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} \left| k(x'_0 + sv'_0, v'_0, v) - \tilde{k}(x'_0 + sv'_0, v'_0, v) \right| E(x'_0, v'_0, s, v) ds dv \\ & \geq e^{-2\text{diam}(X) D_3(d, \tilde{r}) M} \int_{\mathbb{S}^{d-1}} \int_0^{\tau_+(x'_0, v'_0)} |k - \tilde{k}|(x'_0 + sv'_0, v'_0, v) ds dv, \end{aligned} \quad (5.37)$$

for any $(x'_0, v'_0) \in \Gamma_-$.

As $(\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$ we have $\|\tilde{\sigma}_p\|_\infty \leq M$. Using the latter estimate and (3.7), we obtain

$$\begin{aligned} & \sup_{s \in (0, \tau_+(x'_0, v'_0))} \sigma_p(x'_0 + sv'_0, v'_0) \sup_{\substack{(x'_0, v'_0) \in \Gamma_- \\ s \in (0, \tau_+(x'_0, v'_0))}} |E - \tilde{E}|(x'_0, v'_0, s, v) \leq M e^{2\text{diam}(X) D_3(d, \tilde{r}) M} \\ & \times \sup_{\substack{(x'_0, v'_0) \in \Gamma_- \\ s \in (0, \tau_+(x'_0, v'_0))}} \left[\int_0^s |\sigma - \tilde{\sigma}|(x - pv, v) dp + \int_0^{\tau_+(x'_0 + sv'_0, v)} |\sigma - \tilde{\sigma}|(x'_0 + sv'_0 + pv, v) dp \right] \end{aligned}$$

$$\leq 2\text{diam}(X)Me^{2\text{diam}(X)D_3(d,\tilde{r})M}\|\sigma - \tilde{\sigma}\|_\infty, \quad (5.38)$$

for any $(x'_0, v'_0) \in \Gamma_-$. (We also used $|e^u - e^{\tilde{u}}| \leq e^{\max(|u|, |\tilde{u}|)}|u - \tilde{u}|$ where $u = -\int_0^s \sigma(x'_0 + pv'_0, v'_0)dp - \int_0^{\tau_+(x'_0 + sv'_0, v)} \tilde{\sigma}(x'_0 + sv'_0 + pv, v)dp$ and \tilde{u} denotes the real number obtained by replacing σ by $\tilde{\sigma}$ on the right-hand side of the latter equality that defines u ; using (5.33) (for σ and for $\tilde{\sigma}$) we obtain $\max(|u|, |\tilde{u}|) \leq 2\text{diam}(X)D_3(d, \tilde{r})M$.) Note that $\|\sigma - \tilde{\sigma}\|_\infty \leq D_3(d, r)\|\sigma - \tilde{\sigma}\|_{H^{\frac{d}{2}+r}}$ for $0 < r < \tilde{r}$ (see (5.33)). Therefore, combining (5.37), (5.38), (3.21) and (3.23), we obtain (3.24).

Let us finally prove (3.25). Let $0 < r < \tilde{r}$ and let $\kappa = \frac{2(\tilde{r}-r)}{d+1+2\tilde{r}}$. From (3.24) it follows that

$$\int_{\Gamma_-} \int_0^{\tau_+(x'_0, v'_0)} \int_{\mathbb{S}^{d-1}} \left| (k - \tilde{k})(x'_0 + sv'_0, v'_0, v) \right| dv ds d\xi(x'_0, v'_0) \leq D_4 \|A - \tilde{A}\|_{\eta, T}^\kappa \left(1 + \|A - \tilde{A}\|_{\eta, T}^{1-\kappa} \right), \quad (5.39)$$

where $D_4 = C_2 \int_{\Gamma_-} d\xi(x'_0, v'_0)$ and C_2 is the constant that appears on the right-hand side of (3.24). From (5.39) and Lemma 4.1, we obtain (3.25). \square

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