# The Erdös-Falconer distance problem on the unit sphere in vector spaces over finite fields 

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#### Abstract

Hart, Iosevich, Koh and Rudnev (2007) show, using Fourier analysis method, that the finite Erdös-Falconer distance conjecture holds for subsets of the unit sphere in $\mathbb{F}_{q}^{d}$. In this note, we give a graph theoretic proof of this result.


## 1 Introduction

The Erdös Distance Problem is perhaps the best known problem in combinatorial geometry. How many distinct distances can occur among $n$ points in the plane? Although this problem has received considerable attention, we are still far from the solution. The Falconer distance conjecture says that if $E \subset \mathbb{R}^{d}, d \geq 2$, has Hausdroff dimension greater than $\frac{d}{2}$, then the set of distances occur in $E$ has positive Lebesgue measure. See [9] for the connections between the Erdös and Falconer distance conjectures.

In the finite field setting, the distance problem turns out to have features of both the Erdös and Falconer distance problems in real spaces. Let $\mathbb{F}_{q}$ denote the finite field with $q$ elements where $q \gg 1$ is an odd prime power. For any $x, y \in \mathbb{F}_{q}^{d}$, the distance between $x, y$ is defined as $\|x-y\|=\left(x_{1}-y_{1}\right)^{2}+\ldots+\left(x_{d}-y_{d}\right)^{2}$. Let $E \subset \mathbb{F}_{q}^{d}, d \geqslant 2$. Then the finite analog of the classical Erdös distance problem is to determine the smallest possible cardinality of the set

$$
\begin{equation*}
\Delta(E)=\{\|x-y\|: x, y \in E\} \tag{1.1}
\end{equation*}
$$

viewed as a subset of $\mathbb{F}_{q}$. Bourgain, Katz and Tao ([4), showed, using intricate incidence geometry, that for every $\varepsilon>0$, there exists $\delta>0$, such that if $E \in \mathbb{F}_{q}^{2}$ and $C_{\varepsilon}^{1} q^{\varepsilon} \leqslant|E| \leqslant$ $C_{\varepsilon}^{2} q^{2-\varepsilon}$, then $|\Delta(E)| \geqslant C_{\delta}|E|^{\frac{1}{2}+\delta}$ for some constants $C_{\varepsilon}^{1}, C_{\varepsilon}^{2}$ and $C_{\delta}$. The relationship between $\varepsilon$ and $\delta$ in their argument is difficult to determine. Going up to higher dimension
using arguments of Bourgain, Katz and Tao is quite subtle. Iosevich and Rudnev [8] establish the following result using Fourier analytic method.

Theorem 1.1 ([8]) Let $E \subset \mathbb{F}_{q}^{d}$ such that $|E| \gtrsim C q^{d / 2}$ for $C$ sufficiently large. Then

$$
\begin{equation*}
|\Delta(E)| \gtrsim \min \left\{q, \frac{|E|}{q^{(d-1) / 2}}\right\} \tag{1.2}
\end{equation*}
$$

In view of this reslut, Iosevich and Rudnev ([8]) formulated the Erdös-Falconer conjecture as follows.

Conjecture 1.2 Let $E \subset \mathbb{F}_{q}^{d}$ such that $|E| \geq C_{\epsilon} q^{\frac{d}{2}+\epsilon}$. Then there exists $c>0$ such that $|\Delta(E)| \geq c q$.

By modifying the proof of Theorem [1.1 slightly, Iosevich and Rudnev ([8]) obtain the following stronger conclusion.

Theorem 1.3 ([8]) Let $E \subset \mathbb{F}_{q}^{d}$ such that $|E| \geq C q^{\frac{d+1}{2}}$ for sufficient large constant $C$. Then $\Delta(E)=\mathbb{F}_{q}$.

In [7], the authors show that Theorem 1.3 is essentially sharp, which implies that Conjecture 1.2 is not true in general. They show however, that the exponent predicted by Conjecture 1.2 does hold for subsets of the sphere $S^{d-1}=\left\{x \in \mathbb{F}_{q}^{d}: x_{1}^{2}+\ldots x_{d}^{2}=1\right\}$.

Theorem 1.4 ([7]) Let $E \subset \mathbb{F}_{q}^{d}$, $d \geq 3$, be a subset of the sphere

$$
S^{d-1}=\left\{x \in \mathbb{F}_{q}^{d}:\|x\|=1\right\}
$$

Suppose that $|E| \geq C q^{d / 2}$ with a sufficiently large constant $C$. Then there exists $c>0$ such that $|\Delta(E)|>c q$.

In this note, we will give a graph theoretic proof of this result. The rest of this note is organized as follows. In Section 2, we construct our main tools to study the Erdös-Falconer distance problem over subsets of the sphere, the graphs associated to the projective spaces over finite fields. Our construction follows one of Bannai, Shimabukuro and Tanaka in [3]. We then prove Theorem 1.4 in Section 3. We also call the reader's attention that this note is a subsequent of an earlier paper [11].

Remark 1.5 If d is even, Hart, Iosevich, Koh and Rudnev showed that all the ditances can be obtained under the same assumption and the size condition on $E$ cannot be relaxed. If $d$ is odd then we cannot in general get all the distances if $|E| \ll q^{\frac{d+1}{2}}$. Interested readers can see [7] for a detailed discussion and related results.

## 2 Finite non-Euclidean graphs

In this section, we give a construction of graphs from the action of simple orthogonal group on the set of non-isotropic square-type of projective spaces over finite fields. Our construction follows one of Bannai, Shimabukuro and Tanaka in [3]. Let $V=\mathbb{F}_{q}^{d}$ be the $d$-dimensional vector space over the finite field $\mathbb{F}_{q}(q$ is an odd prime power). For each element $x$ of $V$, we denote the 1-dimensional subspace containing $x$ by $[x]$. Let $\Omega$ be the set of all square type non-isotropic 1-dimensional subspaces of $V$ with respect to the quadratic form $Q(x)=x_{1}^{2}+\ldots+x_{d}^{2}$. The simple orthogonal group $O_{d}\left(\mathbb{F}_{q}\right)$ acts transtively on $\Omega$, and yields a symmetric association scheme $\Psi\left(O_{d}\left(\mathbb{F}_{q}\right), \Omega\right)$ of class $(q+1) / 2$. We have two cases.

Case I. Suppose that $d=2 m+1$. The relations of $\Psi\left(O_{2 m+1}\left(\mathbb{F}_{q}\right), \Omega\right)$ are given by

$$
\begin{aligned}
R_{1} & =\{([U],[V]) \in \Omega \times \Omega \mid(U+V) \cdot(U+V)=0\}, \\
R_{i} & =\left\{([U],[V]) \in \Omega \times \Omega \mid(U+V) \cdot(U+V)=2+2 \nu^{-(i-1)}\right\}(2 \leqslant i \leqslant(q-1) / 2) \\
R_{(q+1) / 2} & =\{([U],[V]) \in \Omega \times \Omega \cdot(U+V) \cdot(U+V)=2\},
\end{aligned}
$$

where $\nu$ is a generator of the field $\mathbb{F}_{q}$ and we assume $U \cdot U=1$ for all $[U] \in \Omega$ (see [2]).
Case II. Suppose that $d=2 m$. The relations of $\Psi\left(O_{2 m}\left(\mathbb{F}_{q}\right), \Omega\right)$ are given by

$$
\begin{aligned}
R_{i} & =\left\{([U],[V]) \in \Omega \times \Omega \mid(U+V) \cdot(U+V)=2+2^{-1} \nu^{i}\right\}(1 \leqslant i \leqslant(q-1) / 2) \\
R_{(q+1) / 2} & =\{([U],[V]) \in \Omega \times \Omega \cdot(U+V) \cdot(U+V)=2\}
\end{aligned}
$$

where $\nu$ is a generator of the field $\mathbb{F}_{q}$ and we assume $U \cdot U=1$ for all $[U] \in \Omega$ (see [2]).
The graphs $\left(\Omega, R_{i}\right)$ are not Ramanujan in general, but fortunately, they are asymptotic Ramanujan for large $q$. The following theorem summaries the results from [3] in a rough form.

Theorem 2.1 ([3]) The graphs $\left(\Omega, R_{i}\right)(1 \leq i \leq(q+1) / 2)$ are regular of order $q^{d-1}(1+$ $o(1)) / 2$ and valency $K q^{d-2}(1+o(1))$. Let $\lambda$ be any eigenvalue of the graph $\left(\Omega, R_{i}\right)$ with $\lambda \neq$ valency of the graph then

$$
|\lambda| \leq k(1+o(1)) q^{(d-2) / 2}
$$

for some $k, K>0$ (In fact, we can show that $k=2$ and $K=1$ or $1 / 2$ ).

## 3 Graph theoretic proof of Theorem 1.4

Let $E$ be a subset of the unit sphere $S^{d-1}=\left\{x \in \mathbb{F}_{q}^{d}:||x||=1\right\}$ with $|E| \geq C q^{d / 2}$. Let $E_{1}=\{[x]: x \in E\} \subset \Omega$ (where $\Omega$ is the set of all square type non-isotropic 1-dimensional subspaces of $V$ with respect to the quadratic form $Q(x)=x_{1}^{2}+\ldots+x_{d}^{2}$. Since each line through origin in $\mathbb{F}_{q}^{d}$ intersects the unit sphere $S^{d-1}$ at two points, $\left|E_{1}\right| \geq C q^{d / 2} / 2$. Suppose that $([U],[V]) \in E_{1} \times E_{1}$ is an edge of $\left(\Omega, R_{i}\right)$. Then

$$
(U+V) \cdot(U+V)=2+\alpha_{i},
$$

where $\alpha_{i}=2 \nu^{(-(i-1))}$ if $d$ is odd and $\alpha_{i}=2^{-1} \nu^{i}$ if $d$ is even. Since $U \cdot U=V \cdot V=1$, we have $(U-V) \cdot(U-V)=2-\alpha_{i}$. The distance between $U$ and $V$ (in $E$ ) is either $(U+V) \cdot(U+V)$ or $(U-V) \cdot(U+V)$, so

$$
\left|\Delta(E) \cap\left\{2+\alpha_{i}, 2-\alpha_{i}\right\}\right| \geq 1
$$

Therefore, it is sufficient to show that $E_{1} \times E_{1}$ contains edges of at least $c q$ graphs among $\left(\Omega, R_{i}\right), 1 \leq i \leq(q+1) / 2$.

To complete the proof, we need the following result from spectral graph theory. We call a graph $G(n, d, \lambda)$-regular if $G$ is a $d$-regular graph on $n$ vertices with the absolute value of each of its eigenvalues but the largest one is at most $\lambda$. It is well-known that if $\lambda \ll d$ then a $(n, d, \lambda)$-regular graph behaves similarly as a random graph $G_{n, d / n}$. Presicely, we have the following result (see Corollary 9.2.5 and Corollary 9.2.6 in [1]).

Theorem 3.1 ([1]) Let $G$ be a $(n, d, \lambda)$-regular graph. For every set of vertices $B$ of $G$, we have

$$
\begin{equation*}
\left.\left.\left|e_{G}(B)-\frac{d}{2 n}\right| B\right|^{2}\left|\leqslant \frac{1}{2} \lambda\right| B \right\rvert\, \tag{3.1}
\end{equation*}
$$

where $e_{G}(B)$ is number of edges in the induced subgraph of $G$ on $B$.
From Theorem 3.1, we have

$$
\left.\left.\left|e_{\left(\Omega, R_{i}\right)}\left(E_{1}\right)-\frac{K q^{d-2}(1+o(1))}{q^{d-1}(1+o(1)) / 2}\right| E_{1}\right|^{2}\left|\leq \frac{1}{2} k(1+o(1)) q^{(d-2) / 2}\right| E_{1} \right\rvert\,
$$

Since $\left|E_{1}\right| \geq C q^{d / 2} / 2$ for $C$ sufficiently large, the left hand term $\frac{1}{2} k(1+o(1)) q^{(d-2) / 2}\left|E_{1}\right|$ is neglected by $\frac{K q^{d-2}(1+o(1))}{q^{d-1}(1+o(1)) / 2}\left|E_{1}\right|^{2}$. Thus, we have $e_{\left(\Omega, R_{i}\right)}\left(E_{1}\right)=O\left(\left|E_{1}\right|^{2} / q\right)$. Besides $E_{1} \times E_{1}$ is edge-decomposed into $(q+1) / 2$ graphs $\left(\Omega, R_{i}\right), 1 \leq i \leq(q+1) / 2$. This implies that $E_{1} \times E_{1}$ contains edges of at lesat $c q$ graphs among $\left(\Omega, R_{i}\right), 1 \leq i \leq(q+1) / 2$ for some constant $c>0$. The theorem follows.

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