Results on the diffusion equation with rough coefficients

Burak Aksoylu
Louisiana State University
Department of Mathematics &
Center for Computation and Technology
Baton Rouge, LA 70803, USA

Horst R. Beyer
Louisiana State University
Center for Computation and Technology
Baton Rouge, LA 70803, USA

September 7, 2021

Abstract

We study the behaviour of the solutions of the stationary diffusion equation as a function of a possibly rough (L^{∞}-) diffusivity. This includes the boundary behaviour of the solution maps, associating to each diffusivity the solution corresponding to some fixed source function, when the diffusivity approaches infinite values in parts of the medium. In n-dimensions, $n \geqslant 1$, by assuming a weak notion of convergence on the set of diffusivities, we prove the strong sequential continuity of the solution maps. In 1-dimension, we prove a stronger result, i.e., the unique extendability of the map of solution operators, associating to each diffusivity the corresponding solution operator, to a sequentially continuous map in the operator norm on a set containing 'diffusivities' assuming infinite values in parts of the medium. In this case, we also give explicit estimates on the convergence behaviour of the map.

Mathematics Subject Classification (2000) 35J25, 47F05, 65J10, 65N99.
Keywords: Diffusion equation, diffusion operator, rough coefficients, first-order formulation, mixed formulation, dependence on diffusivity.

1 Motivation

Numerical methods for the diffusion equation with rough coefficients have been studied extensively [3, 4, 5, 12, 13, 14, 16] in the preconditioning (multigrid, domain decomposition, and related iterative methods) literature starting the early eighties and still continue to be an active area of research in various preconditioning efforts [20, 21]. This article came about out of a need of deeper understanding of the performance of preconditioners and their connection to the underlying PDE.

In a recent article [2], the first author constructed a new preconditioning strategy with rigorous justification which is comparable to algebraic multigrid. It is shown in [2] that analytical tools such as singular perturbation analysis gives valuable insight about the asymptotic behavior of the solution of the underlying PDE, hence, provides feedback for preconditioner construction.

According to experience, the performance of a preconditioner depends essentially on the degree to which the preconditioner operator approximates the underlying operator. Then, the fundamental need is to explain the effectiveness of the preconditioner and to justify that rigorously. In that respect, one can view the tools in this article as steps towards adding tools to the arsenal of methods of analysis for rigorous justification at the interface of preconditioning and operator theory. Direct connections from the results here to preconditioning will be the subject for future research.

2 Introduction

The diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\,(\,p\,\operatorname{grad} u) + f \tag{2.0.1}$$

describes general diffusion processes, including the propagation of heat, and flows through porous media. Here u is the density of the diffusing material, p is the diffusivity of the material, and the function f describes the distribution of 'sources' and 'sinks'. This paper focuses on stationary solutions of (2.0.1) satisfying

$$-\operatorname{div}\,(\,p\,\operatorname{grad} u) = f\ . \tag{2.0.2}$$

For instance, the fictitious domain method and composite materials are sources of rough coefficients; see the references in [14]. Important current applications deal with composite materials whose components have nearly constant diffusivity, but vary by several orders of magnitude. In composite material applications, it is quite common to idealize the diffusivity by a piecewise constant function and also to consider limits where the values of that function approach zero or infinity in parts of the material.

Results of such study were given first by J. L. Lions [15]. In his lecture notes, he considers the limit of the solution of (2.0.2) where the limit is associated to a one-parameter family of piecewise constant diffusivities approaching zero on a subdomain. The same piecewise constant one-parametric approach was used in [4, 13], but with diffusivities approaching infinity on a subdomain. The limitation of the one-parametric approach is its dependence on the particular approximating sequence. To the knowledge of the authors, this paper is the first to address these questions in the necessary generality. Hence, we consider general families of diffusivities that are not necessarily piecewise constant. In addition, due to the atomistic structure of matter, the physical treatment of diffusion involves regular (C^∞-) diffusivity. It is unclear to what extent the idealization of diffusivity by piecewise constant coefficients has the capability to capture the underlying physics. Mathematically, the severe contrast in diffusivity should be represented by a regular function whose size is changing drastically over small distances in interface regions. In this paper, we demonstrate that the assumption of piecewise constant diffusivities is meaningful by showing a continuous dependence of the solutions on the diffusivity.

Furthermore, the diffusion equation is meaningless if the ‘diffusivity’ is infinite or zero in regions of the material. Physics requires nowhere vanishing diffusivity in the interior of the material. As a consequence, only the relative size of diffusivities should be significant. Therefore, physically, one might expect that both types of the above limits are equivalent, but mathematically there are differences. The limit of the solution as diffusivity approaches infinite values exists. However, only the limit of the scaled solution exists as diffusivity approaches zero values (see Example 4.0.11 for both cases). That is why, we choose to work with diffusivity approaching infinity. We will refer these cases as ‘asymptotic cases’.

Also, the treatment in [4, 13] considers only limits on specific parts of the material. In this connection, it should also be remarked that, although (2.0.1), (2.0.2) are linear equations, in general, their solutions depend non-linearly on the coefficients.

For the treatment of these questions, we use methods from operator theory. For this, we use a common approach to give (2.0.1) a well-defined meaning that, in a first step, represents the diffusion operator

$$-\operatorname{div} p \operatorname{grad} \tag{2.0.3}$$

as a densely-defined positive self-adjoint linear operator A_p in a suitable Hilbert space. As a result, (2.0.2) is represented by the equation

$$A_p u = f \ ,$$

where f is an element of the Hilbert space, and u is from the domain, $D(A_p)$, of A_p. [1]

Specifically, we treat the class $\mathcal{L}$ of diffusivities $p \in L^{\infty}(\Omega)$ that are almost everywhere $\geqslant \varepsilon$ on Ω for some $\varepsilon > 0$, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}^*$, is some non-empty open subset. By use of Dirichlet boundary conditions, it defines A_p as an operator in the complex Hilbert space $L^2_{\mathbb{C}}(\Omega)$. For non-smooth p, the domain of A_p depends heavily on p. This fact significantly complicates the study of sequences of functions of A_p.

In this paper, we turn to a first-order formulation of (2.0.2) which is often referred as mixed formulation in the discretization literature [7]. The first-order formulation was popularized in the least squares finite element community by the so-called FOSLS pioneering paper [9]. Here, we provide the self-adjointness of the corresponding operator $\hat{A}_p$ in a Hilbert space. The key property of $\hat{A}_p$ is that its domain, $D(\hat{A}_p)$, is independent of p. This property is exploited in establishing the continuity of the solutions $A_p^{-1} f$ as a function of p. Moreover, $\hat{A}_p$ remains defined for the asymptotic cases when (2.0.1), (2.0.2) are ill-defined. This fact is used in the study of the asymptotic cases.

[1] After that, the abstract theory of strongly continuous one-parameter semigroups of operators can be used to associate a rigorous formulation of a well-posed initial value problem to (2.0.1) [6, 10, 17]. In this, A_p becomes the infinitesimal generator of time evolution. This last step will not be detailed here.

Specifically, for $p \in \mathcal{L}$ and by assuming a weak notion of convergence in $\mathcal{L}$, we show that the maps that associate p to the operator A_p^{-1} and $-p\,\nabla A_p^{-1}$, respectively, are strongly sequentially continuous, see Theorem 5.0.21 and Corollary 5.0.22. In particular, this shows in these cases that the approximation by discontinuous coefficients to physical diffusivity is indeed meaningful. In addition, for the case $n = 1$ and bounded open intervals of $\mathbb{R}$, we show stronger results that include also the asymptotic cases, except that where the asymptotic 'diffusivity' is almost everywhere infinite on the interval. In this case, the maps that associate $\bar{p}$ to the operator $A_{1/\bar{p}}^{-1}$ and $-(1/\bar{p})\,\nabla A_{1/\bar{p}}^{-1}$, respectively, have unique extensions to sequentially continuous maps in the operator norm on the set of a.e. positive elements of $L^\infty(\Omega)\backslash\{0\}$, see Corollary 6.0.26, 6.0.27. In addition, an explicit estimate of the convergence behaviour of the maps is given, see Theorem 6.0.25. It is still an open problem, whether the last results are generalizable to dimensions $n \geqslant 2$.

3 Basic notation

Mainly, this section introduces basic notation. In particular, an operator theoretic definition of Sobolev spaces is given that is based on weak derivative operators, instead of distributions. In such formulation, the completeness of the Sobolev spaces is an obvious consequence of the cussedness of these operators. Also, we give some basic results that are connected to this formulation. For the convenience to the reader, corresponding proofs are given in the appendix.

General Assumption 3.0.1. In the following, let $n \in \mathbb{N}^*$ and Ω be a non-empty open subset of $\mathbb{R}^n$.

We follow common usage and do not differentiate between a function f which is almost everywhere defined (with respect to a chosen measure) on some set and the associated equivalence class consisting of all functions which are almost everywhere defined on that set and differ from f only on a set of measure zero. The following definitions need to be understood in this sense.

Definition 3.0.2. (**Complex L^p-spaces**)

(i) For $p > 0$, the symbol $L^p_{\mathbb{C}}(\Omega)$ denotes the vector space of all complex-valued measurable functions f which are a.e. defined on Ω and such that

$|f|^p$ is integrable with respect to the Lebesgue measure v^n. For every such f, we define the L^p-norm $\|f\|_p$ corresponding to f by

$$\|f\|_p := \left(\int_\Omega |f|^p \, dv^n \right)^{1/p} .$$

In addition, for the special case $p = 2$, we define a scalar product $\langle \, | \, \rangle_2$ on $L^2_{\mathbb{C}}(\Omega)$ by

$$\langle f|g\rangle_2 := \int_\Omega f^* g \, dv^n ,$$

for all $f, g \in L^2_{\mathbb{C}}(\Omega)$. Here * denotes complex conjugation on $\mathbb{C}$. As a consequence, $\langle \, | \, \rangle_2$ is antilinear in the first argument and linear in its second. This convention will be used for sesquilinear forms in general.

(ii) $L^\infty_{\mathbb{C}}(\Omega)$ denotes the vector space of complex-valued measurable bounded functions on Ω. For every $f \in L^\infty_{\mathbb{C}}(\Omega)$, we define

$$\|f\|_\infty := \sup_{x \in \Omega} |f(x)| .$$

(iii) For every $k \in \mathbb{N}^*$ and $f, g \in (L^2_{\mathbb{C}}(\Omega))^k$, we define

$$\langle f|g\rangle_{2,k} := \sum_{j=1}^{k} \langle f_j|g_j\rangle_2 \;, \; \|f\|_{2,k} := \left(\sum_{j=1}^{k} \|f_j\|_2^2 \right)^{1/2} .$$

Definition 3.0.3. (**Weak derivatives and Sobolev spaces**) We define

(i) for every multi-index $\alpha \in \mathbb{N}^n$ the densely-defined linear operator ∂^α in $L^2_{\mathbb{C}}(\Omega)$ by

$$\partial^\alpha := (-1)^{|\alpha|} . \left(C_0^\infty(\Omega, \mathbb{C}) \to L^2_{\mathbb{C}}(\Omega), f \mapsto \frac{\partial^\alpha f}{\partial x^\alpha} \right)^* ,$$

where $*$ denotes the adjoint operation and

$$|\alpha| := \sum_{j=1}^{n} \alpha_j .$$

(iii) for every $k \in \mathbb{N}$ the Sobolev space $W^k_{\mathbb{C}}(\Omega)$ of order k by

$$W^k_{\mathbb{C}}(\Omega) := \bigcap_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant k} D(\partial^{\alpha}) \ .$$

Equipped with the scalar product

$$\langle \, , \rangle_k \ : \ (W^k_{\mathbb{C}}(\Omega))^2 \to \mathbb{C} \ ,$$

defined by

$$\langle f, g \rangle_k \ := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant k} \langle \partial^{\alpha} f | \partial^{\alpha} g \rangle_2$$

for all $f, g \in W^k_{\mathbb{C}}(\Omega)$, $W^k_{\mathbb{C}}(\Omega)$ becomes a Hilbert space.

(iv) $W^k_{0,\mathbb{C}}(\Omega)$ as the closure of $C^{\infty}_0(\Omega, \mathbb{C})$ in $(W^k_{\mathbb{C}}(\Omega), \| \ \|_k)$, where $\| \ \|_k$ denotes the norm that is induced on $W^k_{\mathbb{C}}(\Omega)$ by $\langle \, , \rangle_k$.

We note that

Lemma 3.0.4. (**Partial integration**)

$$\langle f | \partial^{e_k} g \rangle_2 = - \langle \partial^{e_k} f | g \rangle_2 \tag{3.0.4}$$

for all $(f, g) \in W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega)$ and $k \in \mathbb{N}^*$, where e_k denotes the k-th canonical unit vector of $\mathbb{R}^n$.

The next defines gradient operators.

Definition 3.0.5. (**Gradient operators**) We define the $(L^2_{\mathbb{C}}(\Omega))^n$-valued densely-defined linear operators in $L^2_{\mathbb{C}}(\Omega)$

$$\nabla_0 : C^{\infty}_0(\Omega, \mathbb{C}) \to (L^2_{\mathbb{C}}(\Omega))^n \ , \ \nabla_w : W^1_{\mathbb{C}}(\Omega) \to (L^2_{\mathbb{C}}(\Omega))^n$$

by

$$\nabla_0 f := {}^t\!\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \ , \ \nabla_w g := {}^t(\partial^{e_1} g, \ldots, \partial^{e_n} g)$$

for all $f \in C^{\infty}_0(\Omega, \mathbb{C})$ and $g \in W^1_{\mathbb{C}}(\Omega)$.

Then the following holds.

Lemma 3.0.6. (**Adjoints of gradient operators**)

$$(\nabla_0{}^*)^* = \nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)} \ , \ \left(\nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)} \right)^* = \nabla_0{}^* \ . \tag{3.0.5}$$

4 Basic properties of the diffusion operator

This section provides the basis of the paper. It defines the diffusion operator as operator in $L^2_{\mathbb{C}}(\Omega)$ and gives basic properties.

Definition 4.0.7. Let $\bar{p} : \Omega \to \mathbb{R}$ be measurable and such that $1/\bar{p}$ is a.e. defined on Ω. We define the linear operator $A : D(A) \to L^2_{\mathbb{C}}(\Omega)$ in $L^2_{\mathbb{C}}(\Omega)$ by

$$D(A) := \{u \in W^1_{0,\mathbb{C}}(\Omega) : (1/\bar{p})\nabla_w u \in D(\nabla_0{}^*)\}$$

and

$$Au := \nabla_0{}^*(1/\bar{p})\,\nabla_w u$$

for every $u \in D(A)$.

Diffusion operators corresponding to diffusivities from the following large subset $\mathcal{L}$ of $L^\infty(\Omega)$ will turn out to be densely-defined self-adjoint linear operators.

Definition 4.0.8. We define the subset $\mathcal{L}$ of $L^\infty(\Omega)$ to consist of those elements $\bar{p}$ for which there are real C_1, C_2 satisfying $C_2 \geqslant C_1 > 0$ and such that $C_1 \leqslant \bar{p} \leqslant C_2$ a.e. on Ω. Note that the last also implies that $1/\bar{p} \in \mathcal{L}$ and in particular that $1/C_2 \leqslant 1/\bar{p} \leqslant 1/C_1$ a.e. on Ω.

The next proves the self-adjointness of diffusion operators corresponding to diffusivities from $\mathcal{L}$. For this, so called ‘form methods’ from operator theory are used. For these methods, see [11].

Theorem 4.0.9. Let $\bar{p} \in \mathcal{L}$. Then A is a densely-defined linear self-adjoint operator in $L^2_{\mathbb{C}}(\Omega)$.

Proof. For this, we define a positive Hermitian sesquilinear form $s : (W^1_{0,\mathbb{C}}(\Omega))^2 \to \mathbb{C}$ by

$$s(u,v) := \langle \nabla_w u \,|\, (1/\bar{p})\,\nabla_w v \rangle_{2,n}$$

for all $u, v \in W^1_{0,\mathbb{C}}(\Omega)$. Then $\langle\,|\,\rangle_s : (W^1_{0,\mathbb{C}}(\Omega))^2 \to \mathbb{C}$, defined by

$$\langle u|v\rangle_s := s(u,v) + \langle u|v\rangle_2$$

for every $u, v \in W^1_{0,\mathbb{C}}(\Omega)$, defines a scalar product on $W^1_{0,\mathbb{C}}(\Omega)$ with induced norm $\|\ \|_s : W^1_{0,\mathbb{C}}(\Omega) \to \mathbb{R}$ given by

$$\|u\|_s^2 = \langle \nabla_w u \,|\, (1/\bar{p})\,\nabla_w u \rangle_{2,n} + \|u\|_2^2$$

for all $u \in W^1_{0,\mathbb{C}}(\Omega)$. In particular, s is closable. For the proof, let $u_1, u_2, \dots$ be a Cauchy sequence in $(W^1_{0,\mathbb{C}}(\Omega), \| \ \|_s)$ and such that

$$\lim_{\nu\to\infty} \|u_\nu\|_2 = 0 \ .$$

We note that

$$\begin{aligned}\min\{1, 1/C_2\} \, \|\!| u \|\!|_1^2 &\leqslant \frac{1}{C_2} \, \|\nabla_w u\|_{2,n} + \|u\|_2^2 \leqslant \|u\|_s^2 \\ &\leqslant \frac{1}{C_1} \, \|\nabla_w u\|_{2,n} + \|u\|_2^2 \leqslant \max\{1, 1/C_1\} \, \|\!| u \|\!|_1^2 \ ,\end{aligned}$$

where $C_1, C_2 \in \mathbb{R}$ satisfy $C_2 \geqslant C_1 > 0$ and are such that $C_1 \leqslant \bar{p} \leqslant C_2$ a.e. on Ω, and hence that $\| \ \|_s$ and the restriction of $\|\!| \ \|\!|_1$ to $W^1_{0,\mathbb{C}}(\Omega)$ are equivalent. Hence it follows that

$$\lim_{\nu\to\infty} \|u_\nu\|_s = 0 \ .$$

Since $(W^1_{0,\mathbb{C}}(\Omega), \| \ \|_s)$ is in particular complete, it follows that s coincides with its closure. As a consequence, there is a unique densely-defined linear self-adjoint operator $A : D(A) \to L^2_{\mathbb{C}}(\Omega)$ in $L^2_{\mathbb{C}}(\Omega)$ such that $D(A)$ is a dense subspace of $(W^1_{0,\mathbb{C}}(\Omega), \|\!| \ \|\!|_1)$ and such that

$$\langle u | Au \rangle_2 = s(u,u) = \langle \nabla_w u \,|\, (1/\bar{p}) \, \nabla_w u \rangle_{2,n}$$

for all $u \in D(A)$. In particular, $D(A)$ consists of all $u \in W^1_{0,\mathbb{C}}(\Omega)$ for which there is $f \in L^2_{\mathbb{C}}(\Omega)$ such that

$$\langle \, f \,|\, \dots \rangle_2 \big|_{W^1_{0,\mathbb{C}}(\Omega)} = \langle \, (1/\bar{p}) \, \nabla_w u \,|\, \nabla_w \dots \rangle_{2,n} \big|_{W^1_{0,\mathbb{C}}(\Omega)} \ .$$

Further, if u and f satisfy these requirements, then

$$Au = f \ .$$

Hence $u \in D(A)$ if and only if

$$(1/\bar{p}) \, \nabla_w u \in D\left(\left(\nabla_w \big|_{W^1_{0,\mathbb{C}}(\Omega)} \right)^* \right) = D({\nabla_0}^*)$$

and in this case

$$Au = {\nabla_0}^* \, (1/\bar{p}) \, \nabla_w u \ .$$

$\square$

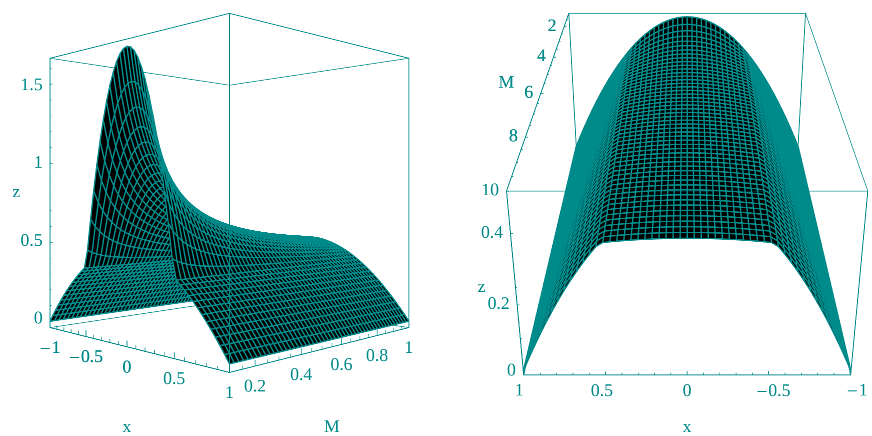

Fig. 1: Graphs of u from Example 4.0.11 as a function of M.

For completeness, the next gives the proof that diffusion operators corresponding to diffusivities from $\mathcal{L}$ have a purely discrete spectrum, i.e., that their spectrum is a discrete subset of the real numbers consisting of eigenvalues of finite multiplicity and that there is a Hilbert basis consisting of eigenvectors. This result is not used in the following.

Corollary 4.0.10. Let $\bar{p} \in \mathcal{L}$ and, in addition, Ω be bounded. Then A has a purely discrete spectrum.

Proof. According to the proof of Theorem 4.0.9, $\| \ \|_s : W^1_{0,\mathbb{C}}(\Omega) \to \mathbb{R}$ defines a norm which is equivalent to the restriction of $\| \ \|_1$ to $W^1_{0,\mathbb{C}}(\Omega)$. Hence the closed unit ball B in $(W^1_{0,\mathbb{C}}(\Omega), \| \ \|_s)$ is contained in a closed ball of $(W^1_{0,\mathbb{C}}(\Omega), \| \ \|_1)$. The last is relatively compact in $L^2_{\mathbb{C}}(\Omega)$. From this, it follows also that B is relatively compact in $L^2_{\mathbb{C}}(\Omega)$. Hence it follows, see, e.g., [18] Vol. IV, that A has a purely discrete spectrum. $\square$

Example 4.0.11. The following example illustrates the influence of discontinuities of the diffusivity on the regularity of the elements in $D(A)$. Consider the case that $\Omega = I := (-1, 1)$ and a piecewise constant diffusivity $p : I \to \mathbb{R}$ given by

$$p(x) := \begin{cases} 1 & \text{if } -1 < x < -1/2 \\ M & \text{if } -1/2 \leqslant x \leqslant 1/2 \\ 1 & \text{if } 1/2 < x < 1 \end{cases}$$

for $x \in I$, where $M > 0$. Then $Au = f$, where $u : I \to \mathbb{R}$ is defined by

$$u(x) := \begin{cases} (1-x^2)/2 & \text{if } -1 < x \leqslant -1/2 \\ (1-4x^2+3M)/(8M) & \text{if } -1/2 < x < 1/2 \\ (1-x^2)/2 & \text{if } 1/2 \leqslant x < 1 \end{cases}$$

and f is the constant function on I of value 1. We note that u' has no extension to a continuous function on I if $M \neq 1$. In general, discontinuities in the diffusivity cause low regularity of elements in $D(A)$. Also, see the concluding remarks.

There is a unique solution u_f to the equation

$$Au_f = f$$

for every $f \in L^2_{\mathbb{C}}(\Omega)$ if and only if A is bijective or equivalently, if and only if 0 is not part of the spectrum of A. In general, A is not bijective. For instance, the operator A that is associated to $\Omega = \mathbb{R}^n$ and the diffusivity $p(x) = 1$ for every $x \in \mathbb{R}^n$ is not surjective. Below, we place a restriction on Ω that leads to bijective diffusion operators.

General Assumption 4.0.12. In the following, we assume that Ω is in addition such that the following Poincare inequality is valid

$$\| \partial^{e_j} f \|_2 \geqslant c \, \|f\|_2 \tag{4.0.6}$$

for some $j \in \{1, \dots, n\}$ and every $f \in W^1_{0,\mathbb{C}}(\Omega)$, where $c > 0$. In the remainder, such c is considered chosen.

Remark 4.0.13. It is known that Ω of the assumed type are not necessarily bounded. For instance, every non-trivial open set, for which there is $\mathbf{n} \in \mathbb{R}^n \backslash \{0\}$ along with real numbers a, b such that

$$a < x \cdot \mathbf{n} < b$$

for all $x \in \Omega$, is of this type.

In particular, the following proves that diffusion operators corresponding to diffusivities from $\mathcal{L}$ are bijective.

Theorem 4.0.14. Let $\bar{p} \in \mathcal{L}$. The spectrum $\sigma(A)$ of A satisfies

$$\sigma(A) \subset [\, c^2/C, \infty) \ , \tag{4.0.7}$$

where $j \in \{1, \dots, n\}$ is such that $\mathbf{n}_j \neq 0$ and $C > 0$ is such that $\bar{p} \leqslant C$ a.e. on Ω.

Proof. For this, let $j \in \{1, \ldots, n\}$ be such that $\mathbf{n}_j \neq 0$. For $u \in D(A)$, it follows that

$$\langle u | Au \rangle_2 = \langle \nabla_w u \,|\, (1/\bar{p})\, \nabla_w u \rangle_{2,n} \geqslant C^{-1} \, \| \nabla_w u \|_{2,n}^2 \geqslant c^2 \, C^{-1} \, \| u \|_2^2 \ ,$$

where $C > 0$ is such that $\bar{p} \leqslant C$ a.e. on Ω. Hence it follows the validity of (4.0.7). $\square$

5 Properties of a first order operator connected to the diffusion operator

As indicated by Example 4.0.11, for non-smooth diffusivities p, the condition that $p \nabla_w u \in W^1_{\mathbb{C}}(\Omega)$ in the definition of the domain of A leads to a strong dependence of that domain on the diffusivity. This fact poses an obstacle to the study of the map, associating to every diffusivity $p \in \mathcal{L}$ the corresponding operator A^{-1}, by the notion of strong resolvent convergence, see, [18, Volume I, Section VIII.7], [11, Section VIII, §1]. By use of the following vector partial differential operator of the first order $\hat{A}$, this problem can be circumvented. Its domain is independent of the diffusivity. The connection of the resolvents of A and $\hat{A}$ is given in Theorem 5.0.20.

Definition 5.0.15. Let $\bar{p} \in L^\infty(\Omega)$. We define the densely-defined, linear operator $\hat{A} : W^1_{0,\mathbb{C}}(\Omega) \times D({\nabla_0}^*) \to L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$ in $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$ by

$$\hat{A}(u, q) := (\, {\nabla_0}^* q \,,\, \nabla_w u - \bar{p}\, q \,)$$

for every $(u, q) \in W^1_{0,\mathbb{C}}(\Omega) \times D({\nabla_0}^*)$.

Theorem 5.0.16. The operator $\hat{A}$ is self-adjoint.

Proof. The statement is a consequence of Lemma 3.0.6. $\square$

The following gives a characterization of the kernel of $\hat{A}$. In particular, the result implies that $\hat{A}$ is bijective for diffusivities from $\mathcal{L}$.

Theorem 5.0.17. Let $\bar{p} \in L^\infty(\Omega)$ be a.e. positive. Then

$$\ker \hat{A} = \{0\} \times (\, \ker {\nabla_0}^* \cap \ker T^n_{\bar{p}} \,) \ ,$$

where $T_{\bar{p}} \in L(L^2_{\mathbb{C}}(\Omega), L^2_{\mathbb{C}}(\Omega))$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\Omega)$ that is associated to $\bar{p}$.

Proof. '$\subset$': Let $q \in \ker \nabla_0{}^* \cap \ker T^n_{\bar{p}}$. Then $(0, q) \in D(\hat{A})$ and

$$\nabla_0{}^* q = 0\,,\ -\bar{p}\,q = 0\ .$$

Hence it follows that $(0, q) \in \ker \hat{A}$.
'$\supset$': Let $(u, q) \in \ker \hat{A}$. Then

$$\nabla_0{}^* q = 0\,,\ \nabla_w u - \bar{p}\,q = 0 \tag{5.0.8}$$

and hence

$$\begin{aligned} 0 &= \langle\, q \,|\, \nabla_w u - \bar{p}\,q \,\rangle_{2,n} = \langle\, q \,|\, \nabla_w u \,\rangle_{2,n} - \langle\, q \,|\, \bar{p}\,q \,\rangle_{2,n} \\ &= \langle\, \nabla_0{}^* q \,|\, u \,\rangle_2 - \|\, \bar{p}^{1/2} q \,\|_{2,n} = -\|\, \bar{p}^{1/2} q \,\|_{2,n}\ . \end{aligned}$$

The last implies that

$$q \in \ker T^n_{\bar{p}^{1/2}}\ ,$$

where $T_{\bar{p}^{1/2}} \in L(L^2_{\mathbb{C}}(\Omega), L^2_{\mathbb{C}}(\Omega))$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\Omega)$ that is associated to $\bar{p}^{1/2}$, and hence also that

$$q \in \ker T^n_{\bar{p}}\ .$$

Further, by (5.0.8)-2), it follows that

$$\nabla_w u = 0\ .$$

The last implies that

$$\nabla_0{}^* \nabla_w u = 0$$

and hence by Theorem 4.0.14 that $u = 0$. □

The following example shows that the kernel of $\hat{A}$ is non-trivial if $\bar{p}$ vanishes on some open subset of Ω. The vanishing of $\bar{p}$ on non-empty subsets of Ω corresponds to the asymptotic cases mentioned in the introduction.

Example 5.0.18. In the following, we give $q \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^n) \cap \ker \nabla_0{}^*$ for $n \geqslant 2$. For this, let h be an element of $C^\infty_0(\mathbb{R})$ with support contained in $[-1, 1]$. In addition, let α be a non-zero antisymmetric $n \times n$-matrix. We define $q \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^n)$ by

$$q(x) := \frac{h(|x|^2)}{2} \sum_{i,j=1}^{n} \alpha_{ij} x_j e_i$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\sum_{i=1}^{n} \frac{\partial q_i}{\partial x_i}(x) = h'(|x|^2) \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j = 0$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and hence $q \in \ker \nabla_0{}^*$.

The following lemma prepares the subsequent theorem which estimates the size of the gap around 0 in the spectrum of A and gives a representation of the resolvent of $\hat{A}$ in terms of the resolvent of A, i.e., (5.0.9). The main tool in the proof is the Closed Graph Theorem in the form of see Theorem 3.1.9 in [6].

Lemma 5.0.19. Let $\bar{p} \in \mathcal{L}$, $\sigma(A)$ the spectrum of A, $\lambda < \min\{\sigma(A)\}$ and $A_\lambda := A - \lambda$. Then

(i) $\nabla_w A_\lambda^{-1} \in L(L^2_{\mathbb{C}}(\Omega), (L^2_{\mathbb{C}}(\Omega))^n)$,

(ii) $\overline{A_\lambda^{-1} \nabla_0{}^*} = (\, \nabla_w A_\lambda^{-1}\,)^*$,

(iii) $D(A_\lambda^{1/2}) = W^1_{0,\mathbb{C}}(\Omega)$ and $A_\lambda^{1/2} : W^1_{0,\mathbb{C}}(\Omega) \to L^2_{\mathbb{C}}(\Omega)$ is continuous,

(iv) $\overline{\nabla_w A_\lambda^{-1} \nabla_0{}^*}$ is a positive self-adjoint element of $L((L^2_{\mathbb{C}}(\Omega))^n, (L^2_{\mathbb{C}}(\Omega))^n)$.

Proof. '(i)': Since $\lambda \in \mathbb{C} \setminus \sigma(A)$, A_λ is densely-defined, linear and bijective. Further, A_λ is self-adjoint and strictly positive. As a consequence of its self-adjointness, A_λ is in particular closed. Further, according to Lemma 3.0.6, the restriction of ∇_w to $W^1_{0,\mathbb{C}}(\Omega)$ is a closed linear operator in $L^2_{\mathbb{C}}(\Omega)$ with values in $(L^2_{\mathbb{C}}(\Omega))^n$. Since $D(A) \subset W^1_{0,\mathbb{C}}(\Omega)$ it follows from the closed graph theorem, e.g., see Theorem 3.16 in [6], the existence of $C \in [0, \infty)$ such that

$$\|\nabla_w f\|_{2,n} \leqslant C\, \|A_\lambda f\|_2$$

for all $f \in D(A)$. As a consequence, it follows for every $f \in L^2_{\mathbb{C}}(\Omega)$ that

$$\|\nabla_w A_\lambda^{-1} f\|_{2,n} \leqslant C\, \|A_\lambda A_\lambda^{-1} f\|_2 = C\, \|f\|_2$$

and hence that $\nabla_w A_\lambda^{-1} \in L(L^2_{\mathbb{C}}(\Omega), (L^2_{\mathbb{C}}(\Omega))^n)$.
'(ii)': $A_\lambda^{-1} \nabla_0{}^*$ is a densely-defined, linear operator in $(L^2_{\mathbb{C}}(\Omega))^n$ with values in $L^2_{\mathbb{C}}(\Omega)$. Further, it follows for $f \in L^2_{\mathbb{C}}(\Omega)$ that

$$\langle\, f \,|\, A_\lambda^{-1} \nabla_0{}^* q \,\rangle_2 = \langle\, A_\lambda^{-1} f \,|\, \nabla_0{}^* q \,\rangle_2 = \langle\, \nabla_w A_\lambda^{-1} f \,|\, q \,\rangle_{2,n}$$

for every $q \in D(\nabla_0{}^*)$ and hence that $f \in D((A_\lambda^{-1}\nabla_0{}^*)^*)$ as well as that

$$(A_\lambda^{-1}\nabla_0{}^*)^* f = \nabla_w A_\lambda^{-1} f \ .$$

As a consequence,

$$(A_\lambda^{-1}\nabla_0{}^*)^* = \nabla_w A_\lambda^{-1} \ .$$

In particular, $A_\lambda^{-1}\nabla_0{}^*$ is closable and

$$\overline{A_\lambda^{-1}\nabla_0{}^*} = (\,\nabla_w A_\lambda^{-1}\,)^* \ .$$

'(iii)': In a first step, we prove the statement for the case $\lambda = 0$. For this, we note that, as a consequence of Theorem 4.0.14, $0 < \min\{\sigma(A)\}$. Further, we note that $D(A)$ is a core $A^{1/2}$. For instance, this follows by Theorem 3.1.9 in [6]. Hence $D(A)$ is dense in the Banach space $(D(A^{1/2}), \|\ \|_{A^{1/2}})$, where

$$\|f\|_{A^{1/2}} := \left[\, \|f\|_2^2 + \|A^{1/2}f\|_2^2 \,\right]^{1/2}$$

for every $f \in D(A^{1/2})$. Further, it follows for $f \in D(A)$ that

$$\|A^{1/2}f\|_2^2 = \langle\, f \mid Af\,\rangle_2 = \langle\, \nabla_w f \mid (1/\bar{p})\, \nabla_w f\,\rangle_{2,n} = s(f,f) \ ,$$

where the real numbers C_1, C_2 and the sesquilinear form s are as in the proof of Theorem 4.0.9, and hence that

$$\min\{1, 1/C_2\}\, |\!|\!| f |\!|\!|_1^2 \leqslant \|f\|_{A^{1/2}}^2 \leqslant \max\{1, 1/C_1\}\, |\!|\!| f |\!|\!|_1^2 \ .$$

As a consequence, the restrictions of $\|\ \|_{A^{1/2}}$ and $|\!|\!|\ |\!|\!|_1$ to $D(A)$ are equivalent. Since $D(A)$ is dense in $(D(A^{1/2}), \|\ \|_{A^{1/2}})$, it follows for $f \in D(A^{1/2})$ the existence of a sequence $f_1, f_2, \dots$ in $D(A)$ such that

$$\lim_{\nu\to\infty} \|f_\nu - f\|_{A^{1/2}} = 0 \ .$$

Since the inclusion of $(D(A^{1/2}), \|\ \|_{A^{1/2}})$ into $L^2_{\mathbb{C}}(\Omega)$ is continuous, this implies also that

$$\lim_{\nu\to\infty} \|f_\nu - f\|_2 = 0 \ .$$

Since the restrictions of $\|\ \|_{A^{1/2}}$ and $|\!|\!|\ |\!|\!|_1$ to $D(A)$ are equivalent, it follows that $f_1, f_2, \dots$ is a Cauchy sequence in $W^1_{0,\mathbb{C}}(\Omega)$ and hence convergent to some $\bar{f} \in$

$W_0^1(\Omega)$. Since the embedding of $(W^1_{\mathbb{C}}(\Omega), \|\ \|_1)$ into $L^2_{\mathbb{C}}(\Omega)$ is continuous, it follows also that

$$\lim_{\nu\to\infty} \|f_\nu - \bar{f}\,\|_2 = 0$$

and hence that $f = \bar{f} \in W_0^1(\Omega)$. Further, it follows that

$$\min\{1, 1/C_2\}\, \|f\|_1^2 \leqslant \|f\|^2_{A^{1/2}} \leqslant \max\{1, 1/C_1\}\, \|f\|_1^2$$

and hence that $\|\ \|_{A^{1/2}}$ and the restriction of $\|\ \|_1$ to $D(A^{1/2})$ are equivalent. Since according to the proof of Theorem 4.0.9, $D(A)$ is a dense subspace of $(W^1_{0,\mathbb{C}}(\Omega), \|\ \|_1)$, we conclude that $D(A^{1/2}) = W^1_{0,\mathbb{C}}(\Omega)$ and that $A^{1/2} : W^1_{0,\mathbb{C}}(\Omega) \to L^2_{\mathbb{C}}(\Omega)$ is continuous. From this, we conclude that statement of (ii) as follows. For this, let $\Lambda \in \mathbb{R}\backslash\sigma(A)$ such that $\Lambda > \max\{0, \lambda\}$. Since $\mathbb{R}\backslash\sigma(A)$ is open, such Λ exists. We note that $D(A)$ is a core also for $A_\lambda^{1/2}$ and $A_\Lambda^{1/2}$. For instance, this follows by Theorem 3.1.9 in [6]. Hence $D(A)$ is dense in the Banach spaces $(D(A_\lambda^{1/2}), \|\ \|_{A_\lambda^{1/2}})$, $(D(A_\Lambda^{1/2}), \|\ \|_{A_\Lambda^{1/2}})$, where

$$\|f\|_{A_\lambda^{1/2}} := \left[\, \|f\|_2^2 + \|A_\lambda^{1/2} f\|_2^2 \,\right]^{1/2} \ , \ \|g\|_{A_\Lambda^{1/2}} := \left[\, \|g\|_2^2 + \|A_\Lambda^{1/2} g\|_2^2 \,\right]^{1/2} \ ,$$

for all $f \in D(A_\lambda^{1/2})$ and $g \in D(A_\Lambda^{1/2})$. Further, it follows for every $f \in D(A)$ that

$$\begin{aligned} \|f\|^2_{A_\lambda^{1/2}} &= \|A_\lambda^{1/2} f\|_2^2 + \|f\|_2^2 = \langle f|A_\lambda f\rangle_2 + \|f\|_2^2 \\ &= \langle f|A_\Lambda f\rangle_2 + \|f\|_2^2 + (\Lambda - \lambda)\|f\|_2^2 = \|f\|^2_{A_\Lambda^{1/2}} + (\Lambda - \lambda)\|f\|_2^2 \end{aligned}$$

and hence that

$$\|f\|^2_{A_\lambda^{1/2}} \geqslant \|f\|^2_{A_\Lambda^{1/2}}$$

as well as that

$$\|f\|^2_{A_\lambda^{1/2}} \leqslant [1 + (\Lambda - \lambda)]\, \|f\|^2_{A_\Lambda^{1/2}} \ .$$

Since $D(A)$ is dense in the Banach spaces $(D(A_\lambda^{1/2}), \|\ \|_{A_\lambda^{1/2}})$ and $(D(A_\Lambda^{1/2}), \|\ \|_{A_\Lambda^{1/2}})$, it follows that

$$D(A_\lambda^{1/2}) = D(A_\Lambda^{1/2})$$

as well as the equivalence of the norms $\| \, \|_{A_\lambda^{1/2}}$ and $\| \, \|_{A_\Lambda^{1/2}}$. In particular, this implies that

$$D(A_\lambda^{1/2}) = D(A^{1/2})$$

and the equivalence of the norms $\| \, \|_{A_\lambda^{1/2}}$ and $\| \, \|_{A^{1/2}}$. By this, the statement (ii) follows from the corresponding statement of (ii) for the special case that $\lambda = 0$. '(iv)': In a first step, we conclude that

$$\nabla_w A_\lambda^{-1/2} \in L(L^2_{\mathbb{C}}(\Omega), (L^2_{\mathbb{C}}(\Omega))^n) \ .$$

As a consequence of the analogous properties of A_λ, $A_\lambda^{1/2}$ is densely-defined, linear, self-adjoint and bijective. Since $A_\lambda^{1/2}$ is self-adjoint, $A_\lambda^{1/2}$ is in particular closed. Further, according to Lemma 3.0.6, the restriction of ∇_w to $W^1_{0,\mathbb{C}}(\Omega)$ is a closed linear operator in $L^2_{\mathbb{C}}(\Omega)$ with values in $(L^2_{\mathbb{C}}(\Omega))^n$. Since $D(A_\lambda^{1/2}) = W^1_{0,\mathbb{C}}(\Omega)$, it follows from the closed graph theorem, e.g., see Theorem 3.16 in [6], the existence of $C \in [0, \infty)$ such that

$$\|\nabla_w f\|_{2,n} \leqslant C \, \|A_\lambda^{1/2} f\|_2$$

for all $f \in D(A_\lambda^{1/2})$. As a consequence, it follows for every $f \in L^2_{\mathbb{C}}(\Omega)$ that

$$\|\nabla_w A_\lambda^{-1/2} f\|_{2,n} \leqslant C \, \|A_\lambda^{1/2} A_\lambda^{-1/2} f\|_2 = C \, \|f\|_2$$

and hence that $\nabla_w A_\lambda^{-1/2} \in L(L^2_{\mathbb{C}}(\Omega), (L^2_{\mathbb{C}}(\Omega))^n)$. In a second step, we conclude that

$$\overline{A_\lambda^{-1/2} \nabla_0{}^*} \in L((L^2_{\mathbb{C}}(\Omega))^n, L^2_{\mathbb{C}}(\Omega)) \ .$$

$A_\lambda^{-1/2} \nabla_0{}^*$ is a densely-defined, linear operator in $(L^2_{\mathbb{C}}(\Omega))^n$ with values in $L^2_{\mathbb{C}}(\Omega)$. Further, it follows for $f \in L^2_{\mathbb{C}}(\Omega)$ that

$$\langle \, f \, | \, A_\lambda^{-1/2} \nabla_0{}^* q \, \rangle_2 = \langle \, A_\lambda^{-1/2} f \, | \, \nabla_0{}^* q \, \rangle_2 = \langle \, \nabla_w A_\lambda^{-1/2} f \, | \, q \, \rangle_{2,n}$$

for every $q \in D(\nabla_0{}^*)$ and hence that $f \in D((A_\lambda^{-1/2} \nabla_0{}^*)^*)$ as well as that

$$(A_\lambda^{-1/2} \nabla_0{}^*)^* f = \nabla_w A_\lambda^{-1/2} f \ .$$

As a consequence,

$$(A_\lambda^{-1/2} \nabla_0{}^*)^* = \nabla_w A_\lambda^{-1/2} \ .$$

In particular, $A_\lambda^{-1/2}\nabla_0{}^*$ is closable and

$$\overline{A_\lambda^{-1/2}\nabla_0{}^*} = (\,\nabla_w A_\lambda^{-1/2}\,)^* \in L((L^2_{\mathbb{C}}(\Omega))^n, L^2_{\mathbb{C}}(\Omega))\ .$$

Further, we note that

$$\nabla_w A_\lambda^{-1}\nabla_0{}^* f = \nabla_w A_\lambda^{-1/2} A_\lambda^{-1/2}\nabla_0{}^* f = \nabla_w A_\lambda^{-1/2}\,(\,\nabla_w A_\lambda^{-1/2}\,)^* f$$

for every $f \in D(\nabla_0{}^*)$. Hence it follows that $\overline{\nabla_w A_\lambda^{-1}\nabla_0{}^*}$ is a positive self-adjoint element of $L((L^2_{\mathbb{C}}(\Omega))^n, (L^2_{\mathbb{C}}(\Omega))^n)$. ☐

By help of the previous lemma, we can now estimate the size of the gap around 0 in the spectrum of A and give a representation of the resolvent of $\hat{A}$ in terms of the resolvent of A, i.e., (5.0.9).

Theorem 5.0.20. Let $\bar{p} \in \mathcal{L}$, $C_1, C_2 \in \mathbb{R}$ satisfy $C_2 \geqslant C_1 > 0$ and be such that $C_1 \leqslant \bar{p} \leqslant C_2$ a.e. on Ω. Further, let $j \in \{1, \dots, n\}$ be such that $\mathbf{n}_j \neq 0$. Then the interval

$$J := (\,-C_1\,,\,c^2/(c + C_2)\,)$$

is contained in the resolvent set of $\hat{A}$. In particular for $\lambda \in J$, $(\hat{A} - \lambda)^{-1}$ is given by

$$\begin{aligned}
&(\hat{A} - \lambda)^{-1}(f, g) = \\
&\Big((A_{\bar{p}_\lambda} - \lambda)^{-1} f + \overline{(A_{\bar{p}_\lambda} - \lambda)^{-1}\nabla_0{}^*}\, p_\lambda g\,, \\
&-p_\lambda g + p_\lambda \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f + p_\lambda \overline{\nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1}\nabla_0{}^*}\, p_\lambda g\Big)
\end{aligned} \tag{5.0.9}$$

for all $(f, g) \in L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$, where $\bar{p}_\lambda := \bar{p} + \lambda, p_\lambda := 1/\bar{p}_\lambda$, and $A_{\bar{p}_\lambda}$ is the operator corresponding to $\bar{p}_\lambda$ according to Definition 4.0.7.

Proof. For this, let $\lambda \in J$. Then,

$$0 < \lambda + C_1 \leqslant \bar{p} + \lambda \leqslant \lambda + C_2$$

and $\bar{p}_\lambda := \bar{p} + \lambda \in \mathcal{L}$. Further, we denote by $A_{\bar{p}_\lambda}$ the operator corresponding to $\bar{p}_\lambda$ according to Definition 4.0.7. As a consequence of Theorem 4.0.14, the spectrum of $A_{\bar{p}_\lambda} - \lambda$ is contained in the interval

$$\big[c^2(\lambda + C_2)^{-1} - \lambda, \infty\big)\ ,$$

The inequality

$$c^2(\lambda + C_2)^{-1} - \lambda > 0$$

is equivalent to

$$\left(\lambda + \frac{C_2}{2}\right)^2 - \frac{C_2^2}{4} = \lambda(\lambda + C_2) < c^2 \ .$$

The last is equivalent to

$$-\sqrt{c^2 + \frac{C_2^2}{4}} - \frac{C_2}{2} < \lambda < \sqrt{c^2 + \frac{C_2^2}{4}} - \frac{C_2}{2} \ .$$

We note that

$$\sqrt{c^2 + \frac{C_2^2}{4}} - \frac{C_2}{2} = \frac{c^2}{\sqrt{c^2 + \frac{C_2^2}{4}} + \frac{C_2}{2}} \geqslant \frac{c^2}{c + C_2}$$

and that

$$-\sqrt{c^2 + \frac{C_2^2}{4}} - \frac{C_2}{2} \leqslant -C_2 \leqslant -C_1 \ .$$

Hence it follows that $A_{\bar{p}_\lambda} - \lambda$ is self-adjoint, strictly positive and bijective. We define the bounded linear operator $B \in L(L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n, L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n)$ by

$$\begin{aligned} B(f, g) := (&(A_{\bar{p}_\lambda} - \lambda)^{-1} f + \overline{(A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g\,, \\ &- p_\lambda g + p_\lambda \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f + p_\lambda \overline{\nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g) \end{aligned}$$

for all $(f, g) \in L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$, where $p_\lambda := 1/\bar{p}_\lambda \in L^\infty(\Omega)$. Further, we define the subspace D of $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$ by

$$D := \{(f, g) \in L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n : p_\lambda g \in D(\nabla_0{}^*)\} \ .$$

We note that the subspace

$$\{\bar{p}_\lambda g : g \in C_0^\infty(\Omega, \mathbb{C})\}$$

of $L^2_{\mathbb{C}}(\Omega)$ is dense in $L^2_{\mathbb{C}}(\Omega)$. For the proof, let $f \in L^2_{\mathbb{C}}(\Omega)$. Since $p_\lambda \in L^\infty(\Omega)$, $p_\lambda f \in L^2_{\mathbb{C}}(\Omega)$. Further, since $C_0^\infty(\Omega, \mathbb{C})$ is dense in $L^2_{\mathbb{C}}(\Omega)$, there exists a sequence $f_1, f_2, \ldots$ in $C_0^\infty(\Omega, \mathbb{C})$ such that

$$\lim_{\nu \to \infty} \|f_\nu - p_\lambda f\|_2 = 0 \ .$$

Since for every $\nu \in \mathbb{N}^*$

$$\|\bar{p}_\lambda f_\nu - f\|_2 = \|\bar{p}_\lambda (f_\nu - p_\lambda f)\|_2 \leqslant \|\bar{p}_\lambda\|_\infty \|f_\nu - p_\lambda f\|_2 \ ,$$

it follows that

$$\lim_{\nu \to \infty} \|\bar{p}_\lambda f_\nu - f\|_2 = 0 \ .$$

Hence it follows also that D is dense in $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$. Further, for $(f, g) \in D$, it follows that $B(f, g) \in D(\hat{A})$ and that

$$\begin{aligned}
&\nabla_0{}^*[\, -p_\lambda g + p_\lambda \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f + p_\lambda \overline{\nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g \,] \\
&- \lambda[\, (A_{\bar{p}_\lambda} - \lambda)^{-1} f + \overline{(A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g \,] \\
&= -\nabla_0{}^* p_\lambda g + A_{\bar{p}_\lambda} (A_{\bar{p}_\lambda} - \lambda)^{-1} f + A_{\bar{p}_\lambda} (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*\, p_\lambda g \\
&\quad - \lambda (A_{\bar{p}_\lambda} - \lambda)^{-1} f - \lambda (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*\, p_\lambda g \,] = f
\end{aligned}$$

and that

$$\begin{aligned}
&\nabla_w \,[\, (A_{\bar{p}_\lambda} - \lambda)^{-1} f + \overline{(A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g \,] \\
&- \bar{p}_\lambda [\, -p_\lambda g + p_\lambda \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f + p_\lambda \overline{\nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*}\, p_\lambda g \,] \\
&= \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f + \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*\, p_\lambda g \\
&\quad g - \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} f - \nabla_w (A_{\bar{p}_\lambda} - \lambda)^{-1} \nabla_0{}^*\, p_\lambda g = g \ .
\end{aligned}$$

Hence it follows that

$$(\hat{A} - \lambda) B(f, g) = (f, g) \ . \tag{5.0.10}$$

Further, since D is dense in $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$, for $(f, g) \in L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$, there is a sequence $(f_1, g_1), (f_2, g_2), \dots$ in D that is convergent to (f, g). Since B is bounded, the corresponding sequence $B(f_1, g_1), B(f_2, g_2), \dots$ is convergent to $B(f, g)$. Since $\hat{A}$ is in particular closed, it follows that $B(f, g) \in D(\hat{A})$ and that

$$(\hat{A} - \lambda) B(f, g) = (f, g) \ . \tag{5.0.11}$$

Therefore, $\hat{A} - \lambda$ is surjective and hence also bijective. □

By help of the previous theorem, the next result follows by application of a well-known criterion for the strong resolvent convergence of sequences of self-adjoint operators.

Theorem 5.0.21. Let $\bar{p}_1, \bar{p}_2, \dots$ be a uniformly bounded sequence in $\mathcal{L}$ for which there is $\varepsilon > 0$ such that $\bar{p}_\nu \geqslant \varepsilon$ for all $\nu \in \mathbb{N}^*$ and which converges a.e. pointwise on Ω to $\bar{p}_\infty \in \mathcal{L}$. In addition, let $\hat{A}_1, \hat{A}_2, \dots$ be the associated sequence of self-adjoint linear operators and $\hat{A}_\infty$ be the self-adjoint linear operator associated to $\bar{p}_\infty$. Then

$$s-\lim_{\nu\to\infty} \hat{A}_\nu^{-1} = \hat{A}_\infty^{-1} \ .$$

Proof. By application of Lebesgue's dominated convergence theorem, it follows that

$$\lim_{\nu\to\infty} \|\hat{A}_\nu(u,q) - \hat{A}_\infty(u,q)\| = 0 \ ,$$

for all $(u,q) \in L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$, where $\| \ \|$ denotes the norm on $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$. From this, the statement follows from a well-known criterion for strong resolvent convergence of a sequence of self-adjoint linear operators, e.g., see part (i) of Theorem 9.16 in [19]. □

Corollary 5.0.22. Let $\bar{p}_1, \bar{p}_2, \dots$ be a uniformly bounded sequence in $\mathcal{L}$ for which there is $\varepsilon > 0$ such that $\bar{p}_\nu \geqslant \varepsilon$ for all $\nu \in \mathbb{N}^*$ and which converges a.e. pointwise on Ω to $\bar{p}_\infty \in \mathcal{L}$. In addition, let $A_1, A_2, \dots$ be the associated sequence of self-adjoint linear operators and A_∞ be the self-adjoint linear operator associated to $\bar{p}_\infty$. Finally, let $f \in L^2_{\mathbb{C}}(\Omega)$. Then

$$\lim_{\nu\to\infty} \|A_\nu^{-1} f - A_\infty^{-1} f\|_2 = \lim_{\nu\to\infty} \|p_\nu \nabla_w A_\nu^{-1} f - p_\infty \nabla_w A_\infty^{-1} f\|_{2,n} = 0 \ , \quad (5.0.12)$$

where $p_\nu := 1/\bar{p}_\nu$ for every $\nu \in \mathbb{N}^*$ and $p_\infty := 1/\bar{p}_\infty$.

Proof. By Theorem 5.0.21, it follows that

$$\lim_{\nu\to\infty} \hat{A}_\nu^{-1}(f,0) = \hat{A}_\infty^{-1}(f,0) \ ,$$

where $\hat{A}_1, \hat{A}_2, \dots$ is the associated sequence of self-adjoint linear operators to $\bar{p}_1, \bar{p}_2, \dots$ and $\hat{A}_\infty$ is the self-adjoint linear operator associated to $\bar{p}$. Hence (5.0.12) follows by Theorem 5.0.20. □

6 The one-dimensional case

In the special cases that Ω is given by a non-empty bounded open interval of $\mathbb{R}$, $\hat{A}^{-1}$ can be explicitly calculated. This is somewhat surprising since in this case the corresponding A is a Sturm-Liouville operator and the standard method of calculating its inverse, e.g., see Theorem 8.26 in [19], seems not directly applicable for general $p \in \mathcal{L}$. Still, by a direct calculation of $\hat{A}^{-1}$, one can give an explicit expression of A^{-1} by using (5.0.9).

Theorem 6.0.23. Let $a, b \in \mathbb{R}$ such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\bar{p} \in L^\infty(\Omega)\backslash\{0\}$ be a.e. positive. Then $\hat{A}$ is bijective and has a purely discrete spectrum. In particular, $\hat{A}^{-1}$ is given by

$$(u, q) := \hat{A}^{-1}(f, g)$$

for every $(f, g) \in (L^2_{\mathbb{C}}(I))^2$, where

$$\begin{aligned} u(x) &= \int_a^x g(y)\,dy + \int_a^x \left[\int_y^x \bar{p}(u)\,du\right] f(y)\,dy \\ &\quad + \|\bar{p}\|_1^{-1} \int_a^x \bar{p}(y)\,dy\, \left\{\int_a^b \left[\int_y^b \bar{p}(x)\,dx\right] f(y)\,dy - \int_a^b g(y)\,dy\right\} \\ q(x) &= -\int_a^x f(y)\,dy + \|\bar{p}\|_1^{-1} \left\{\int_a^b \left[\int_y^b \bar{p}(x)\,dx\right] f(y)\,dy - \int_a^b g(y)\,dy\right\} \end{aligned}$$

for every $x \in I$. Also, $\hat{A}^{-1}$ satisfies

$$\|\hat{A}^{-1}\| \leqslant 2\,(b-a)\,\|\bar{p}\|_1^{-1}(\,1 + \|\bar{p}\|_1)^2\ .$$

Proof. For this, we define the derivative operator

$$D_I : C_0^\infty(I, \mathbb{C}) \to L^2_{\mathbb{C}}(I)$$

by $D_I f := f'$ for every $f \in C_0^\infty(I, \mathbb{C})$. In a first step, we prove an auxiliary result. For this, let $f \in L^2_{\mathbb{C}}(I)$ and $h \in C(\bar{I}, \mathbb{C})$ be defined by

$$h(x) := \int_a^x f(y)\,dy$$

for every $x \in I$. Further, let $\varphi \in C_0^\infty(I, \mathbb{C})$. By Fubini's theorem and change of variables, it follows that

$$\langle\, h \,|\, D_I\varphi\,\rangle_2 = \langle\, h \,|\, \varphi'\,\rangle_2 = \int_a^b h^*(x)\,\varphi'(x)\,dx = \int_a^b \left[\,\int_a^x \varphi'(x) f^*(y)\,dy\right] dx$$

$$
\begin{aligned}
&= \int_{\{(x,y)\in\mathbb{R}^2 : a\leqslant x\leqslant b \wedge a\leqslant y\leqslant x\}} \varphi'(x) f^*(y)\, dxdy \\
&= \int_{\{(x,y)\in\mathbb{R}^2 : a\leqslant y\leqslant b \wedge y\leqslant x\leqslant b\}} \varphi'(x) f^*(y)\, dxdy \\
&= \int_a^b \left[\int_y^b \varphi'(x) f^*(y)\, dx \right] dy = -\int_a^b \varphi(y) f^*(y)\, dy = -\langle f \,|\, \varphi \rangle_2
\end{aligned}
$$

and hence that

$$
h \in W^1_{\mathbb{C}}(I) \text{ and } D_I^* h = -f \ .
$$

With the help of the previous auxiliary result, we proceed in the proof of the lemma. For this, we define for every $(f,g) \in (L^2_{\mathbb{C}}(I))^2$, a corresponding $B(f,g) = (u,q)$ by

$$
q(x) := q_0(x) + c \ , \ u(x) := \int_a^x [\, g(y) + \bar{p}(y)(q_0(y) + c)]\, dy \ ,
$$

where

$$
q_0(x) := -\int_a^x f(y)\, dy \ , \ c := -\|\bar{p}\|_1^{-1} \int_a^b [\, g(y) + \bar{p}(y) q_0(y)\,]\, dy
$$

for every $x \in I$. By help of the auxiliary result above, it follows that $(u,q) \in (W^1_{\mathbb{C}}(I) \cap C(\bar{I},\mathbb{C})) \times D(D_I^*)$ and that

$$
D_I^* q = f \ , \ -D_I^* u - \bar{p} q = g + \bar{p} q - \bar{p} q = g \ .
$$

In addition,

$$
\begin{aligned}
u_b &= \int_a^b [\, g(y) + \bar{p}(y)(q_0(y) + c)\,]\, dy \\
&= \int_a^b (\, g(y) + \bar{p}(y) q_0(y)\,)\, dy + c \int_a^b \bar{p}(y)\, dy = 0 \ .
\end{aligned}
$$

As a consequence,

$$
u_a = u_b = 0 \ .
$$

From the last, it follows also that $u \in W^1_{0,\mathbb{C}}(I)$. For the proof, let $\varphi \in C^\infty(\mathbb{R})$ be such that

$$
\varphi(\,(-\infty,0]\,) \subset \{0\} \ , \ \varphi(\,[1,\infty)\,) \subset \{1\} \ , \ \mathrm{Ran}\,\varphi \subset [0,1] \ .
$$

Such a function is easy to construct. For $\nu \in \mathbb{N}^*$, we define $\varphi_\nu \in C^\infty(\mathbb{R})$ by

$$\varphi_\nu(x) := \varphi(\,\nu(x-a)(b-x)-1\,)$$

for every $x \in I$. Then it follows for every $\nu \in \mathbb{N}^*$ satisfying $\nu \geqslant b-a$ and $x \in I$ that

$$\varphi_\nu(x) = \begin{cases} 0 & \text{if } x \in I \setminus (a+\nu^{-2}, b-\nu^{-2}) \\ 1 & \text{if } (x-a)(b-x) \geqslant 2\,\nu^{-1} \end{cases}$$

and hence that $\varphi_\nu \in C_0^\infty(I,\mathbb{R})$ as well as $\mathrm{Ran}(\varphi_\nu) \subset [0,1]$. In particular,

$$\begin{aligned} & |\,(x-a)(b-x)\,\varphi_\nu'(x)\,| \\ & \leqslant 3\nu\,(a+b)\,(x-a)(b-x)\,\cdot|\,\varphi'(\,\nu(x-a)(b-x)-1\,)\,| \\ & \leqslant 3\nu\,(a+b)\,(x-a)(b-x)\cdot\|\varphi'\|_\infty\cdot\chi_{\{x\in I:(x-a)(b-x)\leqslant 2/\nu\}}(x) \\ & \leqslant 6\,(a+b)\cdot\|\varphi'\|_\infty\cdot\chi_{\{x\in I:(x-a)(b-x)\leqslant 2/\nu\}}(x) \end{aligned}$$

for all $x \in I$. An application of Lebesgue's dominated convergence theorem leads to

$$\lim_{\nu\to\infty}\|\varphi_\nu u - u\|_2 = \lim_{\nu\to\infty}\|D_I^*\varphi_\nu u - D_I^* u\|_2 = \lim_{\nu\to\infty}\|\varphi_\nu D_I^* u + \varphi_\nu' u - D_I^* u\|_2 = 0\ .$$

Hence it follows that $u \in W^1_{0,\mathbb{C}}(I)$ and further that $(u,q) \in D(\hat{A})$ and

$$\hat{A}B(f,g) = \hat{A}(u,q) = (f,g)\ .$$

Further, we conclude by Fubini's theorem that

$$\begin{aligned} c &= -\|\bar{p}\|_1^{-1}\int_a^b g(y)\,dy + \|\bar{p}\|_1^{-1}\int_a^b \bar{p}(x)\left[\int_a^x f(y)\,dy\right]dx \\ &= \|\bar{p}\|_1^{-1}\left\{\int_a^b\left[\int_y^b \bar{p}(x)\,dx\right]f(y)\,dy - \int_a^b g(y)\,dy\right\}\ . \end{aligned}$$

This implies that

$$q(x) = -\int_a^x f(y)\,dy + \|\bar{p}\|_1^{-1}\left\{\int_a^b\left[\int_y^b \bar{p}(x)\,dx\right]f(y)\,dy - \int_a^b g(y)\,dy\right\}$$

for every $x \in I$. Further, again by Fubini's theorem, it follows that

$$u(x) = \int_a^x g(y)\,dy + \int_a^x \bar{p}(y)\,q_0(y)\,dy + c\int_a^x \bar{p}(y)\,dy$$

$$
\begin{aligned}
&= \int_a^x g(y)\,dy + \int_a^x \bar{p}(u)\left[\int_a^u f(y)\,dy\right] du + c\int_a^x \bar{p}(y)\,dy \\
&= \int_a^x g(y)\,dy + \int_a^x \left[\int_y^x \bar{p}(u)\,du\right] f(y)\,dy \\
&\quad + \|\bar{p}\|_1^{-1}\int_a^x \bar{p}(y)\,dy\left\{\int_a^b \left[\int_y^b \bar{p}(x)\,dx\right] f(y)\,dy - \int_a^b g(y)\,dy\right\}
\end{aligned}
$$

for every $x \in I$. In addition, by Hoelder's inequality, we conclude that

$$
\begin{aligned}
|u(x)| &\leqslant 2\,(b-a)^{1/2}\,[\,\|\bar{p}\|_1\,\|f\|_2 + \|g\|_2\,] \leqslant 2\,(b-a)^{1/2}\,(\,1 + \|\bar{p}\|_1)\,\|(f,g)\| \\
|q(x)| &\leqslant (b-a)^{1/2}\left[\,2\,\|f\|_2 + \|\bar{p}\|_1^{-1}\|g\|_2\,\right] \\
&\leqslant 2\,(b-a)^{1/2}\,\|\bar{p}\|_1^{-1}\,(\,1 + \|\bar{p}\|_1)\,\|(f,g)\|_2
\end{aligned}
$$

for every $x \in I$. The last implies

$$
\|u\|_2 \leqslant 2\,(b-a)\,(\,1 + \|\bar{p}\|_1)\,\|(f,g)\|\ ,\ \ \|q\|_2 \leqslant 2\,(b-a)\,\|\bar{p}\|_1^{-1}\,(\,1 + \|\bar{p}\|_1)\,\|(f,g)\|_2
$$

and

$$
\|(u,q)\| \leqslant 2\,(b-a)\,\|\bar{p}\|_1^{-1}(\,1 + \|\bar{p}\|_1)^2\,\|(f,g)\|_2\ .
$$

As consequence, by $((L^2_{\mathbb{C}}(I))^2 \to (L^2_{\mathbb{C}}(I))^2, (f,g) \to B(f,g))$, there is defined a compact bounded linear operator B. Since

$$
\hat{A}B(f,g) = (f,g)
$$

for every $(f,g) \in (L^2_{\mathbb{C}}(I))^2$, the bijectivity of $\hat{A}$ follows as well as that $\hat{A}^{-1} = B$. Finally, since $\hat{A}^{-1}$ is compact, $\hat{A}$ has a purely discrete spectrum. □

Corollary 6.0.24. Let $a, b, \bar{p}$ as in Theorem 6.0.23. Then $U_r(0)$, where

$$
r := 2^{-1}(b-a)^{-1}\,\|\bar{p}\|_1\,(\,1 + \|\bar{p}\|_1)^{-2}\ ,
$$

is contained in the resolvent set of $\hat{A}$.

Proof. For this, let $\lambda \in U_r(0)$. Then

$$
\hat{A} - \lambda = (1 - \lambda\,\hat{A}^{-1})\hat{A}\ .
$$

By help of the previous Theorem 6.0.23, it follows that

$$
|\lambda| \cdot \|\hat{A}^{-1}\| \leqslant |\lambda|/r < 1
$$

and hence that $\hat{A} - \lambda$ is bijective. □

Theorem 6.0.25. Let $a, b \in \mathbb{R}$ such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\bar{p}_1, \bar{p}_2 \in L^\infty(\Omega)\backslash\{0\}$ be a.e. positive and $\hat{A}_1, \hat{A}_2$ be the corresponding operators. Then

$$\|\hat{A}_1^{-1} - \hat{A}_2^{-1}\| \leqslant \frac{2(b-a)}{\|\bar{p}_1\|_1} \left(2 + \|\bar{p}_1\|_1 + \|\bar{p}_2\|_1 + \frac{1}{\|\bar{p}_2\|_1}\right) \|\,\bar{p}_2 - \bar{p}_1\|_1 \ .$$

Proof. Proceeds by direct calculation. ☐

Corollary 6.0.26. Let $a, b \in \mathbb{R}$ such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\bar{p}_\infty \in L^\infty(\Omega)\backslash\{0\}$ be a.e. positive. Let $\bar{p}_1, \bar{p}_2, \dots$ be a sequence of a.e. positive elements of $L^\infty(\Omega)\backslash\{0\}$ such that

$$\lim_{\nu\to\infty} \|\bar{p}_\nu - \bar{p}_\infty\|_1 = 0 \ .$$

In addition, let $\hat{A}_1, \hat{A}_2, \dots$ be the associated sequence of self-adjoint linear operators and $\hat{A}_\infty$ be the self-adjoint linear operator associated to $\bar{p}_\infty$. Then

$$\lim_{\nu\to\infty} \|\hat{A}_\nu^{-1} - \hat{A}_\infty^{-1}\| = 0 \ .$$

Proof. The statement is a simple consequence of Theorem 6.0.23. ☐

Corollary 6.0.27. Let $a, b \in \mathbb{R}$ such that $a < b$ and $\Omega := I := (a, b)$ and $f \in L^2_{\mathbb{C}}(I)$. Further, let $\bar{p}_\infty \in L^\infty(\Omega)\backslash\{0\}$ be a.e. positive and $\bar{p}_1, \bar{p}_2, \dots$ be a sequence in $\mathcal{L}$ such that

$$\lim_{\nu\to\infty} \|\bar{p}_\nu - \bar{p}_\infty\|_1 = 0 \ .$$

In addition, let $A_1, A_2, \dots$ be the sequence of self-adjoint linear operators that is associated to $\bar{p}_1, \bar{p}_2, \dots$ and $p_\nu := 1/\bar{p}_\nu$ for $\nu \in \mathbb{N}^*$. Then $A_1^{-1}, A_2^{-1}, \dots$ and $-p_1 D_I^* A_1^{-1}, -p_2 D_I^* A_2^{-1}, \dots$ are convergent in $L(L^2_{\mathbb{C}}(I), L^2_{\mathbb{C}}(I))$ to $B, C \in L(L^2_{\mathbb{C}}(I), L^2_{\mathbb{C}}(I))$, respectively. In particular, B and C are given by

$$\begin{aligned}
(Bf)(x) &= \int_a^x \left[\int_y^x \bar{p}_\infty(u)\,du\right] f(y)\,dy \\
&\qquad + \|\bar{p}_\infty\|_1^{-1} \int_a^x \bar{p}_\infty(y)\,dy \int_a^b \left[\int_y^b \bar{p}_\infty(x)\,dx\right] f(y)\,dy \ , \\
(Cf)(x) &= -\int_a^x f(y)\,dy + \|\bar{p}_\infty\|_1^{-1} \int_a^b \left[\int_y^b \bar{p}_\infty(x)\,dx\right] f(y)\,dy
\end{aligned}$$

for all $x \in I$ and every $f \in L^2_{\mathbb{C}}(I)$.

Proof. The statement is a simple consequence of Theorem 5.0.20 and Theorem 6.0.23. ☐

7 Concluding remarks

It is unclear whether results similar to those of the previous section can be expected to hold in dimensions greater than 1. According to Theorem 5.0.17 and the subsequent example, and differently to the situation in one dimension, $\hat{A}$ is not injective when $\bar{p}$ vanishes on non-empty open subsets of the material. Hence there does not seem to be an obvious candidate for a limit of a sequence of $\hat{A}^{-1}$ that is associated to a sequence in $\mathcal{L}$ approaching such $\bar{p}$. Therefore, it is conceivable that such limits show a wider variety of phenomena than those in one dimension. This problem deserves further study.

A final remark concerns the fact that it cannot be expected that general 'elliptic regularity theorems' hold for operators A corresponding to discontinuous diffusivities p as a consequence of the condition that every element u from the domain of such operator satisfies $p\,\nabla_w u \in D(\nabla_0^*)$. For instance, the source function f in Example 4.0.11 is in $W^k_{\mathbb{C}}(I)$ for every $k \in \mathbb{N}$, but $u = A^{-1}f \notin W^2_{\mathbb{C}}(I)$, where I is the open interval $(-1, 1)$ of $\mathbb{R}$.

8 Appendix

In the following, proofs of the Lemmata 3.0.4, 3.0.6 from Section 3 are given.

Lemma 8.0.28. (**Partial integration**)

$$\langle f|\partial^{\,e_k} g\rangle_2 = -\langle \partial^{\,e_k} f|g\rangle_2$$

for all $(f, g) \in W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega)$ and $k \in \mathbb{N}^*$, where e_k denotes the k-th canonical unit vector of $\mathbb{R}^n$.

Proof. For this, let $k \in \mathbb{N}^*$. We define the sesquilinear form $s : W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega) \to \mathbb{C}$ by

$$s(f, g) := \langle f|\partial^{\,e_k} g\rangle_2 + \langle \partial^{\,e_k} f|g\rangle_2$$

for all $(f, g) \in W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega)$. By the continuity of $\partial^{\,e_k}$, it follows the continuity of s and by partial integration that $s(f, g) = 0$ for all $f \in C_0^\infty(\Omega, \mathbb{C})$ and

$f \in C^{\infty}(\Omega, \mathbb{C}) \cap W^1_{\mathbb{C}}(\Omega)$. Since $C_0^{\infty}(\Omega, \mathbb{C}) \times ((C^{\infty}(\Omega, \mathbb{C}) \cap W^1_{\mathbb{C}}(\Omega))$ is dense in $W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega)$, this implies the vanishing of s and hence the validity of (3.0.4) for all $(f, g) \in W^1_{0,\mathbb{C}}(\Omega) \times W^1_{\mathbb{C}}(\Omega)$. □

Lemma 8.0.29. (Adjoints of gradient operators)

$$(\nabla_0{}^*)^* = \nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)} \ , \ \left(\nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)}\right)^* = \nabla_0{}^* \ .$$

Proof. Since $\nabla_0{}^*$ is densely-defined, it follows that

$$(\nabla_0{}^*)^* = \overline{\nabla_0} \ .$$

For $f \in D(\overline{\nabla_0})$, there exists a sequence $f_1, f_2, \dots$ in $C_0^{\infty}(\Omega, \mathbb{C})$ such that

$$\lim_{\nu \to \infty} \|f_\nu - f\|_2 = 0 \ , \lim_{\nu \to \infty} \|\nabla_0 f_\nu - \overline{\nabla_0} f\|_{2,n} = \lim_{\nu \to \infty} \|\nabla_w f_\nu - \overline{\nabla_0} f\|_{2,n} = 0 \ .$$

Hence it follows that $f \in W^1_{0,\mathbb{C}}(\Omega)$ and that $\overline{\nabla_0} f = \nabla_w f$. As a consequence,

$$\overline{\nabla_0} \subset \nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)} \ .$$

Further, for $f \in W^1_{0,\mathbb{C}}(\Omega)$, there is a sequence $f_1, f_2, \dots$ in $C_0^{\infty}(\Omega, \mathbb{C})$ such that

$$\lim_{\nu \to \infty} \|f_\nu - f\|_2 = 0 \ , \lim_{\nu \to \infty} \|\nabla_0 f_\nu - \nabla_w f\|_2 = 0 \ .$$

Hence it follows that

$$(f, \nabla_w f) \in G(\overline{\nabla_0}) \ .$$

As a consequence,

$$\nabla_w\big|_{W^1_{0,\mathbb{C}}(\Omega)} \subset \overline{\nabla_0} \ .$$

Finally, it follows the validity of (3.0.5, 1). The validity of (3.0.5, 2) is a simple consequence of (3.0.5, 1) and the closedness of $\nabla_0{}^*$. □

9 Acknowledgment

B. Aksoylu would like to thank Center for Computation Technology at Louisiana State University for generous support of the research, and A. Knyazev for his hospitality during a visit at University of Colorado at Denver.

References

[1] Adams R A, Fournier J J F 2003, *Sobolev spaces*, 2nd ed., Academic Press: Amsterdam.

[2] Aksoylu B, Graham I G, Klie H, and Scheichl R 2008, *Towards a rigorously justified algebraic preconditioner for high-contrast diffusion problems*, Comput. Vis. Sci., **11**, 319-331, doi:10.1007/s00791-008-0105-1.

[3] Alcouffe R E, Brandt A, Dendy J E, Painter J W 1981, *The multi–grid methods for the diffusion equation with strongly discontinuous coefficients*, SIAM J. Sci. Stat. Comput., **2**, 430-454.

[4] Bakhvalov N S, Knyazev A V 1990, *A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations*, Soviet Math. Dokl. **41**, 481-485.

[5] Berndt M, Manteuffel T A, McCormick S F, Starke G 2005, *Analysis of first-order system least squares (FOSLS) for elliptic problems with discontinuous coefficients: Part I*, SIAM J. Numer. Anal., **43**, 386-408.

[6] Beyer H R 2007, *Beyond partial differential equations: A course on linear and quasi-linear abstract hyperbolic evolution equations*, Lecture Notes in Math., **1898**, Springer: Berlin.

[7] Braess D 2007, *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*, 3rd ed., Cambridge University Press: Cambridge.

[8] Brannick J, Brezina M, Falgout R, Manteuffel T, McCormick S, Ruge J, Sheehan B, Xu J, Zikatanov L 2006, *Extending the applicability of multigrid methods*, Journal of Physics: Conference Series, **46**, 443-452, SciDAC 2006.

[9] Cai Z, Lazarov R D, Manteuffel T A, McCormick S F 1994, *First-order system least squares for second-order partial differential equations: Part I*, SIAM J. Numer. Anal., **31**, 1785-1802.

[10] Engel K-J, Nagel R 2000, *One-parameter semigroups for linear evolution equations*, Springer: New York.

[11] Kato T 1966, *Perturbation theory for linear operators*, Springer: New York.

[12] Klawonn A, Widlund O B, Dryja M 2002, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., **40**, 159-179.

[13] Knyazev A V 1992, *Iterative solution of PDE with strongly varying coefficients: Algebraic version*, in: Beauwens R, De Groen P (eds) 1992, *Iterative methods in linear algebra*, Elsevier: New York, 85-89.

[14] Knyazev A and Widlund O 2003, *Lavrentiev regularization + Ritz approximation = uniform finite element error estimates for differential equations with rough coefficients*, Math. Comp., **72**, 17-40.

[15] Lions J L 1973, *Perturbations Singulieres Dans Les Problemes Aux Limites Et En Controle Optimal*, Lecture Notes in Math., **323**, Springer: Berlin.

[16] Oswald P 1999, *On the robustness of the BPX-preconditioner with respect to jumps in the coefficients*, Math. Comp., **68**, 633-650.

[17] Pazy A 1983, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York: Springer.

[18] Reed M and Simon B, 1980, 1975, 1978, *Methods of modern mathematical physics*, Volume I, II, IV, Academic: New York.

[19] Weidmann J 1980, *Linear Operators in Hilbert Spaces*, Springer: New York.

[20] Xu J and Zhu Y 2008, *Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients*, Math. Models Methods Appl. Sci., **18**, 77-105.

[21] Zhu Y 2008, *Domain decomposition preconditioners for elliptic equations with jump coefficients*, Numer. Linear Algebra Appl., **15**, 271-289.