## SMALL GAIN THEOREMS FOR LARGE SCALE SYSTEMS AND CONSTRUCTION OF ISS LYAPUNOV FUNCTIONS\*

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Abstract. We consider interconnections of n nonlinear subsystems in the input-to-state stability (ISS) framework. For each subsystem an ISS Lyapunov function is given that treats the other subsystems as independent inputs. A gain matrix is used to encode the mutual dependencies of the systems in the network. Under a small gain assumption on the monotone operator induced by the gain matrix, a locally Lipschitz continuous ISS Lyapunov function is obtained constructively for the entire network by appropriately scaling the individual Lyapunov functions for the subsystems. The results are obtained in a general formulation of ISS, the cases of summation, maximization and separation with respect to external gains are obtained as corollaries.

Key words. Nonlinear systems, input-to-state stability, interconnected systems, large scale systems, Lipschitz ISS Lyapunov function, small gain condition

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1. Introduction. In many applications large scale systems are obtained through the interconnection of a number of smaller components. The stability analysis of such interconnected systems may be a difficult task especially in the case of a large number of subsystems, arbitrary interconnection topologies, and nonlinear subsystems.

One of the earliest tools in the stability analysis of feedback interconnections of nonlinear systems are small gain theorems. Such results have been obtained by many authors starting with [36]. These results are classically built on the notion of  $L^p$  gains, see [3] for a recent, very readable account of the developments in this area. While most small gain results for interconnected systems yield only sufficient conditions, in [3] it has been shown in a behavioral framework how the notion of gains can be modified so that the small gain condition is also necessary for robust stability.

Small gain theorems for large scale systems have been developed, e.g., in [26, 34, 23]. In [26] the notions of connective stability and stabilization are introduced for interconnections of linear systems using the concept of vector Lyapunov functions. In [23] stability conditions in terms of Lyapunov functions of subsystems have been derived. For the linear case characterizations of quadratic stability of large scale interconnections have been obtained in [16]. A common feature of these references is that the gains describing the interconnection are essentially linear. With the introduction of the concept of input-to-state stability in [28], it has become a common approach to consider gains as a nonlinear functions of the norm of the input. In this nonlinear case small gain results have been derived first for the interconnection of two systems

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in [18, 32]. A Lyapunov version of the same result is given in [17]. A general small gain condition for large scale ISS systems has been presented in [6]. Recently, such arguments have been used in the stability analysis of observers [1], in the stability analysis of decentralized model predictive control [22] and in the stability analysis of groups of autonomous vehicles.

During the revision of this paper it came to our attention that, following the first general small gain theorems for networks [21, 33, 5, 8, 7, 6], other generalizations of small gain results based on similar ideas have been obtained very recently using the maximization formulation of ISS: A generalized small gain theorem for output-Lagrange-input-to-output stable systems in network interconnections has been obtained in [19]. In this reference the authors study ISS in the maximization framework and conclude ISS from a small gain condition in the cycle formulation. It has been noted in [8] that in the maximum case the cycle condition is equivalent to the operator condition examined here. An extension of generalized small gain results to retarded functional differential equations based on the more general cycle condition and vector Lyapunov functions has recently been obtained in [20]. In this reference a construction of a Lyapunov function is shown which takes a different approach to the construction of an overall Lyapunov function. This construction depends vitally on the use of the maximum formulation of ISS.

In this paper we present sufficient conditions for the existence of an ISS Lyapunov function for a system obtained as the interconnection of many subsystems. The results are of interest in two ways. First, it is shown that a small gain condition is sufficient for input-to-state stability of the large scale system in the Lyapunov formulation. Secondly, an explicit formula for an overall Lyapunov function is given. As the dimensions of the subsystems are essentially lower than the dimension of their interconnection, finding Lyapunov functions for them may be an easier task than for the whole system.

Our approach is based on the notion of *input-to-state stability* (ISS) introduced in [28] for nonlinear systems with inputs. A system is ISS if, roughly speaking, it is globally asymptotically stable in the absence of inputs (so-called 0-GAS) and if any trajectory eventually enters a ball centered at the equilibrium point and with radius given by a monotone continuous function, the gain, of the size of the input (the so-called *asymptotic gain property*), cf. [31].

The concept of ISS turned out to be particularly well suited to the investigation of interconnections. For example, it is known that cascades of ISS systems are again ISS [28] and small gain results have been obtained. We briefly review the results of [18, 17] in order to explain the motivation for the approach of this paper. Both papers study a feedback interconnection of two ISS systems as represented in Figure 1.1.

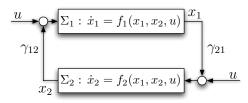


FIG. 1.1. Feedback interconnection of two ISS systems with gains  $\gamma_{12}$  from  $\Sigma_2$  to  $\Sigma_1$  and  $\gamma_{21}$  from  $\Sigma_1$  to  $\Sigma_2$ .

The small gain condition in [18] is that the composition of the gain functions

 $\gamma_{12}, \gamma_{21}$  is less than identity in a robust sense. We denote the composition of functions f, g by  $\circ$ , that is,  $(f \circ g)(x) := f(g(x))$ . The small gain condition then is that if on  $(0, \infty)$  we have

$$(\mathrm{id} + \alpha_1) \circ \gamma_{12} \circ (\mathrm{id} + \alpha_2) \circ \gamma_{21} < \mathrm{id} \,, \tag{1.1}$$

for suitable  $\mathcal{K}_{\infty}$  functions  $\alpha_1, \alpha_2$  then the feedback system is ISS with respect to the external inputs.

In this paper we concentrate on the equivalent definition of ISS in terms of ISS Lyapunov functions [31]. The small gain theorem for ISS Lyapunov functions from [17] states that if on  $(0, \infty)$  the small gain condition

$$\gamma_{12} \circ \gamma_{21} < \mathrm{id} \tag{1.2}$$

is satisfied then an ISS Lyapunov function may be explicitly constructed as follows. Condition (1.2) is equivalent to  $\gamma_{12} < \gamma_{21}^{-1}$  on  $(0, \infty)$ . This permits to construct a function  $\sigma_2 \in \mathcal{K}_{\infty}$  such that  $\gamma_{21} < \sigma_2 < \gamma_{12}^{-1}$  on  $(0, \infty)$ , see Figure 1.2. An ISS Lyapunov function is then defined by scaling and taking the maximum, that is, by setting  $V(x) = \max\{V_1(x_1), \sigma_2^{-1}(V_2(x_2))\}$ . This ISS Lyapunov function describes stability properties of the whole interconnection. In particular, given an input u, it can be seen how fast the corresponding trajectories converge to the neighborhood and how large this neighborhood is.

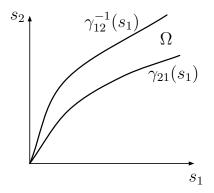


FIG. 1.2. Two gain functions satisfying (1.2).

At first sight the difference between the small gain conditions in (1.1) from [18] and (1.2) from [17] appears surprising. This might lead to the impression that the difference comes from studying the problem in a trajectory based or Lyapunov based framework. This, however, is not the case; the reason for the difference in the conditions is a result of the formulation of the ISS condition. In [18] a summation formulation was used for the trajectory based case. In the maximization formulation of the trajectory case the small gain condition is again (1.2), [6]. In [17] the Lyapunov formulation is investigated using maximization, the corresponding result for summation is Corollary 5.6 below requiring condition (1.1).

In order to generalize the existing results it is useful to reinterpret the approach of [17]: note that the gains may be used to define a matrix

$$\Gamma := \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}$$

which defines in a natural way a monotone operator on  $\mathbb{R}^2_+$ . In this way an alternative characterization of the area between  $\gamma_{21}$  and  $\gamma_{12}^{-1}$  in Figure 1.2 is that it is the area where  $\Gamma(s) < s$  (with respect to the natural ordering in  $\mathbb{R}^2_+$ ). Thus the problem of finding  $\sigma_2$  may be interpreted as the problem of finding a path  $\sigma : r \mapsto (r, \sigma_2(r)), r \in (0, \infty)$  such that  $\Gamma \circ \sigma < \sigma$ .

We generalize this constructive procedure for a Lyapunov function in several directions. First the number of subsystems entering the interconnection will be arbitrary. Secondly, the way in which the gains of subsystem *i* affect subsystem *j* will be formulated in a general manner using the concept of *monotone aggregation functions*. This class of functions allows for a unified treatment of summation, maximization or other ways of formulating ISS conditions. Following the matrix interpretation this leads to a monotone operator  $\Gamma_{\mu}$  on  $\mathbb{R}^{n}_{+}$ . The crucial thing to find is a sufficiently regular path  $\sigma$  such that  $\Gamma_{\mu} \circ \sigma < \sigma$ . This allows for a scaling of the Lyapunov functions for the individual subsystems to obtain one for the large scale system.

Small gain conditions on  $\Gamma_{\mu}$  as in [5, 6] yield sufficient conditions that guarantee that the construction of  $\sigma$  can be performed. However, in [5, 6] the trajectory formulation of ISS has been studied, and the main technical ingredient was, essentially, to prove bounds on  $(id - \Gamma_{\mu})^{-1}$ . The sufficient condition for the existence of the path  $\sigma$  turns out to be the same, but the path itself had not been used in [5, 6]. In fact, the line of argument used there is completely different. It is shown in [24] that the results of [6] also hold for the more general ISS formulation using monotone aggregation functions. The condition requires essentially that the operator is not greater or equal to the identity in a robust sense. The construction of  $\sigma$  then relies on a rather delicate topological argument. What is obvious for the interconnection of two systems is not that clear in higher dimensions. It can be seen that the small gain condition imposed on the interconnection is actually a sufficient condition that allows for the application of the Knaster-Kuratowski-Mazurkiewicz theorem, see [6, 24] for further details. We show in Section 9 how the construction works for three subsystems, but it is fairly clear that this methodology is not something one would like to carry out in higher dimensions. In the maximization formulation a viable alternative is the approach pursued by [20].

The construction of the Lyapunov function is explicit once the scaling function  $\sigma$  is known. Thus to have a really constructive procedure a way of constructing  $\sigma$  is required. We do not study this problem here, but note that based on an algorithm by Eaves [11] it actually possible to turn this mere existence result into a (numerically) constructive method [24, 9]. Using the algorithm by Eaves and the technique of Proposition 8.8, it is then possible to construct such a vector function (but of finite-length) numerically, see [24, Chapter 4]. This will be treated in more detail in future work.

The paper is organized as follows. The next section introduces the necessary notation and basic definitions, in particular the notion of monotone aggregation functions (MAFs) and different formulations of ISS. Section 3 gives some motivating examples that also illustrate the definitions of the Section 2 and explain how different MAFs occur naturally for different problems. In Section 4 we introduce small gain conditions given in terms of monotone operators that naturally appear in the definition of ISS. Section 5 contains the main results, namely the existence of the vector scaling function  $\sigma$  and the construction of an ISS Lyapunov function. In this section we concentrate on strongly connected networks which are easier to deal with from a technical point of view. Once this case has been resolved it is shown in Section 6 how simply connected networks may be treated by studying the strongly connected components.

The actual construction of  $\sigma$  is given in Section 8 to postpone the topological considerations until after applications to interconnected ISS systems have been considered in Section 7. Since the topological difficulties can be avoided in the case n = 3 we treat this case briefly in Section 9 to show a simple construction for  $\sigma$ . Section 10 concludes the paper.

## 2. Preliminaries.

**2.1. Notation and conventions.** Let  $\mathbb{R}$  be the field of real numbers and  $\mathbb{R}^n$  the vector space of real column vectors of length n. We denote the set of nonnegative real numbers by  $\mathbb{R}_+$  and  $\mathbb{R}^n_+ := (\mathbb{R}_+)^n$  denotes the positive orthant in  $\mathbb{R}^n$ . On  $\mathbb{R}^n_+$  the standard partial order is defined as follows. For vectors  $v, w \in \mathbb{R}^n$  we denote

$$v \ge w : \iff v_i \ge w_i \text{ for } i = 1, \dots, n,$$
  

$$v > w : \iff v_i > w_i \text{ for } i = 1, \dots, n,$$
  

$$v \ge w : \iff v \ge w \text{ and } v \ne w.$$

The maximum of two vectors or matrices is to be understood component-wise. By  $|\cdot|$  we denote the 1-norm on  $\mathbb{R}^n$  and by  $S_r$  the induced sphere of radius r in  $\mathbb{R}^n$  intersected with  $\mathbb{R}^n_+$ , which is an (n-1)-simplex. On  $\mathbb{R}^n_+$  we denote by  $\pi_I : \mathbb{R}^n_+ \to \mathbb{R}^{\#I}_+$  the *projection* of the coordinates in  $\mathbb{R}^n_+$  corresponding to the indices in  $I \subset \{1, \ldots, n\}$  onto  $\mathbb{R}^{\#I}$ .

The standard scalar product in  $\mathbb{R}^n$  is denoted by  $\langle \cdot, \cdot \rangle$ . By  $U_{\varepsilon}(x)$  we denote the open ball of radius  $\varepsilon$  around x with respect to the Euclidean norm  $\|\cdot\|$ . The induced operator norm, i.e. the spectral norm, of matrices is also denoted by  $\|\cdot\|$ .

The space of measurable and essentially bounded functions is denoted by  $L^{\infty}$ with norm  $\|\cdot\|_{\infty}$ . To state the stability definitions that we are interested in three sets of comparison functions are used:  $\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+, \gamma \text{ is continuous, strictly}$ increasing, and  $\gamma(0) = 0\}$  and  $\mathcal{K}_{\infty} = \{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}$ . A function  $\beta$  :  $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$  is of class  $\mathcal{KL}$ , if it is of class  $\mathcal{K}$  in the first argument and strictly decreasing to zero in the second argument. We will call a function  $V : \mathbb{R}^N \to \mathbb{R}_+$ proper and positive definite if there are  $\psi_1, \psi_2 \in \mathcal{K}_{\infty}$  such that

$$\psi_1(\|x\|) \le V(x) \le \psi_2(\|x\|), \quad \forall x \in \mathbb{R}^N.$$

A function  $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$  is called *positive definite* if it is continuous and satisfies  $\alpha(r) = 0$  if and only if r = 0.

**2.2. Problem Statement.** We consider a finite set of interconnected systems with state  $x = (x_1^T, \ldots, x_n^T)^T$ , where  $x_i \in \mathbb{R}^{N_i}$ ,  $i = 1, \ldots, n$  and  $N := \sum N_i$ . For  $i = 1, \ldots, n$  the dynamics of the *i*-th subsystem is given by

$$\Sigma_i: \dot{x}_i = f_i(x_1, \dots, x_n, u), \quad x \in \mathbb{R}^N, \ u \in \mathbb{R}^M, \ f_i: \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}.$$
(2.1)

For each *i* we assume unique existence of solutions and forward completeness of  $\Sigma_i$  in the following sense. If we interpret the variables  $x_j, j \neq i$ , and *u* as unrestricted inputs, then this system is assumed to have a unique solution defined on  $[0, \infty)$  for any given initial condition  $x_i(0) \in \mathbb{R}^{N_i}$  and any  $L^{\infty}$ -inputs  $x_j : [0, \infty) \to \mathbb{R}^{N_j}, j \neq i$ , and  $u : [0, \infty) \to \mathbb{R}^M$ . This can be guaranteed for instance by suitable Lipschitz and growth conditions on the  $f_i$ . It will be no restriction to assume that all systems have the same (augmented) external input u.

We write the interconnection of subsystems (2.1) as

$$\Sigma: \dot{x} = f(x, u), \quad f: \mathbb{R}^{N+M} \to \mathbb{R}^N.$$
(2.2)

Associated to such a network is a directed graph, with vertices representing the

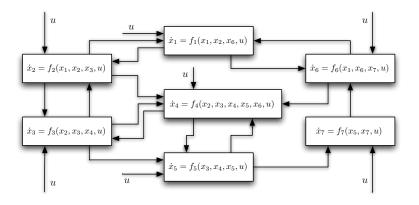


FIG. 2.1. An example of a network of interconnected systems and the associated graph.

subsystems and where the directed edges (i, j) correspond to inputs going from system j to system i, see Figure 2.1. We will call the network strongly connected if its interconnection graph has the same property.

For networks of the type that has been just described we wish to construct Lyapunov functions as they are introduced now.

**2.3.** Stability. An appropriate stability notion to study nonlinear systems with inputs is input-to-state stability, introduced in [28]. The standard definition is as follows.

A forward complete system  $\dot{x} = f(x, u)$  with  $x \in \mathbb{R}^N, u \in \mathbb{R}^M$  is called input-tostate stable if there are  $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$  such that for all initial conditions  $x_0 \in \mathbb{R}^N$ and all  $u \in L^{\infty}(\mathbb{R}_+, \mathbb{R}^M)$  we have

$$\|x(t;x_0,u(\cdot))\| \le \beta(\|x_0\|,t) + \gamma(\|u\|_{\infty}).$$
(2.3)

It is known to be an equivalent requirement to ask for the existence of an ISS Lyapunov function, [30]. These functions can be chosen to be smooth. For our purposes, however, it will be more convenient to have a broader class of functions available for the construction of a Lyapunov function. Thus we will call a function a Lyapunov function candidate, if the following assumption is met.

ASSUMPTION 2.1. The function  $V : \mathbb{R}^N \to \mathbb{R}_+$  is continuous, proper and positive definite and locally Lipschitz continuous on  $\mathbb{R}^N \setminus \{0\}$ . Note that by Rademacher's Theorem (e.g., [12, Theorem 5.8.6, p.281]) locally Lipschitz continuous functions on  $\mathbb{R}^N \setminus \{0\}$  are differentiable almost everywhere in  $\mathbb{R}^N$ .

DEFINITION 2.2. We will call a function satisfying Assumption 2.1 an ISS Lyapunov function for  $\dot{x} = f(x, u)$ , if there exist  $\gamma \in \mathcal{K}$ , and a positive definite function  $\alpha$  such that in all points of differentiability of V we have

$$V(x) \ge \gamma(\|u\|) \implies \nabla V(x) f(x, u) \le -\alpha(\|x\|).$$
(2.4)

ISS and ISS Lyapunov functions are related in the expected manner:

THEOREM 2.3. A system is ISS if and only if it admits an ISS Lyapunov function in the sense of Definition 2.2.

This has been proved for smooth ISS Lyapunov functions in the literature [30]. So the hard converse statement is clear, as it is even possible to find smooth ISS Lyapunov functions, which satisfy Definition 2.2. The sufficiency proof for the Lipschitz continuous case goes along the lines presented in [30, 31] using the necessary tools from nonsmooth analysis, cf. [4, Theorem. 6.3].

Merely continuous ISS Lyapunov functions have been studied in [14, Ch. 3], arising as viscosity supersolutions to certain partial differential inequalities. Here we work with the Clarke generalized gradient  $\partial V(x)$  of V at x. For functions V satisfying Assumption 2.1 Clarke's generalized gradient satisfies for  $x \neq 0$  that

$$\partial V(x) = \operatorname{conv} \left\{ \zeta \in \mathbb{R}^n : \exists x_k \to x : \nabla V(x_k) \text{ exists and } \nabla V(x_k) \to \zeta \right\}.$$
(2.5)

An equivalent formulation to (2.4) is given by

 $V(x) \ge \gamma(\|u\|) \quad \Longrightarrow \quad \forall \zeta \in \partial V(x) \colon \langle \zeta, f(x, u) \rangle \le -\alpha(\|x\|) \,. \tag{2.6}$ 

Note that (2.6) is also applicable in points where V is not differentiable.

The gain  $\gamma$  in (2.3) is in general different from the ISS Lyapunov gain in (2.4). In the sequel we will always assume that gains are of class  $\mathcal{K}_{\infty}$ .

**2.4.** Monotone aggregation. In this paper we concentrate on the construction of ISS Lyapunov functions for the interconnected system  $\Sigma$ . For a single subsystem (2.1), in a similar manner to (2.4), we wish to quantify the combined effect of the inputs  $x_j$ ,  $j \neq i$ , and u on the evolution of the state  $x_i$ . As we will see in the examples given in Section 3 it depends on the system under consideration how this combined effect can be expressed, through the sum of individual effects, using the maximum of individual effects or by other means. In order to be able to give a general treatment of this we introduce the notion of monotone aggregation functions (MAFs).

DEFINITION 2.4. A continuous function  $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$  is called a monotone aggregation function if the following three properties hold

(M1) positivity:  $\mu(s) \ge 0$  for all  $s \in \mathbb{R}^n_+$  and  $\mu(s) > 0$  if  $s \ge 0$ ;

(M2) strict increase<sup>1</sup>: if x < y, then  $\mu(x) < \mu(y)$ ;

(M3) unboundedness: if  $||x|| \to \infty$  then  $\mu(x) \to \infty$ .

The space of monotone aggregation functions is denoted by  $MAF_n$  and  $\mu \in MAF_n^m$ denotes a vector MAF, i.e.,  $\mu_i \in MAF_n$ , for i = 1, ..., m.

A direct consequence of (M2) and continuity is the weaker monotonicity property (M2') monotonicity:  $x \leq y \implies \mu(x) \leq \mu(y)$ .

In [24, 25] MAFs have additionally been required to satisfy another property, (M4) subadditivity:  $\mu(x+y) \leq \mu(x) + \mu(y)$ ,

which we do not need for the constructions provided in this paper, since we take a different approach, see Section 6.

Standard examples of monotone aggregation functions satisfying (M1)—(M4) are

$$\mu(s) = \sum_{i=1}^{n} s_{i}^{l}, \text{ where } l \ge 1, \text{ or } \mu(s) = \max_{i=1,\dots,n} s_{i} \text{ or } \mu(s_{1}, s_{2}, s_{3}, s_{4}) = \max\{s_{1}, s_{2}\} + \max\{s_{3}, s_{4}\}.$$

<sup>&</sup>lt;sup>1</sup>Cf. Assumption (2.10), where for the purposes of this paper (M2) is further restricted.

On the other hand, the following function is not a MAF, since (M1) and (M3) are not satisfied;  $\nu(s) = \prod_{i=1}^{n} s_i$ .

Using this definition we can define a notion of ISS Lyapunov function for systems with multiple inputs. In this case  $\Sigma_i$  in (2.1) will have several gains  $\gamma_{ij}$  corresponding to the inputs  $x_j$ . For notational simplicity, we will include the gain  $\gamma_{ii} \equiv 0$  throughout this paper. The following definition requires only Lipschitz continuity of the Lyapunov function.

DEFINITION 2.5. Consider the interconnected system (2.2) and assume that for each subsystem  $\Sigma_j$  there is a given function  $V_j : \mathbb{R}^{N_j} \to \mathbb{R}_+$  satisfying Assumption 2.1.

For i = 1, ..., n the function  $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$  is called an ISS Lyapunov function for  $\Sigma_i$ , if there exist  $\mu_i \in MAF_{n+1}, \gamma_{ij} \in \mathcal{K}_{\infty} \cup \{0\}, j \neq i, \gamma_{iu} \in \mathcal{K} \cup \{0\}$ , and a positive definite function  $\alpha_i$  such that at all points of differentiability of  $V_i$ 

$$V_i(x_i) \ge \mu_i \left( \gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(||u||) \right)$$
  
$$\implies \nabla V_i(x_i) f_i(x, u) \le -\alpha_i(||x_i||).$$
(2.7)

The functions  $\gamma_{ij}$  and  $\gamma_{iu}$  are called ISS Lyapunov gains.

Several examples of ISS Lyapunov functions are given in the next section.

Let us call  $x_j$ ,  $j \neq i$ , the *internal inputs* to  $\Sigma_i$  and u the *external input*. Note that the role of functions  $\gamma_{ij}$  and  $\gamma_{iu}$  is essentially to indicate whether there is any influence of different inputs on the corresponding state. In case  $f_i$  does not depend on  $x_j$  there is no influence of  $x_j$  on the state of  $\Sigma_i$ . In this case we define  $\gamma_{ij} \equiv 0$ , in particular always  $\gamma_{ii} \equiv 0$ . This allows us to collect the internal gains into a matrix

$$\Gamma := (\gamma_{ij})_{i,j=1,\dots,n} \,. \tag{2.8}$$

If we add the external gains as the last column into this matrix then we denote it by  $\overline{\Gamma}$ . The function  $\mu_i$  describes how the internal and external gains interactively enter in a common influence on  $x_i$ . The above definition motivates the introduction of the following nonlinear map

$$\overline{\Gamma}_{\mu}: \mathbb{R}^{n+1}_{+} \to \mathbb{R}^{n}_{+}, \quad \begin{bmatrix} s_{1} \\ \vdots \\ s_{n} \\ r \end{bmatrix} \mapsto \begin{bmatrix} \mu_{1}(\gamma_{11}(s_{1}), \dots, \gamma_{1n}(s_{n}), \gamma_{1u}(r)) \\ \vdots \\ \mu_{n}(\gamma_{n1}(s_{1}), \dots, \gamma_{nn}(s_{n}), \gamma_{nu}(r)) \end{bmatrix}.$$
(2.9)

Similarly we define  $\Gamma_{\mu}(s) := \overline{\Gamma}_{\mu}(s, 0)$ . The matrices  $\Gamma$  and  $\overline{\Gamma}$  are from now on referred to as gain matrices,  $\Gamma_{\mu}$  and  $\overline{\Gamma}_{\mu}$  as gain operators.

REMARK 2.6 (general assumption). Given  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$  and  $\mu \in MAF^n$ , we will from now on assume that  $\Gamma$  and  $\mu$  are compatible in the following sense: For each i = 1, ..., n, let  $I_i$  denote the set of indices corresponding to the nonzero entries in the ith row of  $\Gamma$ . Then it is understood that also the restriction of  $\mu_i$  to the indices  $I_i$  satisfies (M2), i.e.,

$$\mu_i(x|_{I_i}) < \mu_i(y|_{I_i}) \quad if \quad x|_{I_i} < y|_{I_i}.$$
(2.10)

In particular we assume that the function

$$s \mapsto \mu(s_1, \ldots, s_n, 0), \quad s \in \mathbb{R}^n_+,$$

for  $\mu \in MAF_{n+1}$  satisfies (M2). Note that (M1) and (M3) are automatically satisfied.

The examples in the next section show explicitly how the introduced functions, matrices and operators may look like for some particular cases. Clearly, the gain operators will have to satisfy certain conditions if we want to be able to deduce that (2.2) is ISS with respect to external inputs, see Section 5.

**3. Examples for monotone aggregation.** In this section we show how different MAFs may appear in different applications, for further examples see [10]. We begin with a purely academic example and discuss linear systems and neural networks later in this section. Consider the system

$$\dot{x} = -x - 2x^3 + \frac{1}{2}(1 + 2x^2)u^2 + \frac{1}{2}w$$
(3.1)

where  $x, u, w \in \mathbb{R}$ . Take  $V(x) = \frac{1}{2}x^2$  as a Lyapunov function candidate. It is easy to see that if  $|x| \ge u^2$  and  $|x| \ge |w|$  then

$$\dot{V} \le -x^2 - 2x^4 + \frac{1}{2}x^2(1+2x^2) + \frac{1}{2}x^2 = -x^4 < 0$$

if  $x \neq 0$ . The conditions  $|x| \ge u^2$  and  $|x| \ge |w|$  translate into  $|x| \ge \max\{u^2, |w|\}$  and in terms of V this becomes

$$V(x) \ge \max\{u^4/2, w^2/2\} \implies \dot{V}(x) \le -x^4$$

This is a Lyapunov ISS estimate where the gains are aggregated using a maximum, i.e., in this case we can take  $\mu(s_1, s_2) = \max\{s_1, s_2\}$  and  $\gamma_u(r) = r^4/2$  and  $\gamma_w(r) = r^2/2$ .

Note that there is a certain arbitrariness in the choice of  $\mu$  and  $\gamma_{ij}$ . In the example one could as well take  $\gamma_u(r) = \gamma_w(r) = r$  and  $\mu(s_1, s_2) = \max\{s_1^4/2, s_2^2/2\}$ , giving exactly the same condition, but different gains and a different monotone aggregation function. At the end of the day the small gain condition comes down to mapping properties of  $\Gamma_{\mu}$ . Different choices of  $\Gamma$  and  $\mu$  may lead to the same operator  $\Gamma_{\mu}$ . However, as we will see at a later stage, certain choices of  $\mu$  can be computationally more convenient than others. In particular, if we can choose  $\mu = \max$ , the task of checking the small gain condition reduces to checking a cycle condition, cf. Section 8.4.

3.1. Linear systems. Consider linear interconnected systems

$$\Sigma_i: \ \dot{x}_i = A_i x_i + \sum_{j=1}^n \Delta_{ij} x_j + B_i u_i, \quad i = 1, \dots, n,$$
(3.2)

with  $x_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{M_i}$ , and matrices  $A_i, B_i, \Delta_{ij}$  of appropriate dimensions. Each system  $\Sigma_i$  is ISS from  $(x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_n^T, u_i^T)^T$  to  $x_i$  if and only if  $A_i$  is Hurwitz. It is known that  $A_i$  is Hurwitz if and only if for any given symmetric positive definite  $Q_i$  there is a unique symmetric positive definite solution  $P_i$  of  $A_i^T P_i + P_i A_i =$  $-Q_i$ , see, e.g., [15, Cor. 3.3.47 and Rem. 3.3.48, p.284f]. Thus we choose the Lyapunov function  $V_i(x_i) = x_i^T P_i x_i$ , where  $P_i$  is the solution corresponding to a symmetric positive definite  $Q_i$ . In this case, along trajectories of the autonomous system

$$\dot{x}_i = A_i x_i$$

we have

$$\dot{V}_i = x_i^T P_i A_i x_i + x_i^T A_i^T P_i x_i = -x_i^T Q_i x_i \le -c_i \|x_i\|^2$$

for  $c_i := \lambda_{min}(Q_i) > 0$ , the smallest eigenvalue of  $Q_i$ . For system (3.2) we obtain

$$\dot{V}_{i} = 2x_{i}^{T}P_{i}\left(A_{i}x_{i} + \sum_{j \neq i} \Delta_{ij}x_{j} + B_{i}u_{i}\right)$$

$$\leq -c_{i}\|x_{i}\|^{2} + 2\|x_{i}\|\|P_{i}\|\left(\sum_{j \neq i} \|\Delta_{ij}\|\|x_{j}\| + \|B_{i}\|\|u_{i}\|\right) \leq -\varepsilon c_{i}\|x_{i}\|^{2}, \qquad (3.3)$$

where the last inequality (3.3) is satisfied for a given  $0 < \varepsilon < 1$  if

$$\|x_i\| \ge \frac{2\|P_i\|}{c_i(1-\varepsilon)} \Big(\sum_{j \ne i} \|\Delta_{ij}\| \|x_j\| + \|B_i\| \|u\|\Big)$$
(3.4)

with  $u := (u_1^T, \ldots, u_n^T)^T$ . To write this implication in the form (2.7) we note that  $\lambda_{\min}(P_i) \|x_i\|^2 \leq V_i(x_i) \leq \lambda_{\max}(P_i) \|x_i\|^2$ . Let us denote  $a_i^2 = \lambda_{\min}(P_i)$ ,  $b_i^2 = \lambda_{\max}(P_i) = \|P_i\|$ , then the inequality (3.4) is satisfied if

$$||P_i|| \cdot ||x_i||^2 \ge V_i(x_i) \ge ||P_i||^3 \left(\frac{2}{c_i(1-\varepsilon)}\right)^2 \left(\sum_{j \ne i} \frac{||\Delta_{ij}||}{a_j} \sqrt{V_j(x_j)} + ||B_i|| ||u||\right)^2.$$

This way we see that the function  $V_i$  is an ISS Lyapunov function for  $\Sigma_i$  with gains given by

$$\gamma_{ij}(s) = \left(\frac{2b_i^3}{c_i(1-\varepsilon)} \frac{\|\Delta_{ij}\|}{a_j}\right) \sqrt{s}$$

for  $i = 1, \ldots, n, i \neq j$ , and

$$\gamma_{iu}(s) = \frac{2\|B_i\|b_i^3}{c_i(1-\varepsilon)} s$$

for  $i = 1, \ldots, n$ , and  $s \ge 0$ . Further we have

$$\mu_i(s,r) = \left(\sum_{j=1}^n s_j + r\right)^2$$

for  $s \in \mathbb{R}^n_+$  and  $r \in \mathbb{R}_+$ . This  $\mu_i$  satisfies (M1), (M2), and (M3), but not (M4). By defining  $\gamma_{ii} \equiv 0$  for i = 1, ..., n we can write

$$\overline{\Gamma} = \begin{pmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} & \gamma_{1u} \\ \gamma_{21} & \ddots & \cdots & \gamma_{2n} & \gamma_{2u} \\ \vdots & & \ddots & \vdots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{n,n-1} & 0 & \gamma_{nu} \end{pmatrix}$$

and have

$$\overline{\Gamma}_{\mu}(s,r) = \begin{pmatrix} \left(\frac{2b_1^3}{c_1(1-\varepsilon)}\right)^2 \left(\sum_{j\neq 1} \frac{\|\Delta_{1j}\|}{a_j} \sqrt{s_j} + \|B_1\|r\right)^2 \\ \vdots \\ \left(\frac{2b_n^3}{c_n(1-\varepsilon)}\right)^2 \left(\sum_{j\neq n} \frac{\|\Delta_{nj}\|}{a_j} \sqrt{s_j} + \|B_n\|r\right)^2 \end{pmatrix}.$$
(3.5)

Interestingly, the choice of quadratic Lyapunov functions for the subsystems naturally leads to a nonlinear mapping  $\overline{\Gamma}_{\mu}$  with a useful homogeneity property, see Proposition 7.1.

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**3.2.** Neural networks. As the next example consider a Cohen-Grossberg neural network as in [35]. The dynamics of each neuron is given by

NN<sub>i</sub>: 
$$\dot{x}_i(t) = -a_i(x_i(t)) \Big( b_i(x_i(t)) - \sum_{j=1}^n t_{ij} s_j(x_j(t)) + J_i \Big),$$
 (3.6)

 $i = 1, ..., n, n \ge 2$ , where  $x_i$  denotes the state of the *i*-th neuron and  $a_i$  is a strictly positive amplification function. As in [35] we assume that the fixed point is shifted to the origin. Then the function  $b_i$  typically satisfies the sign condition  $b_i(x_i)x_i \ge 0$ and satisfies furthermore  $|b_i(x_i)| > \tilde{b}_i(|x_i|)$  for some  $\tilde{b}_i \in \mathcal{K}_{\infty}$ . The activation function  $s_i$  is typically assumed to be sigmoid. The matrix  $T = (t_{ij})_{i,j=1,...,n}$  describes the interconnection of neurons in the network and  $J_i$  is a given constant input from outside. However for our consideration we allow  $J_i$  to be an arbitrary measurable function in  $L_{\infty}$ .

In applications the matrix T is usually the result of training using some learning algorithm and appropriate training data. The specifics depend on the type of network architecture and learning algorithm chosen and on the particular application. Such considerations are beyond the scope of the current paper. We simply assume that Tis given and concern ourselves solely with stability considerations.

Note that for any sigmoid function  $s_i$  there exists a  $\gamma_i \in \mathcal{K}$  such that  $|s_i(x_i)| < \gamma_i(|x_i|)$ . Following [35] we assume  $0 < \underline{\alpha}_i < a_i(x_i) < \overline{\alpha}_i, \underline{\alpha}_i, \overline{\alpha}_i \in \mathbb{R}$ .

Recall the triangle inequality for  $\mathcal{K}_{\infty}$ -functions: For any  $\gamma, \rho \in \mathcal{K}_{\infty}$  and any  $a, b \geq 0$  it holds

$$\gamma(a+b) \le \gamma \circ (\mathrm{id} + \rho)(a) + \gamma \circ (\mathrm{id} + \rho^{-1})(b).$$

We claim that  $V_i(x_i) := |x_i|$  is an ISS Lyapunov function for NN<sub>i</sub> in (3.6). Fix an arbitrary function  $\rho \in \mathcal{K}_{\infty}$  and some  $\varepsilon$  satisfying  $\underline{\alpha}_i > \varepsilon > 0$ . Then by the triangle inequality we have

$$\begin{aligned} |x_i| > \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho) \left( \frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} \sum_{j=1}^n |t_{ij}| \gamma_j(|x_j|) \right) + \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho^{-1}) \left( \frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} |J_i| \right) \\ \\ \tilde{b}_i^{-1} \left( \frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} \left( \sum_{j=1}^n |t_{ij}| \gamma_j(|x_j|) + |J_i| \right) \right) \\ \implies \dot{V}_i = -a_i(x_i) \left( |b_i(x_i)| - \operatorname{sign} x_i \sum_{j=1}^n t_{ij} s_j(x_j) + \operatorname{sign} x_i J_i \right) < -\varepsilon |b_i(x_i)| \end{aligned}$$

In this case we have

$$\mu_i(s,r) = \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho)(s_1 + \dots + s_n) + \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho^{-1})(r)$$

additive with respect to the external input and

$$\gamma_{ij} = \frac{\overline{\alpha}_i |t_{ij}|}{\underline{\alpha}_i - \varepsilon} \gamma_j(|x_j|), \quad \gamma_{iu} = \frac{\overline{\alpha}_i \mathrm{id}}{\underline{\alpha}_i - \varepsilon}.$$

The MAF  $\mu_i$  satisfies (M1), (M2), and (M3). It satisfies (M4) if and only if  $(\tilde{b}_i)^{-1}$  is subadditive.

4. Monotone Operators and generalized small gain conditions. In Section 2.4 we saw that in the ISS context the mutual influence between subsystems (2.1) and the influence from external inputs to the subsystems can be quantified by the gain matrices  $\Gamma$  and  $\overline{\Gamma}$  and gain operators  $\Gamma_{\mu}$  and  $\overline{\Gamma}_{\mu}$ . The interconnection structure of the subsystems naturally leads to a weighted, directed graph, where the weights are the nonlinear gain functions, and the vertices are the subsystems. There is an edge from the vertex *i* to the vertex *j* if and only if there is an influence of the state  $x_i$  on the state  $x_j$ , i.e., there is a nonzero gain  $\gamma_{ji}$ .

Connectedness properties of the interconnection graph together with mapping properties of the gain operators will yield a generalized small gain condition. In essence we need a nonlinear version of a Perron vector for the construction of a Lyapunov function for the interconnected system. This will be made rigorous in the sequel. But first we introduce some further notation.

The adjacency matrix  $A_{\Gamma} = (a_{ij})$  of a matrix  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$  is defined by  $a_{ij} = 0$  if  $\gamma_{ij} \equiv 0$  and  $a_{ij} = 1$  otherwise. Then  $A_{\Gamma} = (a_{ij})$  is also the adjacency matrix of the graph representing an interconnection.

We say that a matrix  $\Gamma$  is *primitive*, *irreducible* or *reducible* if and only if  $A_{\Gamma}$  is primitive, irreducible or reducible, respectively. Recall (and see [2] for more on this subject) that a nonnegative matrix A is

- primitive if there exists a  $k \ge 1$  such that  $A^k$  is positive;
- *irreducible* if for every pair (i, j) there exists a  $k \ge 1$  such that the (i, j)th entry of  $A^k$  is positive; obviously, primitivity implies irreducibility;
- *reducible* if it is not irreducible.

A network or a graph is strongly connected if and only if the associated adjacency matrix is irreducible, see also [2].

For  $\mathcal{K}_{\infty}$  functions  $\alpha_1, \ldots, \alpha_n$  we define a diagonal operator  $D: \mathbb{R}^n_+ \to \mathbb{R}^n_+$  by

$$D(s) := (s_1 + \alpha_1(s_1), \dots, s_n + \alpha_n(s_n))^T, \quad s \in \mathbb{R}^n_+.$$
(4.1)

For an operator  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ , the condition  $T \not\geq id$  means that for all  $s \neq 0$ ,  $T(s) \not\geq s$ . In words, at least one component of T(s) has to be strictly less than the corresponding component of s.

DEFINITION 4.1 (Small gain conditions). Let a gain matrix  $\Gamma$  and a monotone aggregation  $\mu$  be given. The operator  $\Gamma_{\mu}$  is said to satisfy the small gain condition (SGC), if

$$\Gamma_{\mu} \geq id,$$
 (SGC)

Furthermore,  $\Gamma_{\mu}$  satisfies the strong small gain condition (sSGC), if there exists a D as in (4.1) such that

$$D \circ \Gamma_{\mu} \geq id.$$
 (sSGC)

It is not difficult to see that (sSGC) can equivalently be stated as

$$\Gamma_{\mu} \circ D \ngeq \text{id.}$$
 (sSGC')

Also for (sSGC) or (sSGC') to hold it is sufficient to assume that the function  $\alpha_1, \ldots, \alpha_n$  are all identical. This can be seen by defining  $\alpha(s) := \min_i \alpha_i(s)$ . We abbreviate this in writing  $D = \text{diag}(\text{id} + \alpha)$  for some  $\alpha \in \mathcal{K}_{\infty}$ .

For maps  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  we define the following sets:

$$\Omega(T) := \{ s \in \mathbb{R}^n_+ : T(s) < s \} = \bigcap_{i=1}^n \Omega_i(T), \text{ where}$$
$$\Omega_i(T) := \{ s \in \mathbb{R}^n_+ : T(s)_i < s_i \}.$$

If no confusion arises we will omit the reference to T. Topological properties of the introduced sets are related to the small gain conditions (SGC), cf. also [5, 6, 25]. They will be used in the next section for the construction of an ISS Lyapunov function for the interconnection.

5. Lyapunov functions. In this section we present the two main results of the paper. The first is a topological result on the existence of a jointly unbounded path in the set  $\Omega$ , provided that  $\Gamma_{\mu}$  satisfies the small gain condition. This path will be crucial in the construction of a Lyapunov function, which is the second main result of this section.

DEFINITION 5.1. A continuous path  $\sigma \in \mathcal{K}_{\infty}^{n}$  will be called an  $\Omega$ -path with respect to  $\Gamma_{\mu}$  if

- (i) for each i, the function  $\sigma_i^{-1}$  is locally Lipschitz continuous on  $(0,\infty)$ ; (ii) for every compact set  $K \subset (0,\infty)$  there are constants 0 < c < C such that for all i = 1, ..., n and all points of differentiability of  $\sigma_i^{-1}$  we have

$$0 < c \le (\sigma_i^{-1})'(r) \le C, \quad \forall r \in K;$$

$$(5.1)$$

(iii)  $\sigma(r) \in \Omega(\Gamma_{\mu})$  for all r > 0, i.e.

$$\Gamma_{\mu}(\sigma(r)) < \sigma(r), \quad \forall r > 0.$$
 (5.2)

Now we can state the first of our two main results, which regards the existence of  $\Omega$ -paths.

THEOREM 5.2. Let  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$  be a gain matrix and  $\mu \in \mathrm{MAF}_{n}^{n}$ . Assume that one of the following assumptions is satisfied

- (i)  $\Gamma_{\mu}$  is linear and the spectral radius of  $\Gamma_{\mu}$  is less than one;
- (ii)  $\Gamma$  is irreducible and  $\Gamma_{\mu} \geq id$ ;
- (iii)  $\mu = \max and \Gamma_{\mu} \geq id;$
- (iv) alternatively assume that  $\Gamma_{\mu}$  is bounded, i.e.,  $\Gamma \in ((\mathcal{K} \setminus \mathcal{K}_{\infty}) \cup \{0\})^{n \times n}$ , and satisfies  $\Gamma_{\mu} \geq id$ .

Then there exists an  $\Omega$ -path  $\sigma$  with respect to  $\Gamma_{\mu}$ .

We will postpone the proof of this rather topological result to Section 8 and reap the fruits of Theorem 5.2 first. Note, however, that for (iii) there exists a "cycle gain < id"-type equivalent formulation, cf. Theorem 8.14 and see [21, 33, 6, 20].

In addition to the above result, the existence of  $\Omega$ -paths can also be asserted for reducible  $\Gamma$  and  $\Gamma$  with mixed, bounded and unbounded, class  $\mathcal{K}$  entries, see Theorem 8.12 and Proposition 8.13, respectively.

THEOREM 5.3. Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$ , the corresponding gain matrix is given by (2.8), and  $\mu = (\mu_1, \ldots, \mu_n)^T$  is given by (2.7). Assume there are an  $\Omega$ -path  $\sigma$  with respect to  $\Gamma_{\mu}$  and a function  $\varphi \in \mathcal{K}_{\infty}$  such that

$$\overline{\Gamma}_{\mu}(\sigma(r),\varphi(r)) < \sigma(r), \quad \forall \ r > 0$$
(5.3)

is satisfied, then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)).$$
(5.4)

In particular, for all points of differentiability of V we have the implication

$$V(x) \ge \max\{\varphi^{-1}(\gamma_{iu}(||u||)) \mid i = 1, \dots n\} \implies \nabla V(x)f(x, u) \le -\alpha(||x||), \quad (5.5)$$

where  $\alpha$  is a suitable positive definite function.

Note that by construction the Lyapunov function V is not smooth, even if the functions  $V_i$  for the subsystems are. This is why it is appropriate in this framework to consider Lipschitz continuous Lyapunov functions, which are differentiable almost everywhere.

*Proof.* We will show the assertion in the Clarke gradient sense. For x = 0 there is nothing to show. So let  $0 \neq x = (x_1^T, \ldots, x_n^T)^T$ . Denote by I the set of indices i for which

$$V(x) = \sigma_i^{-1}(V_i(x_i)) \ge \max_{j \ne i} \sigma_j^{-1}(V_j(x_j)).$$
(5.6)

Then  $x_i \neq 0$ , for  $i \in I$ . Also as V is obtained through maximization we have because of [4, p.83] that

$$\partial V(x) \subset \operatorname{conv}\left\{\bigcup_{i \in I} \partial [\sigma_i^{-1} \circ V_i \circ \pi_i](x)\right\}.$$
 (5.7)

Fix  $i \in I$  and assume without loss of generality i = 1. Then if we assume  $V(x) \geq \max_{i=1,...,n} \{\varphi^{-1}(\gamma_{iu}(||u||))\}$  it follows in particular that  $\gamma_{1u}(||u||) \leq \varphi(V(x))$ . Using the abbreviation r := V(x), denoting the first component of  $\overline{\Gamma}_{\mu}$  by  $\overline{\Gamma}_{\mu,1}$  and using assumption (5.3) we have

$$\begin{aligned} V_{1}(x_{1}) &= \sigma_{1}(r) > \overline{\Gamma}_{\mu,1}(\sigma(r),\varphi(r)) \\ &= \mu_{1} \left[ \gamma_{11}(\sigma_{1}(r)), \dots, \gamma_{1n}(\sigma_{n}(r)), \varphi(r) \right] \\ &\geq \mu_{1} \left[ \gamma_{11}(\sigma_{1}(r)), \dots, \gamma_{1n}(\sigma_{n}(r)), \gamma_{1u}(\|u\|) \right] \\ &= \mu_{1} \left[ \gamma_{11} \circ \sigma_{1} \circ \sigma_{1}^{-1}(V_{1}(x_{1})), \dots, \gamma_{1n} \circ \sigma_{n} \circ \sigma_{1}^{-1}(V_{1}(x_{1})), \gamma_{1u}(\|u\|) \right] \\ &\geq \mu_{1} \left[ \gamma_{11} \circ V_{1}(x_{1}), \dots, \gamma_{1n} \circ V_{n}(x_{n}), \gamma_{1u}(\|u\|) \right], \end{aligned}$$

where we have used (5.6) and (M2') in the last inequality. Thus the ISS condition (2.7) is applicable and we have for all  $\zeta \in \partial V_1(x_1)$  that

$$\langle \zeta, f_1(x, u) \rangle \le -\alpha_1(\|x_1\|) . \tag{5.8}$$

By the chain rule for Lipschitz continuous functions [4, Theorem 2.5] we have

$$\partial(\sigma_i^{-1} \circ V_i)(x_i) \subset \{ c\zeta : c \in \partial \sigma_i^{-1}(y), y = V_i(x_i), \zeta \in \partial V_i(x_i) \}.$$

Note that in the previous equation the number c is bounded away from zero because of (5.1). We set for  $\rho > 0$ 

$$\tilde{\alpha}_i(\rho) := c_{\rho,i} \,\alpha_i(\rho) > 0 \,,$$

where  $c_{\rho,i}$  is the constant corresponding to the set  $K := \{x_i \in \mathbb{R}^{N_i} : \rho/2 \leq \|x_i\| \leq 2\rho\}$  given by (5.1) in the definition of an  $\Omega$ -path. With the convention  $x = (x_1^T, \ldots, x_n^T)^T$  we now define for r > 0

$$\alpha(r) = \min\{\tilde{\alpha}_i(\|x_i\|) \mid \|x\| = r, V(x) = \sigma_i^{-1}(V_i(x_i)))\} > 0.$$

Here we have used that for a given r > 0 and ||x|| = r the norm of  $||x_i||$  such that  $V(x) = \sigma_i^{-1}(V_i(x_i)))$  is bounded away from 0.

It now follows from (5.8) that if  $V(x) \ge \max_{i=1,...,n} \{\varphi^{-1}(\gamma_{iu}(||u||))\}$ , then we have for all  $\zeta \in \partial [\sigma_1^{-1} \circ V_1](x_1)$  that

$$\langle \zeta, f_1(x, u) \rangle \le -\alpha(\|x\|) . \tag{5.9}$$

In particular, the right hand side depends on x and not only on  $x_1$ . The same argument applies for all  $i \in I$ . Now for any  $\zeta \in \partial V(x)$  we have by (5.7) that  $\zeta = \sum_{i \in I} \lambda_i c_i \zeta_i$  for suitable  $\lambda_i \geq 0$ ,  $\sum_{i \in I} \lambda_i = 1$  and with  $\zeta_i \in \partial (V_i \circ \pi_i)(x)$  and  $c_i \in \partial \sigma_i^{-1}(V_i(x_i))$ . It follows that

$$\begin{split} \langle \zeta, f(x,u) \rangle &= \sum_{i \in I} \lambda_i \langle c_i \zeta_i, f(x,u) \rangle = \sum_{i \in I} \lambda_i \langle c_i \pi_i(\zeta_i), f_i(x,u) \rangle \\ &\leq -\sum_{i \in I} \lambda_i \alpha(\|x\|) = -\alpha(\|x\|) \,. \end{split}$$

This shows the assertion.  $\Box$ 

In the absence of external inputs, ISS is the same as 0-GAS (cf. [29, 30, 31]). We note the following consequence in the case that only global asymptotic stability is of interest.

COROLLARY 5.4 (0-GAS for strongly interconnected networks). In the setting of Theorem 5.3, assume that the external inputs satisfy  $u \equiv 0$  and that the network of interconnected systems is strongly connected. If  $\Gamma_{\mu} \geq id$  then the network is 0-GAS.

*Proof.* By Theorem 5.2 (ii) there exists an  $\Omega$ -path and a nonsmooth Lyapunov for the network is given by (5.4), hence the origin of the externally unforced composite system is GAS.  $\Box$ 

REMARK 5.5. At first sight it might seem that the previous corollary is stronger than [18, Cor. 2.1], as no robustness term D is needed in the assumptions. However, the result here is formulated for Lyapunov functions whereas the result in [18] is based on the trajectory formulation of ISS in summation form. The proof in the trajectory version essentially requires bounds on  $(id - \Gamma_{\mu})^{-1}$ , which relies heavily on D unless  $\mu = \max$ , [18, 6, 24]. In contrast, for 0-GAS the D is not needed in the Lyapunov setting, because for irreducible  $\Gamma$  it is possible to construct the path  $\sigma$  without D by Theorem 5.2 (ii).

We now specialize the Theorem 5.3 to particular cases of interest. Namely, when the gain with respect to the external input u enters the ISS condition (i) additively, (ii) via maximization and (iii) as a factor.

COROLLARY 5.6 (Additive gain of external input **u**). Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$  and the corresponding gain matrix is given by (2.9). Assume that the ISS condition is additive in the gain of u, that is,

$$\overline{\Gamma}_{\mu}(V_1(x_1), \dots, V_n(x_n), \|u\|) = \Gamma_{\mu}(V_1(x_1), \dots, V_n(x_n)) + \gamma_u(\|u\|), \quad (5.10)$$

where  $\gamma_u(||u||) = (\gamma_{1u}(||u||), \dots, \gamma_{nu}(||u||))^T$ . If  $\Gamma_\mu$  is irreducible and if there exists an  $\alpha \in \mathcal{K}_\infty$  such that for  $D = \text{diag}(id + \alpha)$  the gain operator  $\Gamma_\mu$  satisfies the strong small gain condition

$$D \circ \Gamma_{\mu}(s) \not\geq s$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (5.4), where  $\sigma \in \mathcal{K}_{\infty}^{n}$  is an arbitrary  $\Omega$ -path with respect to  $D \circ \Gamma_{\mu}$ .

*Proof.* By Theorem 5.2 an  $\Omega(D \circ \Gamma_{\mu})$ -path  $\sigma$  exists. Observe that by irreducibility, (M1), and (M3) it follows that  $\Gamma_{\mu}(\sigma)$  is unbounded in all components. Let  $\varphi \in \mathcal{K}_{\infty}$  be such that for all  $r \geq 0$ 

$$\min_{i=1,\dots,n} \{ \alpha(\Gamma_{\mu,i}(\sigma(r))) \} \ge \max_{i=1,\dots,n} \{ \gamma_{iu}(\varphi(r)) \}.$$

Note that this is possible, because on the left we take the minimum of a finite number of  $\mathcal{K}_{\infty}$  functions. Then we have for all  $r > 0, i = 1, \ldots, n$  that

$$\sigma_i(r) > D \circ \Gamma_{\mu,i}(\sigma(r)) = \Gamma_{\mu,i}(\sigma(r)) + \alpha(\Gamma_{\mu,i}(\sigma(r))) \ge \Gamma_{\mu,i}(\sigma(r)) + \gamma_{iu}(\varphi(r)).$$

Thus  $\sigma(r) > \overline{\Gamma}_{\mu}(\sigma(r), \varphi(r))$  and the assertion follows from Theorem 5.3.  $\Box$ 

COROLLARY 5.7 (Maximization w.r.t. external gain). Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$  and the corresponding gain matrix is given by (2.9). Assume that u enters the ISS condition via maximization, that is,

$$\overline{\Gamma}_{\mu}(V_1(x_1), \dots, V_n(x_n), ||u||) = \max\left\{\Gamma_{\mu}(V_1(x_1), \dots, V_n(x_n)), \gamma_u(||u||)\right\}, \quad (5.11)$$

where  $\gamma_u(||u||) = (\gamma_{1u}(||u||), \dots, \gamma_{nu}(||u||))^T$ . Then, if  $\Gamma_{\mu}$  is irreducible and satisfies the small gain condition

$$\Gamma_{\mu}(s) \not\geq s$$

the interconnected system is ISS and an ISS Lyapunov function is given by (5.4), where  $\sigma \in \mathcal{K}_{\infty}^{n}$  is an arbitrary  $\Omega$ -path with respect to  $\Gamma_{\mu}$  and  $\varphi$  is a  $\mathcal{K}_{\infty}$  function with the property

$$\gamma_{iu} \circ \varphi(r) \le \Gamma_{\mu,i}(\sigma(r)), \quad i = 1, \dots, n, \tag{5.12}$$

where  $\Gamma_{\mu,i}$  denotes the *i*-th row of  $\Gamma_{\mu}$ .

*Proof.* By Theorem 5.2 an  $\Omega(\Gamma_{\mu})$ -path  $\sigma$  exists. Note that by irreducibility, (M1), and (M3) it follows that  $\Gamma_{\mu}(\sigma)$  is unbounded in all components. Hence  $\varphi \in \mathcal{K}_{\infty}$  satisfying (5.12) exists and we obtain

$$\sigma(r) > \max \{ \Gamma_{\mu}(\sigma(r)), \gamma_{u}(\varphi(r)) \} .$$

This is (5.3) for the case of maximization of gains in u. The claim follows from Theorem 5.3.  $\Box$ 

In the next result observe that (M3) is not always necessary for the *u*-component of  $\mu$ .

COROLLARY 5.8 (Separation in gains). Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$  and the corresponding gain matrix  $\Gamma$  is given by (2.9). Assume that  $\Gamma$  is irreducible

and that the gains in the ISS condition are separated, that is, there exist  $\mu \in MAF_n^n$ ,  $c \in \mathbb{R}, c > 0$ , and  $\gamma_u \in \mathcal{K}_{\infty}$  such that

$$\overline{\Gamma}_{\mu}(V_1(x_1),\dots,V_n(x_n),\|u\|) = (c + \gamma_u(\|u\|)) \ \Gamma_{\mu}(V_1(x_1),\dots,V_n(x_n)).$$
(5.13)

If there exists an  $\alpha \in \mathcal{K}_{\infty}$  such that for  $D = \operatorname{diag}(c \cdot id + id \cdot \alpha)$  the gain operator  $\Gamma_{\mu}$  satisfies the strong small gain condition

$$D \circ \Gamma_{\mu}(s) \not\geq s$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (5.4), where  $\sigma \in \mathcal{K}_{\infty}^{n}$  is an arbitrary  $\Omega$ -path with respect to  $D \circ \Gamma_{\mu}(s)$ .

*Proof.* If  $\Gamma_{\mu}$  is irreducible, then also  $D \circ \Gamma_{\mu}$  is irreducible and so by Theorem 5.2 (ii) an  $\Omega(D \circ \Gamma_{\mu})$ -path  $\sigma$  exists. Let  $\varphi \in \mathcal{K}_{\infty}$  be such that for all  $r \geq 0$ 

$$\varphi(r) \leq \min_{i=1,\dots,n} \{\gamma_u^{-1} \circ \alpha \circ \Gamma_{\mu,i}(\sigma(r))\},\$$

where as in the previous corollaries we appeal to irreducibility, (M1), and (M3). Then for each i we have

$$\sigma_i(r) > \Gamma_{\mu,i}(\sigma(r))(c + \alpha(\Gamma_{\mu,i}(\sigma(r)))) \ge \Gamma_{\mu,i}(\sigma(r))(c + \gamma_u \circ \varphi(r))$$

and hence

$$\sigma(r) > (c + \gamma_u(\varphi(r)))\Gamma_\mu(\sigma(r)) = \overline{\Gamma}_\mu(\sigma(r), \varphi(r))$$

and the assertion follows from (5.13) and Theorem 5.3.  $\Box$ 

6. The reducible case and scaling. The results that have been obtained so far concern mostly strongly connected networks, that is, networks with an irreducible gain operator. Already in [27] it has been shown that cascades of ISS systems are ISS. Cascades are a special case of networks where the gain matrix is reducible. In this section we briefly explain how a Lyapunov function for a network that is not strongly connected may be constructed based on the construction for the strongly connected components of the network. Another approach would be to construct the  $\Omega$ -path for reducible operators  $\Gamma_{\mu}$  as has been done in [25] using assumption (M4).

It is well known, that if the network is not strongly connected, or equivalently if the gain matrix  $\Gamma$  is reducible, then  $\Gamma$  may be brought in upper block triangular form via a permutation of the vertices of the network as in the nonnegative matrix case [2, 6]. After this transformation  $\overline{\Gamma}$  is of the form

$$\overline{\Gamma} = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \dots & \Upsilon_{1d} & \Upsilon_{1u} \\ 0 & \Upsilon_{22} & \dots & \Upsilon_{2d} & \Upsilon_{2u} \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & \Upsilon_{dd} & \Upsilon_{du} \end{bmatrix},$$
(6.1)

where each of the blocks on the diagonal  $\Upsilon_{jj} \in (\mathcal{K}_{\infty} \cup \{0\})^{d_j \times d_j}$ ,  $j = 1, \ldots, d$ , is either irreducible or 0. Let  $q_j = \sum_{l=1}^{j-1} d_l$ , with the convention that  $q_1 = 0$ . We denote the states corresponding to the strongly connected components by

$$z_j^T = \begin{bmatrix} x_{q_j+1}^T & x_{q_j+2}^T & \dots & x_{q_{j+1}}^T \end{bmatrix}.$$

We will show that in order to obtain an overall ISS Lyapunov function it is sufficient to construct ISS Lyapunov functions for each of the irreducible blocks (where the respective states with higher indices are treated as inputs). The desired result is an iterative application of the following observation.

LEMMA 6.1. Let a gain matrix  $\overline{\Gamma} \in (\mathcal{K}_{\infty} \cup \{0\})^{2 \times 3}$  be given by

$$\overline{\Gamma} = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{1u} \\ 0 & 0 & \gamma_{2u} \end{bmatrix}, \tag{6.2}$$

and let  $\overline{\Gamma}_{\mu}$  be defined by  $\mu \in MAF_3^2$ . Then there exist an  $\Omega$ -path  $\sigma$  and  $\varphi \in \mathcal{K}_{\infty}$  such that (5.3) holds.

*Proof.* By construction the maps  $\eta_1 : r \mapsto \mu_1(\gamma_{12}(r), \gamma_{1u}(r))$  and  $\eta_2 : r \mapsto \mu_2(\gamma_{12}(u))$  are in  $\mathcal{K}_{\infty}$ . Choose a  $\mathcal{K}_{\infty}$ -function  $\tilde{\eta}_1 \geq \eta_1$ , such that  $\tilde{\eta}_1$  satisfies the conditions (i) and (ii) in Definition 5.1. Define  $\sigma(r) = \begin{bmatrix} 2\tilde{\eta}_1(r) & r \end{bmatrix}^T$  and  $\varphi(r) := \min\{r, \eta_2^{-1}(r/2)\}$ . Then it is a straightforward calculation to check that the assertion holds.  $\Box$ 

The result is now as follows.

PROPOSITION 6.2. Consider a simply connected interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$ , the corresponding gain matrix is given by (2.8), and  $\mu = (\mu_1, \ldots, \mu_n)^T$  is given by (2.7). Assume that the gain matrix  $\overline{\Gamma}$  is in the reduced form (6.1). If for each  $j = 1, \ldots, d-1$ there exists an ISS Lyapunov function  $W_j$  for the state  $z_j$  with respect to the inputs  $z_{j+1}, \ldots, z_d, u$  then there exists an ISS Lyapunov function V for the state x with respect to the input u.

*Proof.* By assumption for each j = 1, ..., d-1 there exist gain functions  $\chi_{jk} \in \mathcal{K}_{\infty}$ and  $\chi_{ju} \in \mathcal{K}_{\infty}$  and MAFs  $\tilde{\mu}_j$  such that

$$W_{j}(z_{j}) \geq \tilde{\mu}_{j}(\chi_{jj+1}(W_{j+1}(z_{j+1})), \dots, \chi_{jd}(W_{d}(z_{d})), \chi_{ju}(||u||)) \implies \nabla W_{j}(z_{j})f_{j}(z_{j}, z_{j+1}, \dots, z_{d}, u) < -\tilde{\alpha}_{j}(||z_{j}||).$$

We now argue by induction. If d = 1, there is nothing to show. If the result is shown for d-1 blocks, consider a gain matrix as in (6.1). By assumption there exists an ISS Lyapunov function  $V_{d-1}$  such that

$$V_{d-1}(z_{d-1}) \ge \mu_1(\gamma_{12}(V_d(z_d)), \gamma_{1u}(||u||)) \Longrightarrow \nabla V_{d-1}(z_{d-1}) f_{d-1}(z_{d-1}, z_d, u) \le -\alpha_{d-1}(||z_{d-1}||).$$

As the remaining part has only external inputs, we see that  $\overline{\Gamma}$  is of the form (6.2) and so Lemma 6.1 is applicable. This shows that the assumptions of Theorem 5.3 are met and so a Lyapunov function for the overall system is given by (5.4).  $\Box$ 

It is easy to see that the assumption  $\Gamma_{\mu} \geq id$  (or  $\Gamma_{\mu} \circ D \geq id$ ) is equivalent to the requirement that the blocks  $\Upsilon_{jj}$  on the diagonal satisfy the (strong) small gain condition (SGC)/(sSGC). Thus we immediately obtain the following statements.

COROLLARY 6.3 (Summation of gains). Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$  and the corresponding gain matrix is given by (2.9). Assume that the ISS condition is additive in the gains, that is,

$$\overline{\Gamma}_{\mu,i}(V_1(x_1),\dots,V_n(x_n),\|u\|) = \sum_{j=1}^n \gamma_{ij}(V_j(x_j)) + \gamma_{iu}(\|u\|).$$
(6.3)

If there exists an  $\alpha \in \mathcal{K}_{\infty}$  such that for  $D = \operatorname{diag}(id+\alpha)$  the gain operator  $\Gamma_{\mu}$  satisfies the strong small gain condition

$$D \circ \Gamma_{\mu}(s) \not\geq s$$

then the interconnected system is ISS.

*Proof.* After permutation  $\overline{\Gamma}$  is of the form (6.1). For each of the diagonal blocks Corollary 5.6 is applicable and the result follows from Proposition 6.2.  $\Box$ 

COROLLARY 6.4 (Maximization of gains). Consider the interconnected system  $\Sigma$  given by (2.1), (2.2) where each of the subsystems  $\Sigma_i$  has an ISS Lyapunov function  $V_i$  and the corresponding gain matrix is given by (2.9). Assume that the gains enter the ISS condition via maximization, that is,

$$\overline{\Gamma}_{\mu,i}(V_1(x_1),\dots,V_n(x_n),\|u\|) = \max\left\{\gamma_{i1}(V_1(x_1)),\dots,\gamma_{in}(V_n(x_n)),\gamma_{iu}(\|u\|)\right\}.$$
 (6.4)

If the gain operator  $\Gamma_{\mu}$  satisfies the small gain condition

 $\Gamma_{\mu}(s) \geq s$ 

then the interconnected system is ISS.

*Proof.* After permutation  $\overline{\Gamma}$  is of the form (6.1). For each of the diagonal blocks Corollary 5.7 is applicable and the result follows from Proposition 6.2.  $\Box$ 

7. Applications of the general small gain theorem. In Section 3 we have presented several examples of functions  $\mu_i$ ,  $\gamma_i$  and gain operators  $\Gamma_{\mu}$ ,  $\overline{\Gamma}_{\mu}$ . Here we will show how our main results apply to these examples. Before we proceed, let us consider the special case of homogeneous  $\Gamma_{\mu}$  (of degree 1) [13]. Here  $\Gamma_{\mu}$  is homogeneous of degree one if for any  $s \in \mathbb{R}^n_+$  and any r > 0 we have  $\Gamma_{\mu}(rs) = r\Gamma_{\mu}(s)$ .

PROPOSITION 7.1 (Explicit paths and Lyapunov functions for homogeneous gain operators). Let  $\Sigma$  in (1.2) be a strongly connected network of subsystems (1.1) and  $\Gamma_{\mu}$ ,  $\overline{\Gamma}_{\mu}$  be the corresponding gain operators. Let  $\Gamma_{\mu}$  be homogeneous and let  $\overline{\Gamma}_{\mu}$  satisfy one of the conditions (6.3), (6.4), or (5.13). If  $\Gamma_{\mu}$  satisfies the strong small gain condition (sSGC) ((SGC) in case of (6.4)) then the interconnection  $\Sigma$  is ISS, moreover there exists a (nonlinear) eigenvector  $0 < s \in \mathbb{R}^n$  of  $\Gamma_{\mu}$  such that  $\Gamma_{\mu}(s) = \lambda s$  with  $\lambda < 1$ and an ISS Lyapunov function for the network is given by

$$V(x) = \max\{V_i(x_i)/s_i\}.$$
(7.1)

*Proof.* First note that one of the Corollaries 6.3, 6.4, or 5.8 can be applied and the ISS property follows immediately. By the assumptions of the proposition we have an irreducible monotone homogeneous operator  $\Gamma_{\mu}$  on the positive orthant  $\mathbb{R}^{n}_{+}$ . By the generalized Perron-Frobenius Theorem [13] there exists a positive eigenvector  $s \in \mathbb{R}^{n}_{+}$ . Its eigenvalue  $\lambda$  is less than one, otherwise we have a contradiction to the small gain condition. The ray defined by this vector s is a corresponding  $\Omega$ -path and by Theorem 5.3 we obtain (7.1).  $\Box$ 

One type of homogeneous operators arises from linear operators through multiplicative coordinate transforms. In this case we can further specialize the assumptions of the previous result.

LEMMA 7.2. Let  $\alpha \in \mathcal{K}_{\infty}$  satisfy<sup>2</sup>  $\alpha(ab) = \alpha(a)\alpha(b)$  for all  $a, b \geq 0$ . Let  $D = \text{diag}(\alpha), G \in \mathbb{R}^{n \times n}_+$ , and  $\Gamma_{\mu}$  be given by

$$\Gamma_{\mu}(s) = D^{-1}(GD(s)) \,.$$

<sup>&</sup>lt;sup>2</sup>In other words,  $\alpha(r) = r^c$  for some c > 0.

Then  $\Gamma_{\mu}$  is homogeneous. Moreover,  $\Gamma_{\mu} \geq id$  if and only if the spectral radius of G is less than one.

*Proof.* If the spectral radius of G is less than one, then there exists a positive vector  $\tilde{s}$  satisfying  $G\tilde{s} < \tilde{s}$ : Just add a small  $\delta > 0$  to every entry of G, so that the spectral radius  $\rho(\tilde{G})$  of  $\tilde{G}$  is still less than one, due to continuity of the spectrum. Then there exists a Perron vector  $\tilde{s}$  such that  $G\tilde{s} < \tilde{G}\tilde{s} = \rho(\tilde{G})\tilde{s} < \tilde{s}$ . Define  $\hat{s} = D^{-1}(\tilde{s}) > 0$  and observe that  $\alpha^{-1}(ab) = \alpha^{-1}(a)\alpha^{-1}(b)$ . Then we have

$$\Gamma_{\mu}(r\hat{s}) = D^{-1}(GD(r\hat{s})) = D^{-1}(\alpha(r)GD(\hat{s})) = rD^{-1}(G\tilde{s})$$

$$< rD^{-1}(\tilde{s}) = r\hat{s},$$
(7.2)

for all r > 0. So an  $\Omega$ -path for  $\Gamma_{\mu}$  is given by  $\sigma(r) = r\hat{s}$  for  $r \ge 0$ . Existence of an  $\Omega$ -path implies the small gain condition: The origin in  $\mathbb{R}^n_+$  is globally attractive with respect to the system  $s^{k+1} = \Gamma_{\mu}(s^k)$ , as can be seen by a monotonicity argument. By [6, Theorem 23] or [25, Prop. 4.1] we have  $\Gamma_{\mu} \ge id$ .

Assuming that the spectral radius of G is greater or equal to one there exists  $\tilde{s} \in \mathbb{R}^n_+$ ,  $\tilde{s} \neq 0$ , such that  $G\tilde{s} \geq \tilde{s}$ . Defining  $\hat{s} = D^{-1}(\tilde{s})$  we have  $\Gamma_{\mu}(\hat{s}) = D^{-1}(GD(\hat{s})) = D^{-1}(G\tilde{s}) \geq D^{-1}(\tilde{s}) = \hat{s}$ . Hence  $\Gamma_{\mu} \not\geq id$  if and only if the spectral radius of G is less than one.

Homogeneity of  $\Gamma_{\mu}$  is obtained as in (7.2).

**7.1.** Application to linear interconnected systems. Consider the interconnection (3.2) of linear systems from Section 3.1.

PROPOSITION 7.3. Let each  $\Sigma_i$  in (3.2) be ISS with a quadratic ISS Lyapunov function  $V_i$ , so that the corresponding operator  $\Gamma_{\mu}$  can be taken to be as in (3.5). If the spectral radius r(G) of the associated matrix

$$G = \left(\frac{2b_i^3 \|\Delta_{ij}\|}{c_i(1-\varepsilon)a_j}\right)_{ij} \tag{7.3}$$

is less than 1, then the interconnection

$$\Sigma: \quad \dot{x} = (A + \Delta)x + Bu$$

is ISS and its (nonsmooth) ISS Lyapunov function can be taken as

$$V(x) = \max_{i} \frac{1}{s_i} x_i^T P_i x_i$$

for some positive vector  $s \in \mathbb{R}^n_+$ .

Proof. We have  $\Gamma_{\mu} = D^{-1}(GD(\cdot))$ , where  $D = \operatorname{diag}(\alpha)$  for  $\alpha(r) = \sqrt{r}$ . Now  $\alpha$  satisfies the assumptions of Lemma 7.2, which yields that  $\Gamma_{\mu}$  satisfies the small gain condition  $\Gamma_{\mu} \not\geq \operatorname{id}$  if and only if r(G) < 1. If G or equivalently  $\Gamma_{\mu}$  is irreducible then there exists by Proposition 7.1 an s > 0 such that  $\Gamma_{\mu}(s) < s$ . By (3.5) we see, that there exists an  $r^* \in (0, \infty)$  such that  $\overline{\Gamma}_{\mu}(s, r^*) < s$ . Then defining  $\sigma(r) = rs, \varphi(r) = \sqrt{rr^*}$  we obtain for all r > 0 that

$$\overline{\Gamma}_{\mu}(\sigma(r), \varphi(r)) = r\overline{\Gamma}_{\mu}(s, r^*) < rs = \sigma(r).$$

Thus the conditions of Theorem 5.3 are satisfied and an ISS Lyapunov function can be taken as  $V(x) = \max_i \frac{1}{s_i} x_i^T P_i x_i$ .

If G is reducible the previous construction has to be performed for every irreducible blocks and then the scaling techniques of Section 6 need to be applied.  $\Box$ 

**7.2.** Application to neural networks. Consider the neural network (3.6) discussed in Section 3.2. This is a system of coupled nonlinear equations, and we have seen that each subsystem is ISS. Note that so far we have not imposed any restrictions on the coefficients  $t_{ij}$ . Moreover the assumptions imposed on  $a_i$ ,  $b_i$ ,  $s_i$  are essentially milder then in [35]. However to obtain the ISS property of the network we need to require more. The small gain condition can be used for this purpose. It will impose restrictions on the coupling terms  $t_{ij}s(x_j)$ . From Corollary 5.6 it follows:

THEOREM 7.4. Consider the Cohen-Grossberg neural network (3.6). Let  $\Gamma_{\mu}$  be given by  $\gamma_{ij}$  and  $\mu_i$ , i, j = 1, ..., n, as calculated for the interconnection in Section 3.2. Assume that  $\Gamma_{\mu}$  satisfies the strong small gain condition  $D \circ \Gamma_{\mu} \geq id$  for  $s \in \mathbb{R}^n_+ \setminus 0$ . Then this network is ISS from  $(J_1, ..., J_n)^T$  to x.

REMARK 7.5. In [35] the authors have proved that there exists a unique equilibrium point for the network and given constant external inputs. They have also proved the exponential stability of this equilibrium. We have considered arbitrary external inputs to the network and proved the ISS property for the interconnection.

8. Path construction. This section explains the relation between the small gain condition for  $\Gamma_{\mu}$  and its mapping properties. Then we construct a strictly increasing  $\Omega$ -path and prove Theorem 5.2 and some extensions. Let us first consider some simple particular cases to explain the main ideas, as depicted in Figure 8.1. In the following subsections we then proceed to the main path construction results.

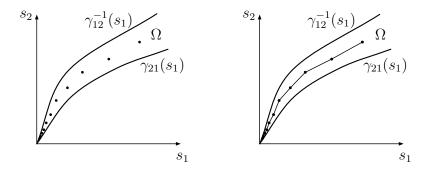


FIG. 8.1. A sequence of points  $\{\Gamma_{\mu}^{k}(s)\}_{k\geq 0}$  for some  $s \in \Omega(\Gamma_{\mu})$ , where  $\Gamma_{\mu} : \mathbb{R}^{2}_{+} \to \mathbb{R}^{2}_{+}$  is given by  $\Gamma_{\mu}(s) = (\gamma_{12}(s_{2}), \gamma_{21}(s_{1}))^{T}$  and satisfies  $\Gamma_{\mu} \not\geq id$ , or, equivalently,  $\gamma_{21} \circ \gamma_{12} < id$ , and the corresponding linear interpolation, cf. Lemmas 8.1, 8.2, and 8.3.

A map  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  is monotone if  $x \leq y$  implies  $T(x) \leq T(y)$ . Clearly any matrix  $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$  together with an aggregation  $\mu \in \mathrm{MAF}^n_n$  induces a monotone map  $\Gamma_{\mu}$ .

LEMMA 8.1. Let  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  and  $\mu \in MAF_n^n$ , such that  $\Gamma_{\mu}$  satisfies (SGC). If  $s \in \Omega(\Gamma_{\mu})$ , then  $\lim_{k \to \infty} \Gamma_{\mu}^k(s) = 0$ .

Proof. If  $s \in \Omega$ , then  $\Gamma_{\mu}(s) < s$  and by monotonicity  $\Gamma_{\mu}^{2}(s) \leq \Gamma_{\mu}(s)$ . By induction  $\Gamma_{\mu}^{k}(s)$  is a monotonically decreasing sequence bounded from below by 0. Thus  $\lim_{k\to\infty}\Gamma_{\mu}^{k}(s) =: s^{*}$  exists and by continuity we have  $\Gamma_{\mu}(s^{*}) = s^{*}$ . By the small gain condition it follows  $s^{*} = 0$ .  $\Box$ 

LEMMA 8.2. Assume that  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  has no zero rows and let  $\mu \in MAF_n^n$ . If  $0 < s \in \Omega(\Gamma_\mu)$ , then

- (i)  $0 < \Gamma_{\mu}(s) \in \Omega$
- (ii) for all  $\lambda \in [0,1]$  the convex combination  $s_{\lambda} := \lambda s + (1-\lambda)\Gamma_{\mu}(s) \in \Omega$ .

*Proof.* (i) By assumption  $\Gamma_{\mu}(s) < s$  and so by the monotonicity assumption (M2) we have  $\Gamma_{\mu}(\Gamma_{\mu}(s)) < \Gamma_{\mu}(s)$ . Furthermore, as s > 0 and the matrix  $\Gamma$  has no zeros rows, we have that  $\Gamma_{\mu}(s) > 0$  by assumption (M1).

(ii) As  $\Gamma_{\mu}(s) < s$  it follows for all  $\lambda \in (0,1)$  that  $\Gamma_{\mu}(s) < s_{\lambda} < s$ . Hence by monotonicity and using (i)

$$0 < \Gamma_{\mu}(\Gamma_{\mu}(s)) < \Gamma_{\mu}(s_{\lambda}) < \Gamma_{\mu}(s) < s_{\lambda}.$$

This implies  $s_{\lambda} \in \Omega$  as desired.  $\Box$ 

LEMMA 8.3. Assume that  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  has no zero rows and let  $\mu \in MAF_n^n$ be such that  $\Gamma_{\mu}$  satisfies the small gain condition (SGC). Let  $s \in \Omega(\Gamma_{\mu})$ . Then there exists a path in  $\Omega \cup \{0\}$  connecting the origin and s.

Proof. By Lemma 8.2, the line segment  $\{\lambda\Gamma_{\mu}(s) + (1-\lambda)s\} \subset \Omega$ . By induction all the line segments  $\{\lambda\Gamma_{\mu}^{k+1}(s) + (1-\lambda)\Gamma_{\mu}^{k}(s)\} \subset \Omega$  for  $k \geq 1$ . Using Lemma 8.1 we see that  $\Gamma_{\mu}^{k}(s) \to 0$  as  $k \to \infty$ . This constructs an  $\Omega$ -path with respect to  $\Gamma_{\mu}$  from 0 to s.  $\Box$ 

The following result applies to  $\Gamma$  whose entries are bounded, i.e., in  $(\mathcal{K} \setminus \mathcal{K}_{\infty}) \cup \{0\}$ .

PROPOSITION 8.4. Assume that  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  has no zero rows and let  $\mu \in MAF_n^n$  be such that  $\Gamma_{\mu}$  satisfies the small gain condition (SGC). Assume furthermore that  $\Gamma_{\mu}$  is bounded, then there exists an  $\Omega$ -path with respect to  $\Gamma_{\mu}$ .

Proof. By assumption the set  $\Gamma_{\mu}(\mathbb{R}^{n}_{+})$  is bounded, so pick  $s > \sup \Gamma_{\mu}(\mathbb{R}^{n}_{+})$ . Then clearly,  $\Gamma_{\mu}(s) < s$  and so  $s \in \Omega$ . By the same argument  $\eta s \in \Omega$  for all  $\eta \in [1, \infty)$ . Thus a path in  $\Omega$  through the point s exists, if we find a path from s to 0 contained in  $\Omega$ . The remainder of the result is given by Lemma 8.3.  $\Box$ 

The difficulty now arises if  $\Gamma_{\mu}$  happens to be unbounded, i.e.,  $\Gamma$  contains entries of class  $\mathcal{K}_{\infty}$ . In the unbounded case the simple construction above is not possible. In the following we will first consider the case that all nonzero entries of  $\Gamma$  are of class  $\mathcal{K}_{\infty}$ . Beforehand we introduce a few technical lemmas.

**8.1. Technical lemmas.** Throughout this subsection  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  denotes a continuous, monotone map, i.e., T satisfies  $T(v) \leq T(w)$  whenever  $v \leq w$ . We start with a few observations.

LEMMA 8.5. Let  $\rho \in \mathcal{K}_{\infty}$ . Then there exists a  $\tilde{\rho} \in \mathcal{K}_{\infty}$  such that  $(id + \rho)^{-1} = id - \tilde{\rho}$ .

*Proof.* Just define  $\tilde{\rho} = \rho \circ (\mathrm{id} + \rho)^{-1}$ . Then  $(\mathrm{id} - \tilde{\rho}) \circ (\mathrm{id} + \rho) = (\mathrm{id} + \rho) - \tilde{\rho} \circ (\mathrm{id} + \rho) = \mathrm{id} + \rho - \rho \circ (\mathrm{id} + \rho)^{-1} \circ (\mathrm{id} + \rho) = \mathrm{id} + \rho - \rho = \mathrm{id}$ , which proves the lemma. LEMMA 8.6.

(i) Let  $D = \operatorname{diag}(\rho)$  for some  $\rho \in \mathcal{K}_{\infty}$  such that  $\rho > id$ . Then for any  $k \ge 0$  there exist  $\rho_1^{(k)}, \rho_2^{(k)} \in \mathcal{K}_{\infty}$  satisfying  $\rho_i^{(k)} > id$ , such that for  $D_i^{(k)} = \operatorname{diag}(\rho_i^{(k)}), i = 1, 2$ ,

$$D = D_1^{(k)} \circ D_2^{(k)}.$$

Moreover,  $D_2^{(k)}$ ,  $k \ge 0$ , can be chosen such that for all  $0 < s \in \mathbb{R}^n_+$  we have

$$D_2^{(k)}(s) < D_2^{(k+1)}(s).$$

(ii) Let  $D = \text{diag}(id + \alpha)$  for some  $\alpha \in \mathcal{K}_{\infty}$ . Then there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ , such that for  $D_i = \text{diag}(id + \alpha_i)$ , i = 1, 2,

$$D = D_1 \circ D_2$$

For maps  $T: \mathbb{R}^n_+ \to \mathbb{R}^n_+$  define the *decay set* 

$$\Psi(T) := \left\{ s \in \mathbb{R}^n_+ : T(s) \le s \right\},\$$

where we again omit the reference to T if this is clear from the context.

LEMMA 8.7. Let  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  be monotone and  $D = \operatorname{diag}(\rho)$  for some  $\rho \in \mathcal{K}_{\infty}, \rho > id$ . Then

(i)  $T^{k+1}(\Psi) \subset T^k(\Psi)$  for all  $k \ge 0$ ;

(ii)  $\Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s > 0\} \subset \Omega(T), \text{ if } T \text{ satisfies } T(v) < T(w) \text{ whenever } v < w; \text{ the same is true for } D \circ T \text{ replaced by } T \circ D;$ 

The proofs of the lemmas are simple and thus omitted for reasons of space. Nevertheless they can be found in [24, p.10, p.29].

We will need the following connectedness property in the sequel.

PROPOSITION 8.8. Let  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  and  $\mu \in MAF_n^n$  be such that  $\Gamma_{\mu}$  satisfies the small gain condition (SGC). Then  $\Psi$  is nonempty and pathwise connected. Moreover, if  $\Gamma_{\mu}$  satisfies  $\Gamma_{\mu}(v) < \Gamma_{\mu}(w)$  whenever v < w, then for any  $s \in \Omega(\Gamma_{\mu})$  there exists a strictly increasing  $\Omega$ -path connecting 0 and s.

*Proof.* Note that always  $0 \in \Psi$ , hence  $\Psi$  cannot be empty. Along the lines the proof of Lemma 8.3 it follows that each point in  $\Psi$  is pathwise connected to the origin.  $\Box$ 

Another crucial step, which is of topological nature, regards preimages of points in the decay set  $\Psi$ . In general it is not guaranteed, that for  $s \in \mathbb{R}^n_+$  with  $T(s) \in \Psi$ , we also have  $s \in \Psi$ . The set of points in  $\Psi$  for which preimages of arbitrary order are also in  $\Psi$  is the set

$$\Psi_{\infty}(T) := \bigcap_{k=0}^{\infty} T^{k}(\Psi).$$

Of course, this set might be empty or bounded. We will use it to construct  $\Omega$ -paths for operators  $\Gamma_{\mu}$  satisfying the small gain condition.

PROPOSITION 8.9 ([25, Prop. 5.4]). Let  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  be monotone and continuous and satisfy  $T(s) \geq s$  for all  $s \neq 0$ . Assume that T satisfies the property

$$||s_k|| \to \infty \implies ||T(s_k)|| \longrightarrow \infty$$
 (8.1)

as  $k \to \infty$  for any sequence  $\{s_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n_+$ .

Then  $\Psi_{\infty}(T) \subset \Psi(T)$ ,  $\Psi_{\infty}(T) \cap S_r \neq \emptyset$  for all  $r \geq 0$ , and  $\Psi_{\infty}(T)$  is unbounded.

A result based on the topological fixed point theorem due to Knaster, Kuratowski, and Mazurkiewicz allows to relate  $\Omega$  and the small gain condition. It is essential for the proof of Proposition 8.9.

PROPOSITION 8.10. Let  $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$  be monotone and continuous. If  $T(s) \geq s$  for all  $s \in \mathbb{R}^n_+$  then the set  $\Omega \cap S_r$  is nonempty for all r > 0.

In particular,  $s \in \Omega \cap S_r$  for r > 0 implies s > 0. The proof for this result can be found in [24, Prop. 1.5.3, p.26] or in a slightly different form in [6].

8.2. Paths for  $\mathcal{K}_{\infty} \cup \{0\}$  gain matrices. In this subsection we consider matrices  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$ , i.e., all nonzero entries of  $\Gamma$  are assumed to be unbounded functions.

In this setting we assume and utilize that the graph associated to  $\Gamma$  is strongly connected, i.e.,  $\Gamma$  is irreducible. So that if we consider powers  $\Gamma^k_{\mu}(x)$ , for each com-

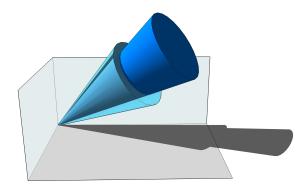


FIG. 8.2. A sketch of the set  $\Psi_{\infty} \subset \Psi \subset \mathbb{R}^n_+$  in Proposition 8.9.

ponents i and j there exists a k = k(i, j) such that  $t \mapsto \Gamma^k_{\mu} (t \cdot e_j)_i$  is an unbounded function.

THEOREM 8.11. Let  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$  be irreducible,  $\mu \in \mathrm{MAF}_{n}^{n}$ , and assume  $\Gamma_{\mu} \not\geq id$ . Then there exists a strictly increasing path  $\sigma \in \mathcal{K}_{\infty}^{n}$  satisfying

$$\Gamma_{\mu}(\sigma(r)) < \sigma(r), \quad \forall r > 0$$

The main technical difficulty in the proof is to construct the path in the unbounded direction, the other case has already been dealt with in Proposition 8.8.

The proof comprises the following steps: First due to [25, Prop. 5.8] we may choose a  $\mathcal{K}_{\infty}$  function  $\varphi > \text{id}$  so that for  $D = \text{diag}(\varphi)$  we have  $\Gamma_{\mu} \circ D \not\geq \text{id}$ . Then we construct a monotone (but not necessarily strictly monotone) sequence  $\{s^k\}_{k\geq 0}$ in  $\Psi(\Gamma_{\mu} \circ D)$ , satisfying  $s^k = \Gamma_{\mu}(D(s^{k+1})) \not\leq s^{k+1}$ , so that each component sequence is unbounded. At this point a linear interpolation of the sequence points may not yield a *strictly increasing* path. So finally we use the "extra space" provided by D in the set  $\Omega(\Gamma_{\mu}) \supset \Omega(\Gamma_{\mu} \circ D)$  to obtain a strictly increasing sequence  $\{\tilde{s}^k\}_{k\geq 0}$  in  $\Omega(\Gamma_{\mu})$ which we can linearly interpolate to obtain the desired  $\Omega$ -path.

*Proof.* Since Γ is irreducible, it has no zero rows and hence Γ<sub>μ</sub> satisfies Γ<sub>μ</sub>(v) < Γ<sub>μ</sub>(w) whenever v < w. By [25, Prop. 5.8] there exists a  $\varphi > \text{id}$  so that for  $D = \text{diag}(\varphi)$  we have Γ<sub>μ</sub>  $\circ D \ngeq$  id. Now we construct a nondecreasing sequence  $\{s^k\}$  in  $\Psi(\Gamma_\mu \circ D)$ :

Let  $T := \Gamma_{\mu} \circ D$ . Then T and by induction also all powers  $T^{l}$ ,  $l \geq 1$ , satisfy (8.1).

By Proposition 8.9 the set  $\Psi_{\infty}(T)$  is unbounded, so we may pick an  $0 \neq s^0 \in \Psi_{\infty}(T)$ . We can actually choose  $s^0 > 0$ , since the sequence  $\{s^k\}$  we are going to construct will be unbounded in every component, at which point we may replace  $s^0$  by some  $s^k > 0$  for k large enough.

Due to irreducibility of  $\Gamma$  (and Remark 2.6) the following property holds: For any pair  $1 \leq i, j \leq n$  there exists an  $l \geq 1$  such that

$$r \mapsto (\Gamma^l_\mu(re_j))_i \tag{8.2}$$

is an unbounded and increasing function, where  $e_j$  is the *j*-th unit vector. By monotonicity the same holds when T is considered instead of  $\Gamma_{\mu}$ . Now define a sequence  $\{s^k\}_{k>0}$  by choosing

$$s^{k+1} \in T^{-1}(s^k) \cap \Psi_{\infty}(T)$$

for  $k \ge 0$ . This is possible, since by definition  $\Psi_{\infty}(T)$  is backward invariant under T.

This sequence  $\{s^k\}$  satisfies  $s^k \lneq s^{k+1}$  by definition. We claim that it is unbounded, and also unbounded in every component: To this end assume first that it is bounded. Then by monotonicity there exists a limit  $s^* = \lim_{k \to \infty} s^k$ . By continuity of T and since  $s^k = T(s^{k+1})$  we have

$$s^* = \lim_{k \to \infty} s^k = \lim_{k \to \infty} T(s^{k+1}) = T\left(\lim_{k \to \infty} s^{k+1}\right) = T(s^*)$$

contradicting  $T(s) \geq s$  for all  $s \neq 0$ . Hence the sequence  $\{s^k\}$  must be unbounded.

Let j be an index such that  $\{s_j^k\}_{k\in\mathbb{N}}$  is unbounded, let  $i \in \{1, \ldots, n\}$  be arbitrary and choose l such that the function in (8.2) is unbounded for i, j, l. Choose real numbers  $r_k \to \infty$  such that  $r_k e_j \leq s^k$  for all  $k \in \mathbb{N}$ . Then we have

$$(T^{l}(r_{k}e_{j}))_{i} \leq (T^{l}(s^{k}))_{i} = s_{i}^{k-l}.$$

As the term on the left goes to  $\infty$  for  $k \to \infty$ , so does  $s_i^k$ . Hence  $\{s^k\}$  is unbounded in every component.

Now by Lemma 8.7(ii) the sequence  $\{s^k\}$  is contained in  $\Omega(\Gamma_{\mu})$ , but it may not be strictly increasing, as we only know  $s^k \leq s^{k+1}$  for all  $k \geq 0$ . We define a strictly increasing sequence  $\{\tilde{s}^k\}$  as follows: By Lemma 8.6 for any  $k \geq 0$  we may factorize  $D = D_1^{(k)} \circ D_2^{(k)}$  so that  $D_1^{(k)}, D_2^{(k)} > \text{id}$  and  $D_2^{(k)}(s) < D_2^{(k+1)}(s)$  for all  $k \geq 0$  and all s > 0. Using this factorization we define

$$\tilde{s}^k := D_2^{(k)}(s^k)$$

for all  $k \ge 0$ . By the definition of  $D_2^{(k)}$ , this sequence is clearly strictly increasing and inherits from  $\{s^k\}$  the unboundedness in all components.

We claim that  $\{\tilde{s}^k\} \subset \Omega(\Gamma_\mu)$ : This follows from

$$\tilde{s}^k > s^k \ge \Gamma_{\mu} \circ D(s^k) = \Gamma_{\mu} \circ D_1^{(k)} \circ D_2^{(k)}(s^k) = \Gamma_{\mu} \circ D_1^{(k)}(\tilde{s}^k) > \Gamma_{\mu}(\tilde{s}^k).$$

Now we prove that for  $\lambda \in (0,1)$  we have  $(1-\lambda)\tilde{s}^k + \lambda \tilde{s}^{k+1} \in \Omega(\Gamma_{\mu})$ . Clearly

$$\tilde{s}^k < (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} < \tilde{s}^{k+1}$$

and application of the strictly increasing operator  $\Gamma_{\mu}$  yields

$$\begin{split} \Gamma_{\mu}((1-\lambda)\tilde{s}^{k}+\lambda\tilde{s}^{k+1}) &< \Gamma_{\mu}(\tilde{s}^{k+1}) \\ &= \Gamma_{\mu} \circ D_{2}^{(k+1)}(s^{k+1}) < \Gamma_{\mu} \circ D_{1}^{(k+1)} \circ D_{2}^{(k+1)}(s^{k+1}) \\ &= s^{k} < \tilde{s}^{k} < (1-\lambda)\tilde{s}^{k}+\lambda\tilde{s}^{k+1}. \end{split}$$

Hence  $(1 - \lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_{\mu}).$ 

Now we may define  $\sigma$  as a parametrization of the linear interpolation of the points  $\{\tilde{s}^k\}_{k\geq 0}$  in the unbounded direction and utilize the construction from Lemma 8.3 for the other direction. Clearly this function  $\sigma$  is an  $\Omega$ -path as it has component functions of class  $\mathcal{K}_{\infty}$  and is piecewise linear on every compact interval contained in  $(0, \infty)$ .  $\Box$ 

It is possible to consider the reducible case in a similar fashion. The argument is essentially an induction over the number of irreducible and zero blocks on the diagonal of the reducible operator. We cite the following result from [25, Theorem 5.10]. However, for the construction of an ISS Lyapunov function in the case of reducible  $\Gamma$ , we take a different route as described in Section 6, thus avoiding the use of assumption (M4).

THEOREM 8.12. Let  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$  be reducible,  $\mu \in \mathrm{MAF}_{n}^{n}$  satisfying  $(M_{4}), D = \mathrm{diag}(id + \alpha)$  for some  $\rho \in \mathcal{K}_{\infty}$ , and assume  $\Gamma_{\mu} \circ D \ngeq id$ . Then there exists a monotone and continuous operator  $\tilde{D} : \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}^{n}$  and a strictly increasing path  $\sigma : \mathbb{R}_{+} \to \mathbb{R}_{+}^{n}$  whose component functions are all unbounded, such that  $\Gamma_{\mu} \circ \tilde{D}(\sigma) < \sigma$ .

**8.3.** General  $\Gamma_{\mu}$ . In the preceding subsections we have seen that it is possible to construct  $\Omega$ -paths for matrices  $\Gamma$  whose nonzero entries are either all bounded, or all unbounded. It remains to consider the case that the nonzero entries of  $\Gamma$  are partly of class  $\mathcal{K}_{\infty}$  and partly of class  $\mathcal{K} \setminus \mathcal{K}_{\infty}$ . We can state the following result.

PROPOSITION 8.13. Let  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$  and let  $\mu \in MAF_n^n$  satisfy (M4). Assume  $\Gamma_{\mu}$  satisfies (sSGC). Then there exists an  $\Omega$ -path for  $\Gamma_{\mu}$ . Proof. Write

$$\Gamma = \Gamma_U + \Gamma_B$$

with  $\Gamma_U \in (\mathcal{K}_{\infty} \cup \{0\})^{n \times n}$ ,  $\Gamma_B \in (\mathcal{K} \setminus \mathcal{K}_{\infty} \cup \{0\})^{n \times n}$ . Clearly we have  $(\Gamma_U)_{\mu} \leq \Gamma_{\mu}$ and  $(\Gamma_B)_{\mu} \leq \Gamma_{\mu}$  and hence both maps satisfy

$$(\Gamma_{\bullet})_{\mu} \not\geq \mathrm{id},$$

where  $\bullet$  serves as a placeholder for the subscripts U and B.

The map  $(\Gamma_B)_{\mu}$  is bounded. Hence  $s^* := \sup(\Gamma_B)_{\mu}(\mathbb{R}^n_+)$  is a finite vector. By Theorem 8.12. for  $(\Gamma_U)_{\mu}$  there exists a  $\mathcal{K}_{\infty}$  function  $\tilde{\rho}$  and a  $\mathcal{K}_{\infty}$ -path  $\sigma_U$  so that for the diagonal operator  $\tilde{D} = \operatorname{diag}(\operatorname{id} + \tilde{\rho})$  we have

$$((\Gamma_U)_{\mu} \circ D)(\sigma_U(r)) < \sigma_U(r), \text{ for all } r > 0$$

Similarly, by Proposition 8.4, there exists a  $\mathcal{K}_{\infty}$ -path  $\sigma_B$  such that  $(\Gamma_B)_{\mu}(\sigma_B(r)) < \sigma_B(r)$  for all r > 0. In fact, and this is the key to this proof, it is possible to choose  $\sigma_B$  in the region where  $\sigma_B(r) > s^*$  to grow arbitrarily slowly: For any  $\alpha, \beta \in \mathcal{K}_{\infty}$  we can find a  $\kappa \in \mathcal{K}_{\infty}$ , such that

$$(\alpha \circ \kappa)(r) < \beta(r), \quad r > 0,$$

e.g., by choosing  $\kappa \in \mathcal{K}_{\infty}$  satisfying  $\kappa(r) < (\alpha^{-1} \circ \beta)(r)$ . This is always possible. Denote  $\overline{D} = \operatorname{diag}(\tilde{\rho})$ , (so that  $\tilde{D} = \operatorname{id} + \overline{D}$ ) and choose  $r^*$ , such that  $\overline{D}(\sigma_U(r^*)) > s^*$ . Then after reparametrization we may assume that

$$\sigma_B(r) < \overline{D}(\sigma_U(r))$$
 and  $\sigma_B(r) > s^*$ 

for all  $r \geq r^*$ . Using Lemma 8.3, we let  $\sigma_L : [0, r^*] \to \mathbb{R}^n_+$  be a finite-length path satisfying

$$\Gamma_{\mu}(\sigma_{L}(r)) < \sigma_{L}(r), \quad \forall r \in (0, r^{*}],$$
  
$$\sigma_{L} \text{ is strictly increasing}$$
  
$$\sigma_{L}(0) = 0 \text{ and } \sigma_{L}(r^{*}) = \sigma_{B}(r^{*}) + \sigma_{U}(r^{*}).$$

Now define  $\sigma$  by

$$\sigma(r) = \begin{cases} \sigma_B(r) + \sigma_U(r) & \text{if } r > r^* \\ \sigma_L(r) & \text{if } r < r^*. \end{cases}$$

It remains to check that  $\sigma$  satisfies  $\Gamma_{\mu}(\sigma(r)) < \sigma(r)$  for  $r \ge r^*$ . Indeed, for  $r \ge r^*$  we have

$$\sigma(r) = \sigma_U(r) + \sigma_B(r) > ((\Gamma_U)_\mu \circ D)(\sigma_U(r)) + s^*$$
  
>  $(\Gamma_U)_\mu(\sigma_U(r) + \sigma_B(r)) + (\Gamma_B)_\mu(\sigma_U(r) + \sigma_B(r))$   
\ge \Gamma\_\mu(\sigma\_U(r) + \sigma\_B(r)),

where the last inequality is due to (M4). This completes the proof.  $\Box$ 

**8.4.** Special case: Maximization. The case when the aggregation is the maximum, i.e.,  $\mu = \max$ , is indeed a special case, since not only the small gain condition can be formulated in simpler manner, but also the path construction can be achieved without the need of the diagonal operator D as before.

A cycle in a matrix  $\Gamma$  is finite sequence of nonzero entries of  $\Gamma$  of the form

$$(\gamma_{i_1,i_2},\gamma_{i_2,i_3},\ldots,\gamma_{i_K,i_1}).$$

A cycle is called *subordinated* if  $i_1 > \max\{i_2, \ldots, i_K\}$ , and it is called a *contraction*, if

$$\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \ldots \circ \gamma_{i_K,i_1} < \mathrm{id}.$$

It is an easy exercise to show that when all subordinated cycles are contractions then already all cycles are contractions.

THEOREM 8.14. Let  $\mu = \max$  and  $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$ . If all subordinated cycles of  $\Gamma$  are contractions, then there exists an  $\Omega$ -path with respect to  $\Gamma_{\mu}$ .

The proof is composed of the following steps. The first step is to show that the cycle condition (all cycles being contractions) is equivalent to  $\Gamma_{\mu} \geq id$ . Note that  $\mu = \max$  automatically satisfies (M4), but (M4) is actually not needed for the proof. Then the path-construction can essentially be done as before, replacing sums by maximization, and one can even avoid the use of  $D = \text{diag}(id + \rho)$ . Cf. also[25].

**8.5.** Proof of Theorem 5.2. We now come to the easiest part of this section, which is to combine all the preceding results to one general theorem for matrices with entries of class  $\mathcal{K}$ , namely Theorem 5.2.

Proof of Theorem 5.2.

- (i) In the linear case we can identify  $\Gamma_{\mu}$  with a real matrix with nonnegative entries. Then there exists a positive vector v > 0 so that  $\Gamma_{\mu}v < v$  if the spectral radius  $\rho(\Gamma_{\mu}) < 1$ , cf. [2] or [24, Lemma 2.0.1, p.33]. For r > 0 this gives  $\Gamma_{\mu}rv < rv$ , i.e., a  $\mathcal{K}_{\infty}$ -path is given by  $\sigma(r) = rv$ .
- (ii) This is Theorem 8.11.
- (iii) This is Theorem 8.14.
- (iv) This is Proposition 8.4.  $\Box$

9. Remarks for the case of three subsystems. Recall that a construction of an  $\Omega$ -path  $\sigma$  for the case of two subsystem was given in [17]. We have seen that in a general case of  $n \in \mathbb{N}$  subsystems the construction involves more theory and topological properties of  $\Gamma_{\mu}$  that follow from the small gain condition. However in case

of three subsystems  $\sigma$  can be found by rather simple considerations. Here we provide this illustrative construction. Let us consider the special case  $\Gamma \in (\mathcal{K}_{\infty} \cup \{0\})^{3 \times 3}$ ,  $\mu_i(s) = s_1 + s_2 + s_3$ , i = 1, 2, 3, and for simplicity assume that  $\gamma_{ij} \in \mathcal{K}_{\infty}$  for all  $i \neq j$ , so that

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{bmatrix}, \quad \Gamma_{\mu}(s) = \begin{pmatrix} \gamma_{12}(s_2) + \gamma_{13}(s_3) \\ \gamma_{21}(s_1) + \gamma_{23}(s_3) \\ \gamma_{31}(s_1) + \gamma_{32}(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$
(9.1)

Fix  $s_1 \geq 0$ , then it follows that there is exactly one  $s_2$  satisfying

$$\gamma_{13}^{-1}(s_1 - \gamma_{12}(s_2)) = \gamma_{23}^{-1}(s_2 - \gamma_{21}(s_1)), \qquad (9.2)$$

since, for a fixed  $s_1$  the left side of (9.2) is a strictly decreasing function of  $s_2$  while the right side of (9.2) is strictly increasing one. The small gain condition (9.1) in particular assures that  $\gamma_{12}^{-1}(\gamma_{21}^{-1}(r)) > r$  for any r > 0. Let  $s_2^*$  be the solution of  $s_1 - \gamma_{12}(s_2) = 0$  and  $s_2^{**}$  be the solution of  $s_2 - \gamma_{21}(s_1) = 0$  then

$$s_2^* = \gamma_{12}^{-1}(s_1) = \gamma_{12}^{-1}(\gamma_{21}^{-1}(s_2^{**})) > s_2^{**}.$$

Hence the root of the left side of (9.2) is greater than the root of the right side of (9.2). This proves that for any  $s_1$  there is always exactly one  $s_2$  satisfying (9.2), see Figure 9.1.

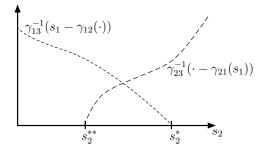


FIG. 9.1. Visualization of (9.2).

By the continuity and monotonicity of  $\gamma_{12}, \gamma_{21}, \gamma_{13}, \gamma_{23}$  it follows that  $s_2$  depends continuously on  $s_1$  and is strictly increasing with  $s_1$ . We can define  $\sigma_1(r) = r$  for  $r \ge 0$  and  $\sigma_2(r)$  to be the unique  $s_2$  solving (9.2) for  $s_1 = r$ .

continuously on  $s_1$  and is strictly increasing with  $s_1$ . We can define  $\sigma_1(r) = r$  for  $r \ge 0$  and  $\sigma_2(r)$  to be the unique  $s_2$  solving (9.2) for  $s_1 = r$ . Denote  $h(r) = \gamma_{31}(\sigma_1(r)) + \gamma_{32}(\sigma_2(r))$  and  $g(r) = \gamma_{13}^{-1}(\sigma_1(r) - \gamma_{12}(\sigma_2(r))) = \gamma_{23}^{-1}(\sigma_2(r) - \gamma_{21}(\sigma_1(r)))$ , and define  $M(r) := \{s_3 : h(r) < s_3 < g(r)\}$ . Let us show that  $M(r) \neq \emptyset$  for all r > 0. If this is not true then there exists  $r^* > 0$  such that  $s_3^* := h(r^*) \ge g(r^*)$  holds. Consider the point  $s^* := (s_1^*, s_2^*, s_3^*) := (r^*, \sigma_2(r^*), s_3^*)$ . Then  $s_3^* \ge g(r^*) = \gamma_{13}^{-1}(s_1^* - \gamma_{12}(s_2^*)), s_3^* \ge g(r^*) = \gamma_{23}^{-1}(s_2^* - \gamma_{21}(s_1^*))$ , and  $s_3^* = h(r^*) = \gamma_{31}(s_1^*) + \gamma_{32}(s_2^*)$ . In other words,

$$\Gamma(s^*) = \begin{pmatrix} \gamma_{12}(s_2^*) + \gamma_{13}(s_3^*) \\ \gamma_{21}(s_1^*) + \gamma_{23}(s_3^*) \\ \gamma_{31}(s_1^*) + \gamma_{32}(s_2^*) \end{pmatrix} \ge \begin{pmatrix} s_1^* \\ s_2^* \\ s_3^* \end{pmatrix},$$

contradicting (2.1). Hence M(r) is not empty for all r > 0.

Consider the functions h(r) and g(r). The question is how to choose  $\sigma_3(r) \in M(r)$  such that  $\sigma_3 \in \mathcal{K}_{\infty}$ . Note that  $h(r) \in \mathcal{K}_{\infty}$ . Let  $g^*(r) := \min_{u \ge r} g(u)$ , so that  $g^*(r) \le g(r)$  for all  $r \ge 0$ . Since h(r) is unbounded, for all r > 0 the set  $C(r) := \arg\min_{u \ge r} g(u)$  is compact and for all points  $p \in C(r)$  the relation  $g^*(r) \ge g(p) > h(p) \ge h(r)$  holds. We have  $h(r) < g^*(r) \le g(r)$  for all r > 0 where  $g^*$  is a (not necessarily strictly) increasing function. Now take  $\sigma_3(r) := \frac{1}{2}(g^*(r) + h(r))$  and observe that  $\sigma_3 \in \mathcal{K}_{\infty}$  and  $h(r) < \sigma_3(r) < g^*(r)$  for all r > 0. Hence  $\sigma := (\sigma_1, \sigma_2, \sigma_3)^T$  satisfies  $\Gamma_{\mu}(\sigma(r)) < \sigma(r)$  for all r > 0.

The case where one of  $\gamma_{ij}$  is not a  $\mathcal{K}_{\infty}$  function but zero can be treated similarly.

**10.** Conclusions. In this paper we have provided a method for the construction of ISS Lyapunov functions for interconnections of nonlinear ISS systems. The method applies for an interconnection of an arbitrary finite number of subsystems interconnected in an arbitrary way and satisfying a small gain condition. The small gain condition is imposed on the nonlinear gain operator  $\Gamma_{\mu}$  that we have introduced here. This operator contains the information of the topological structure of the network and the interactions between its subsystems. An ISS Lyapunov function for such a network is given in terms of ISS Lyapunov functions of subsystems and some auxiliary functions. We have shown how this construction is related to the small gain condition and mapping properties of the gain operator  $\Gamma_{\mu}$  and its invariant sets. Namely the small gain condition guarantees the existence of an unbounded vector function with path in an invariant set  $\Omega$  of the operator  $\Gamma_{\mu}$ . This auxiliary function can be used to rescale the ISS Lyapunov functions of the individual subsystems and aggregate them into an ISS Lyapunov function for the entire network. The construction technique for this vector function has been detailed as well as the construction of the composite Lyapunov function. The constructed Lyapunov function is only locally Lipschitz continuous, so that methods from nonsmooth analysis had to be used. The proposed method has been exemplified for linear systems and neural networks.

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