# Mortar Boundary Elements 

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#### Abstract

We establish a mortar boundary element scheme for hypersingular boundary integral equations representing elliptic boundary value problems in three dimensions. We prove almost quasi-optimal convergence of the scheme in broken Sobolev norms of order 1/2. Sub-domain decompositions can be geometrically non-conforming and meshes must be quasi-uniform only on sub-domains. Numerical results confirm the theory.


Key words: boundary element method, domain decomposition, mortar method, non-conforming Galerkin method
AMS Subject Classification: 65N55, 65N38

## 1 Introduction and model problem

In the finite element framework, mortar methods are used to discretize a given problem independently on sub-domains. It is a non-overlapping domain decomposition method. Necessary continuity requirements on interfaces of the sub-domains are implemented via Lagrangian multipliers. The motivation is to facilitate the construction of finite element meshes on complicated domains and to allow for parallelization. Bernardi, Maday and Patera introduced this technique and gave first analyses in [3, 4]. Later, geometrically non-conforming sub-domain decompositions and problems in $\mathbf{R}^{3}$ have been studied by Ben Belgacem and Maday [2, 1]. There is a large number of publications on mortar methods, all dealing with the discretization of differential equations of different types and with related numerical linear algebra. The first papers, just mentioned, derive a priori error estimates in the framework of non-conforming methods involving a Strang type estimate.

In this paper we establish a mortar setting for the boundary element method (BEM) and prove almost quasi-optimal convergence for a model problem involving the hypersingular operator of the Laplacian. The advantages of this domain decomposition scheme (easier construction of meshes and availability of parallel techniques) also apply to the BEM. To be precise, we apply the mortar technique directly to the boundary element discretization, not as a coupling procedure between boundary and finite elements as in [9]. The analysis of finite elements for the discretization of boundary integral equations of the first kind goes back to Nédélec and Planchard [21], and Hsiao and Wendland [17]. Stephan [22] studied

[^0]boundary elements for singular problems on open surfaces. Hypersingular boundary integral equations are well posed in fractional Sobolev spaces of order $1 / 2$ and conforming Galerkin discretizations require continuous basis functions. Due to the non-existence of a trace operator in these Sobolev spaces, needed for the analysis of interface conditions, mortar boundary elements give rise to a variational crime. Indeed, it turns out that there is no well-defined continuous variational formulation of the mortar setting for the BEM. Instead we will analyze the discrete mortar scheme as a non-conforming method for the original un-decomposed integral equation. We follow the analysis presented in 1 where projection and extension operators are used to bound the approximation error in the kernel space (of functions satisfying the Lagrangian multiplier condition). Note that there is a shorter presentation by Braess, Dahmen and Wiener [7] where the simpler argument [6, Remark III.4.6] is used to bound this error by a standard approximation error (in un-restricted spaces). Nevertheless, in our case the Strang type error estimate has a more complicated structure and it is not straightforward to follow the argument [6, Remark III.4.6].

We will make use of some preliminary results in [13, 16]. In 13 we studied the discretization of hypersingular operators on open surfaces using functions that vanish only in a discrete weak sense on the boundary of the surface. Such functions in general do not belong to the energy space of the operator and require a different variational setting. This setting will be used also for the mortar boundary elements. In [16] this setting served to establish (non-conforming) Crouzeix-Raviart boundary elements and to prove their quasi-optimal convergence. Main tool in that paper is a discrete fractional-order PoincaréFriedrichs inequality. It serves to show ellipticity of the principal bilinear form of the discrete scheme. In this paper we generalize this inequality to the geometrically non-conforming case, needed for general mortar decompositions. Again, it is needed to prove (quasi-) ellipticity of the principal bilinear form. Our model problem is defined on an open flat surface $\Gamma$ with polygonal boundary. We prove that, up to logarithmical terms, the mortar boundary element method converges quasi-optimally, subject to a compatibility condition of the boundary meshes and the meshes on the interfaces for the Lagrangian multipliers. Here we rely on the known Sobolev regularity of the exact solution leading to almost $O\left(h^{1 / 2}\right)$ convergence where $h$ is the maximum mesh size. Our techniques are applicable also to polyhedral surfaces and include meshes of shape-regular triangles and quadrilaterals.

An overview of this paper is as follows. In the rest of this section we recall definitions of fractional order Sobolev norms and formulate the model problem. In Section 2 we define the mortar scheme and present the main result (Theorem 2.1) establishing almost quasi-optimal convergence of the mortar boundary element method. Technical details and proofs are given in Section 3, In Section 4 we present some numerical results that underline the stated convergence of the mortar BEM.

First let us briefly define the needed Sobolev spaces. We consider standard Sobolev spaces where the following norms are used: For a bounded domain $S \subset \mathbf{R}^{n}$ and $0<s<1$ we define

$$
\|u\|_{H^{s}(S)}^{2}:=\|u\|_{L^{2}(S)}^{2}+|u|_{H^{s}(S)}^{2}
$$

with semi-norm

$$
|u|_{H^{s}(S)}:=\left(\int_{S} \int_{S} \frac{|u(x)-u(y)|^{2}}{|x-y|^{2 s+n}} d x d y\right)^{1 / 2} .
$$

For $0<s<1$ the space $\tilde{H}^{s}(S)$ is defined as the completion of $C_{0}^{\infty}(S)$ under the norm

$$
\|u\|_{\tilde{H}^{s}(S)}:=\left(|u|_{H^{s}(S)}^{2}+\int_{S} \frac{u(x)^{2}}{(\operatorname{dist}(x, \partial S))^{2 s}} d x\right)^{1 / 2} .
$$

For $s \in(0,1 / 2),\|\cdot\|_{\tilde{H}^{s}(S)}$ and $\|\cdot\|_{H^{s}(S)}$ are equivalent norms whereas for $s \in(1 / 2,1)$ there holds $\tilde{H}^{s}(S)=H_{0}^{s}(S)$, the latter space being the completion of $C_{0}^{\infty}(S)$ with norm in $H^{s}(S)$. Also we note that functions from $\tilde{H}^{s}(S)$ are continuously extendable by zero onto a larger domain. For details see, e.g., [18, 14]. For $s>0$, the spaces $H^{-s}(S)$ and $\tilde{H}^{-s}(S)$ are the dual spaces of $\tilde{H}^{s}(S)$ and $H^{s}(S)$, respectively.

Let $\Gamma$ be a plane open surface with polygonal boundary. In the following we will identify $\Gamma$ with a domain in $\mathbf{R}^{2}$, thus referring to sub-domains of $\Gamma$ rather than sub-surfaces. The boundary of $\Gamma$ is denoted by $\partial \Gamma$.

Our model problem is: For a given $f \in L^{2}(\Gamma)$ find $u \in \tilde{H}^{1 / 2}(\Gamma)$ such that

$$
\begin{equation*}
W u(x):=-\frac{1}{4 \pi} \frac{\partial}{\partial \mathbf{n}_{x}} \int_{\Gamma} u(y) \frac{\partial}{\partial \mathbf{n}_{y}} \frac{1}{|x-y|} d S_{y}=f(x), \quad x \in \Gamma \tag{1.1}
\end{equation*}
$$

Here, $\mathbf{n}$ is a normal unit vector on $\Gamma$, e.g. $\mathbf{n}=(0,0,1)^{T}$. We note that $W$ maps $\tilde{H}^{1 / 2}(\Gamma)$ continuously onto $H^{-1 / 2}(\Gamma)($ see [23] $)$. We have the following weak formulation of (1.1). Find $u \in \tilde{H}^{1 / 2}(\Gamma)$ such that

$$
\begin{equation*}
\langle W u, v\rangle_{\Gamma}=\langle f, v\rangle_{\Gamma} \quad \forall v \in \tilde{H}^{1 / 2}(\Gamma) \tag{1.2}
\end{equation*}
$$

Here, $\langle\cdot, \cdot\rangle_{\Gamma}$ denotes the duality pairing between $H^{-1 / 2}(\Gamma)$ and $\tilde{H}^{1 / 2}(\Gamma)$. Throughout, this generic notation will be used for other dualities as well, the domain mentioned by the index.

A standard boundary element method (BEM) for the approximate solution of (1.2) is to select a piecewise polynomial subspace $\tilde{X}_{h} \subset \tilde{H}^{1 / 2}(\Gamma)$ and to define an approximant $\tilde{u}_{h} \in \tilde{X}_{h}$ by

$$
\left\langle W \tilde{u}_{h}, v\right\rangle_{\Gamma}=\langle f, v\rangle_{\Gamma} \quad \forall v \in \tilde{X}_{h}
$$

Such a scheme is known to converge quasi-optimally in the energy norm, cf. Remark 2.1 below. In the numerical section we will compare such a conforming approximation with a mortar approximation, for the case where the meshes are globally conforming.

## 2 Mortar method and main result

In this section we introduce the mortar boundary element method for the approximate solution of the model problem (1.2). First we discuss the decomposition of $\Gamma$ into sub-domains. Then we introduce the discrete approximation spaces. The main result of this paper is given at the end of this section.

### 2.1 Sub-domain decomposition

We consider a decomposition of $\Gamma$ into non-intersecting sub-domains $\Gamma_{i}, i=1, \ldots, N$, giving rise to a coarse mesh

$$
\mathcal{T}:=\left\{\Gamma_{1}, \ldots, \Gamma_{N}\right\}
$$

For ease of presentation we assume that each $\Gamma_{i}$ is either a triangle or quadrilateral. More general decompositions into polygonal sub-domains can be dealt with by further decomposing into triangles and quadrilaterals and by considering conforming interface conditions on additional interfaces. The mesh $\mathcal{T}$ can be non-conforming but must satisfy the assumption (A1) below. The diameter of a sub-domain $\Gamma_{i}$ is denoted by $H_{i}$, and $H:=\max _{i=1, \ldots, N} H_{i}$. The interface between two neighboring sub-domains $\Gamma_{i}, \Gamma_{j}$
$\left(i \neq j, \bar{\Gamma}_{i} \cap \bar{\Gamma}_{j}\right.$ contains more than a point) is denoted by $\gamma_{i j}$. For our analysis below we need the following assumption.
(A1) Each interface $\gamma_{i j}$ consists of an entire edge of $\Gamma_{i}$ or $\Gamma_{j}$.
The (relatively) open edges of a sub-domain $\Gamma_{i}$ are $\gamma_{i}^{j}, j=1, \ldots, m$. Here, $m$ is a generic number ( $m=3$ if $\Gamma_{i}$ is a triangle and $m=4$ otherwise). Using the symbol $\partial \Gamma$ for the boundary of $\Gamma$, and similarly $\partial \Gamma_{i}$ for the boundary of $\Gamma_{i}$, the skeleton of the sub-domain decomposition is

$$
\gamma:=\cup_{i=1}^{N} \partial \Gamma_{i} \backslash \partial \Gamma .
$$

According to assumption (A1) the skeleton is covered by a set of non-intersecting edges $\gamma_{i j}$. We number the edges like $\gamma_{1}, \ldots, \gamma_{L}$, giving a decomposition of the skeleton like $\bar{\gamma}=\cup_{l=1}^{L} \bar{\gamma}_{l}$. In the following we will denote this decomposition of the skeleton by

$$
\tau:=\left\{\gamma_{1}, \ldots, \gamma_{L}\right\}
$$

We will refer to these edges as interface edges. Each interface edge $\gamma_{l}$ is the interface between two subdomains $\Gamma_{i}, \Gamma_{j}$ and is an entire edge of one or both of them. Given an integer $l(1 \leq l \leq L)$ we denote by $l_{\text {lag }}$ (respectively, $l_{\text {mor }}$ ) the number of a sub-domain which has $\gamma_{l}$ as an edge (respectively, the number of the other sub-domain),

$$
\gamma_{l}=\gamma_{l_{\text {lag }}, l_{\text {mor }}}
$$

As mentioned before, the selection of the index pair $\left(l_{\text {lag }}, l_{\text {mor }}\right)$ for $l \in\{1, \ldots, L\}$ is not unique but will be fixed for a specific sub-domain decomposition of $\Gamma$. Below, we will introduce a Lagrangian multiplier on the interfaces and on $\gamma_{l}$ we will use a mesh related to the mesh on $\Gamma_{l_{\text {lag }}}$. The side of $\gamma_{l}$ stemming from $\Gamma_{l_{\text {mor }}}$ is usually called mortar side in the finite element literature and this explains our notation. The side defining the Lagrangian multiplier is often called non-mortar side.

Corresponding to the decomposition of $\Gamma$ we will need the product Sobolev space

$$
H^{s}(\mathcal{T}):=\prod_{K \in \mathcal{T}} H^{s}(K)=\prod_{i=1}^{N} H^{s}\left(\Gamma_{i}\right)
$$

with usual product norm.

### 2.2 Meshes and discrete spaces

On each of the sub-domains $\Gamma_{i}(i \in\{1, \ldots, N\})$ we consider a (sequence of) regular, quasi-uniform meshes $\mathcal{T}_{i}$ consisting of shape regular triangles or quadrilaterals, $\bar{\Gamma}_{i}=\cup_{T \in \mathcal{T}_{i}} \bar{T}$. The maximum diameter of the elements of $\mathcal{T}_{i}$ is denoted by $h_{i}$ and we use the symbols

$$
\underline{h}:=\min _{i=1, \ldots, N} h_{i}, \quad h:=\max _{i=1, \ldots, N} h_{i} .
$$

Throughout the paper we assume without loss of generality that $h<1$. This makes the writing of logarithmic terms in $h$ easier.

In the case of $\Gamma$ being a square, Figure 2.1 shows a conforming sub-domain decomposition (a) and a non-conforming sub-domain decomposition (b), both with globally non-conforming meshes.


Figure 2.1: Sub-domain decompositions with non-conforming meshes.
Now we introduce discrete spaces on sub-domains consisting of piecewise (bi)linear functions,

$$
X_{h, i}:=\left\{v \in C^{0}\left(\Gamma_{i}\right) ;\left.v\right|_{T} \text { is a polynomial of degree one } \forall T \in \mathcal{T}_{i},\left.v\right|_{\partial \Gamma \cap \partial \Gamma_{i}}=0\right\}, \quad i=1, \ldots, N .
$$

The global discrete space on $\Gamma$ is

$$
X_{h}:=\prod_{i=1}^{N} X_{h, i} .
$$

Note that functions $v \in X_{h}$ do satisfy the homogeneous boundary condition along $\partial \Gamma$ but are in general discontinuous across interfaces. Therefore, $X_{h}$ is not a subspace of the energy space $\tilde{H}^{1 / 2}(\Gamma)$. Functions from different sub-domains will be coupled via a discrete Lagrangian multiplier on the skeleton. To this end we introduce a mesh on the skeleton $\gamma$ as follows.

On each interface edge $\gamma_{l}$ there is a trace mesh $\mathcal{T}_{l_{\text {lag }}} \mid \gamma_{l}$ inherited from the mesh $\mathcal{T}_{l_{\text {lag }}}$ on the sub-domain $\Gamma_{l_{\text {lag }}}$. (We recall that by definition, $\gamma_{l}$ is an entire edge of the sub-domain with number $l_{\text {lag }}$.) This trace mesh is quasi-uniform with mesh width $h_{l_{\text {lag }}}$. Now we introduce a new (coarser) quasi-uniform mesh $\mathcal{G}_{l}$ on $\gamma_{l}$ in such a way that the following assumption is satisfied.
(A2) For any $l \in\{1, \ldots, L\}$ there holds: the mesh $\mathcal{G}_{l}$ is a strict coarsening of the trace mesh $\left.\mathcal{T}_{\text {lag }}\right|_{\gamma_{l}}$. In particular, any interior node of $\mathcal{T}_{l_{\text {lag }}} \mid \gamma_{l}$ together with its two neighboring elements (intervals) is covered by one element of $\mathcal{G}_{l}$.

The mesh width (length of longest element) of $\mathcal{G}_{l}$ is denoted by $k_{l}$, and $k:=\max _{l=1, \ldots, L} k_{l}$. On each interface edge we define a space of piecewise constant functions,

$$
M_{k, l}:=\left\{v \in L^{2}\left(\gamma_{l}\right) ;\left.v\right|_{J} \text { is constant } \forall J \in \mathcal{G}_{l}\right\}, \quad l=1, \ldots, L .
$$

The space for the discrete Lagrangian multiplier then is

$$
M_{k}:=\prod_{l=1}^{L} M_{k, l} .
$$

Notations. The symbols " $\lesssim$ " and " $\gtrsim$ " will be used in the usual sense. In short, $a_{h}(v) \lesssim b_{h}(v)$ when there exists a constant $C>0$ independent of the discretization parameter $h$ and the involved function $v$ such that $a_{h}(v) \leq C b_{h}(v)$ for any $v$ of the given set. The double inequality $a_{h}(v) \lesssim b_{h}(v) \lesssim a_{h}(v)$ is simplified to $a_{h}(v) \simeq b_{h}(v)$. The generic constant $C$ above is usually also independent of appearing fractional Sobolev indexes $\epsilon>0$, but this will be mentioned. We note that these notations usually do not mean independence of involved constants on the decomposition $\mathcal{T}$ of $\Gamma$. In this paper we consider a generic decomposition $\mathcal{T}$ which is fixed and estimates will in general depend on $\mathcal{T}$.

Throughout the paper we also will use the notation $v_{j}$ for the restriction of a function $v$ to the sub-domain $\Gamma_{j}$.

### 2.3 Setting of the mortar boundary element method and main result

For the setup of the mortar boundary element method we need some operators. We introduce the surface differential operators

$$
\operatorname{curl} \varphi:=\left(\partial_{x_{2}} \varphi,-\partial_{x_{1}} \varphi, 0\right), \quad \operatorname{curl} \varphi:=\partial_{x_{1}} \varphi_{2}-\partial_{x_{2}} \varphi_{1} \quad \text { for } \quad \varphi=\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right) .
$$

The definitions of the surface curl operators are appropriate just for flat surfaces (as in our case) but can be extended to open and closed Lipschitz surfaces, cf. [8, 13]. We define corresponding piecewise differential operators $\operatorname{curl}_{H} v$ and $\operatorname{curl}_{H} \phi$ by

$$
\operatorname{curl}_{H} v:=\sum_{i=1}^{N}\left(\operatorname{curl}_{\Gamma_{i}} v_{i}\right)^{0}, \quad \operatorname{curl}_{H} \phi:=\sum_{i=1}^{N}\left(\operatorname{curl}_{\Gamma_{i}} \phi_{i}\right)^{0} .
$$

The notations $\operatorname{curl}_{\Gamma_{i}}$ and $\operatorname{curl}_{\Gamma_{i}}$ refer to the restrictions of curl and curl, respectively, onto $\Gamma_{i}$, and $(\cdot)^{0}$ indicates extension by zero to $\Gamma$. We made use of the notation introduced before, $v_{i}=\left.v\right|_{\Gamma_{i}}, \phi_{i}=\left.\phi\right|_{\Gamma_{i}}$. Furthermore, we need the single layer potential operator $V$ defined by

$$
V \boldsymbol{\varphi}(x):=\frac{1}{4 \pi} \int_{\Gamma} \frac{\varphi(y)}{|x-y|} d S_{y}, \quad \boldsymbol{\varphi} \in\left(\tilde{H}^{-1 / 2}(\Gamma)\right)^{3}, x \in \Gamma
$$

For the formulation of the mortar boundary element method we define, for sufficiently smooth functions $v, w, \mu$, the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ by

$$
\begin{aligned}
a(v, w) & :=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} w\right\rangle_{\mathcal{T}}:=\sum_{i=1}^{N}\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{\Gamma_{i}} w\right\rangle_{\Gamma_{i}}, \\
b(v, \mu) & :=\langle[v], \mu\rangle_{\tau}:=\sum_{l=1}^{L}\langle[v], \mu\rangle_{\gamma_{l}} .
\end{aligned}
$$

Here, as mentioned before, for a domain $S \subset \Gamma$ or an arc $S,\langle\cdot, \cdot\rangle_{S}$ denotes the $L^{2}(S)$-inner product and its extension by duality, and $[v]$ is the jump of $v$ across $\gamma$, more precisely

$$
\left.[v]\right|_{\gamma_{l}}=\left.v_{l_{\text {lag }}}\right|_{\gamma_{l}}-\left.v_{l_{\text {mor }}}\right|_{\gamma_{l}}, \quad l=1, \ldots, L
$$

Of course, for sufficiently smooth functions $v, w, \mu$ there holds

$$
a(v, w)=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} w\right\rangle_{\Gamma}, \quad b(v, \mu)=\langle[v], \mu\rangle_{\gamma} .
$$

Note that we will use the introduced notations $\langle\cdot, \cdot\rangle_{\mathcal{T}}$ and $\langle\cdot, \cdot\rangle_{\tau}$ for duality pairings of product spaces corresponding to the given decompositions ( $\mathcal{T}$ and $\tau$ ). We also define, for a sufficiently smooth function $v$, the linear form

$$
F(v):=\sum_{i=1}^{N}\left\langle f_{i}, v_{i}\right\rangle_{\Gamma_{i}}=\langle f, v\rangle_{\mathcal{T}}
$$

where $f \in L^{2}(\Gamma)$ is the function given in (1.1). The mortar boundary element method for the approximate solution of (1.2) then reads: Find $u_{h} \in X_{h}$ and $\lambda_{k} \in M_{k}$ such that

$$
\begin{array}{lll}
a\left(u_{h}, v\right)+b\left(v, \lambda_{k}\right) & =F(v) & \forall v \in X_{h} \\
b\left(u_{h}, \psi\right) & =0 & \forall \psi \in M_{k} \tag{2.1}
\end{array}
$$

This scheme is equivalent to: Find $u_{h} \in V_{h}$ such that

$$
a\left(u_{h}, v\right)=F(v) \quad \forall v \in V_{h}
$$

where

$$
\begin{equation*}
V_{h}=\left\{v \in X_{h} ; b(v, \psi)=0 \quad \forall \psi \in M_{k}\right\} . \tag{2.2}
\end{equation*}
$$

The main result of this paper is as follows.
Theorem 2.1. There exists a unique solution $\left(u_{h}, \lambda_{k}\right)$ of (2.1). Assume that the solution $u$ of (1.2) satisfies $u \in \tilde{H}^{1 / 2+r}(\Gamma)(r \in(0,1 / 2])$. Then there holds

$$
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})} \lesssim\left(|\log \underline{h}|^{2} h^{r}+|\log \underline{h}|^{3 / 2} k^{r}\right)\|u\|_{\tilde{H}^{1 / 2+r}(\Gamma)} .
$$

For proportional mesh sizes $h$ and $k$ this means that

$$
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})} \lesssim|\log \underline{h}|^{2} h^{r}\|u\|_{\tilde{H}^{1 / 2+r}(\Gamma)}
$$

The appearing constants in the estimates above are independent of $h$ and $k$ provided that the assumptions on the meshes, in particular (A1) and (A2), are satisfied.

A proof of this theorem is given at the end of Section 3,
Remark 2.1. As in [13] we note that, in our case of an open surface $\Gamma$, the solution $u$ of (1.1) has strong corner and corner-edge singularities which cannot be exactly described by standard Sobolev regularity. It is well known that $u \in \tilde{H}^{s}(\Gamma)$ for any $s<1$ (see, e.g., [24]) so that the error estimate by Theorem [2.1] holds for any $r<1 / 2$. In general $u \notin H_{0}^{1}(\Gamma)$ but a more specific error analysis for the conforming BEM yields for quasi-uniform meshes the optimal error estimate

$$
\left\|u-u_{h}\right\|_{\tilde{H}^{1 / 2}(\Gamma)} \lesssim h^{1 / 2}
$$

see [5]. The logarithmical perturbations in $\underline{h}$ of our error estimate are due to the non-conformity of the mortar method. They stem from the non-existence of a trace operator within $H^{1 / 2}(\Gamma)$ and from non-local properties of the fractional order Sobolev norms (the difference between $\tilde{H}^{1 / 2}$ and $H^{1 / 2}$-spaces).

## 3 Technical details and proof of the main result

We start by citing some technical results (Lemmas 3.1-3.4) which are needed to deal with the fractional order Sobolev norms. Afterwards we study a discrete Poincaré-Friedrichs inequality (Proposition 3.1) which will be applied to prove ellipticity of the bilinear form $a(\cdot, \cdot)$ on $V_{h}$. Afterwards an integration-byparts formula for the hypersingular operator is recalled from [13] and adapted to our situation of many sub-domains. Then, Lemma 3.5 states the well-posedness of integration by parts. Lemmas 3.6 3.13 study requirements for the Babuška-Brezzi theory and provide details for a Strang-type error estimate which is given by Theorem 3.1. Later, Lemmas 3.14 and 3.15 are needed to analyze the bound of the Strang-type estimate and lead to Theorem 3.2 which gives a general a priori error estimate for the mortar BEM. The section is finished by giving a proof of the main result (Theorem 2.1).
Lemma 3.1. [15, Lemma 5] Let $R \subset \mathbf{R}^{2}$ be a Lipschitz domain. There exists $C>0$ such that

$$
\|v\|_{\tilde{H}^{s}(R)} \leq \frac{C}{1 / 2-|s|}\|v\|_{H^{s}(R)} \quad \forall s \in(-1 / 2,1 / 2), \quad \forall v \in H^{s}(R)
$$

Lemma 3.2. Let $R \subset \mathbf{R}^{2}$ be a Lipschitz domain, and let $v$ be a piecewise linear function defined on a quasi-uniform mesh on $R$ with mesh size $h<1$. There exists a constant $C>0$ which is independent of $h$ (but may depend on $R$ ) such that there holds

$$
\|v\|_{\tilde{H}^{-1 / 2}(R)} \leq C|\log h|\|v\|_{H^{-1 / 2}(R)}
$$

Proof. By [15, Lemma 6] there holds for a piecewise polynomial function of degree $p$ the estimate

$$
\|v\|_{\tilde{H}^{-1 / 2}(R)} \leq C \log \left(\frac{p+1}{h}\right)\|v\|_{H^{-1 / 2}(R)}, \quad p \geq 0, h<1
$$

Fixing $p$ gives the claimed bound. The proof of [15, Lemma 6] gives full details for rectangular meshes. For triangular meshes the proof applies as well by making use of Schmidt's inequality for triangles, cf. [11, Lemma 5.1]. Nevertheless, we are considering only polynomials of low degrees where Schmidt's inequality is not needed.

Lemma 3.3. [13, Lemma 4.3] Let $R \subset \mathbf{R}^{2}$ be a bounded Lipschitz domain. There exists $C>0$ such that, for any $\epsilon \in(0,1 / 2)$, there holds

$$
\|v\|_{L^{2}(\partial R)} \leq \frac{C}{\epsilon^{1 / 2}}\|v\|_{H^{1 / 2+\epsilon}(R)} \quad \forall v \in H^{1 / 2+\epsilon}(R)
$$

Here $\partial R$ is the boundary of $R$.
Lemma 3.4. For $S$ being one of the sub-domains $\Gamma_{i} \in \mathcal{T}$ or $\Gamma$ there holds

$$
\begin{equation*}
|v|_{H^{1 / 2}(S)} \lesssim\left\|\operatorname{curl}_{S} v\right\|_{\boldsymbol{H}_{\mathrm{t}}^{-1 / 2}(S)} \quad \forall v \in H^{1 / 2}(S) \tag{3.1}
\end{equation*}
$$

The restriction of $\operatorname{curl}_{S}$ onto $\tilde{H}^{1 / 2}(S)$ is continuous,

$$
\begin{equation*}
\operatorname{curl}_{S}: \tilde{H}^{1 / 2}(S) \rightarrow \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(S) \tag{3.2}
\end{equation*}
$$

Moreover, there holds the continuity

$$
\begin{equation*}
\operatorname{curl}_{S}: H^{1 / 2+s}(S) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{-1 / 2+s}(S) \quad \forall s \in[0,1 / 2] \tag{3.3}
\end{equation*}
$$

Proof. The bounds (3.1) and (3.2) are proved by Lemmas 4.1 and 2.2 in [13], respectively. By Lemma 2.1 in [13], $\operatorname{curl}_{S}: H^{1 / 2}(S) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{-1 / 2}(S)$ is continuous, and $\operatorname{curl}_{S}: H^{1}(S) \rightarrow \boldsymbol{L}_{\mathrm{t}}^{2}(S)=\boldsymbol{H}_{\mathrm{t}}^{0}(S)$ is continuous as well. Estimate (3.3) then follows by interpolation.

The following result is a generalized version of a discrete Poincaré-Friedrichs inequality in fractional order Sobolev spaces, cf. Theorem 8 in [16.

Proposition 3.1. There exists a constant $C>0$, independent of the decomposition $\mathcal{T}$ as long as subdomains are shape-regular, such that for all $\epsilon \in(0,1 / 2]$ there holds

$$
\|v\|_{L^{2}(\Gamma)}^{2} \leq C\left(\epsilon^{-1}|v|_{H^{1 / 2+\epsilon}(\mathcal{T})}^{2}+\sum_{l=1}^{L}\left|\gamma_{l}\right|^{-1-2 \epsilon}\left(\int_{\gamma_{l}}[v] d s\right)^{2}\right) \quad \forall v \in H^{1 / 2+\epsilon}(\mathcal{T}),\left.v\right|_{\partial \Gamma}=0 .
$$

Here, $\left|\gamma_{l}\right|$ denotes the length of $\gamma_{l}$.
Proof. For the case of conforming decompositions $\tilde{\mathcal{T}}$ of $\Gamma$ into triangles, [16, Theorem 8] proves that there holds

$$
\begin{equation*}
\|v\|_{L^{2}(\Gamma)}^{2} \leq C\left(\epsilon^{-1}|v|_{H^{1 / 2+\epsilon}(\tilde{\mathcal{T}})}^{2}+\sum_{l=1}^{L}\left|\gamma_{l}\right|^{-1-2 \epsilon}\left(\int_{\gamma_{l}}[v] d s\right)^{2}+\left|\int_{\Gamma} v d x\right|^{2}\right) \quad \forall v \in H^{1 / 2+\epsilon}(\tilde{\mathcal{T}}) . \tag{3.4}
\end{equation*}
$$

It is easy to see that the mean zero term can be avoided by assuming the homogeneous boundary condition for $v$. To obtain the result for our non-conforming decomposition $\mathcal{T}$ including quadrilaterals we introduce further edges to reduce quadrilateral sub-domains to triangles and to transform $\mathcal{T}$ into a conforming decomposition $\tilde{\mathcal{T}}$. By definition of the Sobolev-Slobodeckij semi-norm there holds

$$
|v|_{H^{1 / 2+\epsilon}(\tilde{\mathcal{T}})} \leq|v|_{H^{1 / 2+\epsilon}(\mathcal{T})} .
$$

We note that for new edges $\gamma^{\prime}$ there holds $\left.[v]\right|_{\gamma^{\prime}}=0$ by the trace theorem and the regularity $v \in H^{1 / 2+\epsilon}(T)$ for any $T \in \mathcal{T}$. The result then follows from (3.4).

Following [13] we now examine an integration-by-parts formula for the hypersingular operator. For a smooth scalar function $v$ and a smooth tangential vector field $\varphi$, integration by parts gives

$$
\left\langle\operatorname{curl}_{\Gamma_{i}} v, \boldsymbol{\varphi}\right\rangle_{\Gamma_{i}}=\left\langle v, \operatorname{curl}_{\Gamma_{i}} \boldsymbol{\varphi}\right\rangle_{\Gamma_{i}}-\left\langle v, \boldsymbol{\varphi} \cdot \mathbf{t}_{i}\right\rangle_{\partial \Gamma_{i}}, \quad i=1, \ldots, N .
$$

Here, $\mathbf{t}_{i}$ is the unit tangential vector on $\partial \Gamma_{i}$ (oriented mathematically positive when identifying $\Gamma_{i}$ with a subset of $\mathbf{R}^{2}$ which is compatible with the identification of $\Gamma$ as a subset of $\mathbf{R}^{2}$ ). Applying this formula to $\varphi=\left.\left(V \operatorname{curl}_{\Gamma} u\right)\right|_{\Gamma_{i}}$, we obtain for smooth functions $v$ and $u$

$$
\left\langle\mathbf{t}_{i} \cdot V \operatorname{curl}_{\Gamma} u, v_{i}\right\rangle_{\partial \Gamma_{i}}=\left\langle\operatorname{curl}_{\Gamma_{i}} V \operatorname{curl}_{\Gamma} u, v_{i}\right\rangle_{\Gamma_{i}}-\left\langle V \operatorname{curl}_{\Gamma} u, \operatorname{curl}_{\Gamma_{i}} v_{i}\right\rangle_{\Gamma_{i}} .
$$

Now we sum over $i$ and take into account that $t_{i}=-t_{j}$ on $\gamma_{i j}$. Further we let $\gamma_{0}:=\partial \Gamma$, use the convention for the jump $\left.[v]\right|_{\gamma_{0}}=\left.v\right|_{\gamma_{0}}$, denote by $\mathbf{t}_{0}$ the unit tangential vector along $\partial \Gamma$ (again mathematically positive
oriented) and let $0_{\text {lag }}:=0$ (remember the notation $l_{\text {lag }}$ and $l_{\text {mor }}$ for the numbers of the Lagrangian multiplier side and mortar side of $\gamma_{l}$, respectively). This yields

$$
\begin{aligned}
\sum_{l=0}^{L}\left\langle\mathbf{t}_{l_{\text {lag }}} \cdot V \operatorname{curl}_{\Gamma} u,[v]\right\rangle_{\gamma_{l}} & =\sum_{i=1}^{N}\left\langle\operatorname{curl}_{\Gamma_{i}} V \operatorname{curl}_{\Gamma} u, v_{i}\right\rangle_{\Gamma_{i}}-\sum_{i=1}^{N}\left\langle V \operatorname{curl}_{\Gamma} u, \operatorname{curl}_{\Gamma_{i}} v_{i}\right\rangle_{\Gamma_{i}} \\
& =\left\langle\operatorname{curl}_{\Gamma} V \operatorname{curl}_{\Gamma} u, v\right\rangle_{\mathcal{T}}-\left\langle V \operatorname{curl}_{\Gamma} u, \operatorname{curl}_{H} v\right\rangle_{\mathcal{T}}
\end{aligned}
$$

for a piecewise (with respect to $\mathcal{T}$ ) smooth function $v$ on $\Gamma$ with $v_{i}:=\left.v\right|_{\Gamma_{i}}$, as defined before. In the last step we used the fact that

$$
\operatorname{curl}_{H} w=\operatorname{curl}_{\Gamma} w \quad \forall w \in \boldsymbol{H}_{\mathrm{t}}^{1 / 2}(\Gamma)
$$

which holds by a density argument and the continuity of $\operatorname{curl}_{\Gamma}: \boldsymbol{H}_{\mathrm{t}}^{1 / 2}(\Gamma) \rightarrow H^{-1 / 2}(\Gamma)$ as the adjoint operator of $\boldsymbol{c u r l}_{\Gamma}: \tilde{H}^{1 / 2}(\Gamma) \rightarrow \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)$, cf. (3.2).

Now we use the relation

$$
W u=\operatorname{curl}_{\Gamma} V \operatorname{curl}_{\Gamma} u \quad\left(u \in \tilde{H}^{1 / 2}(\Gamma)\right)
$$

see [19, 20] and [13, Lemma 2.3]. Then choosing a piecewise smooth function $v$ with $\left.v\right|_{\partial \Gamma}=0$ we obtain

$$
\begin{equation*}
\langle\lambda,[v]\rangle_{\tau}=\sum_{l=1}^{L}\langle\lambda,[v]\rangle_{\gamma_{l}}=\langle W u, v\rangle_{\mathcal{T}}-\left\langle V \operatorname{curl}_{\Gamma} u, \operatorname{curl}_{H} v\right\rangle_{\mathcal{T}} \tag{3.5}
\end{equation*}
$$

Here, $\lambda$ denotes our Lagrangian multiplier on the skeleton $\gamma$ defined by

$$
\begin{equation*}
\left.\lambda\right|_{\gamma_{l}}:=\left.\mathbf{t}_{l_{\mathrm{lag}}} \cdot\left(V \operatorname{curl}_{\Gamma} u\right)\right|_{\gamma_{l}}, \quad l=1, \ldots, L \tag{3.6}
\end{equation*}
$$

Relation (3.5) does not extend to $v \in H^{1 / 2}(\mathcal{T})$ since the trace of such a function $v$ onto $\gamma$ is not well defined. However, there holds the following lemma.
Lemma 3.5. For $u \in \tilde{H}^{1 / 2}(\Gamma)$ with $W u=f \in L^{2}(\Gamma)$, (3.5) defines $\lambda \in \prod_{l=1}^{L} H^{-s}\left(\gamma_{l}\right)$ for any $s \in(0,1 / 2]$.
Remark 3.1. The above lemma can be extended to values of s larger than $1 / 2$. Though small values of $s$ represent the interesting cases, the limit $s=0$ being excluded. Also, the condition on $f$ can be relaxed but excluding the case $f \in H^{-1 / 2}(\Gamma)$ which is the standard regularity using the mapping properties of the hypersingular operator.

Proof of Lemma 3.5. We must show that $\lambda$ defined by (3.5) is a bounded linear functional on $\prod_{l=1}^{L} \tilde{H}^{s}\left(\gamma_{l}\right)$, the dual space of $\prod_{l=1}^{L} H^{-s}\left(\gamma_{l}\right)$.

Let $v \in \prod_{l=1}^{L} \tilde{H}^{s}\left(\gamma_{l}\right)$ be given. We continuously extend $v$ to an element $\tilde{v} \in H^{s+1 / 2}(\mathcal{T})$ with $\tilde{v}=0$ on $\partial \Gamma$ such that $\left.[\tilde{v}]\right|_{\gamma_{l}}=\left.v\right|_{\gamma_{l}}$. (Simply extend $v$ on each interface edge $\gamma_{l}$ to a function in $H^{s+1 / 2}\left(\Gamma_{l_{\text {lag }}}\right)$ vanishing on $\partial \Gamma_{l_{\text {lag }}} \backslash \gamma_{l}$ and extend by zero to the rest of $\Gamma$. Then sum up with respect to $l$.) The definition of $\lambda$ is independent of the particular extension $\tilde{v}$, see [13] for details in the case of one sub-domain. Using a duality estimate we obtain from (3.5)

$$
\begin{align*}
\sum_{l=1}^{L}\langle\lambda,[v]\rangle_{\gamma_{l}} & =\langle f, \tilde{v}\rangle_{\mathcal{T}}-\left\langle V \operatorname{curl}_{\Gamma} u, \operatorname{curl}_{H} \tilde{v}\right\rangle_{\mathcal{T}} \\
& \leq\|f\|_{L^{2}(\Gamma)}\|\tilde{v}\|_{L^{2}(\Gamma)}+\sum_{i=1}^{N}\left\|V \operatorname{curl}_{\Gamma} u\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{1 / 2-s}\left(\Gamma_{i}\right)}\left\|\operatorname{curl}_{\Gamma_{i}} \tilde{v}_{i}\right\|_{\boldsymbol{H}_{\mathrm{t}}^{s-1 / 2}\left(\Gamma_{i}\right)} \tag{3.7}
\end{align*}
$$

Now, for $s \in(0,1 / 2]$ the norms in $\tilde{\boldsymbol{H}}_{\mathrm{t}}^{1 / 2-s}\left(\Gamma_{i}\right)$ and $\boldsymbol{H}_{\mathrm{t}}^{1 / 2-s}\left(\Gamma_{i}\right)$ are equivalent (cf., e.g., [18]) so that together with the mapping property of $V$ [10],

$$
V: \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2-s}(\Gamma) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{1 / 2-s}(\Gamma),
$$

and (3.2) we obtain

$$
\sum_{i=1}^{N}\left\|V \operatorname{curl}_{\Gamma} u\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{1 / 2-s}\left(\Gamma_{i}\right)}^{2} \lesssim \sum_{i=1}^{N}\left\|V \operatorname{curl}_{\Gamma} u\right\|_{\boldsymbol{H}_{\mathrm{t}}^{1 / 2-s}\left(\Gamma_{i}\right)}^{2} \lesssim\left\|V \operatorname{curl}_{\Gamma} u\right\|_{\boldsymbol{H}_{\mathrm{t}}^{1 / 2-s}(\Gamma)}^{2} \lesssim\|u\|_{\tilde{H}^{1 / 2}(\Gamma)}^{2} .
$$

Here, the appearing constants are independent of $u$ but may depend on $s$. Also, using (3.3) we are able to bound (with constant independent of $\tilde{v}$ )

$$
\sum_{i=1}^{N}\left\|\operatorname{curl}_{\Gamma_{i}} \tilde{v}_{i}\right\|_{\boldsymbol{H}_{\mathrm{t}}^{s-1 / 2}\left(\Gamma_{i}\right)}^{2} \lesssim \sum_{i=1}^{N}\left\|\tilde{v}_{i}\right\|_{H^{s+1 / 2}\left(\Gamma_{i}\right)}^{2}=\|\tilde{v}\|_{H^{s+1 / 2}(\mathcal{T})}^{2}
$$

Taking the last two estimates into account, (3.7) proves that

$$
\sum_{l=1}^{L}\langle\lambda,[v]\rangle_{\gamma_{l}} \lesssim\left(\|f\|_{L^{2}(\Gamma)}+\|u\|_{\tilde{H}^{1 / 2}(\Gamma)}\right)\|\tilde{v}\|_{H^{s+1 / 2}(\mathcal{T})}
$$

Using the continuity of the extension (with constant independent of $v$ )

$$
\|\tilde{v}\|_{H^{s+1 / 2}(\mathcal{T})}^{2} \lesssim \sum_{l=1}^{L}\|v\|_{\tilde{H}^{s}\left(\gamma_{l}\right)}^{2}
$$

finishes the proof.

## Lemma 3.6.

$$
\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}^{2} \gtrsim|\log \underline{h}|^{-1}\|v\|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in V_{h}
$$

Proof. By (3.1) there holds

$$
\begin{equation*}
\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}^{2} \gtrsim \sum_{i=1}^{N}\left\|\operatorname{curl}_{\Gamma_{i}} v_{i}\right\|_{\boldsymbol{H}_{\mathrm{t}}^{-1 / 2}\left(\Gamma_{i}\right)}^{2} \gtrsim \sum_{i=1}^{N}\left|v_{i}\right|_{H^{1 / 2}\left(\Gamma_{i}\right)}^{2}=|v|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in H^{1 / 2}(\mathcal{T}) . \tag{3.8}
\end{equation*}
$$

For $v \in V_{h}$ there holds $\int_{\gamma_{l}}[v] d s=0$ for any interface edge $\gamma_{l}$ since by construction $M_{k}$ contains the piecewise constant function which has the value 1 on $\gamma_{l}$ and vanishes on $\gamma \backslash \gamma_{l}$. For the definition of $V_{h}$ (the discrete kernel of $b(\cdot, \cdot)$ ) see (2.2). Therefore, Proposition 3.1 proves that

$$
\|v\|_{L^{2}(\Gamma)}^{2} \lesssim \epsilon^{-1}|v|_{H^{1 / 2+\epsilon}(\mathcal{T})}^{2} \quad \forall v \in V_{h}
$$

Here, the appearing constant is independent of $\epsilon \in(0,1 / 2]$. Making use of the inverse property we bound

$$
\sum_{i=1}^{N}|v|_{H^{1 / 2+\epsilon}\left(\Gamma_{i}\right)}^{2} \lesssim \sum_{i=1}^{N} h_{i}^{-2 \epsilon}|v|_{H^{1 / 2}\left(\Gamma_{i}\right)}^{2} \quad \forall v \in V_{h}
$$

so that, with the previous estimate,

$$
\begin{equation*}
\|v\|_{L^{2}(\Gamma)}^{2} \lesssim \epsilon^{-1} \underline{h}^{-2 \epsilon}|v|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in V_{h} . \tag{3.9}
\end{equation*}
$$

Selecting $\epsilon=|\log \underline{h}|^{-1}$ (for $\underline{h}$ being small enough) and combining (3.9) with (3.8) proves the statement.

Lemma 3.7. The bilinear form $a(\cdot, \cdot)$ is almost uniformly $V_{h}$-elliptic. More precisely there hold the lower bounds

$$
a(v, v) \gtrsim|\log \underline{h}|^{-1}\|v\|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in V_{h}
$$

and

$$
\begin{equation*}
a(v, v) \gtrsim|\log \underline{h}|^{-1 / 2}\|v\|_{H^{1 / 2}(\mathcal{T})}\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)} \quad \forall v \in V_{h} \tag{3.10}
\end{equation*}
$$

Proof. First we note that for $v \in V_{h} \subset L^{2}(\Gamma)$ there holds $V \operatorname{curl}_{H} v \in \boldsymbol{L}_{\mathrm{t}}^{2}(\Gamma)$ so that

$$
\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} v\right\rangle_{\mathcal{T}}=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} v\right\rangle_{\Gamma} .
$$

Using the ellipticity of $V: \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{1 / 2}(\Gamma)$ and Lemma 3.6 we then obtain for $v \in V_{h}$

$$
a(v, v)=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} v\right\rangle_{\Gamma} \gtrsim\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}^{2} \gtrsim|\log \underline{h}|^{-1}\|v\|_{H^{1 / 2}(\mathcal{T})}^{2},
$$

which is the first assertion. The estimate (3.10) is obtained by bounding $\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}{ }_{(\Gamma)}}$ only once with the help of Lemma 3.6.

Lemma 3.8. The bilinear form $a(\cdot, \cdot)$ is almost uniformly continuous on $X_{h}$. More precisely there holds

$$
a(v, w) \lesssim|\log \underline{h}|^{2}\|v\|_{H^{1 / 2}(\mathcal{T})}\|w\|_{H^{1 / 2}(\mathcal{T})} \quad \forall v, w \in X_{h}
$$

Proof. As in the proof of Lemma 3.7 we note that for $v, w \in X_{h} \subset L^{2}(\Gamma)$ there holds

$$
\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} w\right\rangle_{\mathcal{T}}=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} w\right\rangle_{\Gamma}
$$

Then, using the continuity of $V: \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{1 / 2}(\Gamma)$ and the estimate for fractional order Sobolev norms $\|\cdot\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}{ }_{(\Gamma)}} \lesssim\|\cdot\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})}$, we obtain for $v, w \in X_{h}$

$$
\begin{align*}
a(v, w)=\left\langle V \operatorname{curl}_{H} v, \operatorname{curl}_{H} w\right\rangle_{\Gamma} & \lesssim\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)} \\
& \lesssim\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})}\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}}^{-1 / 2}(\mathcal{T}) \tag{3.11}
\end{align*}
$$

Now making use of Lemma 3.2 and (3.3) we bound

$$
\left\|\operatorname{curl}_{\Gamma_{i}} v_{i}\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}\left(\Gamma_{i}\right)}^{2} \lesssim\left|\log h_{i}\right|^{2}\left\|\operatorname{curl}_{\Gamma_{i}} v_{i}\right\|_{\boldsymbol{H}_{\mathrm{t}}^{-1 / 2}\left(\Gamma_{i}\right)}^{2} \lesssim\left|\log h_{i}\right|^{2}\left\|v_{i}\right\|_{H^{1 / 2}\left(\Gamma_{i}\right)}^{2},
$$

giving

$$
\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})} \lesssim|\log \underline{h}|\|v\|_{H^{1 / 2}(\mathcal{T})} \quad \forall v \in X_{h} .
$$

Combination with (3.11) proves the statement.

For the eventual error estimate we need the boundedness of the bilinear form $a(\cdot, \cdot)$. However, Lemma 3.8 is not applicable to non-discrete functions. Instead we will use the next lemma.

Lemma 3.9. Assume that $u \in \tilde{H}^{1 / 2+r}(\Gamma)(r>0)$. Then there holds

$$
a(u-v, w) \lesssim s^{-1}\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})} \quad \forall v, w \in X_{h}, \forall s \in(0, \min \{r, 1 / 2\}] .
$$

In particular, the appearing constant is independent of $s$.
Proof. First we note that (3.11) holds also for continuous functions, so that

$$
a(u-v, w) \lesssim\left\|\operatorname{curl}_{H}(u-v)\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})}\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\mathcal{T})} \quad \forall v, w \in X_{h} .
$$

Using the continuous injection $\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2+s}\left(\Gamma_{i}\right) \rightarrow \tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}\left(\Gamma_{i}\right)$ and Lemma 3.1 we bound for $i \in\{1, \ldots, N\}$

$$
\left\|\operatorname{curl}_{\Gamma_{i}}\left(u_{i}-v_{i}\right)\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}\left(\Gamma_{i}\right)} \leq\left\|\operatorname{curl}_{\Gamma_{i}}\left(u_{i}-v_{i}\right)\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2+s}\left(\Gamma_{i}\right)} \lesssim s^{-1}\left\|\operatorname{curl}_{\Gamma_{i}}\left(u_{i}-v_{i}\right)\right\|_{\boldsymbol{H}_{\mathrm{t}}^{-1 / 2+s}\left(\Gamma_{i}\right)} .
$$

The continuity of $\operatorname{curl}_{\Gamma_{i}}: \quad H^{1 / 2+s}\left(\Gamma_{i}\right) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{-1 / 2+s}\left(\Gamma_{i}\right)$ for any $i \in\{1, \ldots, N\}$ by (3.3) finishes the proof.

In order to analyze the error bound of the Strang-type estimate by Theorem 3.1 below, we need to extend functions from interface edges to sub-domains. This is also required to prove an inf-sup condition for the bilinear form $b(\cdot, \cdot)$.

To this end let us define extension operators that extend piecewise linear functions from interface edges to piecewise (bi)linear functions on the corresponding Lagrangian sub-domain,

$$
\begin{equation*}
E_{l}:\left.X_{h, l_{\text {lag }}}\right|_{\bar{\gamma}_{l}} \rightarrow X_{h, l_{\text {lag }}}, \quad l=1, \ldots, L \tag{3.12}
\end{equation*}
$$

Here, for $v \in X_{h, l_{\text {lag }}}{\overline{\gamma_{\gamma}}}$ the extension $E_{l} v$ is defined as the function of $X_{h, l_{\text {lag }}}$ that coincides with $v$ in the nodes on $\bar{\gamma}_{l}$ stemming from the mesh $\mathcal{T}_{l_{\text {lag }}}$ and is zero in the remaining nodes of $\mathcal{T}_{l_{\text {lag }}}$.
Lemma 3.10.

$$
\left\|E_{l} v\right\|_{H^{s}\left(\Gamma_{l_{\operatorname{lag}}}\right)} \lesssim h_{l_{\operatorname{lag}}}^{1 / 2-s}\|v\|_{L^{2}\left(\gamma_{l}\right)} \quad \forall v \in X_{h, l_{\operatorname{lag}}} \mid \bar{\gamma}_{l}, \forall s \in[0,1], \quad l=1, \ldots, L .
$$

In particular, the appearing constant is independent of $s$.
Proof. Using the equivalence of norms in finite dimensional spaces and scaling properties of the $L^{2}$-norm one obtains, by taking into account the construction of $E_{l}$,

$$
\left\|E_{l} v\right\|_{L^{2}\left(\Gamma_{l_{\text {lag }}}\right)}^{2} \lesssim h_{l_{\text {lag }}}\|v\|_{L^{2}\left(\gamma_{l}\right)}^{2} \quad \forall v \in X_{h, l_{\text {lag }}} \mid \bar{\gamma}_{l} .
$$

Analogously we find

$$
\begin{aligned}
\left\|E_{l} v\right\|_{H^{1}\left(\Gamma_{l_{\text {lag }}}\right)}^{2} & =\left\|E_{l} v\right\|_{L^{2}\left(\Gamma_{l_{\text {lag }}}\right)}^{2}+\left|E_{l} v\right|_{H^{1}\left(\Gamma_{l_{\text {lag }}}\right)}^{2} \\
& \lesssim h_{l_{\text {lag }}}\|v\|_{L^{2}\left(\gamma_{l}\right)}^{2}+h_{l_{\text {lag }}}^{-1}\|v\|_{L^{2}\left(\gamma_{l}\right)}^{2} \lesssim h_{l_{\text {lag }}}^{-1}\|v\|_{H^{1}\left(\gamma_{l}\right)}^{2} \quad \forall v \in X_{h, l_{\text {lag }}} \mid \bar{\gamma}_{l} .
\end{aligned}
$$

The result then follows by interpolation.

Lemma 3.11. The bilinear form $b(\cdot, \cdot)$ satisfies the discrete inf-sup condition

$$
\exists \beta>0: \quad \sup _{v \in X_{h} \backslash\{0\}} \frac{b(v, \mu)}{\|v\|_{H^{1 / 2}(\mathcal{T})}} \geq \beta\|\mu\|_{L^{2}(\gamma)} \quad \forall \mu \in M_{k} .
$$

Here, the constant $\beta$ is independent of $h$ and $k$ subject to the assumptions made on the meshes.
Proof. Let $\mu \in M_{k}$ be given. On each interface edge $\gamma_{l}, \mu$ is a piecewise constant function on $\mathcal{G}_{l}$, a mesh that is coarser than the trace mesh $\mathcal{T}_{l_{\text {lag }}} \mid \bar{\gamma}_{l}$ stemming from the Lagrangian side $\Gamma_{l_{\text {lag }}}$, cf. assumption (A2). On $\gamma_{l}$ we construct a piecewise linear function $\left.w_{l} \in X_{h, l_{\text {ag }}}\right|_{\gamma_{l}}$ in the following way. For each element $J \in \mathcal{G}_{l}, w_{l}$ vanishes at the endpoints of $J$, coincides with $\mu$ at one interior node of $J$ and is linearly interpolated elsewhere on $\gamma_{l}$. See Figure 3.1 for an example where $\mu$ is represented by the dashed line and $w_{l}$ by the solid line. The bullets indicate the nodes of the mesh for the Lagrangian multiplier and the dashes indicate additional nodes of the trace mesh (from the Lagrangian multiplier side).

We then extend $w_{l}$ to $\tilde{w}_{l}$ in $X_{h}$ by first extending to $E_{l} w_{l} \in X_{h, l_{\text {lag }}}$, cf. (3.12), and then further by zero onto $\Gamma$. Eventually we define $v:=\sum_{l=1}^{L} \tilde{w}_{l}$.

Note that $\tilde{w}_{l}$ vanishes on all interface edges except $\gamma_{l}$. The trace of $\tilde{w}_{l}$ onto $\gamma_{l}$ from $\Gamma_{l_{\text {lag }}}$ equals $w_{l}$ whereas the trace coming from the other side $\Gamma_{l_{\text {mor }}}$ vanishes. This yields

$$
\begin{equation*}
[v]=\left[\tilde{w}_{l}\right]=w_{l} \quad \text { on } \quad \gamma_{l}, \quad l=1, \ldots, L \tag{3.13}
\end{equation*}
$$

By the construction of $w_{l}$ there holds, uniformly for $\mu \in M_{k}$,

$$
\begin{equation*}
\|\mu\|_{L^{2}\left(\gamma_{l}\right)}^{2} \simeq\left\langle w_{l}, \mu\right\rangle_{\gamma_{l}} \simeq\left\|w_{l}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}, \quad l=1, \ldots, L . \tag{3.14}
\end{equation*}
$$

Also, taking into account that each sub-domain $\Gamma_{i}$ has a limited number of (interface) edges, determined by the relation $l \in\{1, \ldots, L\}: l_{\text {lag }}=i$, Lemma 3.10 yields

$$
\begin{align*}
\|v\|_{H^{1 / 2}(\mathcal{T})}^{2} & =\sum_{i=1}^{N}\left\|\sum_{l \in\{1, \ldots, L\}: l_{\operatorname{lag}}=i} E_{l} w_{l}\right\|_{H^{1 / 2}\left(\Gamma_{i}\right)}^{2} \\
& \lesssim \sum_{i=1}^{N} \sum_{l \in\{1, \ldots, L\}: l_{\operatorname{lag}}=i}\left\|E_{l} w_{l}\right\|_{H^{1 / 2}\left(\Gamma_{i}\right)}^{2}=\sum_{l=1}^{L}\left\|E_{l} w_{l}\right\|_{H^{1 / 2}\left(\Gamma_{\left.l_{\text {lag }}\right)}\right.}^{2} \lesssim \sum_{l=1}^{L}\left\|w_{l}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2} \tag{3.15}
\end{align*}
$$

Now, using (3.13), (3.14) and (3.15), we finish the proof by bounding

$$
b(v, \mu)=\sum_{l=1}^{L}\langle[v], \mu\rangle_{\gamma_{l}}=\sum_{l=1}^{L}\left\langle w_{l}, \mu\right\rangle_{\gamma_{l}} \simeq\|\mu\|_{L^{2}(\gamma)}\left(\sum_{l=1}^{L}\left\|w_{l}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}\right)^{1 / 2} \gtrsim\|\mu\|_{L^{2}(\gamma)}\|v\|_{H^{1 / 2}(\mathcal{T})} .
$$

Lemma 3.12.

$$
\|[v]\|_{L^{2}(\gamma)}^{2} \lesssim|\log \underline{h}|\|v\|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in X_{h}
$$



Figure 3.1: Construction of $w_{l}$ in the proof of Lemma 3.11.

Proof. By the triangle inequality and Lemma 3.3 there holds uniformly for $\epsilon \in(0,1 / 2)$

$$
\begin{aligned}
\|[v]\|_{L^{2}(\gamma)}^{2} & \lesssim \sum_{l=1}^{L}\left(\left\|v_{l_{\text {lag }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}+\left\|v_{l_{\text {mor }}}\right\|_{L^{2}(\gamma)}^{2}\right) \\
& \lesssim \epsilon^{-1} \sum_{l=1}^{L}\left(\left\|v_{l_{\text {lag }}}\right\|_{H^{1 / 2+\epsilon\left(\Gamma_{\left.l_{\text {lag }}\right)}\right.},}^{2}+\left\|v_{l_{\text {mor }}}\right\|_{H^{1 / 2+\epsilon\left(\Gamma_{\left.l_{\text {mor }}\right)}\right)}}^{2}\right) \lesssim \epsilon^{-1}\|v\|_{H^{1 / 2+\epsilon}(\mathcal{T})}^{2} \quad \forall v \in X_{h} .
\end{aligned}
$$

The inverse property, applied separately to $v_{i}=\left.v\right|_{\Gamma_{i}}$, yields

$$
\|v\|_{H^{1 / 2+\epsilon}(\mathcal{T})}^{2} \lesssim \underline{h}^{-2 \epsilon}\|v\|_{H^{1 / 2}(\mathcal{T})}^{2} \quad \forall v \in X_{h}
$$

and selecting $\epsilon=|\log \underline{h}|^{-1}$ (for $\underline{h}$ being small enough) finishes the proof.
Lemma 3.13. The bilinear form $b(\cdot, \cdot)$ is almost uniformly discretely continuous, in the sense that

$$
\begin{equation*}
b(v, \mu) \lesssim|\log \underline{h}|^{1 / 2}\|v\|_{H^{1 / 2}(\mathcal{T})}\|\mu\|_{L^{2}(\gamma)} \quad \forall v \in X_{h}, \forall \mu \in M_{k} . \tag{3.16}
\end{equation*}
$$

Moreover, for given $\psi \in L^{2}(\gamma)$, there holds

$$
\begin{equation*}
b(v, \psi) \lesssim|\log \underline{h}| \inf _{\mu \in M_{k}}\|\psi-\mu\|_{L^{2}(\gamma)}\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)} \quad \forall v \in V_{h} . \tag{3.17}
\end{equation*}
$$

Proof. There holds

$$
b(v, \mu)=\sum_{l=1}^{L}\langle[v], \mu\rangle_{\gamma_{l}} \leq\|[v]\|_{L^{2}(\gamma)}\|\mu\|_{L^{2}(\gamma)} \quad \forall v \in X_{h}, \forall \mu \in M_{k} .
$$

Estimate (3.16) follows with the help of Lemma 3.12. To prove (3.17) we start as before and note that by definition of $V_{h}$ there holds $b(v, \mu)=0 \forall \mu \in M_{k}, \forall v \in V_{h}$. Therefore, for any $\mu \in M_{k}$ and $v \in V_{h}$ we find that

$$
\begin{equation*}
b(v, \psi) \leq\|[v]\|_{L^{2}(\gamma)}\|\psi-\mu\|_{L^{2}(\gamma)} . \tag{3.18}
\end{equation*}
$$

The proof of (3.17) is finished by noting that combination of Lemmas 3.12 and 3.6 yields

$$
\|[v]\|_{L^{2}(\gamma)} \lesssim|\log \underline{h}|^{1 / 2}\|v\|_{H^{1 / 2}(\mathcal{T})} \lesssim|\log \underline{h}|\left\|\operatorname{curl}_{H} v\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)} \quad \forall v \in V_{h} .
$$

We are now ready to prove the following Strang-type error estimate.
Theorem 3.1. System (2.1) is uniquely solvable. Let $u$ and $u_{h}$ be the solutions of (1.2) and (2.1), respectively. Assuming that $u \in \tilde{H}^{1 / 2+r}(\Gamma)(r \in(0,1 / 2])$ there holds

$$
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})} \lesssim|\log \underline{h}|^{1 / 2}\left(s^{-1} \inf _{v \in V_{h}}\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}+\sup _{w \in V_{h} \backslash\{0\}} \frac{a\left(u-u_{h}, w\right)}{\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}}\right)
$$

uniformly for $s \in(0, \min \{1 / 2, r\}]$.
Proof. The existence and uniqueness of $\left(u_{h}, \lambda_{k}\right) \in X_{h} \times M_{k}$ follows from the Babuška-Brezzi theory. Indeed, the bilinear form $a(\cdot, \cdot)$ is continuous on $X_{h}$ by Lemma 3.8 and $V_{h}$-elliptic by Lemma 3.7, and the bilinear form $b(\cdot, \cdot)$ is continuous on $X_{h} \times M_{k}$ by (3.16) and satisfies a discrete inf-sup condition by Lemma 3.11. The continuity and ellipticity bounds depend on $h$ but that does not influence the unique solvability of the discrete scheme.

The error estimate is obtained by the usual steps. Combining the triangle inequality, the non-standard ellipticity and continuity properties of $a(\cdot, \cdot)$, cf. (3.10) and Lemma 3.9, we obtain for any $v \in V_{h}$

$$
\begin{aligned}
& \| u-u_{h}\left\|_{H^{1 / 2}(\mathcal{T})} \leq\right\| u-v\left\|_{H^{1 / 2}(\mathcal{T})}+\right\| v-u_{h} \|_{H^{1 / 2}(\mathcal{T})} \\
& \lesssim\|u-v\|_{H^{1 / 2}(\mathcal{T})}+|\log \underline{h}|^{1 / 2} \sup _{w \in V_{h} \backslash\{0\}} \frac{a\left(v-u_{h}, w\right)}{\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}} \\
& \quad \leq\|u-v\|_{H^{1 / 2}(\mathcal{T})}+|\log \underline{h}|^{1 / 2}\left(\sup _{w \in V_{h} \backslash\{0\}} \frac{a(v-u, w)}{\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}}+\sup _{w \in V_{h} \backslash\{0\}} \frac{a\left(u-u_{h}, w\right)}{\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}}\right) \\
& \lesssim\|u-v\|_{H^{1 / 2}(\mathcal{T})}+s^{-1}|\log \underline{h}|^{1 / 2}\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}+|\log \underline{h}|^{1 / 2} \sup _{w \in V_{h} \backslash\{0\}} \frac{a(u-u, w)}{\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)}} .
\end{aligned}
$$

This proves the stated error bound.
In order to analyze the upper bound provided by Theorem 3.1 we need, apart from the extension operators $E_{l}$ defined before, projection operators $\pi_{l}$ acting on $L^{2}\left(\gamma_{l}\right)$ and mapping onto special continuous, piecewise linear functions on $\gamma_{l}, l=1, \ldots, L$. We recall that on each $\gamma_{l}$ we have two meshes: the trace mesh $\mathcal{T}_{l_{\text {lag }}} \mid \gamma_{l}$ stemming from the mesh on the sub-domain $\Gamma_{l_{\text {lag }}}$ of the Lagrangian side, and the mesh $\mathcal{G}_{l}$ for the Lagrangian multiplier. For each element $J \in \mathcal{G}_{l}$ we consider a hat function $\phi_{l, J}$ that vanishes at the endpoints of $J$ and has the tip at a node of $\mathcal{T}_{l_{\text {lag }}} \mid \gamma_{\gamma_{l}}$ that is interior to $J$. This choice is not unique if $J$ contains more than two elements of the trace mesh. In that case we select an arbitrary but fixed node for the definition of $\phi_{l, J}$. Using this notation we define

$$
\begin{equation*}
\pi_{l}: L^{2}\left(\gamma_{l}\right) \rightarrow \operatorname{span}\left\{\phi_{l, J} ; J \in \mathcal{G}_{l}\right\} \subset X_{h, l_{\text {lag }}} \mid \bar{\gamma}_{l}, \quad l=1, \ldots, L \tag{3.19}
\end{equation*}
$$

such that the integral mean zero conditions

$$
\left\langle v-\pi_{l} v, 1\right\rangle_{J}=0 \quad \forall J \in \mathcal{G}_{l}, l=1, \ldots, L,
$$

hold. This operator satisfies the following properties.

Lemma 3.14. For any $v \in L^{2}\left(\gamma_{l}\right), \pi_{l} v$ vanishes at the endpoints of $\gamma_{l}, l=1, \ldots, L$, and there holds

$$
\begin{gather*}
\left\langle v-\pi_{l} v, \mu\right\rangle_{\gamma_{l}}=0 \quad \forall v \in L^{2}\left(\gamma_{l}\right), \quad \forall \mu \in M_{k, l}, \quad l=1, \ldots, L,  \tag{3.20}\\
\left\|\pi_{l} v\right\|_{L^{2}\left(\gamma_{l}\right)} \lesssim\|v\|_{L^{2}\left(\gamma_{l}\right)} \quad \forall v \in L^{2}\left(\gamma_{l}\right), l=1, \ldots, L . \tag{3.21}
\end{gather*}
$$

Proof. For $l \in\{1, \ldots, L\}$ let $v \in L^{2}\left(\gamma_{l}\right)$ be given. By definition of $\pi_{l}, \pi_{l} v$ vanishes at the endpoints of $\gamma_{l}$, and the orthogonality (3.20) follows by noting that any $\mu \in M_{k, l}$ is constant on any $J \in \mathcal{G}_{l}$.

To show (3.21) let $J \in \mathcal{G}_{l}$ be given. With $\phi_{l, J}$ being the hat function defined previously (with height 1) there holds

$$
\pi_{l} v=\frac{2}{|J|}\left(\int_{J} v d s\right) \phi_{l, J} \quad \text { on } \quad J
$$

so that

$$
\left\|\pi_{l} v\right\|_{L^{2}(J)}^{2}=\frac{4}{3|J|}\left(\int_{J} v d s\right)^{2} \leq \frac{4}{3}\|v\|_{L^{2}(J)}^{2}
$$

Summing over $J \in \mathcal{\mathcal { G } _ { l }}$ finishes the proof.
We are now ready to analyze the first term of the upper bound provided by Theorem 3.1.
Lemma 3.15. For $r \in(0,1 / 2]$ let $u \in H^{1 / 2+r}(\Gamma)$. There holds

$$
\inf _{v \in V_{h}}\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}^{2} \lesssim\|u-w\|_{H^{1 / 2+s}(\mathcal{T})}^{2}+\sum_{l=1}^{L} h_{l_{\text {lag }}}^{-2 s}\left(\left\|u-w_{l_{\text {lag }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}+\left\|u-w_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}\right) \quad \forall w \in X_{h}
$$

uniformly for $s \in(0, \min \{1 / 2, r\}]$.
Proof. Let $w \in X_{h}$ be given. We adapt $w$ such that the new function satisfies the jump conditions defining $V_{h}$, cf. (2.2). We set

$$
v:=w+\sum_{l=1}^{L} r^{l} \in X_{h}
$$

with

$$
r^{l}:= \begin{cases}E_{l} \pi_{l}\left(\left.w_{l_{\text {lag }}}\right|_{\gamma_{l}}-\left.w_{l_{\text {mor }}}\right|_{\gamma_{l}}\right) & \text { on } \bar{\Gamma}_{l_{\text {lag }}} \\ \text { elsewhere. }\end{cases}
$$

Here, $E_{l}$ and $\pi_{l}$ are the extension and projection operators specified in (3.12) and (3.19), respectively. Note that, since $\pi_{l}\left(w_{l_{\text {lag }}}\left|\gamma_{l}-w_{l_{\text {mor }}}\right| \gamma_{l}\right)$ vanishes at the endpoints of $\gamma_{l}$, the extension $E_{l} \pi_{l}\left(w_{l_{\text {lag }}}\left|\gamma_{l}-w_{l_{\text {mor }}}\right| \gamma_{\gamma_{l}}\right)$ vanishes on $\partial \Gamma_{l_{\text {lag }}} \backslash \gamma_{l}$. Therefore, using (3.20) one obtains

$$
\begin{aligned}
\langle[v], \mu\rangle_{\gamma_{l}} & =\left\langle v_{l_{\text {lag }}}-v_{l_{\text {mor }}}, \mu\right\rangle_{\gamma_{l}}=\left\langle w_{l_{\text {lag }}}+r^{l}-w_{l_{\text {mor }}}, \mu\right\rangle_{\gamma_{l}} \\
& =\left\langle w_{l_{\text {lag }}}-w_{l_{\text {mor }}}+\pi_{l}\left(w_{l_{\text {lag }}}\left|\gamma_{\gamma_{l}}-w_{l_{\text {mor }}}\right|_{\gamma_{l}}\right), \mu\right\rangle_{\gamma_{l}}=0 \quad \forall \mu \in M_{k, l}, l=1, \ldots, L .
\end{aligned}
$$

That is, $v \in V_{h}$. We start bounding the error by

$$
\begin{align*}
\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}^{2} & =\sum_{i=1}^{N}\left\|u_{i}-w_{i}-\sum_{l \in\{1, \ldots, L\}: l_{\text {lag }}=i} r^{l}\right\|_{H^{1 / 2+s}\left(\Gamma_{i}\right)}^{2} \\
& \lesssim \sum_{i=1}^{N}\left\|u_{i}-w_{i}\right\|_{H^{1 / 2+s}\left(\Gamma_{i}\right)}^{2}+\sum_{l=1}^{L}\left\|r^{l}\right\|_{H^{1 / 2+s}\left(\Gamma_{l_{\text {lag }}}\right)}^{2} . \tag{3.22}
\end{align*}
$$

Applying Lemma 3.10 (3.21) and the triangle inequality we find that there holds

$$
\begin{align*}
\left\|r^{l}\right\|_{H^{1 / 2+s}\left(\Gamma_{l_{\text {lag }}}\right)} & \lesssim h_{l_{\text {lag }}}^{-s}\left\|\pi_{l}\left(\left.w_{l_{\text {lag }}}\right|_{\gamma_{l}}-\left.w_{l_{\text {mor }}}\right|_{\gamma_{l}}\right)\right\|_{L^{2}\left(\gamma_{l}\right)} \lesssim h_{l_{\text {lag }}}^{-s}\left\|w_{l_{\text {lag }}}-w_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)} \\
& \lesssim h_{l_{\text {lag }}^{-s}}^{-s}\left(\left\|u-w_{l_{\text {lag }}}\right\|_{L^{2}\left(\gamma_{l}\right)}+\left\|u-w_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)}\right) . \tag{3.23}
\end{align*}
$$

Combining (3.22) and (3.23) one obtains the assertion.
The next result provides an a priori error estimate for the mortar BEM.
Theorem 3.2. Let $u$ and $u_{h}$ be the solutions of (1.2) and (2.1), respectively. Assuming that $u \in$ $\tilde{H}^{1 / 2+r}(\Gamma)(r \in(0,1 / 2])$ there holds $\lambda \in \prod_{l=1}^{L} H^{r}\left(\gamma_{l}\right)$ and we have the a priori error estimate

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})}^{2} \lesssim & s^{-2}|\log \underline{h}|\left(\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}^{2}+\underline{h}^{-2 s} \sum_{l=1}^{L}\left(\left\|u-v_{l_{\text {lag }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}+\left\|u-v_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}\right)\right) \\
& +|\log \underline{h}|^{3}\|\lambda-\mu\|_{L^{2}(\gamma)}^{2} \quad \forall v \in X_{h}, \quad \forall \mu \in M_{k}
\end{aligned}
$$

uniformly for $s \in(0, r]$. Here, $\lambda$ is the Lagrangian multiplier defined by (3.5), (3.6).
Proof. Since $u \in \tilde{H}^{1 / 2+r}(\Gamma)$ there holds

$$
\left.\lambda\right|_{\gamma_{l}}=\left.\mathbf{t}_{l_{\text {lag }}} \cdot\left(V \operatorname{curl}_{\Gamma} u\right)\right|_{\gamma_{l}} \in H^{r}\left(\gamma_{l}\right), \quad l=1, \ldots, L
$$

To this end note that $\operatorname{curl}_{\Gamma}: \tilde{H}^{1 / 2+r}(\Gamma) \rightarrow \tilde{\boldsymbol{H}}_{\mathrm{t}}^{r-1 / 2}(\Gamma)$ (combine (3.2) with the continuity $\operatorname{curl}_{\Gamma}$ : $\left.H_{0}^{1}(\Gamma) \rightarrow \boldsymbol{L}_{\mathrm{t}}^{2}(\Gamma)\right)$ and $V: \tilde{\boldsymbol{H}}_{\mathrm{t}}^{r-1 / 2}(\Gamma) \rightarrow \boldsymbol{H}_{\mathrm{t}}^{1 / 2+r}(\Gamma)$. The trace theorem concludes the claimed regularity of $\lambda$. In particular there holds $\lambda \in L^{2}(\gamma)$.

By definition of $V_{h}$, and making us of Lemma 3.5, we find

$$
a\left(u-u_{h}, w\right)=a(u, w)-F(w)=-b(w, \lambda) \quad \forall w \in V_{h}
$$

Application of (3.17) yields

$$
a\left(u-u_{h}, w\right) \lesssim \mid \log \underline{h} \inf _{\mu \in M_{k}}\|\lambda-\mu\|_{L^{2}(\gamma)}\left\|\operatorname{curl}_{H} w\right\|_{\tilde{\boldsymbol{H}}_{\mathrm{t}}^{-1 / 2}(\Gamma)} \quad \forall w \in V_{h} .
$$

Therefore, combining Theorem 3.1 with Lemma 3.15 we obtain

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})}^{2} \lesssim|\log \underline{h}|\{ & s^{-2}\left(\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}^{2}+\sum_{l=1}^{L} h_{l_{\text {lag }}}^{-2 s}\left(\left\|u-v_{l_{\text {lag }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}+\left\|u-v_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)}^{2}\right)\right) \\
& \left.+|\log \underline{h}|^{2}\|\lambda-\mu\|_{L^{2}(\gamma)}^{2}\right\} \quad \forall v \in X_{h}, \quad \forall \mu \in M_{k} .
\end{aligned}
$$

This proves the statement.
Proof of Theorem [2.1. By Theorem [3.1, system (2.1) is uniquely solvable. We employ the general a priori estimate by Theorem 3.2 to show the given error bound. By standard approximation theory there exist $v \in X_{h}$ and $\mu \in M_{k}$ such that

$$
\|u-v\|_{H^{1 / 2+s}(\mathcal{T})}^{2} \lesssim h^{2(r-s)}\|u\|_{\tilde{H}^{1 / 2+r}(\Gamma)}^{2} \quad \text { and } \quad\|\lambda-\mu\|_{L^{2}(\gamma)}^{2} \lesssim k^{2 r} \sum_{l=1}^{L}\|\lambda\|_{H^{r}\left(\gamma_{l}\right)}^{2}
$$

and as in the proof of Theorem 3.2 one concludes that $\sum_{l=1}^{L}\|\lambda\|_{H^{r}\left(\gamma_{l}\right)}^{2} \lesssim\|u\|_{\tilde{H}^{1 / 2+r}(\Gamma)}^{2}$. By Lemma 3.3 one bounds

$$
\left\|u-v_{l_{\operatorname{lag}}}\right\|_{L^{2}\left(\gamma_{l}\right)} \lesssim s^{-1 / 2}\|u-v\|_{H^{1 / 2+s}\left(\Gamma_{\operatorname{lag}}\right)} \lesssim s^{-1 / 2} h^{r-s}\|u\|_{\tilde{H}^{1 / 2+r}(\Gamma)},
$$

and accordingly the mortar part $\left\|u-v_{l_{\text {mor }}}\right\|_{L^{2}\left(\gamma_{l}\right)}$. Using these bounds in Theorem 3.2 and selecting $s=|\log \underline{h}|^{-1}$ one obtains the assertion.

## 4 Numerical results

We consider the model problem (1.2) with $\Gamma=(0,1) \times(0,1)$ and $f=1$. In this case there holds $u \in \tilde{H}^{1 / 2+r}(\Gamma)$ for any $r<1 / 2$ so that by Theorem [2.1] we expect a convergence of the mortar method close to $h^{1 / 2}$, the convergence of the conforming BEM, cf. Remark 2.1. This assumes that the mesh sizes $h$ (of the sub-domain meshes) and $k$ (of the meshes for the Lagrangian multiplier on the skeleton) are proportional, which will be the case in all our experiments. In fact, the elements of the mesh for the Lagrangian multiplier will always consist of two or three elements of the trace mesh.

Since the exact solution $u$ to (1.1) is unknown we approximate an upper bound for the semi-norm $\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})}$. Here, we follow the strategy from [13]. Let us recall the procedure and discussion.

By the ellipticity of $V$ and (3.8) there holds

$$
\begin{equation*}
a\left(u-u_{h}, u-u_{h}\right) \gtrsim\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})}^{2} \tag{4.1}
\end{equation*}
$$

On the other hand, using that $u$ solves (1.1) and $u_{h} \in V_{h}$ solves (2.1), one finds

$$
a\left(u-u_{h}, u-u_{h}\right)=a(u, u)-2 a\left(u, u_{h}\right)+a\left(u_{h}, u_{h}\right)=\langle W u, u\rangle_{\Gamma}-2 a\left(u, u_{h}\right)+F\left(u_{h}\right) .
$$

By (3.5) there holds

$$
a\left(u, u_{h}\right)=F\left(u_{h}\right)-\left\langle\left[u_{h}\right], \lambda\right\rangle_{\gamma}
$$

such that, with the previous relation,

$$
\begin{equation*}
a\left(u-u_{h}, u-u_{h}\right)=\langle W u, u\rangle_{\Gamma}-F\left(u_{h}\right)+2\left\langle\left[u_{h}\right], \lambda\right\rangle_{\gamma} \leq\langle W u, u\rangle_{\Gamma}-F\left(u_{h}\right)+2\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}\|\lambda\|_{L^{2}(\gamma)} . \tag{4.2}
\end{equation*}
$$

Like in the proof of Theorem 3.2 one sees that $\|\lambda\|_{L^{2}(\gamma)}$ is bounded. Therefore, by (4.1) we find that

$$
\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})}^{2} \lesssim\left|\langle W u, u\rangle_{\Gamma}-F\left(u_{h}\right)\right|+\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)} .
$$

The terms $F\left(u_{h}\right)$ and $\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}$ are directly accessible and $\langle W u, u\rangle_{\Gamma}$ can be approximated by an extrapolated value that we denote by $\|u\|_{\text {ex }}^{2}$ (cf. [12]). Therefore, instead of the relative error

$$
\left\|u-u_{h}\right\|_{H^{1 / 2}(\mathcal{T})} /\|u\|_{H^{1 / 2}(\Gamma)},
$$

we present results for the expression

$$
\begin{equation*}
\left(\left|\|u\|_{\mathrm{ex}}^{2}-F\left(u_{h}\right)\right|+\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}\right)^{1 / 2} /\|u\|_{\mathrm{ex}} \tag{4.3}
\end{equation*}
$$

which is, up to a constant factor, an upper bound for $\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})} /\|u\|_{\mathrm{ex}}$.

In the figures below we show different error curves, indicated by numbers $(n)(n=1, \ldots, 4)$ as follows.

$$
\begin{array}{ll}
\left(\left|\|u\|_{\mathrm{ex}}^{2}-F\left(u_{h}\right)\right|+\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}\right)^{1 / 2} & \text { "mortar BEM" } \\
\left(\left|\|u\|_{\mathrm{ex}}^{2}-F\left(u_{h}\right)\right|\right)^{1 / 2} & \text { "error1" } \\
\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}^{1 / 2} & \text { "error2" } \\
a\left(u-\tilde{u}_{h}, u-\tilde{u}_{h}\right)^{1 / 2} & \text { "conforming BEM" } \tag{4}
\end{array}
$$

Here, $\tilde{u}_{h}$ denotes a conforming boundary element solution. Additionally, all curves are normalized by $\|u\|_{\text {ex }}$.

Therefore, to resume, an error curve (1) represents the upper bound (4.3) for the (normalized) error $\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})}$ of the mortar BEM. Curves (2) and (3) are the two components of (1). Here, (3) controls the non-conformity of the mortar approximant $u_{h}$. Curve (4) represents the error of the conforming BEM. In this case it is equivalent to the error in energy norm $\left\|u-\tilde{u}_{h}\right\|_{\tilde{H}^{1 / 2}(\Gamma)}$.

All results are plotted on double logarithmic scales versus $1 / h$. For our numerical experiments we always use rectangular meshes and in this section, $h_{i}$ refers to the length of the longest edge on $\Gamma_{i}$, and $h:=\max _{i} h_{i}$ as before.

## Conforming sub-domain decomposition.

Experiment 1 (conforming mesh, results in Figure 4.1). First let us consider a conforming decomposition of $\Gamma$ into four sub-domains as indicated in Figure 2.1(a). Moreover, let us first test the case where the separate meshes on the sub-domains form globally conforming meshes (we take uniform meshes consisting of squares). The corresponding results are shown in Figure 4.1. Along with the curves (1), (2), (4) we plot the values of $h^{1 / 2}$. The numerical results indicate a convergence of the order $O\left(h^{1 / 2}\right)$, for the conforming as well as the mortar BEM. According to the discussion above this is the best one can expect. The curves (1) and (2), referring to our upper bound (4.3) and the first term in (4.3), respectively, are almost identical. This means that the second term in (4.3), which in the next plots will be labeled by (3), is negligible in comparison. Indeed, in this symmetric case the jumps $\left[u_{h}\right]$ disappear and the numerical results vanish at the order of single precision. Therefore, in this plot, we do not show the curve (3).

We do not observe a logarithmical perturbation of the convergence in this range of number of unknowns. This may be caused by the fact that we are not including the $L^{2}$-parts in the error since our results are, up to constant factors, upper bounds only for the semi-norm $\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})}$. Also, we do not know whether our bounds including the logarithmic terms are sharp.

Experiment 2 (non-conforming mesh, results in Figure4.2). Now let us test globally non-conforming meshes. Again we use uniform meshes consisting of squares on each sub-domain. We mesh as in Figure 2.1 (a) starting with $2,3,4$, and 5 "slides" on $\Gamma_{1}, \Gamma_{2}, \Gamma_{3}$, and $\Gamma_{4}$, respectively and increase the number of slides in each sub-domain by one in each step of our sequence of meshes. The corresponding results are shown in Figure 4.2. Again, a convergence of the expected order $O\left(h^{1 / 2}\right)$ is confirmed. Curve (3) indicates very fast convergence of the jumps $\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)} \rightarrow 0$. In the experiments below, however, we observe a slower convergence. In this particular sequence of meshes, where we increase the slides on the sub-domains by the same amount, the trace meshes from different sides on a particular interface edge approach each other in a certain sense. We conjecture that this specific situation ("approaching" conforming meshes) causes the fast convergence of the jumps.

Experiment 3 (non-conforming mesh, results in Figure 4.3). For the next experiment we start with a mesh of four squares on each sub-domain (the sub-domains are again as in Figure 2.1(a)), and increase


Figure 4.1: Conforming sub-domain decomposition with conforming meshes.
the numbers of slides on different sub-domains by different steps (increase by $2,3,4,5$ slides on $\Gamma_{1}, \Gamma_{2}$, $\Gamma_{3}, \Gamma_{4}$, respectively). In this case both error parts, curves (2) and (3), behave like $O\left(h^{1 / 2}\right)$, confirming our a priori error estimate and thus the good performance of the mortar BEM. Let us note, however, that the part $\left\|\left[u_{h}\right]\right\|_{\gamma}^{1 / 2}$ of the error expression (4.3) is an overestimation. Indeed, our substitution (4.3) for $\left|u-u_{h}\right|_{H^{1 / 2}(\mathcal{T})} /\|u\|_{\text {ex }}$ is not precise. On the one hand we replaced the term $2\|\lambda\|_{L^{2}(\gamma)}$ in (4.2) by 1 (and the generic constant in (4.1) by 1). On the other hand the term $\left\langle\left[u_{h}\right], \lambda\right\rangle_{\gamma}$ is of higher order than $\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}$. According to (3.18) and by standard approximation theory there holds for any $r<1 / 2$

$$
\left|\left\langle\left[u_{h}\right], \lambda\right\rangle_{\gamma}\right| \lesssim\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)} \inf _{\psi \in M_{k}}\|\lambda-\psi\|_{L^{2}(\gamma)} \lesssim k^{r}\left(\sum_{l=1}^{L}|\lambda|_{H^{r}\left(\gamma_{l}\right)}^{2}\right)^{1 / 2}\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}
$$

This shows that $\left\langle\left[u_{h}\right], \lambda\right\rangle_{\gamma}$ is of higher order than $\left\|\left[u_{h}\right]\right\|_{L^{2}(\gamma)}$. Note that, by the proof of Theorem 3.2 and since $u \in H^{1 / 2+r}(\Gamma)$, one has the regularity $\lambda \in \prod_{l=1}^{L} H^{r}\left(\gamma_{l}\right) \quad \forall r<1 / 2$. Therefore, by (4.2) the term

$$
\left(\left|\|u\|_{\mathrm{ex}}^{2}-F\left(u_{h}\right)\right|\right)^{1 / 2} /\|u\|_{\mathrm{ex}} \quad(\text { curve (2), "error1") }
$$

is asymptotically equal to

$$
a\left(u-u_{h}, u-u_{h}\right)^{1 / 2} /\|u\|_{\mathrm{ex}}
$$

and this dominates the error.


Figure 4.2: Conforming sub-domain decomposition with non-conforming meshes, same refinement steps on sub-domains.

## Non-conforming sub-domain decomposition.

Experiment 4 (non-conforming mesh, results in Figure 4.5). Finally, we consider the fully nonconforming mortar method, i.e. with non-conforming sub-domain decomposition and non-conforming meshes. We decompose $\Gamma$ into three sub-domains as in Figure 4.4 and use the initial mesh given there on the left. Then slides on sub-domains are increased in each direction by $3,2,1$ on $\Gamma_{1}, \Gamma_{2}, \Gamma_{3}$, respectively, in each step. The second mesh is on the right in Figure 4.4. Note that in each second step the cross-point $(0,0)$ between the sub-domains is a hanging node and our theory includes this case. The numerical results are shown in Figure 4.5 and again confirm the expected convergence of the mortar BEM.

In this case, the meshes for the Lagrangian multiplier are coarsenings of the trace meshes from $\Gamma_{2}$ on $\gamma_{12}$ and $\gamma_{23}$, and of the trace mesh from $\Gamma_{1}$ on $\gamma_{13}$. We always join two elements of the respective trace mesh to form an element of the Lagrangian multiplier mesh, except for an odd number of elements of the trace mesh when one set of three elements is joined. The corresponding numbers of unknowns for the steps are listed in Table 1 .


Figure 4.3: Conforming sub-domain decomposition with non-conforming meshes, different refinement steps on sub-domains.


Figure 4.4: Conforming sub-domain decomposition with non-conforming meshes.


Figure 4.5: Non-conforming sub-domain decomposition with non-conforming meshes, different refinement steps on sub-domains.

| $h=h_{3}$ | $h_{1}$ | $h_{2}$ | $\operatorname{dim}\left(X_{h}\right)$ | $\operatorname{dim}\left(M_{k}\right)$ |
| :---: | ---: | ---: | ---: | ---: |
| 0.5000 | 0.1250 | 0.1667 | 27 | 4 |
| 0.3333 | 0.0625 | 0.0833 | 80 | 7 |
| 0.2500 | 0.0417 | 0.0556 | 161 | 11 |
| 0.2000 | 0.0313 | 0.0417 | 270 | 14 |
| 0.1667 | 0.0250 | 0.0333 | 407 | 18 |
| 0.1429 | 0.0208 | 0.0278 | 572 | 21 |
| 0.1250 | 0.0179 | 0.0238 | 765 | 25 |
| 0.1111 | 0.0156 | 0.0208 | 986 | 28 |
| 0.1000 | 0.0139 | 0.0185 | 1235 | 32 |
| 0.0909 | 0.0125 | 0.0167 | 1512 | 35 |
| 0.0833 | 0.0114 | 0.0152 | 1817 | 39 |
| 0.0769 | 0.0104 | 0.0139 | 2150 | 42 |

Table 1: Dimensions and mesh sizes for experiment 4.

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