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COMMENT ON "FUNCTIONS AND DOMAINS HAVING MINIMAL RESISTANCE UNDER A SINGLE-IMPACT ASSUMPTION" [SIAM J. MATH. ANAL., 34 (2002), PP. 101–120]*

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Abstract. Recently Comte and Lachand-Robert [*SIAM J. Math. Anal.*, 34 (2002), pp. 101–120] stated a very interesting and actual problem of minimizing mean specific resistance of infinite surfaces in a parallel flow of noninteracting point particles. They also constructed surfaces having resistance 0.593 and proved that they are minimizers. Unfortunately, their proof is incorrect. In this comment we provide a counterexample showing that the least value of resistance is not attained and is less than 0.581 (but greater than or equal to 0.5). Therefore, the problem remains open.

Key words. surface of minimal resistance, Newton's problem, calculus of variations

AMS subject classifications. 49K30, 49Q10

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Recently Comte and Lachand-Robert [1] considered a problem of minimizing mean specific resistance of infinite surfaces in a parallel stream of point particles: a far-going generalization of Newton's problem of minimal resistance.

The mathematical formulation of the problem is the following. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $u : \Omega \to \mathbb{R}$ be a piecewise smooth function such that $u\rfloor_{\partial\Omega} = 0$ and $u(x) < 0$ for any $x \in \Omega$. It is assumed that Ω tiles the plane; that is, there exists a group G of isometries of $\mathbb{R}^2$ such that $\cup_{g\in G} g(\bar{\Omega}) = \mathbb{R}^2$ and $g_1(\Omega) \cap g_2(\Omega) = \emptyset$ if $g_1 \neq g_2$. In such a case, u can be extended to a function $\bar{u} : \mathbb{R}^2 \to \mathbb{R}$ invariant with respect to G; that is, $\bar{u}\rfloor_\Omega = u$ and $\bar{u} \circ g = \bar{u}$ for any $g \in G$. Besides, u is assumed to satisfy the so-called single impact condition: for any regular point $x \in \Omega$ and any $t > 0$ such that $x - t\nabla u(x) \in \bar{\Omega}$,

$$\frac{u(x - t\nabla u(x)) - u(x)}{t} \leq \frac{1}{2}\left(1 - |\nabla u(x)|^2\right). \tag{1}$$

The paper [1] is devoted to solving the following problem.

PROBLEM 1. *Minimize the functional*

$$F(u;\Omega) = \frac{1}{|\Omega|}\int_\Omega \frac{dx}{1 + |\nabla u(x)|^2} \tag{2}$$

over all domains Ω tiling the plane and all functions u.

The single impact condition (SIC) was first introduced in [2]. It has the following mechanical interpretation. Consider a uniform stream of (mutually noninteracting) particles falling vertically downward on the body in $\{(x, z) \in \mathbb{R}^3 : z \leq \bar{u}(x)\}$. When hitting the body's boundary, the particles are reflected in the perfectly elastic way

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and move uniformly between consecutive reflections. SIC means precisely that no more than one reflection can be made; that is, each particle reflecting from a regular point of the boundary moves freely afterwards. The trajectory of the posterior free motion can contain points of the boundary but not points of the body's interior. The functional (2) equals the mean pressure exerted by the stream on the body.

The main result of the paper states that there are two solutions to Problem 1, up to similitude. In the first solution, the domain is the square $\Omega = (-1/2,\, 1/2) \times (-1/2,\, 1/2)$ and $u(x_1, x_2) = \max\{\varphi(|x_1| + 1/2),\, \varphi(|x_2| + 1/2)\}$, where $\varphi(r) = (r^2 - 1)/2$. In the second solution, Ω is a regular hexagon and u is defined similarly to the case 1. The reported minimal value (the same for both cases) equals $F_{min} = \pi + 4\ln 1.6 - 4\arctan 2 \simeq 0.5930123$.

In my opinion, this problem may be of potential practical interest for highly rarefied hypersonic flows falling on rough surfaces. The solutions given above provide the instructions for designing dimples on the surface in a periodical way. They also indicate that in the limiting case of high flow velocity and rarefaction and perfectly elastic gas-surface interaction, the decrease of pressure can reach 40% as compared to a perfectly smooth surface.

Unfortunately, the result of [1] seems to be incorrect. The point is that the authors implicitly used the assumption that the minimum *does* exist, without proving it. The counterexample given below implies that the infimum in the problem is less than 0.593 and therefore is not attained.

Let us first state the relaxed Problem 2 and prove that the smallest values in Problems 1 and 2 coincide.

PROBLEM 2. *Minimize* $F(u;\Omega)$ *in* (2) *over all domains* Ω (not necessarily tiling the plane) *and all functions* u.

Proof. Denote by inf(P1) the infimum in Problem 1, and by inf(P2) the infimum in Problem 2. Obviously, inf(P2) $\le$ inf(P1). To prove the reverse inequality inf(P1) $\le$ inf(P2), it suffices to show that for any Ω, u, and $\varepsilon > 0$, there exists a function $\tilde u$ on $\tilde\Omega = (-1/2,\, 1/2) \times (-1/2,\, 1/2)$ such that $F(\tilde u; \tilde\Omega) \le F(u;\Omega) + \varepsilon$. (Notice that $\tilde\Omega$ of course tiles the plane.)

Choose a finite number of mutually nonintersecting copies of Ω: $\Omega_1, \Omega_2, \dots$, all contained in $\tilde\Omega$, such that $|\tilde\Omega \setminus (\cup_i \Omega_i)| < \varepsilon/2$. By saying that Ω_i is a copy of Ω we mean that there exist a real value $k_i > 0$ and an isometry f_i of the plane such that $k_i f_i$ takes Ω to Ω_i. The algorithm of choice consists in several steps. On the first step, select $\Omega_1 \subset \tilde\Omega$. On the second step divide $\tilde\Omega$ into n^2 smaller squares of size $1/n \times 1/n$, select those squares that do not intersect with Ω_1, and put a copy of Ω into each of them. On the third step divide each of the small squares into n^2 even smaller ones, etc. Choosing n and the number of steps large enough, one can make the area $|\tilde\Omega \setminus (\cup_i \Omega_i)|$ arbitrarily small.

Define the function u^* on $\tilde\Omega$ as follows: the restriction $u^*\rfloor_{\Omega_i}$ is obtained from u by the transformation that takes Ω to Ω_i, that is, $u^*\rfloor_{\Omega_i}(k_i f_i x) = k_i u(x)$, and $u^* = 0$ on $\tilde\Omega \setminus (\cup_i \Omega_i)$. Then

$$F(u^*; \tilde\Omega) = |\cup_i \Omega_i| \cdot F(u;\Omega) + |\tilde\Omega \setminus (\cup_i \Omega_i)| \le F(u;\Omega) + \varepsilon/2.$$

The function u^* satisfies SIC, $u^*\rfloor_{\partial\tilde\Omega} = 0$, and $u^*(x) \le 0$ for $x \in \tilde\Omega$. This inequality is essentially nonstrict, since $u^*(x) = 0$ for $x \in \partial\Omega_i$. To improve this, take $\tilde u(x) = \min\{-\delta\,\mathrm{dist}(x, \partial\tilde\Omega),\, u^*(x)\}$. Now one has $\tilde u(x) < 0$ for $x \in \tilde\Omega$, and for $\delta > 0$ sufficiently small, $\tilde u$ satisfies SIC and $F(\tilde u; \tilde\Omega) \le F(u^*; \tilde\Omega) + \varepsilon/2$. Therefore $F(\tilde u; \tilde\Omega) \le F(u;\Omega) + \varepsilon$. $\square$

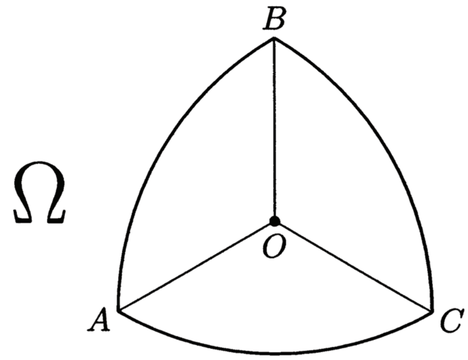

FIG. 1. *Reuleaux triangle.*

Let Ω be the curvilinear triangle ABC shown in Figure 1. The curve BC is an arc of a circumference centered at A; similarly, CA and AB are arcs of circumferences centered at B and C, respectively. This figure is called the Reuleaux triangle. Denote its center by O, and define the function u on Ω symmetric with respect to the lines OA, OB, and OC as follows. Introduce the polar coordinates $r = r(x)$, $\theta = \theta(x)$ centered at A and such that the points B and C have the coordinates $r = 1$, $\theta = \pi/6$ and $r = 1$, $\theta = -\pi/6$, respectively. The restriction of u on the curvilinear triangle OBC is $u(x) := (r^2(x) - 1)/2$. On the rest of Ω, u is defined by the symmetry conditions. One obviously has $u\rfloor_{\partial\Omega} = 0$ and $u(x) < 0$ for $x \in \Omega$. Note that u can be represented as a maximum of three convex functions, and therefore it is convex.

The "dimple" $\{(x, z) \in \Omega \times \mathbb{R} : u(x) \le z \le 0\}$ looks like an "inflated" trihedral pyramid turned down. Its base is Ω, and its lateral faces are pieces of three paraboloids of rotation with vertical axes and with foci at the points $(A, 0)$, $(B, 0)$, and $(C, 0)$. Each particle of the stream falling on one of the lateral faces after reflection will pass through the corresponding focus. Since the function u is convex, the particle trajectory between the point of reflection and the focus is located above the graph of u. Therefore u satisfies SIC.

The curvilinear triangle OBC is given by $1/\sqrt{3} \le r \le 1$, $|\theta| \le \frac{\pi}{3} - \arcsin\frac{1}{2r}$, and therefore its area equals $|\triangle OBC| = 2\int_{1/\sqrt{3}}^{1} \left(\frac{\pi}{3} - \arcsin\frac{1}{2r}\right) r\,dr$. Further, for $x \in \triangle OBC$, $|\nabla u(x)| = r(x)$, and therefore

$$F(u;\Omega) = \frac{\int_{1/\sqrt{3}}^{1} \left(\frac{\pi}{3} - \arcsin\frac{1}{2r}\right) \frac{r\,dr}{1+r^2}}{\int_{1/\sqrt{3}}^{1} \left(\frac{\pi}{3} - \arcsin\frac{1}{2r}\right) r\,dr} \simeq 0.58077812345. \tag{3}$$

Thus, the infimum in both Problems 1 and 2 is less than 0.581 (and of course is greater than or equal to 0.5). Finding its exact value constitutes an open and very intriguing question.

Note in passing that the two dimples reported in [1] look like inflated tetrahedral and hexahedral pyramids turned down; their lateral faces are pieces of parabolic cylinders. The integrand in (2), $(1 + |\nabla u(x)|^2)^{-1}$ ("specific resistance"), in all the three cases discussed here is a monotone increasing function of $\operatorname{dist}(x, \partial\Omega)$; it takes its maximal value at the center of Ω (0.8 in the two cases in [1] and 0.75 in the example presented here) and the minimal value 0.5 on the boundary $\partial\Omega$. Loosely speaking, the fraction of the area of Ω where specific resistance is large is smaller in this example than in the two cases in [1], and therefore the total resistance is also smaller.

I believe the result of [1] can be saved by reasonably reducing the class of admissible functions u. Let us, for instance, impose the additional condition that u is convex; the resulting reduced class of functions u on Ω is compact in C. The class of corresponding gradients ∇u is compact in $L^1(\Omega)$, and therefore the functional $F(u;\Omega)$ attains its minimal value, given that Ω is fixed. Hopefully, this argument can lead to an existence proof for minimizers; in this case the minimizers are exactly those found in [1].

Note in conclusion that the relaxed problem where SIC is replaced with double impact condition, and periodic functions are replaced with periodic surfaces, was considered in [3]. The smallest value in this problem equals 0.5.

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