# Minimal inequalities for an infinite relaxation of integer programs 

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#### Abstract

We show that maximal $S$-free convex sets are polyhedra when $S$ is the set of integral points in some rational polyhedron of $\mathbb{R}^{n}$. This result extends a theorem of Lovász characterizing maximal lattice-free convex sets. Our theorem has implications in integer programming. In particular, we show that maximal $S$-free convex sets are in one-to-one correspondence with minimal inequalities.


## 1 Introduction

Consider a mixed integer linear program, and the optimal tableau of the linear programming relaxation. We select $n$ rows of the tableau, relative to $n$ basic integer variables $x_{1}, \ldots, x_{n}$. Let $s_{1}, \ldots, s_{m}$ denote the nonbasic variables. Let $f_{i} \geq 0$ be the value of $x_{i}$ in the basic solution associated with the tableau, $i=1, \ldots, n$, and suppose $f \notin \mathbb{Z}^{n}$. The tableau restricted to these $n$ rows is of the form

$$
\begin{equation*}
x=f+\sum_{j=1}^{m} r^{j} s_{j}, \quad x \geq 0 \text { integral, } s \geq 0, \text { and } s_{j} \in \mathbb{Z}, j \in I, \tag{1}
\end{equation*}
$$

where $r^{j} \in \mathbb{R}^{n}, j=1, \ldots, m$, and $I$ denotes the set of integer nonbasic variables.

[^0]An important question in integer programming is to derive valid inequalities for (11), cutting off the current infeasible solution $x=f, s=0$. We will consider a simplified model where the integrality conditions are relaxed on all nonbasic variables. On the other hand, we can present our results in a more general context, where the constraints $x \geq 0, x \in \mathbb{Z}^{n}$, are replaced by constraints $x \in S$, where $S$ is the set of integral points in some given rational polyhedron such that $\operatorname{dim}(S)=n$, i.e. $S$ contains $n+1$ affinely independent points. Recall that a polyhedron $A x \leq b$ is rational if the matrix $A$ and vector $b$ have rational entries.

So we study the following model, introduced by Johnson [8].

$$
\begin{equation*}
x=f+\sum_{j=1}^{m} r^{j} s_{j}, \quad x \in S, s \geq 0 \tag{2}
\end{equation*}
$$

where $f \in \operatorname{conv}(S) \backslash \mathbb{Z}^{n}$. Note that every inequality cutting off the point $(f, 0)$ can be expressed in terms of the nonbasic variables $s$ only, and can therefore be written in the form $\sum_{j=1}^{m} \alpha_{j} s_{j} \geq 1$.

In this paper we are interested in "formulas" for deriving such inequalities. More formally, we are interested in functions $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that the inequality

$$
\sum_{j=1}^{m} \psi\left(r^{j}\right) s_{j} \geq 1
$$

is valid for (2) for every choice of $m$ and vectors $r^{1}, \ldots, r^{m} \in \mathbb{R}^{n}$. We refer to such functions $\psi$ as valid functions (with respect to $f$ and $S$ ). Note that, if $\psi$ is a valid function and $\psi^{\prime}$ is a function such that $\psi \leq \psi^{\prime}$, then $\psi^{\prime}$ is also valid, and the inequality $\sum_{j=1}^{m} \psi^{\prime}\left(r^{j}\right) s_{j} \geq 1$ is implied by $\sum_{j=1}^{m} \psi\left(r^{j}\right) s_{j} \geq 1$. Therefore we only need to investigate (pointwise) minimal valid functions.

Andersen, Louveaux, Weismantel, Wolsey [1] characterize minimal valid functions for the case $n=2, S=\mathbb{Z}^{2}$. Borozan and Cornuéjols [6] extend this result to $S=\mathbb{Z}^{n}$ for any $n$. These papers and a result of Zambelli 11 show a one-to-one correspondence between minimal valid functions and maximal lattice-free convex sets with $f$ in the interior. These results have been further generalized in [4]. Minimal valid functions for the case $S=\mathbb{Z}^{n}$ are intersection cuts [2].

Our interest in model (2) arose from a recent paper of Dey and Wolsey [7. They introduce the notion of $S$-free convex set as a convex set without points of $S$ in its interior, and show the connection between valid functions and $S$-free convex sets with $f$ in their interior.

A class of valid functions can be defined as follows. A function $\psi$ is positively homogeneous if $\psi(\lambda r)=\lambda(\psi r)$ for every $r \in \mathbb{R}^{n}$ and every $\lambda \geq 0$, and it is subadditive if $\psi(r)+\psi\left(r^{\prime}\right) \geq$ $\psi\left(r+r^{\prime}\right)$ for all $r, r^{\prime} \in \mathbb{R}^{n}$. A function $\psi$ is sublinear if it is positively homogeneous and subadditive. It is easy to observe that sublinear functions are also convex.

Assume that $\psi$ is a sublinear function such that the set

$$
\begin{equation*}
B_{\psi}=\left\{x \in \mathbb{R}^{n} \mid \psi(x-f) \leq 1\right\} \tag{3}
\end{equation*}
$$

is $S$-free. Note that $B_{\psi}$ is closed and convex because $\psi$ is convex. Since $\psi$ is positively homogeneous, $\psi(0)=0$, thus $f$ is in the interior of $B_{\psi}$. We claim that $\psi$ is a valid function.

Indeed, given any solution $(\bar{x}, \bar{s})$ to (2), we have

$$
\sum_{j=1}^{m} \psi\left(r^{j}\right) \bar{s}_{j} \geq \psi\left(\sum_{j=1}^{m} r^{j} \bar{s}_{j}\right)=\psi(\bar{x}-f) \geq 1
$$

where the first inequality follows from sublinearity and the last one follows from the fact that $\bar{x}$ is not in the interior of $B_{\psi}$.

Dey and Wolsey [7] show that every minimal valid function $\psi$ is sublinear and $B_{\psi}$ is an $S$-free convex set with $f$ in its interior. In this paper, we prove that if $\psi$ is a minimal valid function, then $B_{\psi}$ is a maximal $S$-free convex set.

In Section2, we show that maximal $S$-free convex sets are polyhedra. Therefore a maximal $S$-free convex set $B \subseteq \mathbb{R}^{n}$ containing $f$ in its interior can be uniquely written in the form $B=\left\{x \in \mathbb{R}^{n}: a_{i}(x-f) \leq 1, i=1, \ldots, k\right\}$. Let $\psi_{B}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be the function defined by

$$
\begin{equation*}
\psi_{B}(r)=\max _{i=1, \ldots, k} a_{i} r, \quad \forall r \in \mathbb{R}^{n} \tag{4}
\end{equation*}
$$

It is easy to observe that the above function is sublinear and $B=\left\{x \in \mathbb{R}^{n} \mid \psi_{B}(x-f) \leq 1\right\}$. In Section 3 we will prove that every minimal valid function is of the form $\psi_{B}$ for some maximal $S$-free convex set $B$ containing $f$ in its interior. Conversely, if $B$ is a maximal $S$-free convex set containing $f$ in its interior, then $\psi_{B}$ is a minimal valid function.

## 2 Maximal $S$-free convex sets

Let $S \subseteq \mathbb{Z}^{n}$ be the set of integral points in some rational polyhedron of $\mathbb{R}^{n}$. We say that $B \subset \mathbb{R}^{n}$ is an $S$-free convex set if $B$ is convex and does not contain any point of $S$ in its interior. We say that $B$ is a maximal $S$-free convex set if it is an $S$-free convex set and it is not properly contained in any $S$-free convex set. It follows from Zorn's lemma that every $S$-free convex set is contained in a maximal $S$-free convex set.

When $S=\mathbb{Z}^{n}$, an $S$-free convex set is called a lattice-free convex set. The following theorem of Lovász characterizes maximal lattice-free convex sets. A linear subspace or cone in $\mathbb{R}^{n}$ is rational if it can be generated by rational vectors, i.e. vectors with rational coordinates.

Theorem 1. (Lovász [9) $A$ set $B \subset \mathbb{R}^{n}$ is a maximal lattice-free convex set if and only if one of the following holds:
(i) $B$ is a polyhedron of the form $B=P+L$ where $P$ is a polytope, $L$ is a rational linear space, $\operatorname{dim}(B)=\operatorname{dim}(P)+\operatorname{dim}(L)=n$, $B$ does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of $B$;
(ii) $B$ is a hyperplane of $\mathbb{R}^{n}$ that is not rational.

Lovász only gives a sketch of the proof. A complete proof can be found in [4]. The next theorem is an extension of Lovász' theorem to maximal $S$-free convex sets.

Given a convex set $K \subset \mathbb{R}^{n}$, we denote by $\operatorname{rec}(K)$ its recession cone and by $\operatorname{lin}(K)$ its lineality space. Given a set $X \subseteq \mathbb{R}^{n}$, we denote by $\langle X\rangle$ the linear space generated by $X$. Given a $k$-dimensional linear space $V$ and a subset $\Lambda$ of $V$, we say that $\Lambda$ is a lattice of $V$ if there exists a linear bijection $f: \mathbb{R}^{k} \rightarrow V$ such that $\Lambda=f\left(\mathbb{Z}^{k}\right)$.

Theorem 2. Let $S$ be the set of integral points in some rational polyhedron of $\mathbb{R}^{n}$ such that $\operatorname{dim}(S)=n$. A set $B \subset \mathbb{R}^{n}$ is a maximal $S$-free convex set if and only if one of the following holds:
(i) $B$ is a polyhedron such that $B \cap \operatorname{conv}(S)$ has nonempty interior, $B$ does not contain any point of $S$ in its interior and there is a point of $S$ in the relative interior of each of its facets.
(ii) $B$ is a half-space of $\mathbb{R}^{n}$ such that $B \cap \operatorname{conv}(S)$ has empty interior and the boundary of $B$ is a supporting hyperplane of $\operatorname{conv}(S)$.
(iii) $B$ is a hyperplane of $\mathbb{R}^{n}$ such that $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ is not rational.

Furthermore, if (i) holds, the recession cone of $B \cap \operatorname{conv}(S)$ is rational and it is contained in the lineality space of $B$.

We illustrate case (i) of the theorem in the plane in Figure 2, The question of the polyhedrality of maximal $S$-free convex sets was raised by Dey and Wolsey [7]. They proved that this is the case for a maximal $S$-free convex set $B$, under the assumptions that $B \cap \operatorname{conv}(S)$ has nonempty interior and that the recession cone of $B \cap \operatorname{conv}(S)$ is finitely generated and rational. Theorem 2 settles the question in general.


Figure 1: Two examples of $S$-free sets in the plane (case (i) of Theorem 2). The light gray region indicates conv $(S)$ and the dark grey regions illustrate the $S$-free sets. A jagged line indicates that the region extends to infinity.

To prove Theorem 2 we will need the following lemmas. The first one is proved in [4] and is an easy consequence of Dirichlet's theorem.

Lemma 3. Let $y \in \mathbb{Z}^{n}$ and $r \in \mathbb{R}^{n}$. For every $\varepsilon>0$ and $\bar{\lambda} \geq 0$, there exists an integral point at distance less than $\varepsilon$ from the half line $\{y+\lambda r \mid \lambda \geq \bar{\lambda}\}$.

Lemma 4. Let $B$ be an $S$-free convex set such that $B \cap \operatorname{conv}(S)$ has nonempty interior. For every $r \in \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S)), B+\langle r\rangle$ is $S$-free.

Proof. Let $C=\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ and $r \in C \backslash\{0\}$. Suppose by contradiction that there exists $y \in S \cap \operatorname{int}(B+\langle r\rangle)$. We show that $y \in \operatorname{int}(B)+\langle r\rangle$. If not, $(y+\langle r\rangle) \cap \operatorname{int}(B)=\emptyset$, which implies that there is a hyperplane $H$ separating the line $y+\langle r\rangle$ and $B+\langle r\rangle$, a contradiction. Thus there exists $\bar{\lambda}$ such that $\bar{y}=y+\bar{\lambda} r \in \operatorname{int}(B)$, i.e. there exists $\varepsilon>0$ such that $B$ contains the open ball $B_{\varepsilon}(\bar{y})$ of radius $\varepsilon$ centered at $\bar{y}$. Since $r \in C \subseteq \operatorname{rec}(B)$, it follows that $B_{\varepsilon}(\bar{y})+\{\lambda r \mid \lambda \geq 0\} \subset B$. Since $y \in \mathbb{Z}^{n}$, by Lemma 3 there exists $z \in \mathbb{Z}^{n}$ at distance less than $\varepsilon$ from the half line $\{y+\lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_{\varepsilon}(\bar{y})+\{\lambda r \mid \lambda \geq 0\}$, hence $z \in \operatorname{int}(B)$. Note that the half-line $\{y+\lambda r \mid \lambda \geq \bar{\lambda}\}$ is in $\operatorname{conv}(S)$, since $y \in S$ and $r \in \operatorname{rec}(\operatorname{conv}(S))$. Since $\operatorname{conv}(S)$ is a rational polyhedron, for $\varepsilon>0$ sufficiently small every integral point at distance at most $\varepsilon$ from $\operatorname{conv}(S)$ is in $\operatorname{conv}(S)$. Therefore $z \in S$, a contradiction.

Proof of Theorem 园. The proof of the "if" part is standard, and it is similar to the proof for the lattice-free case (see [4]). We show the "only if" part. Let $B$ be a maximal $S$-free convex set. If $\operatorname{dim}(B)<n$, then $B$ is contained in some affine hyperplane $K$. Since $K$ has empty interior, $K$ is $S$-free, thus $B=K$ by maximality of $B$. Next we show that $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ is not rational. Suppose not. Then the linear subspace $L=\langle\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))\rangle$ is rational. Therefore the projection $\Lambda$ of $\mathbb{Z}^{n}$ onto $L^{\perp}$ is a lattice of $L^{\perp}$ (see, for example, Barvinok [3] p 284 problem 3). The projection $S^{\prime}$ of $S$ onto $L^{\perp}$ is a subset of $\Lambda$. Let $B^{\prime}$ be the projection of $B$ onto $L^{\perp}$. Then $B^{\prime} \cap \operatorname{conv}\left(S^{\prime}\right)$ is the projection of $B \cap \operatorname{conv}(S)$ onto $L^{\perp}$. Since $B$ is a hyperplane, $\operatorname{lin}(B)=\operatorname{rec}(B)$. This implies that $B^{\prime} \cap \operatorname{conv}\left(S^{\prime}\right)$ is bounded : otherwise there is an unbounded direction $d \in L^{\perp}$ in $\operatorname{rec}\left(B^{\prime}\right) \cap \operatorname{rec}\left(\operatorname{conv}\left(S^{\prime}\right)\right)$ and so $d+l \in \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ for some $l \in L$. Since $\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))=\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Fix $\delta>0$. Since $\Lambda$ is a lattice and $S^{\prime} \subseteq \Lambda$, there is a finite number of points at distance less than $\delta$ from the bounded set $B^{\prime} \cap \operatorname{conv}\left(S^{\prime}\right)$ in $L^{\perp}$. It follows that there exists $\varepsilon>0$ such that every point of $S^{\prime}$ has distance at least $\varepsilon$ from $B^{\prime} \cap \operatorname{conv}\left(S^{\prime}\right)$. Let $B^{\prime \prime}=\left\{v+w \mid v \in B, w \in L^{\perp},\|w\| \leq \varepsilon\right\}$. The set $B^{\prime \prime}$ is $S$-free by the choice of $\varepsilon$, but $B^{\prime \prime}$ strictly contains $B$, contradicting the maximality of $B$. Therefore (iii) holds when $\operatorname{dim}(B)<n$. Hence we may assume $\operatorname{dim}(B)=n$. If $B \cap \operatorname{conv}(S)$ has empty interior, then there exists a hyperplane separating $B$ and $\operatorname{conv}(S)$ which is supporting for $\operatorname{conv}(S)$. By maximality of $B$ case ( $i i$ ) follows.

Therefore we may assume that $B \cap \operatorname{conv}(S)$ has nonempty interior. We show that $B$ satisfies ( $i$ ).

Claim 1. There exists a rational polyhedron $P$ such that:
i) $\operatorname{conv}(S) \subset \operatorname{int}(P)$,
ii) The set $K=B \cap P$ is lattice-free,
iii) For every facet $F$ of $P, F \cap K$ is a facet of $K$,
iv) For every facet $F$ of $P, F \cap K$ contains an integral point in its relative interior.

Since $\operatorname{conv}(S)$ is a rational polyhedron, there exist integral $A$ and $b$ such that $\operatorname{conv}(S)=$ $\left\{x \in \mathbb{R}^{n} \mid A x \leq b\right\}$. The set $P^{\prime}=\left\{x \in \mathbb{R}^{n} \left\lvert\, A x \leq b+\frac{1}{2} \mathbf{1}\right.\right\}$ satisfies $i$. The set $B \cap P^{\prime}$ is lattice-free since $B$ is $S$-free and $P^{\prime}$ does not contain any point in $\mathbb{Z}^{n} \backslash S$, thus $P^{\prime}$ also satisfies ii). Let $\bar{A} x \leq \bar{b}$ be the system containing all inequalities of $A x \leq b+\frac{1}{2} \mathbf{1}$ that define facets
of $B \cap P^{\prime}$. Let $P_{0}=\left\{x \in \mathbb{R}^{n} \mid \bar{A} x \leq \bar{b}\right\}$. Then $P_{0}$ satisfies $\left.\left.i\right), i i\right)$, $i i i$ ). See Figure 2 for an illustration.


Figure 2: Illustration for Claim 1
It will be more convenient to write $P_{0}$ as intersection of the half-spaces defining the facets of $P_{0}, P_{0}=\cap_{H \in \mathcal{F}_{0}} H$. We construct a sequence of rational polyhedra $P_{0} \subset P_{1} \subset \ldots \subset P_{t}$ such that $P_{i}$ satisfies $\left.\left.\left.i\right), i i\right), i i i\right), i=1, \ldots, t$, and such that $P_{t}$ satisfies $\left.i v\right)$. Given $P_{i}$, we construct $P_{i+1}$ as follows. Let $P_{i}=\cap_{H \in \mathcal{F}_{i}} H$, where $\mathcal{F}_{i}$ is the set of half spaces defining facets of $P_{i}$. Let $\bar{H}$ be a half-space in $\mathcal{F}_{i}$ defining a facet of $B \cap P_{i}$ that does not contain an integral point in its relative interior; if no such $\bar{H}$ exists, then $P_{i}$ satisfies $i v$ ) and we are done. If $B \cap \bigcap_{H \in \mathcal{F}_{i} \backslash\{\bar{H}\}} H$ does not contain any integral point in its interior, let $\mathcal{F}_{i+1}=\mathcal{F}_{i} \backslash\{\bar{H}\}$. Otherwise, since $P_{i}$ is rational, among all integral points in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i} \backslash\{\bar{H}\}} H$ there exists one, say $\bar{x}$, at minimum distance from $\bar{H}$. Let $H^{\prime}$ be the half-space containing $\bar{H}$ with $\bar{x}$ on its boundary. Let $\mathcal{F}_{i+1}=\mathcal{F}_{i} \backslash\{\bar{H}\} \cap\left\{H^{\prime}\right\}$. Observe that $H^{\prime}$ defines a facet of $P_{i+1}$ since $\bar{x}$ is in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i+1} \backslash\left\{H^{\prime}\right\}} H$ and it is on the boundary of $H^{\prime}$. So $i), i i), i i i)$ are satisfied and $P_{i+1}$ has fewer facets that violate $i v$ ) than $P_{i}$.

Let $T$ be a maximal lattice-free convex set containing the set $K$ defined in Claim 1. As remarked earlier, such a set $T$ exists. By Theorem $\mathbb{1}, T$ is a polyhedron with an integral point in the relative interior of each of its facets. Let $H$ be a hyperplane that defines a facet of $P$. Since $K \cap H$ is a facet of $K$ with an integral point in its relative interior, it follows that $H$ defines a facet of $T$. This implies that $T \subset P$. Therefore we can write $T$ as

$$
\begin{equation*}
T=P \cap \bigcap_{i=1}^{k} H_{i} \tag{5}
\end{equation*}
$$

where $H_{i}$ are halfspaces. Let $\bar{H}_{i}=\mathbb{R}^{n} \backslash \operatorname{int}\left(H_{i}\right), i=1, \ldots, k$.
Claim 2. B is a polyhedron.

We first show that, for $i=1, \ldots, k, \operatorname{int}(B) \cap\left(\bar{H}_{i} \cap \operatorname{conv}(S)\right)=\emptyset$. Consider $y \in \operatorname{int}(B) \cap \bar{H}_{i}$. Since $y \in \bar{H}_{i}$ and $K$ is contained in $T, y \notin \operatorname{int}(K)$. Since $K=B \cap P$ and $y \in \operatorname{int}(B) \backslash \operatorname{int}(K)$, it follows that $y \notin \operatorname{int}(P)$. Hence $y \notin \operatorname{conv}(S)$ because $\operatorname{conv}(S) \subseteq \operatorname{int}(P)$.

Thus, for $i=1, \ldots, k$, there exists a hyperplane separating $B$ and $\bar{H}_{i} \cap \operatorname{conv}(S)$. Hence there exists a halfspace $K_{i}$ such that $B \subset K_{i}$ and $\bar{H}_{i} \cap \operatorname{conv}(S)$ is disjoint from the interior of $K_{i}$. We claim that the set $B^{\prime}=\cap_{i=1}^{k} K_{i}$ is $S$-free. Indeed, let $y \in S$. Then $y$ is not interior of $T$. Since $y \in \operatorname{conv}(S)$ and $\operatorname{conv}(S) \subseteq \operatorname{int}(P), y$ is in the interior of $P$. Hence, by (5), there exists $i \in\{1, \ldots, k\}$ such that $y$ is not in the interior of $H_{i}$. Thus $y \in \bar{H}_{i} \cap \operatorname{conv}(S)$. By construction, $y$ is not in the interior of $K_{i}$, hence $y$ is not in the interior of $B^{\prime}$. Thus $B^{\prime}$ is an $S$-free convex set containing $B$. Since $B$ is maximal, $B^{\prime}=B$.
Claim 3. $\operatorname{lin}(K)=\operatorname{rec}(K)$.
Let $r \in \operatorname{rec}(K)$. We show $-r \in \operatorname{rec}(K)$. By Lemma 4 applied to $\mathbb{Z}^{n}, K+\langle r\rangle$ is latticefree. We observe that $B+\langle r\rangle$ is $S$-free. If not, let $y \in S \cap \operatorname{int}(B+\langle r\rangle)$. Since $S \subseteq \operatorname{int}(P)$, $y \in \operatorname{int}(P+\langle r\rangle)$, hence $y \in \operatorname{int}(K+\langle r\rangle)$, a contradiction. Hence, by maximality of $B$, $B=B+\langle r\rangle$. Thus $-r \in \operatorname{rec}(B)$. Suppose that $-r \notin \operatorname{rec}(P)$. Then there exists a facet $F$ of $P$ that is not parallel to $r$. By construction, $F \cap K$ is a facet of $K$ containing an integral point $\bar{x}$ in its relative interior. The point $\bar{x}$ is then in the interior of $K+\langle r\rangle$, a contradiction. $\diamond$

Claim 4. $\operatorname{lin}(K)$ is rational.
Consider the maximal lattice-free convex set $T$ containing $K$ considered earlier. By Theorem 亿 $\operatorname{lin}(T)=\operatorname{rec}(T)$, and $\operatorname{lin}(T)$ is rational. Clearly $\operatorname{lin}(T) \supseteq \operatorname{lin}(K)$. Hence, if the claim does not hold, there exists a rational vector $r \in \operatorname{lin}(T) \backslash \operatorname{lin}(K)$. By (5), $r \in \operatorname{lin}(P)$. Since $K=B \cap P, r \notin \operatorname{lin}(B)$. Hence $B \subset B+\langle r\rangle$. We will show that $B+\langle r\rangle$ is $S$ free, contradicting the maximality of $B$. Suppose there exists $y \in S \cap \operatorname{int}(B+\langle r\rangle)$. Since $\operatorname{conv}(S) \subseteq \operatorname{int}(P), y \in \operatorname{int}(P) \subseteq \operatorname{int}(P)+\langle r\rangle$. Therefore $y \in \operatorname{int}(B \cap P)+\langle r\rangle$. Since $B \cap P \subseteq T$, then $y \in \operatorname{int}(T)+\langle r\rangle=\operatorname{int}(T)$ where the last equality follows from $r \in \operatorname{lin}(T)$. This contradicts the fact that $T$ is lattice-free.

By Lemma 4 and by the maximality of $B, \operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))=\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$.
Claim 5. $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ is rational.
Since $\operatorname{lin}(K)$ and $\operatorname{rec}(\operatorname{conv}(S))$ are both rational, we only need to show $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))=$ $\operatorname{lin}(K) \cap \operatorname{rec}(\operatorname{conv}(S))$. The " $\supseteq$ " direction follows from $B \supseteq K$. For the other direction, note that, since $\operatorname{conv}(S) \subseteq P$, we have $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) \subseteq \operatorname{lin}(B) \cap \operatorname{rec}(P)=\operatorname{lin}(B \cap P)=$ $\operatorname{lin}(K)$, hence $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) \subseteq \operatorname{lin}(K) \cap \operatorname{rec}(\operatorname{conv}(S))$.

Claim 6. Every facet of $B$ contains a point of $S$ in its relative interior.
Let $L$ be the linear space generated by $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$. By Claim 国, $L$ is rational. Let $B^{\prime}, S^{\prime}, \Lambda$ be the projections of $B, S, \mathbb{Z}^{n}$, respectively, onto $L^{\perp}$. Since $L$ is rational, $\Lambda$ is a lattice of $L^{\perp}$ and $S^{\prime}=\operatorname{conv}\left(S^{\prime}\right) \cap \Lambda$. Also, $B^{\prime}$ is a maximal $S^{\prime}$-free convex set of $L^{\perp}$, since for any $S^{\prime}$-free set $D$ of $L^{\perp}, D+L$ is $S$-free. Note that $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))=$ $\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ implies that $B^{\prime} \cap \operatorname{conv}\left(S^{\prime}\right)$ is bounded. Otherwise there is an unbounded direction $d \in L^{\perp}$ in $\operatorname{rec}\left(B^{\prime}\right) \cap \operatorname{rec}\left(\operatorname{conv}\left(S^{\prime}\right)\right)$ and so $d+l \in \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ for some
$l \in L$. Since $\operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))=\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$, this would imply that $d \in L$ which is a contradiction. Let $B^{\prime}=\left\{x \in L^{\perp} \mid \alpha_{i} x \leq \beta_{i}, i=1, \ldots, t\right\}$. Given $\varepsilon>0$, let $\bar{B}=\left\{x \in L^{\perp} \mid \alpha_{i} x \leq \beta_{i}, i=1, \ldots, t-1, \alpha_{t} x \leq \beta_{t}+\varepsilon\right\}$. The polyhedron $\operatorname{conv}\left(S^{\prime}\right) \cap \bar{B}$ is a polytope since it has the same recession cone as $\operatorname{conv}\left(S^{\prime}\right) \cap B^{\prime}$. The polytope $\operatorname{conv}\left(S^{\prime}\right) \cap \bar{B}$ contains points of $S^{\prime}$ in its interior by the maximality of $B^{\prime}$. Since $\Lambda$ is a lattice of $L^{\perp}$, $\operatorname{int}\left(\operatorname{conv}\left(S^{\prime}\right) \cap \bar{B}\right)$ has a finite number of points in $S^{\prime}$, hence there exists one minimizing $\alpha_{t} x$, say $z$. By construction, the polyhedron $B^{\prime \prime}=\left\{x \in L^{\perp} \mid \alpha_{i} x \leq \beta_{i}, i=1, \ldots, t-1, \alpha_{t} x \leq \alpha_{t} z\right\}$ does not contain any point of $S^{\prime}$ in its interior and contains $B^{\prime}$. By the maximality of $B^{\prime}$, $B^{\prime}=B^{\prime \prime}$ hence $B^{\prime}$ contains $z$ in its relative interior, and $B$ contains a point of $S$ in its relative interior.

Corollary 5. For every maximal $S$-free convex set $B$ there exists a maximal lattice-free convex set $K$ such that, for every facet $F$ of $B, F \cap K$ is a facet of $K$.

Proof. Let $K$ be defined as in Claim 1 in the proof of Theorem 2. It follows from the proof that $K$ is a maximal lattice-free convex set with the desired properties.

## 3 Minimal valid functions

In this section we study minimal valid functions. We find it convenient to state our results in terms of an infinite model introduced by Dey and Wolsey [7].

Throughout this section, $S \subseteq \mathbb{Z}^{n}$ is a set of integral points in some rational polyhedron of $\mathbb{R}^{n}$ such that $\operatorname{dim}(S)=n$, and $f$ is a point in $\operatorname{conv}(S) \backslash \mathbb{Z}^{n}$. Let $R_{f, S}$ be the set of all infinite dimensional vectors $s=\left(s_{r}\right)_{r \in \mathbb{R}^{n}}$ such that

$$
\begin{align*}
& f+\sum_{r \in \mathbb{R}^{n}} r s_{r} \in S \\
& s_{r} \geq 0, \quad r \in \mathbb{R}^{n}  \tag{6}\\
& s \text { has finite support }
\end{align*}
$$

where $s$ has finite support means that $s_{r}$ is zero for all but a finite number of $r \in \mathbb{R}^{n}$.
A function $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is valid (with respect to $f$ and $S$ ) if the linear inequality

$$
\begin{equation*}
\sum_{r \in \mathbb{R}^{n}} \psi(r) s_{r} \geq 1 \tag{7}
\end{equation*}
$$

is satisfied by every $s \in R_{f, S}$. Note that this definition coincides with the one we gave in the introduction.

Given two functions $\psi, \psi^{\prime}$ we say that $\psi^{\prime}$ dominates $\psi$ if $\psi^{\prime}(r) \leq \psi(r)$ for all $r \in \mathbb{R}^{n}$. A valid function $\psi$ is minimal if there is no valid function $\psi^{\prime} \neq \psi$ that dominates $\psi$.

Theorem 6. For every minimal valid function $\psi$, there exists a maximal $S$-free convex set $B$ with $f$ in its interior such that $\psi_{B}$ dominates $\psi$. Furthermore, if $B$ is a maximal $S$-free convex set containing $f$ in its interior, then $\psi_{B}$ is a minimal valid function.

We will need the following lemma.

Lemma 7. Every valid function is dominated by a sublinear valid function.
Sketch of proof. Given a valid function $\psi$, define the following function $\bar{\psi}$. For all $\bar{r} \in \mathbb{R}^{n}$, let $\bar{\psi}(\bar{r})=\inf \left\{\sum_{r \in \mathbb{R}^{n}} \psi(r) s_{r} \mid \sum_{r \in \mathbb{R}^{n}} r s_{r}=\bar{r}, s \geq 0\right.$ with finite support $\}$. Following the proof of Lemma 18 in [4 one can show that $\bar{\psi}$ is a valid sublinear function that dominates $\psi$.

Given a valid sublinear function $\psi$, the set $B_{\psi}=\left\{x \in \mathbb{R}^{n} \mid \psi(x-f) \leq 1\right\}$ is closed, convex, and contains $f$ in its interior. Since $\psi$ is a valid function, $B_{\psi}$ is $S$-free. Indeed the interior of $B_{\psi}$ is $\operatorname{int}\left(B_{\psi}\right)=\left\{x \in \mathbb{R}^{n}: \psi(x-f)<1\right\}$. Its boundary is $\mathbf{b d}\left(B_{\psi}\right)=\left\{x \in \mathbb{R}^{n}: \psi(x-f)=\right.$ $1\}$, and its recession cone is $\operatorname{rec}\left(B_{\psi}\right)=\left\{x \in \mathbb{R}^{n}: \psi(x-f) \leq 0\right\}$.

Before proving Theorem 6, we need the following general theorem about sublinear functions. Let $K$ be a closed, convex set in $\mathbb{R}^{n}$ with the origin in its interior. The polar of $K$ is the set $K^{*}=\left\{y \in \mathbb{R}^{n} \mid r y \leq 1\right.$ for all $\left.r \in K\right\}$. Clearly $K^{*}$ is closed and convex, and since $0 \in \operatorname{int}(K)$, it is well known that $K^{*}$ is bounded. In particular, $K^{*}$ is a compact set. Also, since $0 \in K, K^{* *}=K$. Let

$$
\begin{equation*}
\hat{K}=\left\{y \in K^{*} \mid \exists x \in K \text { such that } x y=1\right\} . \tag{8}
\end{equation*}
$$

Note that $\hat{K}$ is contained in the relative boundary of $K^{*}$. Let $\rho_{K}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
\rho_{K}(r)=\sup _{y \in \hat{K}} r y, \quad \text { for all } r \in \mathbb{R}^{n} . \tag{9}
\end{equation*}
$$

It is easy to show that $\rho_{K}$ is sublinear.
Theorem 8 (Basu et al. [5]). Let $K \subset \mathbb{R}^{n}$ be a closed convex set containing the origin in its interior. Then $K=\left\{r \in \mathbb{R}^{n} \mid \rho_{K}(r) \leq 1\right\}$. Furthermore, for every sublinear function $\sigma$ such that $K=\{r \mid \sigma(r) \leq 1\}$, we have $\rho_{K}(r) \leq \sigma(r)$ for every $r \in \mathbb{R}^{n}$.

Remark 9. Let $K \subset \mathbb{R}^{n}$ be a polyhedron containing the origin in its interior. Let $a_{1}, \ldots, a_{t} \in$ $\mathbb{R}^{n}$ such that $K=\left\{r \in \mathbb{R}^{n} \mid a_{i} r \leq 1, i=1, \ldots, t\right\}$. Then $\rho_{K}(r)=\max _{i=1, \ldots, t} a_{i} r$.

Proof. The polar of $K$ is $K^{*}=\operatorname{conv}\left\{0, a_{1}, \ldots, a_{t}\right\}$ (see Theorem 9.1 in Schrijver [10]). Furthermore, $\hat{K}$ is the union of all the facets of $K^{*}$ that do not contain the origin, therefore

$$
\rho_{K}(r)=\sup _{y \in \hat{K}} y r=\max _{i=1, \ldots, t} a_{i} r
$$

for all $r \in \mathbb{R}^{n}$.
Remark 10. Let $B$ be a closed $S$-free convex set in $\mathbb{R}^{n}$ with $f$ in its interior, and let $K=$ $B-f$. Then $\rho_{K}$ is a valid function.

Proof: Let $s \in R_{f, S}$. Then $x=f+\sum_{r \in \mathbb{R}^{n}} r s_{r}$ is in $S$, therefore $x \notin \operatorname{int}(B)$ because $B$ is $S$-free. By Theorem [8, $\rho_{K}(x-f) \geq 1$. Thus

$$
1 \leq \rho_{K}\left(\sum_{r \in \mathbb{R}^{n}} r s_{r}\right) \leq \sum_{r \in \mathbb{R}^{n}} \rho_{K}(r) s_{r},
$$

where the second inequality follows from the sublinearity of $\rho_{K}$.

Lemma 11. Let $C$ be a closed $S$-free convex set containing $f$ in its interior, and let $K=$ $C-f$. There exists a maximal $S$-free convex set $B=\left\{x \in \mathbb{R}^{n} \mid a_{i}(x-f) \leq 1, i=1, \ldots, k\right\}$ such that $a_{i} \in \mathbf{c l}(\operatorname{conv}(\hat{K}))$ for $i=1, \ldots, k$.

Proof. Since $C$ is an $S$-free convex set, it is contained in some maximal $S$-free convex set $T$. The set $T$ satisfies one of the statements (i)-(iii) of Theorem 2, By assumption, $f \in \operatorname{conv}(S)$ and $f$ is in the interior of $C$. Since $\operatorname{dim}(S)=n, \operatorname{conv}(S)$ is a full dimensional polyhedron, thus $\operatorname{int}(C \cap \operatorname{conv}(S)) \neq \emptyset$. This implies that $\operatorname{int}(T \cap \operatorname{conv}(S)) \neq \emptyset$, hence case (i) applies.

Thus $T$ is a polyhedron and $\operatorname{rec}(T \cap \operatorname{conv}(S))=\operatorname{lin}(T) \cap \operatorname{rec}(\operatorname{conv}(S))$ is rational. Let us choose $T$ such that the dimension of $\operatorname{lin}(T)$ is largest possible.

Since $T$ is a polyhedron containing $f$ in its interior, there exists $D \in \mathbb{R}^{t \times q}$ and $b \in \mathbb{R}^{t}$ such that $b_{i}>0, i=1, \ldots, t$, and $T=\left\{x \in \mathbb{R}^{n} \mid D(x-f) \leq b\right\}$. Without loss of generality, we may assume that $\sup _{x \in C} d_{i}(x-f)=1$ where $d_{i}$ denotes the $i$ th row of $D, i=1, \ldots, t$. By our assumption, $\sup _{r \in K} d_{i} r=1$. Therefore $d_{i} \in K^{*}$, since $d_{i} r \leq 1$ for all $r \in K$. Furthermore $d_{i} \in \operatorname{cl}(\hat{K})$, since $\sup _{r \in K} d_{i} r=1$.

Let $P=\left\{x \in \mathbb{R}^{n} \mid D(x-f) \leq e\right\}$. Note that $\operatorname{lin}(P)=\operatorname{lin}(T)$. By our choice of $T, P+\langle r\rangle$ is not $S$-free for any $r \in \operatorname{rec}(\operatorname{conv}(S)) \backslash \operatorname{lin}(P)$, otherwise $P$ would be contained in a maximal $S$-free convex set whose lineality space contains $\operatorname{lin}(T)+\langle r\rangle$, a contradiction.

Let $L=\langle\operatorname{rec}(P \cap \operatorname{conv}(S))\rangle$. Since $\operatorname{lin}(P)=\operatorname{lin}(T), L$ is a rational space. Note that $L \subseteq \operatorname{lin}(P)$, implying that $d_{i} \in L^{\perp}$ for $i=1, \ldots, t$.

We observe next that we may assume that $P \cap \operatorname{conv}(S)$ is bounded. Indeed, let $\bar{P}, \bar{S}, \Lambda$ be the projections onto $L^{\perp}$ of $P, S$, and $\mathbb{Z}^{n}$, respectively. Since $L$ is a rational space, $\Lambda$ is a lattice of $L^{\perp}$ and $\bar{S}=\operatorname{conv}(\bar{S}) \cap \Lambda$. Note that $\bar{P} \cap \operatorname{conv}(\bar{S})$ is bounded, since $L \supseteq \operatorname{rec}(P \cap \operatorname{conv}(S))$. If we are given a maximal $\bar{S}$-free convex set $\bar{B}$ in $L^{\perp}$ such that $\bar{B}=\left\{x \in L^{\perp} \mid a_{i}(x-f) \leq\right.$ $1, i=1, \ldots, h\}$ and $a_{i} \in \operatorname{conv}\left\{d_{1}, \ldots, d_{t}\right\}$ for $i=1, \ldots, h$, then $B=\bar{B}+L$ is the set $B=\left\{x \in \mathbb{R}^{n} \mid a_{i}(x-f) \leq 1, i=1, \ldots, h\right\}$. Since $\bar{B}$ contains a point of $\bar{S}$ in the relative interior of each of its facets, $B$ contains a point of $S$ in the relative interior of each of its facets, thus $B$ is a maximal $S$-free convex set.

Thus we assume that $P \cap \operatorname{conv}(S)$ is bounded, so $\operatorname{dim}(L)=0$. If all facets of $P$ contain a point of $S$ in their relative interior, then $P$ is a maximal $S$-free convex set, thus the statement of the lemma holds. Otherwise we describe a procedure that replaces one of the inequalities defining a facet of $P$ without any point of $S$ in its relative interior with an inequality which is a convex combination of the inequalities of $D(x-f) \leq e$, such that the new polyhedron thus obtained is $S$-free and has one fewer facet without points of $S$ in its relative interior. More formally, suppose the facet of $P$ defined by $d_{1}(x-f) \leq 1$ does not contain any point of $S$ in its relative interior. Given $\lambda \in[0,1]$, let

$$
P(\lambda)=\left\{x \in \mathbb{R}^{n} \mid\left[\lambda d_{1}+(1-\lambda) d_{2}\right](x-f) \leq 1, \quad d_{i}(x-f) \leq 1 i=2, \ldots, t\right\} .
$$

Note that $P(1)=P$ and $P(0)$ is obtained from $P$ by removing the inequality $d_{1}(x-f) \leq 1$. Furthermore, given $0 \leq \lambda^{\prime} \leq \lambda^{\prime \prime} \leq 1$, we have $P\left(\lambda^{\prime}\right) \supseteq P\left(\lambda^{\prime \prime}\right)$.

Let $r_{1}, \ldots, r_{m}$ be generators of $\operatorname{rec}(\operatorname{conv}(S))$. Note that, since $P \cap \operatorname{conv}(S)$ is bounded, for every $j=1, \ldots, m$ there exists $i \in\{1, \ldots, t\}$ such that $d_{i} r_{j}>0$. Let $r_{1}, \ldots, r_{h}$ be the
generators of $\operatorname{rec}(\operatorname{conv}(S))$ satisfying

$$
\begin{aligned}
d_{1} r_{j} & >0 \\
d_{i} r_{j} & \leq 0 \quad i=2, \ldots, t .
\end{aligned}
$$

Note that, if no such generators exist, then $P(0) \cap \operatorname{conv}(S)$ is bounded. Otherwise $P(\lambda) \cap$ $\operatorname{conv}(S)$ is bounded if and only if, for $j=1, \ldots, h$

$$
\left[\lambda d_{1}+(1-\lambda) d_{2}\right] r_{j}>0
$$

This is the case if and only if $\lambda>\lambda^{*}$, where

$$
\lambda^{*}=\max _{j=1, \ldots, h} \frac{-d_{2} r_{j}}{\left(d_{1}-d_{2}\right) r_{j}} .
$$

Let $r^{*}$ be one of the vectors $r_{1}, \ldots, r_{h}$ attaining the maximum in the previous equation. Then $r^{*} \in \operatorname{rec}\left(P\left(\lambda^{*} \cap \operatorname{conv}(S)\right.\right.$.

Note that $P\left(\lambda^{*}\right)$ is not $S$-free otherwise $P\left(\lambda^{*}\right)+\left\langle r^{*}\right\rangle$ is $S$-free by Lemma 4, and so is $P+\left\langle r^{*}\right\rangle$, a contradiction.

Thus $P\left(\lambda^{*}\right)$ contains a point of $S$ in its interior. That is, there exists a point $\bar{x} \in S$ such that $\left[\lambda^{*} d_{1}+\left(1-\lambda^{*}\right) d_{2}\right](\bar{x}-f)<1$ and $d_{i}(\bar{x}-f)<1$ for $i=2, \ldots, t$. Since $P$ is $S$-free, $d_{1}(\bar{x}-f)>1$. Thus there exists $\bar{\lambda}>\lambda^{*}$ such that $\left[\bar{\lambda} d_{1}+(1-\bar{\lambda}) d_{2}\right](\bar{x}-f)=1$. Note that, since $P(\bar{\lambda}) \cap \operatorname{conv}(S)$ is bounded, there is a finite number of points of $S$ in the interior of $P(\bar{\lambda})$. So we may choose $\bar{x}$ such that $\bar{\lambda}$ is maximum. Thus $P(\bar{\lambda})$ is $S$-free and $\bar{x}$ is in the relative interior of the facet of $P(\bar{\lambda})$ defined by $\left[\bar{\lambda} d_{1}+(1-\bar{\lambda}) d_{2}\right](x-f) \leq 1$.

Note that, for $i=2, \ldots, t$, if $d_{i}(x-f) \leq 1$ defines a facet of $P$ with a point of $S$ in its relative interior, then it also defines a facet of $P(\bar{\lambda})$ with a point of $S$ in its relative interior, because $P \subset P(\bar{\lambda})$. Thus repeating the above construction at most $t$ times, we obtain a set $B$ satisfying the lemma.

Remark 12. Let $C$ and $K$ be as in Lemma 11. Given any maximal $S$-free convex set $B=\left\{x \in \mathbb{R}^{n} \mid a_{i}(x-f) \leq 1, i=1, \ldots, k\right\}$ containing $C$, then $a_{1}, \ldots, a_{k} \in K^{*}$. If $\operatorname{rec}(C)$ is not full dimensional, then the origin is not an extreme point of $K^{*}$. Since all extreme points of $K^{*}$ are contained in $\{0\} \cup \hat{K}$, in this case $\mathbf{c l}(\operatorname{conv}(\hat{K}))=K^{*}$. Therefore, when $\operatorname{rec}(C)$ is not full dimensional, every maximal $S$-free convex set containing $C$ satisfies the statement of Lemma 11 .

## Proof of Theorem [6.

We first show that any valid function is dominated by a function of the form $\psi_{B}$, for some maximal $S$-free convex set $B$ containing $f$ in its interior.

Let $\psi$ be a valid function. By Lemma 7 we may assume that $\psi$ is sublinear. Let $K=\left\{r \in \mathbb{R}^{n} \mid \psi(r) \leq 1\right\}$, and let $\hat{K}$ be defined as in (8). Note that $K=B_{\psi}-f$. Thus, by Remark [10, $\sum_{r \in \mathbb{R}^{n}} \rho_{K}(r) s_{r} \geq 1$ is valid for $R_{f, S}$. Since $\psi$ is sublinear, it follows from Theorem 8 that $\rho_{K}(r) \leq \psi(r)$ for every $r \in \mathbb{R}^{n}$.

By Lemma 11, there exists a maximal $S$-free convex set $B=\left\{x \in \mathbb{R}^{n} \mid a_{i}(x-f) \leq 1, i=\right.$ $1, \ldots, k\}$ such that $a_{i} \in \mathbf{c l}(\operatorname{conv}(\hat{K}))$ for $i=1, \ldots, k$.

Then

$$
\psi(r) \geq \rho_{K}(r)=\sup _{y \in \hat{K}} y r=\max _{y \in \mathbf{c l}(\operatorname{conv}(\hat{K}))} y r \geq \max _{i=1, \ldots, k} a_{i} r=\psi_{B}(r) .
$$

This shows that $\psi_{B}$ dominates $\psi$ for all $r \in \mathbb{R}^{n}$.
To complete the proof of the theorem, we need to show that, given a maximal $S$-free convex set $B$, the function $\psi_{B}$ is minimal. Consider any valid function $\psi$ dominating $\psi_{B}$. Then $B_{\psi} \supseteq B$ and $B_{\psi}$ is $S$-free. By maximality of $B, B=B_{\psi}$. By Theorem 8 and Remark 9 , $\psi_{B}(r) \leq \psi(r)$ for all $r \in \mathbb{R}^{n}$, proving $\psi=\psi_{B}$.


Figure 3: Illustration for Example 13

Example 13. We illustrate the ideas behind the proof in the following two-dimensional example. Consider $f=\left(\frac{1}{4}, \frac{1}{2}\right)$ and $\left.S=\left\{\left(x_{1}, x_{2}\right) \mid x_{1} \geq 0\right)\right\}$. See Figure 圂. Then the function $\psi(r)=\max \left\{4 r_{1}+8 r_{2}, 4 r_{1}-8 r_{2}\right\}$ is a valid linear inequality for $R_{f, S}$. The corresponding $B_{\psi}$ is $\left\{\left(x_{1}, x_{2}\right) \left\lvert\, 4\left(x_{1}-\frac{1}{4}\right)+8\left(x_{2}-\frac{1}{2}\right) \leq 1\right.,4\left(x_{1}-\frac{1}{4}\right)-8\left(x_{2}-\frac{1}{2}\right) \leq 1\right\}$. Note that $B_{\psi}$ is not a maximal $S$-free convex set and it corresponds to $C$ in Lemma 11. Following the procedure outlined in the proof, we obtain the maximal $S$-free convex set $B=\left\{\left(x_{1}, x_{2}\right) \left\lvert\, 4\left(x_{1}-\frac{1}{4}\right)+4\left(x_{2}-\frac{1}{2}\right) \leq\right.\right.$ $\left.1,4\left(x_{1}-\frac{1}{4}\right)-4\left(x_{2}-\frac{1}{2}\right) \leq 1\right\}$. Then, $\psi_{B}(r)=\max \left\{4 r_{1}+4 r_{2}, 4 r_{1}-4 r_{2}\right\}$ and $\psi_{B}$ dominates $\psi$.

Remark 14. Note that $\psi$ is nonnegative if and only if $\operatorname{rec}\left(B_{\psi}\right)$ is not full-dimensional. It follows from Remark 12 that, for every maximal $S$-free convex set $B$ containing $B_{\psi}$, we have $\psi_{B}(r) \leq \psi(r)$ for every $r \in \mathbb{R}^{n}$ when $\psi$ is nonnegative.

A statement similar to the one of Theorem 6 was shown by Borozan-Cornuéjols [6] for a model similar to (6) when $S=\mathbb{Z}^{n}$ and the vectors $s$ are elements of $\mathbb{R}^{\mathbb{Q}^{n}}$. In this case, it is
easy to show that, for every valid inequality $\sum_{r \in \mathbb{Q}^{n}} \psi(r) s_{r} \geq 1$, the function $\psi: \mathbb{Q}^{n} \rightarrow \mathbb{R}$ is nonnegative. Remark 14 explains why in this context it is much easier to prove that minimal inequalities arise from maximal lattice-free convex sets.

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