OPTIMIZING LINEAR EXTENSIONS

BRIDGET EILEEN TENNER

Abstract. The minimum number of elements needed for a poset to have exactly n linear extensions is at most $2\sqrt{n}$. In a special case, the bound can be improved to $\sqrt{n}$.

1. Introduction and definitions

A partially ordered set, or *poset*, $P = (X, \preceq)$ consists of a set X together with a partial ordering $\preceq$ on X. For background on these structures, the reader is encouraged to review [3] and [4].

One statistic that can hint at how much information is missing in a partial ordering is based on the following definition.

Definition 1.1. A *linear extension* of a poset $P = (X, \preceq)$ is a total ordering of the elements of X that is compatible with $\preceq$. The number of linear extensions of P is denoted $e(P)$.

As suggested in [3], the number of linear extensions of a poset gives an indication of the intricacy of the original partial ordering. Thus understanding the function e can provide some insight into the complexity of the structure of partial orderings.

Another poset statistic, the number of order ideals in a poset, is considered in [1], and a bound is given for the minimal number of elements needed to have a particular number of order ideals. Here, the analogous question is answered for the function e.

Definition 1.2. The *size* of a poset $P = (X, \preceq)$, denoted $|P|$, is the cardinality of $|X|$.

Definition 1.3. For any integer $n \geq 1$, set $\lambda(n) = \min\{|P| : e(P) = n\}$.

The main result of this work, Theorem 3.2, is the bound

$$\lambda(n) \leq 2\sqrt{n}.$$

In a certain case, as discussed in Section 4, this bound can be improved further to $\sqrt{n}$. As displayed in Table 1, there are values of n for which $\lambda(n)$ equals $2\sqrt{n}$.

In the next section, the values of $\lambda(n)$ for small n are given, together with examples of the posets that obtain them. Furthermore, the poset operations that give the primary tools for proving Theorem 3.2 are stated. Section 3 consists of the main result, and a special case is treated in the last section.

2. Examples and arithmetic of poset operations

Before describing how basic poset operations affect the function λ, it is instructive to calculate $\lambda(n)$ for some small values of n, and to view the posets that give these values. These examples appear in Table 1, and as sequence A160371 in [2].

2000 *Mathematics Subject Classification.* Primary 06A07; Secondary 05A99, 06A05.

Key words and phrases. poset, linear extension, optimization.

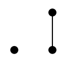
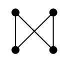
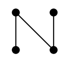
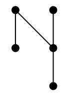
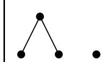
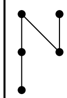
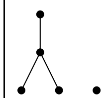
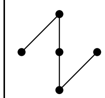
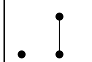
n	1	2	3	4	5	6	7	8	9	10	11	12
$\lambda(n)$	0	2	3	4	4	3	5	4	5	5	5	4
poset example	$\emptyset$											
$\lfloor 2\sqrt{n} \rfloor$	2	2	3	4	4	4	5	5	6	6	6	6

TABLE 1. The values of $\lambda(n)$ for $1 \leq n \leq 12$, together with demonstrative posets, and the upper bound of Theorem 3.2.

Two elementary operations on posets are the *direct sum* and the *ordinal sum*. Note that a poset which can be constructed entirely by these two operations is called *series-parallel*.

Definition 2.1. Let P and Q be posets on the sets X_P and X_Q, respectively, with order relations $\preceq_P$ and $\preceq_Q$, respectively. The direct sum $P+Q$ is the poset defined on $X_P \cup X_Q$, with order relations $\preceq_P \cup \preceq_Q$. The ordinal sum $P \oplus Q$ is the poset defined on $X_P \cup X_Q$, with order relations $\preceq_P \cup \preceq_Q \cup \{x_P \preceq x_Q : x_P \in X_P, x_Q \in X_Q\}$.

The next lemma follows immediately from the definitions.

Lemma 2.2. *For posets P and Q,*

$$e(P+Q) = \binom{|P|+|Q|}{|P|} e(P)e(Q)$$

and

$$e(P \oplus Q) = e(P)e(Q).$$

Definition 2.3. For any $\ell \geq 0$, let the poset C_ℓ be the chain of ℓ elements, where $C_0 = \emptyset$.

Certainly the poset C_ℓ is already a total ordering, so $\lambda(C_\ell) = 1$ for all ℓ. Moreover, it follows from the identities of Lemma 2.2 that

$$e(P + C_\ell) = \binom{|P|+\ell}{|P|} e(P)$$

and

$$e(P \oplus C_\ell) = e(P) \tag{1}$$

for all $\ell \geq 0$. Equation (1) implies that a poset with n linear extensions can have arbitrarily large size. Perhaps unexpectedly, equation (1) will be very helpful in bounding $\lambda(n)$. The key is to employ it as in the following result.

Proposition 2.4. *For all $\ell \geq 0$, $e\left((P \oplus C_\ell) + C_1\right) = (|P| + \ell + 1)e(P)$.*

Proposition 2.4 gives the following initial result for all n.

Corollary 2.5. *If $n = ab$ for $a, b \in \mathbb{Z}^+$ with $a < b$, then $\lambda(n) \leq b$.*

Proof. First note that

$$\lambda(n) \leq n \tag{2}$$

for all $n \in \mathbb{Z}^+$, by considering the n-element poset $C_{n-1} + C_1$, which has n linear extensions.

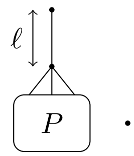
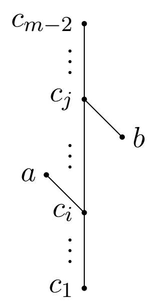


FIGURE 1. The poset $(P \oplus C_\ell) + C_1$ described in Proposition 2.4.

Let P be a poset of size $\lambda(a)$, with $e(P) = a$. Since $a < b$, equation (2) implies $\lambda(a) < b$, and so $b - 1 - |P| \geq 0$. Set $Q = (P \oplus C_{b-1-|P|}) + C_1$. Then $|Q| = |P| + b - 1 - |P| + 1 = b$, and $e(Q) = (|P| + b - 1 - |P| + 1)e(P) = ab = n$. □

3. BOUNDS

The proof of the main result, Theorem 3.2, begins with an analysis of the following m-element poset $Q_{i,j,m}$, where $1 \leq i < j \leq m - 2$. Note that $Q_{i,j,m}$ is not series-parallel.



In any linear extension of $Q_{i,j,m}$, the elements $\{c_1, c_2, \ldots, c_{m-2}\}$ may appear in exactly one order. The element a can appear anywhere after c_i, while the element b can appear anywhere before c_j. The elements a and b are incomparable in $Q_{i,j,m}$, so they can appear in either order if they both appear between c_k and c_{k+1} in a linear extension. Thus

$$e(Q_{i,j,m}) = (m - 1 - i)j + (j - i) = (m - i)j - i,$$

and so

$$\lambda\left((m - i)j - i\right) \leq m.$$

Proposition 3.1. *For all integers $n \geq 1$ and $d \geq 1$,*

$$\lambda(n) \leq \lfloor n/d \rfloor + d.$$

Proof. This is proved by induction on d, where the case $d = 1$ follows from equation (2).

Now suppose that $d \geq 2$ and that the result holds for all $d' \in [1, d)$. The integer n can be written as $n = qd - r$, where $r \in [0, d - 1]$. If $r \geq 1$ and $q + r - 2 \geq d$, then $Q_{r,d,q+r}$ is a poset having n linear extensions and size

$$q + r \leq \lfloor n/d \rfloor + 1 + (d - 1) = \lfloor n/d \rfloor + d.$$

Thus it remains to consider when $r = 0$ or $q + r - 2 < d$.

If $r = 0$, then $n = qd$ and Lemma 2.2 implies that

$$\lambda(n) \leq \lambda(q) + \lambda(d) \leq q + d = \lfloor n/d \rfloor + d.$$

This leaves the case when $r \in [1, d-1]$ and $q + r - 1 \leq d$. The few cases that remain when $d \in \{2, 3\}$ are easy to check (in fact, they concern only $n \leq 12$, and so appear in Table 1). For the conclusion of the argument, suppose $d \geq 4$.

Rewrite n as $n = q'(d-1) + r'$ where $r' \in [0, d-2]$. Because $n = q(d-1) + q + r$, the restrictions on q, r, and d imply that there is at most one extra factor of $d-1$ in $q+r$. That is, $q' \in \{q, q+1\}$. From the induction hypothesis for $d' = d-1$, it follows that $\lambda(n) \leq q' + d - 1 \leq q + d$, which completes the proof. □

Although the bound in Proposition 3.1 is linear, the fact that it holds for all integers $d \geq 1$ indicates that it can be improved further.

Theorem 3.2. *For all $n \geq 1$, $\lambda(n) \leq 2\sqrt{n}$.*

Proof. Apply Proposition 3.1 with $d = \lceil \sqrt{n} \rceil$ and $\varepsilon = \lceil \sqrt{n} \rceil - \sqrt{n}$, where $\varepsilon \in [0, 1)$:

$$\lambda(n) \leq \left\lfloor \frac{n}{\lceil \sqrt{n} \rceil} \right\rfloor + \lceil \sqrt{n} \rceil = \left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor + \sqrt{n} + \varepsilon. \tag{3}$$

If $\varepsilon = 0$, then $d = \sqrt{n}$, and the theorem holds. If $\varepsilon \in (0, .5]$, then $\varepsilon - 1 \leq -\varepsilon$, and

$$\left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor \leq \lfloor \sqrt{n} \rfloor = \sqrt{n} + \varepsilon - 1 \leq \sqrt{n} - \varepsilon.$$

On the other hand, if $\varepsilon \in (.5, 1)$, then $\varepsilon - 2 < -\varepsilon$, and

$$\left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor < \left\lfloor \sqrt{n} - \varepsilon + \frac{1}{2} \right\rfloor \leq \lfloor \sqrt{n} \rfloor.$$

In other words, if $\varepsilon \in (.5, 1)$, then

$$\left\lfloor \sqrt{n} - \varepsilon + \frac{\varepsilon^2}{\sqrt{n} + \varepsilon} \right\rfloor \leq \lfloor \sqrt{n} \rfloor - 1 = \sqrt{n} + \varepsilon - 2 < \sqrt{n} - \varepsilon.$$

Therefore, for any $\varepsilon \in [0, 1)$, it follows from inequality (3) that $\lambda(n) \leq 2\sqrt{n}$. □

4. A special case

As suggested in Corollary 2.5, the number $\lambda(n)$ is influenced by the factorization of n. In particular, primality of n can be a challenge for the function λ. On the other hand, if n factors in a particular way, then the bound on $\lambda(n)$ can be further tightened along the lines of Corollary 2.5.

Corollary 4.1. *If $n = ab$ for $a, b \in \mathbb{Z}^+$ with $2\sqrt{b} < a \leq b$, then $\lambda(n) \leq \sqrt{n}$.*

Proof. Suppose that $n = ab$, where $1 \leq a \leq b < (a/2)^2$. Construct a poset P with $e(P) = b$ and $|P| = \lambda(b) \leq 2\sqrt{b} < a$. Let $Q = (P \oplus C_{a-1-|P|}) + C_1$. Note that $e(Q) = ab = n$ and $|Q| = a$. Since $n = ab$ and $a \leq b$, this implies that $|Q| \leq \sqrt{n}$, and so $\lambda(n) \leq \sqrt{n}$. □

References

[1] K. Ragnarsson and B. E. Tenner, Obtainable sizes of finite topologies, to appear in *J. Combin. Theory, Series A*.

[2] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://www.research.att.com/~njas/sequences/`.

[3] R. P. Stanley, *Enumerative Combinatorics, vol. 1*, Cambridge Studies in Advanced Mathematics, no. 49, Cambridge University Press, Cambridge, 1997.

[4] W. T. Trotter, *Combinatorics and Partially Ordered Sets: Dimension Theory*, Johns Hopkins University Press, Baltimore, 1992.

E-mail address: bridget@math.depaul.edu

Department of Mathematical Sciences, DePaul University, Chicago, Illinois 60614