# A NOTE ON THE PAPER BY ECKSTEIN AND SVAITER ON "GENERAL PROJECTIVE SPLITTING METHODS FOR SUMS OF MAXIMAL MONOTONE OPERATORS" 

Heinz H. Bauschke*

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#### Abstract

In their recent SIAM J. Control Optim. paper from 2009, J. Eckstein and B.F. Svaiter proposed a very general and flexible splitting framework for finding a zero of the sum of finitely many maximal monotone operators. In this short note, we provide a technical result that allows for the removal of Eckstein and Svaiter's assumption that the sum of the operators be maximal monotone or that the underlying Hilbert space be finite-dimensional.


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Throughout, we assume that $\mathcal{H}$ is a real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and induced norm $\|\cdot\|$. We shall assume basic notation and results from Fixed Point Theory and from Monotone Operator Theory; see, e.g., [1, 4, 5, 6, 7, $, 8,9]$. The graph of a maximal monotone operator $A: \mathcal{H} \rightrightarrows \mathcal{H}$ is denoted by gra $A$, and its resolvent $(A+\mathrm{Id})^{-1}$ by $J_{A}$. Weak convergence is indicated by $\rightharpoonup$.

Lemma 1 Let $C$ be a closed linear subspace of $\mathcal{H}$ and let $F: \mathcal{H} \rightarrow \mathcal{H}$ be firmly nonexpansive. Then $P_{C} F+\left(\operatorname{Id}-P_{C}\right)(\mathrm{Id}-F)$ is firmly nonexpansive.

Proof. Since $P_{C}$ and $F$ are firmly nonexpansive, we have that $2 P_{C}$ - Id and $2 F$ - Id are both nonexpansive. Set $T=P_{C} F+\left(\operatorname{Id}-P_{C}\right)(\mathrm{Id}-F)$. Then $2 T-\mathrm{Id}=\left(2 P_{C}-\mathrm{Id}\right)(2 F-\mathrm{Id})$ is nonexpansive, and hence $T$ is firmly nonexpansive.

[^0]Theorem 2 Let $A: \mathcal{H} \rightrightarrows \mathcal{H}$ be maximal monotone, and let $C$ be a closed linear subspace of $\mathcal{H}$. Let $\left(x_{n}, u_{n}\right)_{n \in \mathbb{N}}$ be a sequence in gra $A$ such that $\left(x_{n}, u_{n}\right) \rightharpoonup(x, u) \in \mathcal{H} \times \mathcal{H}$. Suppose that $x_{n}-P_{C} x_{n} \rightarrow 0$ and that $P_{C} u_{n} \rightarrow 0$, where $P_{C}$ denotes the projector onto $C$. Then $(x, u) \in($ gra $A) \cap\left(C \times C^{\perp}\right)$ and $\left\langle x_{n}, u_{n}\right\rangle \rightarrow\langle x, u\rangle=0$.

Proof. Since $P_{C}$ is a bounded linear operator, it is weakly continuous ([2, Theorem VI.1.1]). Thus $x \leftharpoonup x_{n}=\left(x_{n}-P_{C} x_{n}\right)+P_{C} x_{n} \rightharpoonup 0+P_{C} x$ and hence $x=P_{C} x \in C$. Similarly, $0 \leftarrow P_{C} u_{n} \rightharpoonup P_{C} u$; hence $P_{C} u=0$ and so $u \in C^{\perp}$. Altogether,

$$
\begin{equation*}
(x, u) \in C \times C^{\perp} . \tag{1}
\end{equation*}
$$

Since Id $-J_{A}$ is firmly nonexpansive, we see from Lemma 1 that

$$
\begin{equation*}
T=P_{C}\left(\operatorname{Id}-J_{A}\right)+\left(\operatorname{Id}-P_{C}\right) J_{A}=P_{C}+\left(\operatorname{Id}-2 P_{C}\right) J_{A} \tag{2}
\end{equation*}
$$

is also firmly nonexpansive. Now $(\forall n \in \mathbb{N}) u_{n} \in A x_{n}$, i.e.,

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad x_{n}=J_{A}\left(x_{n}+u_{n}\right) . \tag{3}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
x_{n}+u_{n} \rightharpoonup x+u, \tag{4}
\end{equation*}
$$

and (2) and (3) imply that $T\left(x_{n}+u_{n}\right)=P_{C}\left(x_{n}+u_{n}\right)+\left(\operatorname{Id}-2 P_{C}\right) J_{A}\left(x_{n}+u_{n}\right)=P_{C} x_{n}+P_{C} u_{n}+$ $\left(\operatorname{Id}-2 P_{C}\right) x_{n}=x_{n}-P_{C} x_{n}+P_{C} u_{n} \rightarrow 0$, i.e., that

$$
\begin{equation*}
T\left(x_{n}+u_{n}\right) \rightarrow 0 . \tag{5}
\end{equation*}
$$

Since Id $-T$ is (firmly) nonexpansive, the demiclosedness principle (see [4, 5]), applied to the sequence $\left(x_{n}+u_{n}\right)_{n \in \mathbb{N}}$ and the operator Id $-T$, and (4) and (5) imply that $(\operatorname{Id}-(\operatorname{Id}-T))(x+u)=$ 0 , i.e., that $T(x+u)=0$. Using (2), this means that

$$
\begin{equation*}
J_{A}(x+u)=2 P_{C} J_{A}(x+u)-P_{C}(x+u) \in C . \tag{6}
\end{equation*}
$$

Applying $P_{C}$ to both sides of (6), we deduce that $J_{A}(x+u)=P_{C} J_{A}(x+u)$; consequently, (6) simplifies to

$$
\begin{equation*}
J_{A}(x+u)=P_{C} x+P_{C} u \tag{7}
\end{equation*}
$$

However, (1) yields $P_{C} x=x$ and $P_{C} u=0$, hence (7) becomes $J_{A}(x+u)=x$; equivalently, $u \in A x$ or

$$
\begin{equation*}
(x, u) \in \operatorname{gra} A . \tag{8}
\end{equation*}
$$

Combining (1) and (8), we see that $(x, u) \in(\operatorname{gra} A) \cap\left(C \times C^{\perp}\right)$, as claimed. Finally, $\left\langle x_{n}, u_{n}\right\rangle=$ $\left\langle P_{C} x_{n}, P_{C} u_{n}\right\rangle+\left\langle P_{C^{\perp}} x_{n}, P_{C^{\perp}} u_{n}\right\rangle \rightarrow\left\langle P_{C} x, 0\right\rangle+\left\langle 0, P_{C^{\perp}} u\right\rangle=0=\left\langle P_{C} x, P_{C^{\perp}} u\right\rangle=\langle x, u\rangle$.

Corollary 3 Let $A_{1}, \ldots, A_{m}$ be maximal monotone operators $\mathcal{H}$, and let $z_{1}, \ldots, z_{m}$ and $w_{1}, \ldots, w_{m}$ be vectors in $\mathcal{H}$. Suppose that for each $i,\left(x_{i, n}, y_{i, n}\right)_{n \in \mathbb{N}}$ is a sequence in gra $A_{i}$ such that for all $i$ and $j$,

$$
\begin{align*}
\left(x_{i, n}, y_{i, n}\right) & \rightharpoonup\left(z_{i}, w_{i}\right)  \tag{9}\\
\sum_{i=1}^{m} y_{i, n} & \rightarrow 0  \tag{10}\\
x_{i, n}-x_{j, n} & \rightarrow 0 . \tag{11}
\end{align*}
$$

Then $z_{1}=\cdots=z_{n}, w_{1}+\cdots+w_{n}=0$, and each $w_{i} \in A_{i} z_{i}$.

Proof. We work in product Hilbert space $\mathcal{H}=\mathcal{H}^{m}$, and we set

$$
\begin{equation*}
\mathbf{A}=A_{1} \times \cdots \times A_{m}, \text { and } \mathbf{C}=\left\{\left(x_{1}, \ldots, x_{m}\right) \in \mathcal{H} \mid x_{1}=\cdots=x_{m}\right\} \tag{12}
\end{equation*}
$$

Note that $\mathbf{A}$ is maximal monotone on $\mathcal{H}$, and that $\mathbf{C}$ is a closed linear subspace of $\mathcal{H}$. Next, set $\mathbf{x}=\left(z_{1}, \ldots, z_{m}\right), \mathbf{u}=\left(w_{1}, \ldots, w_{m}\right)$, and $(\forall n \in \mathbb{N}) \mathbf{x}_{n}=\left(x_{1, n}, \ldots, x_{m, n}\right)$ and $\mathbf{u}_{n}=\left(y_{1, n}, \ldots, y_{m, n}\right)$. By (9), $\left(\mathbf{x}_{n}, \mathbf{u}_{\mathbf{n}}\right)_{n \in \mathbb{N}}$ is a sequence in gra A such that $\left(\mathbf{x}_{n}, \mathbf{u}_{n}\right) \longrightarrow(\mathbf{x}, \mathbf{u})$. Furthermore, (10) and (11) imply that $P_{\mathbf{C}} \mathbf{u}_{n} \rightarrow 0$ and that $\mathbf{x}_{n}-P_{\mathrm{C}} \mathbf{x}_{n} \rightarrow 0$, respectively. Therefore, by Theorem $2,(\mathbf{x}, \mathbf{u}) \in$ (gra $\mathbf{A}) \cap\left(\mathbf{C} \times \mathbf{C}^{\perp}\right)$, which is precisely the announced conclusion.

Remark 4 Corollary 3 is a considerable strengthening of [3, Proposition A.1], where it was additionally assumed that $A_{1}+\cdots+A_{m}$ is maximal monotone, and where part of the conclusion of Corollary 3, namely $z_{1}=\cdots=z_{m}$, was an additional assumption.

Remark 5 Because of the removal of the assumption that $A_{1}+\cdots+A_{m}$ be maximal monotone (see the previous remark), a second look at the proofs in Eckstein and Svaiter's paper [3] reveals that - in our present notation - the assumption that
"either $\mathcal{H}$ is finite-dimensional or $A_{1}+\cdots+A_{m}$ is maximal monotone"
is superfluous in both [3, Proposition 3.2 and Proposition 4.2]. This is important in the infinitedimensional case, where the maximality of the sum can typically be only guaranteed when a constraint qualification is satisfied; consequently, Corollary 3 helps to widen the scope of the powerful algorithmic framework of Eckstein and Svaiter.

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[^0]:    *Mathematics, Irving K. Barber School, University of British Columbia Okanagan, Kelowna, B.C. V1V 1V7, Canada. E-mail: heinz.bauschke@ubc.ca.

