# A NOTE ON THE PAPER BY ECKSTEIN AND SVAITER ON "GENERAL PROJECTIVE SPLITTING METHODS FOR SUMS OF MAXIMAL MONOTONE OPERATORS"

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#### Abstract

In their recent *SIAM J. Control Optim.* paper from 2009, J. Eckstein and B.F. Svaiter proposed a very general and flexible splitting framework for finding a zero of the sum of finitely many maximal monotone operators. In this short note, we provide a technical result that allows for the removal of Eckstein and Svaiter's assumption that the sum of the operators be maximal monotone or that the underlying Hilbert space be finite-dimensional.

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Throughout, we assume that  $\mathcal{H}$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ . We shall assume basic notation and results from Fixed Point Theory and from Monotone Operator Theory; see, e.g., [1, 4, 5, 6, 7, 8, 9]. The *graph* of a maximal monotone operator  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  is denoted by gra A, and its *resolvent*  $(A + \text{Id})^{-1}$  by  $J_A$ . Weak convergence is indicated by  $\rightharpoonup$ .

**Lemma 1** Let C be a closed linear subspace of  $\mathcal{H}$  and let  $F: \mathcal{H} \to \mathcal{H}$  be firmly nonexpansive. Then  $P_CF + (\mathrm{Id} - P_C)(\mathrm{Id} - F)$  is firmly nonexpansive.

*Proof.* Since  $P_C$  and F are firmly nonexpansive, we have that  $2P_C - \text{Id}$  and 2F - Id are both nonexpansive. Set  $T = P_CF + (\text{Id} - P_C)(\text{Id} - F)$ . Then  $2T - \text{Id} = (2P_C - \text{Id})(2F - \text{Id})$  is nonexpansive, and hence T is firmly nonexpansive.

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**Theorem 2** Let  $A: \mathcal{H} \Rightarrow \mathcal{H}$  be maximal monotone, and let C be a closed linear subspace of  $\mathcal{H}$ . Let  $(x_n, u_n)_{n \in \mathbb{N}}$  be a sequence in gra A such that  $(x_n, u_n) \rightharpoonup (x, u) \in \mathcal{H} \times \mathcal{H}$ . Suppose that  $x_n - P_C x_n \rightarrow 0$  and that  $P_C u_n \rightarrow 0$ , where  $P_C$  denotes the projector onto C. Then  $(x, u) \in (\operatorname{gra} A) \cap (C \times C^{\perp})$  and  $\langle x_n, u_n \rangle \rightarrow \langle x, u \rangle = 0$ .

*Proof.* Since  $P_C$  is a bounded linear operator, it is weakly continuous ([2, Theorem VI.1.1]). Thus  $x \leftarrow x_n = (x_n - P_C x_n) + P_C x_n \rightarrow 0 + P_C x$  and hence  $x = P_C x \in C$ . Similarly,  $0 \leftarrow P_C u_n \rightarrow P_C u$ ; hence  $P_C u = 0$  and so  $u \in C^{\perp}$ . Altogether,

$$(1) (x,u) \in C \times C^{\perp}.$$

Since Id  $-J_A$  is firmly nonexpansive, we see from Lemma 1 that

(2) 
$$T = P_C(\mathrm{Id} - J_A) + (\mathrm{Id} - P_C)J_A = P_C + (\mathrm{Id} - 2P_C)J_A$$

is also firmly nonexpansive. Now  $(\forall n \in \mathbb{N}) u_n \in Ax_n$ , i.e.,

$$(3) \qquad (\forall n \in \mathbb{N}) \quad x_n = J_A(x_n + u_n).$$

Furthermore,

$$(4) x_n + u_n \rightharpoonup x + u,$$

and (2) and (3) imply that  $T(x_n + u_n) = P_C(x_n + u_n) + (Id - 2P_C)J_A(x_n + u_n) = P_Cx_n + P_Cu_n + (Id - 2P_C)x_n = x_n - P_Cx_n + P_Cu_n \rightarrow 0$ , i.e., that

(5) 
$$T(x_n+u_n)\to 0.$$

Since Id -T is (firmly) nonexpansive, the demiclosedness principle (see [4, 5]), applied to the sequence  $(x_n + u_n)_{n \in \mathbb{N}}$  and the operator Id -T, and (4) and (5) imply that (Id - (Id - T))(x + u) = 0, i.e., that T(x + u) = 0. Using (2), this means that

(6) 
$$J_A(x+u) = 2P_C J_A(x+u) - P_C(x+u) \in C.$$

Applying  $P_C$  to both sides of (6), we deduce that  $J_A(x + u) = P_C J_A(x + u)$ ; consequently, (6) simplifies to

(7) 
$$J_A(x+u) = P_C x + P_C u.$$

However, (1) yields  $P_C x = x$  and  $P_C u = 0$ , hence (7) becomes  $J_A(x + u) = x$ ; equivalently,  $u \in Ax$  or

$$(8) (x,u) \in \operatorname{gra} A.$$

Combining (1) and (8), we see that  $(x, u) \in (\text{gra } A) \cap (C \times C^{\perp})$ , as claimed. Finally,  $\langle x_n, u_n \rangle = \langle P_C x_n, P_C u_n \rangle + \langle P_{C^{\perp}} x_n, P_{C^{\perp}} u_n \rangle \rightarrow \langle P_C x, 0 \rangle + \langle 0, P_{C^{\perp}} u \rangle = 0 = \langle P_C x, P_{C^{\perp}} u \rangle = \langle x, u \rangle$ .

**Corollary 3** Let  $A_1, \ldots, A_m$  be maximal monotone operators  $\mathcal{H}$ , and let  $z_1, \ldots, z_m$  and  $w_1, \ldots, w_m$  be vectors in  $\mathcal{H}$ . Suppose that for each i,  $(x_{i,n}, y_{i,n})_{n \in \mathbb{N}}$  is a sequence in gra  $A_i$  such that for all i and j,

(9) 
$$(x_{i,n}, y_{i,n}) \rightharpoonup (z_i, w_i)$$

$$(10) \qquad \qquad \sum_{i=1}^m y_{i,n} \to 0$$

$$(11) x_{i,n} - x_{j,n} \to 0.$$

Then  $z_1 = \cdots = z_n$ ,  $w_1 + \cdots + w_n = 0$ , and each  $w_i \in A_i z_i$ .

*Proof.* We work in product Hilbert space  $\mathcal{H} = \mathcal{H}^m$ , and we set

(12) 
$$\mathbf{A} = A_1 \times \cdots \times A_m, \text{ and } \mathbf{C} = \{(x_1, \dots, x_m) \in \mathcal{H} \mid x_1 = \cdots = x_m\}.$$

Note that **A** is maximal monotone on  $\mathcal{H}$ , and that **C** is a closed linear subspace of  $\mathcal{H}$ . Next, set  $\mathbf{x} = (z_1, \ldots, z_m)$ ,  $\mathbf{u} = (w_1, \ldots, w_m)$ , and  $(\forall n \in \mathbb{N}) \mathbf{x}_n = (x_{1,n}, \ldots, x_{m,n})$  and  $\mathbf{u}_n = (y_{1,n}, \ldots, y_{m,n})$ . By (9),  $(\mathbf{x}_n, \mathbf{u}_n)_{n \in \mathbb{N}}$  is a sequence in gra **A** such that  $(\mathbf{x}_n, \mathbf{u}_n) \rightarrow (\mathbf{x}, \mathbf{u})$ . Furthermore, (10) and (11) imply that  $P_{\mathbf{C}}\mathbf{u}_n \rightarrow 0$  and that  $\mathbf{x}_n - P_{\mathbf{C}}\mathbf{x}_n \rightarrow 0$ , respectively. Therefore, by Theorem 2,  $(\mathbf{x}, \mathbf{u}) \in (\text{gra } \mathbf{A}) \cap (\mathbf{C} \times \mathbf{C}^{\perp})$ , which is precisely the announced conclusion.

**Remark 4** Corollary 3 is a considerable strengthening of [3, Proposition A.1], where it was *additionally assumed* that  $A_1 + \cdots + A_m$  is maximal monotone, and where part of the *conclusion* of Corollary 3, namely  $z_1 = \cdots = z_m$ , was an additional *assumption*.

**Remark 5** Because of the removal of the assumption that  $A_1 + \cdots + A_m$  be maximal monotone (see the previous remark), a second look at the proofs in Eckstein and Svaiter's paper [3] reveals that — in our present notation — the assumption that

"either 
$$\mathcal{H}$$
 is finite-dimensional or  $A_1 + \cdots + A_m$  is maximal monotone"

is superfluous in both [3, Proposition 3.2 and Proposition 4.2]. This is important in the infinitedimensional case, where the maximality of the sum can typically be only guaranteed when a constraint qualification is satisfied; consequently, Corollary 3 helps to widen the scope of the powerful algorithmic framework of Eckstein and Svaiter.

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