A CONVERGENT NONCONFORMING FINITE ELEMENT METHOD FOR COMPRESSIBLE STOKES FLOW

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Abstract. We propose a nonconforming finite element method for isentropic viscous gas flow in situations where convective effects may be neglected. We approximate the continuity equation by a piecewise constant discontinuous Galerkin method. The velocity (momentum) equation is approximated by a finite element method on div–curl form using the nonconforming Crouzeix–Raviart space. Our main result is that the finite element method converges to a weak solution. The main challenge is to demonstrate the strong convergence of the density approximations, which is mandatory in view of the nonlinear pressure function. The analysis makes use of a higher integrability estimate on the density approximations, an equation for the "effective viscous flux", and renormalized versions of the discontinuous Galerkin method.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, with $N = 2$ or 3, be a bounded polygonal domain with Lipschitz boundary $\partial\Omega$ and let $T > 0$ be a fixed final time. In this paper, we consider the

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mixed hyperbolic-elliptic type system

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$-\mu\Delta\boldsymbol{u} - \lambda\nabla \operatorname{div}\boldsymbol{u} + \nabla p(\varrho) = \boldsymbol{f}, \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

with initial data

$$\varrho|_{t=0} = \varrho_0, \quad \text{in } \Omega. \tag{1.3}$$

The unknowns are the density $\varrho = \varrho(t,\boldsymbol{x}) \geq 0$ and the velocity $\boldsymbol{u} = \boldsymbol{u}(t,\boldsymbol{x}) \in \mathbb{R}^N$, with $\boldsymbol{x} \in \Omega$ and $t \in (0,T)$. The source term $\boldsymbol{f}$ is a given function representing body forces such as gravity. We denote by div and ∇ the usual spatial divergence and gradient operators and by Δ the Laplace operator. At the boundary $\partial\Omega$, the system is supplemented with the homogenous Dirichlet condition

$$\boldsymbol{u} = 0, \quad \text{on } (0,T)\times\partial\Omega.$$

The pressure $p(\varrho)$ is governed by the equation of state $p(\varrho) = a\varrho^\gamma$, $a > 0$. Typical values of γ ranges from a maximum of $\frac{5}{3}$ for monoatomic gases, through $\frac{7}{5}$ for diatomic gases *including air*, to lower values close to 1 for polyatomic gases at high temperatures. Throughout this paper, we will always assume that $\gamma > 1$, which is the most difficult case. The viscosity coefficients μ, λ are assumed to be constant and satisfy $\mu > 0$, $N\lambda + 2\mu \geq 0$.

The system (1.1)–(1.2) is a gross simplification of the isentropic compressible Navier–Stokes equations. It provides a reasonable approximation in situations where convective effects may be neglected. Solutions of (1.1)–(1.2) have also been utilized by Lions [12] to construct solutions of the isentropic compressible Navier–Stokes equations. Regarding the mathematical theory, the semi–stationary system (1.1)–(1.3) has been analyzed by Lions [12, Section 8.2], among many others. More precisely, he proves the existence of weak solutions and provide some uniqueness and higher regularity results.

In the literature one can find a variety of numerical methods for the compressible Stokes and Navier–Stokes equations. However, there are few results with reference to the convergence properties of these methods, especially in several dimensions. In one dimension, we refer to the works of Hoff and his collaborators [15, 16, 17]. These results apply to the compressible Navier–Stokes equations written in Lagrangian form and requires the initial density to be of bounded variation. In several dimensions there are a few very recent results. In [7, 8], the authors present a convergent finite element method for a Stokes model. This model is a stationary version of (1.1)–(1.2). In their finite element method the approximation spaces for the density and velocity are the same. Moreover, their method is based on the standard weak formulation of the velocity equation (1.2). Since the finite element space is non-conforming, this approach may not preserve the div–curl structure of the continuous system. This complicates the convergence proof. In [7, 8], additional stabilization terms are needed in the discretization of the continuity equation (1.1). In [11], we construct a convergent mixed finite element method for (1.1)–(1.2). However, this method is based on a vorticity formulation of the velocity equation, which is only valid for the Navier slip boundary condition:

$$\boldsymbol{u}\cdot\nu = 0, \quad \operatorname{curl}\boldsymbol{u}\times\nu = 0, \quad \text{on } \partial\Omega.$$

In addition, the velocity is approximated by a $H(\text{div})$ (Nedelec) element.

We now outline the numerical method proposed in this paper. First of all, the density ϱ is approximated by piecewise constants in the spatial and temporal variables. For the approximation of the velocity $\boldsymbol{u}$ we utilize the *Crouzeix–Raviart* element space [4] in the spatial variable, denoted by $\boldsymbol{V}_h(\Omega)$, and piecewise constants in the temporal variable. Hence, the numerical method is nonconforming in the sense that $\boldsymbol{V}_h \not\subset \boldsymbol{W}_0^{1,2}(\Omega)$. In what follows, we mostly suppress the time variable t and refer to subsequent sections for precise statements. For the continuity equation (1.1) we make use of a discontinuous Galerkin method. To achieve stability, the numerical fluxes are evaluated in the upwind direction dictated by the velocity. However, since the velocity space is not continuous across element faces, average velocities are used in this discretization. Our discontinuous Galerkin method is equivalent to a standard finite volume method for the continuity equation [6, 9]. In [11], we use a similar discontinuous Galerkin method with the velocity in the div conforming Nedelec space of the first order and kind. Since the method used herein only depends on the average normal velocity at faces, the approximations constructed by this method are also solutions to the discrete continuity equation of [11]. More precisely, if the pair $(\varrho_h, \boldsymbol{u}_h)$ solves the discrete continuity equation proposed herein, then $(\varrho_h, \Pi_h^N \boldsymbol{u}_h)$ is a solution to the discrete continuity equation of [11], where Π_h^N is the canonical interpolation operator onto the div conforming Nedelec space of first order and kind. As a consequence, several of the favorable properties of the method in [11] continue to hold for the continuity method herein. In particular, renormalized formulations, weak time-continuity, and consistency bounds are readily obtained by exploiting this connection.

To discretize the velocity equation (1.2) we bring into service a non-standard finite element formulation, which starts off from the identity

$$\int_\Omega D\boldsymbol{u} D\boldsymbol{v} \, dx = \int_\Omega \operatorname{curl} \boldsymbol{u} \operatorname{curl} \boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} \, dx, \tag{1.4}$$

valid for all $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega)$. However, since the velocity space is nonconforming, this identity does not hold discretely, but we insist on utilizing the right-hand side of (1.4) as a starting point for discretizing the velocity equation. Utilizing the form on the right–hand side, it is possible to split the curl part of the Laplacian away from the divergence part. By setting $\boldsymbol{v} = \nabla s$, we obtain the divergence part, while $\boldsymbol{v} = \operatorname{curl} \boldsymbol{\eta}$ gives the curl part. Of course, to satisfy boundary conditions, this argument must be localized. Discretely, this still holds for the element space $\boldsymbol{V}_h$ since this admits the exact orthogonal Hodge decomposition

$$\boldsymbol{V}_h = \operatorname{curl} \boldsymbol{\zeta}_h + \nabla S_h.$$

Hence, the curl and divergence part of the Laplace operator can be separated by using test functions $\boldsymbol{v}_h = \operatorname{curl} \boldsymbol{\zeta}_h$, $\boldsymbol{\zeta}_h \in \boldsymbol{W}_h$ or $\boldsymbol{v}_h = \nabla s_h$, $s_h \in S_h$. This property lies at the heart of the matter in the upcoming convergence analysis.

Contrasting with the standard situation in which the left–hand side of (1.4) is used, a discretization based on the right-hand side of (1.4) does not converge unless additional terms controlling the discontinuities of the velocity are added, cf. Brenner [2]. The standard discretization of the Laplacian (based on the left–hand side of (1.4)) leads to a L^2 bound on $\nabla_h \boldsymbol{u}_h$, where ∇_h is the gradient restricted to each element E. For the velocity space $\boldsymbol{V}_h$, this bound actually controls the jump of $\boldsymbol{u}_h$ across faces. This in turn, is sufficient to conclude that $\nabla_h \boldsymbol{u}_h \rightharpoonup \nabla \boldsymbol{u}$ as $h \to 0$. When discretizing the Laplacian based on the right–hand side of (1.4), one obtains L^2 bounds on $\operatorname{curl}_h \boldsymbol{u}_h$ and $\operatorname{div}_h \boldsymbol{u}_h$, where curl_h and div_h denotes the curl and divergence operators, respectively, restricted to each element E. The jump of $\boldsymbol{u}_h$ across faces is not controlled by these terms. In fact, $\boldsymbol{V}_h$ contains non-zero functions for which both div_h and curl_h are zero. For this reason, extra terms controlling the jump of $\boldsymbol{u}_h$ across faces need to be added.

In choosing these terms we are inspired by the work of Brenner [2], which deals with two-dimensional elliptic operators of the form "$\operatorname{curl}\operatorname{curl} - \beta\nabla \operatorname{div}$". To be more precise, our finite element method for the velocity equation (1.2) seeks $\boldsymbol{u}_h \in \boldsymbol{V}_h(\Omega)$

such that

$$
\begin{aligned}
&\int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h \operatorname{curl}_h \boldsymbol{v}_h + [(\mu+\lambda)\operatorname{div}_h \boldsymbol{u}_h - p(\varrho_h)] \operatorname{div}_h \boldsymbol{v}_h \, dx \\
&\qquad + \mu \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon-1} \int_\Gamma [\![\boldsymbol{u}_h \cdot \nu]\!]_\Gamma [\![\boldsymbol{v}_h \cdot \nu]\!]_\Gamma + [\![\boldsymbol{u}_h \times \nu]\!]_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x) \\
&\quad = \int_\Omega \boldsymbol{f}_h \boldsymbol{v}_h \, dx, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega),
\end{aligned}
\tag{1.5}
$$

for some fixed $\epsilon \in (0,1)$, where ρ_h and $\boldsymbol{f}_h$ are given piecewise functions on Ω with respect to a tetrahedral mesh E_h with elements E. Moreover, Γ_h^I denote the set of internal faces, and $[\![\cdot]\!]_\Gamma$ denotes the jump across a face $\Gamma \in \Gamma_h^I$. The scaling factor h^ϵ is required to prove convergence of the finite element method. Of course, the size of ϵ will affect the accuracy of the method [2] and should be fixed very small in practical computations.

For any fixed $h > 0$, let $(\varrho_h, \boldsymbol{u}_h) = (\varrho_h, \boldsymbol{u}_h)(t,x)$ denote the numerical solution to the compressible Stokes system. Our goal is to prove that $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ converges along a subsequence to a weak solution. The main challenge is to show that the density approximations ϱ_h, which a priori is only weakly compact in L^2, in fact converges strongly. Strong convergence is needed when sending $h \to 0$ in the nonlinear pressure function. It is this issue that motivates the above nonconforming finite element method. Since the finite element space $\boldsymbol{V}_h$ is piecewise linear and totally determined by its value at the faces, Green's theorem yield

$$
\operatorname{div}_h \Pi_h^V \boldsymbol{v} = \Pi_h^Q \operatorname{div} \boldsymbol{v}, \qquad \operatorname{curl}_h \Pi_h^V \boldsymbol{v} = \Pi_h^Q \operatorname{curl} \boldsymbol{v},
$$

where Π_h^V is the canonical interpolation operator onto $\boldsymbol{V}_h$ and Π_h^Q is the L^2 projection onto piecewise constants. Consequently, the projection of a divergence or curl free function is again (piecewise) divergence or curl free. Using this, we see that the function $\boldsymbol{v}_h = \Pi_h^V \nabla \Delta^{-1} \varrho_h$ is a solution to the div–curl problem

$$
\operatorname{div}_h \boldsymbol{v}_h = \varrho_h, \quad \operatorname{curl}_h \boldsymbol{v}_h = 0,
$$

away from the boundary. By using $\boldsymbol{v}_h$ as test function in (1.5), the curl term vanishes, while the remaining terms constitute the so-called effective viscous flux $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) = p(\varrho_h) - (\lambda+\mu)\operatorname{div} \boldsymbol{u}_h$, the source term, and the jump terms. The latter terms are shown to converge to zero. Using this, we are able to prove following weak continuity property:

$$
\lim_{h\to 0} \iint P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\, \varrho_h \phi \, dxdt = \iint \overline{P_{\text{eff}}}\, \varrho \phi \, dxdt \quad (\overline{P_{\text{eff}}}, \varrho \text{ are weak } L^2 \text{ limits}),
\tag{1.6}
$$

for all $\phi \in C_0^\infty$. This is the main ingredient in the strong convergence proof for the density approximations ϱ_h. The argument is inspired by the work of Lions on the compressible Navier-Stokes equations, cf. [12].

If we instead of (1.5), discretize the Laplacian based on the left–hand side of (1.4), then the above analysis becomes more involved. In particular, it seems difficult to establish the key property (1.6). In this case, we would need to establish

$$
\int\int \nabla_h \boldsymbol{u}_h \nabla_h \Pi_h^V \left[\nabla \Delta^{-1} \varrho_h\right] - \operatorname{div}_h \boldsymbol{u}_h \varrho_h \, dxdt \to 0, \quad \text{as } h \to 0,
$$

which is intricate since all the involved quantities are only weakly convergent.

The remaining part of this paper is organized as follows: In Section 2, we first introduce some relevant notation and state a few basic results from analysis. Next, we formulate our notion of a weak solution. Finally, we introduce the finite element spaces and derive some of their basic properties. In Section 3, we present the numerical method and state our main convergence result. This section also provides a result regarding the existence of solutions to the discrete equations. In Section 4, we derive stability and higher integrability results. Section 5 is devoted to proving the convergence result stated in Section 3.

2. Preliminary material

2.1. **Functional spaces and analysis results.** We denote the spatial divergence and curl operators by div and curl, respectively. As usual in the two dimensions, we denote both the rotation operator taking scalars into vectors and the curl operator taking vectors into scalars by curl.

We will make use of the spaces

$$\boldsymbol{W}^{\mathrm{div},2}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{div} \boldsymbol{v} \in L^2(\Omega) \right\},$$
$$\boldsymbol{W}^{\mathrm{curl},2}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{curl} \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\},$$

where ν denotes the unit outward pointing normal vector on $\partial\Omega$. If $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ satisfies $\boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0$, we write $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$. Similarly, $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ means $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{curl},2}(\Omega)$ and $\boldsymbol{v} \times \nu|_{\partial\Omega} = 0$. From [10],

$$\boldsymbol{W}_0^{1,2}(\Omega) = \boldsymbol{W}_0^{\mathrm{curl},2} \cap \boldsymbol{W}_0^{\mathrm{div},2}.$$

The next lemma lists some basic results from functional analysis to be used in subsequent arguments (for proofs, see, e.g., [5]). Throughout the paper we use overbars to denote weak limits, in spaces that should be clear from the context.

Lemma 2.1. *Let O be a bounded and open subset of $\mathbb{R}^M$ with $M \geq 1$. Suppose $g\colon \mathbb{R} \to (-\infty, \infty]$ is a lower semicontinuous convex function and $\{v_n\}_{n\geq 1}$ is a sequence of functions on O for which $v_n \rightharpoonup v$ in $L^1(O)$, $g(v_n) \in L^1(O)$ for each n, $g(v_n) \rightharpoonup \overline{g(v)}$ in $L^1(O)$. Then $g(v) \leq \overline{g(v)}$ a.e. on O, $g(v) \in L^1(O)$, and $\int_O g(v)\, dy \leq \liminf_{n\to\infty} \int_O g(v_n)\, dy$. If, in addition, g is strictly convex on an open interval $(a,b) \subset \mathbb{R}$ and $g(v) = \overline{g(v)}$ a.e. on O, then, passing to a subsequence if necessary, $v_n(y) \to v(y)$ for a.e. $y \in \{y \in O \mid v(y) \in (a,b)\}$.*

Let X be a Banach space and $X^\star$ its dual. The space $X^\star$ equipped with the weak-$\star$ topology is denoted by $X^\star_{\mathrm{weak}}$, while X equipped with the weak topology is denoted by X_{weak}. By the Banach-Alaoglu theorem, bounded balls in $X^\star$ are $\sigma(X^\star, X)$-compact. If X separable, the weak-$\star$ topology is metrizable on bounded sets in $X^\star$, which makes it possible to consider the metric space $C([0,T]; X^\star_{\mathrm{weak}})$ of functions $v : [0,T] \to X^\star$ that are continuous with respect to the weak topology. We have $v_n \to v$ in $C([0,T]; X^\star_{\mathrm{weak}})$ if $\langle v_n(t), \phi\rangle_{X^\star,X} \to \langle v(t), \phi\rangle_{X^\star,X}$ uniformly with respect to t, for any $\phi \in X$. The succeding lemma is a consequence of the Arzelà-Ascoli theorem:

Lemma 2.2. *Let X be a separable Banach space, and suppose $v_n\colon [0,T] \to X^\star$, $n = 1, 2, \dots$, is a sequence for which $\|v_n\|_{L^\infty([0,T];X^\star)} \leq C$, for some constant C independent of n. Suppose the sequence $[0,T] \ni t \mapsto \langle v_n(t), \Phi\rangle_{X^\star,X}$, $n = 1, 2, \dots$, is equi-continuous for every Φ that belongs to a dense subset of X. Then v_n belongs to $C([0,T]; X^\star_{\mathrm{weak}})$ for every n, and there exists a function $v \in C([0,T]; X^\star_{\mathrm{weak}})$ such that along a subsequence as $n \to \infty$ there holds $v_n \to v$ in $C([0,T]; X^\star_{\mathrm{weak}})$.*

Later we frequently obtain a priori estimates for a sequence $\{v_n\}_{n\geq 1}$ that we make known as "$v_n \in_{\mathrm{b}} X$" for a given functional space X. What this really means is that we have a bound on $\|v_n\|_X$ that is independent of n.

The following discrete version of a lemma due to Lions [12, Lemma 5.1] will prove useful in the convergence analysis.

Lemma 2.3. *Given $T > 0$ and a small number $h > 0$, write $(0,T] = \cup_{k=1}^M (t_{k-1}, t_k]$ with $t_k = hk$ and $Mh = T$. Let $\{f_h\}_{h>0}^\infty$, $\{g_h\}_{h>0}^\infty$ be two sequences such that:*

(1) *the mappings* $t \mapsto g_h(t,x)$ *and* $t \mapsto f_h(t,x)$ *are constant on each interval* $(t_{k-1}, t_k]$, $k = 1, \dots, M$.
(2) $\{f_h\}_{h>0}$ *and* $\{g_h\}_{h>0}$ *converge weakly to* f *and* g *in* $L^{p_1}(0,T; L^{q_1}(\Omega))$ *and* $L^{p_2}(0,T; L^{q_2}(\Omega))$, *respectively, where* $1 < p_1, q_1 < \infty$ *and*
$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1.$$
(3) *the discrete time derivative satisfies*
$$\frac{g_h(t,x) - g_h(t-h,x)}{h} \in_b L^1(0,T; W^{-1,1}(\Omega))$$
(4) $\|f_h(t,x) - f_h(t,x-\xi)\|_{L^{p_2}(0,T;L^{q_2}(\Omega))} \to 0$ *as* $|\xi| \to 0$, *uniformly in* h.

Then $g_h f_h \rightharpoonup gf$ *in the sense of distributions on* $(0,T) \times \Omega$.

Proof. Let us introduce an auxiliary piecewise linear function $\widetilde{g}_h$ by setting
$$\widetilde{g}_h(t,\cdot) = g_h(t_k) + h^{-1}(t - t_k)\left(g_h(t_{k+1}) - g_h(t_k)\right), \quad t \in (t_k, t_{k+1}],$$
for $k = 0, \dots, M-1$. Using property (3),
$$\begin{aligned}\left|\int_0^T \int_\Omega (\widetilde{g}_h - g_h)\phi \, dxdt\right| &\leq h \left|\int_0^T \int_\Omega \left(\frac{g_h(t,x) - g_h(t-h,x)}{h}\right)\phi \, dxdt\right| \\ &\leq Ch\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}, \qquad \phi \in C_0^\infty(\Omega).\end{aligned} \tag{2.1}$$
Thus, $(\widetilde{g}_h - g_h) \rightharpoonup 0$ as in the sense of distributions on $(0,T) \times \Omega$ as $h \to 0$.

Next, we write
$$g_h f_h = \widetilde{g}_h f_h + (g_h - \widetilde{g}_h) f_h.$$
By requirement (3), $\partial_t \widetilde{g}_h \in_b L^1(0,T; W^{-1,1}(\Omega))$. This and requirement (4) allow us to apply a lemma due to Lions [12, Lemma 5.1], yielding
$$f_h \widetilde{g}_h \rightharpoonup fg,$$
in the sense of distributions on $(0,T) \times \Omega$ as $h \to 0$.

It only remains to prove that $(g_h - \widetilde{g}_h) f_h \rightharpoonup 0$ in the sense of distributions. For this purpose, set $f_h^\epsilon = f_h \star \kappa_\epsilon$, where κ_ϵ is a standard smoothing kernel and $\star$ denotes the convolution product. We write
$$(g_h - \widetilde{g}_h) f_h = (g_h - \widetilde{g}_h) f_h^\epsilon + (g_h - \widetilde{g}_h)(f_h - f_h^\epsilon).$$
Now, requirement (4) yields
$$\|f_h - f_h^\epsilon\|_{L^{p_2}(0,T;L^{q_2}(\Omega))} \to 0 \quad \text{as } \epsilon \to 0,$$
uniformly in h, and hence
$$\lim_{\epsilon \to 0} \lim_{h \to 0} \int_0^T \int_\Omega (g_h - \widetilde{g}_h)(f_h - f_h^\epsilon)\phi \, dxdt = 0.$$
Thus, the proof is complete provided that
$$\lim_{\epsilon \to 0} \lim_{h \to 0} \int_0^T \int_\Omega (g_h - \widetilde{g}_h) f_h^\epsilon \phi \, dxdt = 0.$$
By a calculation similar to (2.1) we see that
$$\left|\int_0^T \int_\Omega (g_h - \widetilde{g}_h) f_h^\epsilon \, dxdt\right| \leq h^{\frac{p_2 - 1}{p_2}} C \|f_h^\epsilon\|_{L^{p_2}(0,T;W^{1,\infty}(\Omega))},$$
where we have also applied Lemma 2.10 (below) to the time variable. From this we can conclude that $(g_h - \widetilde{g}_h) f_h^\epsilon \rightharpoonup 0$ in the sense of distributions as $h \to 0$. This brings the proof to an end. □

2.2. **Weak and renormalized solutions.**

Definition 2.4 (Weak solutions)**.** A pair of functions $(\varrho, \boldsymbol{u})$ constitutes a weak solution of the semi-stationary compressible Stokes system (1.1)–(1.2) with initial data (1.3) provided that:

(1) $(\varrho, \boldsymbol{u}) \in L^\infty(0,T; L^\gamma(\Omega)) \times L^2(0,T; \boldsymbol{W}_0^{1,2}(\Omega))$,
(2) $\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0$ in the weak sense, i.e, $\forall \phi \in C^\infty([0,T) \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \varrho \left(\phi_t + \boldsymbol{u} D\phi\right) \, dx dt + \int_\Omega \varrho_0 \phi|_{t=0} \, dx = 0; \tag{2.2}$$

(3) $-\mu\Delta \boldsymbol{u} - \lambda D \operatorname{div} \boldsymbol{u} + Dp(\varrho) = \boldsymbol{f}$ in the weak sense, i.e, $\forall \boldsymbol{\phi} \in \boldsymbol{C}_0^\infty([0,T) \times \Omega)$,

$$\int_0^T \int_\Omega \mu \nabla \boldsymbol{u} \nabla \boldsymbol{\phi} + \left[(\mu + \lambda \operatorname{div} \boldsymbol{u} - p(\varrho)\right] \operatorname{div} \boldsymbol{\phi} \, dx dt = \int_0^T \int_\Omega \boldsymbol{f} \boldsymbol{\phi} \, dx dt. \tag{2.3}$$

For the convergence analysis we shall also need the DiPerna-Lions concept of renormalized solutions of the continuity equation.

Definition 2.5 (Renormalized solutions)**.** Given $\boldsymbol{u} \in L^2(0,T; \boldsymbol{W}_0^{1,2}(\Omega))$, we say that $\varrho \in L^\infty(0,T; L^\gamma(\Omega))$ is a renormalized solution of (1.1) provided

$$B(\varrho)_t + \operatorname{div}\left(B(\varrho)\boldsymbol{u}\right) = b(\varrho) \operatorname{div} \boldsymbol{u} \quad \text{in the sense of distributions on } [0,T) \times \overline{\Omega},$$

for any $B \in C[0,\infty) \cap C^1(0,\infty)$ with $B(0) = 0$ and $b(\varrho) := B'(\varrho)\varrho - B(\varrho)$.

We shall need the following well-known lemma [12] stating that square-integrable weak solutions ϱ are also renormalized solutions.

Lemma 2.6. *Suppose $(\varrho, \boldsymbol{u})$ is a weak solution according to Definition 2.4. If $\varrho \in L^2((0,T) \times \Omega))$, then ϱ is a renormalized solution according to Definition 2.5.*

Remark 2.7. Regarding the continuity equation and the definitions of weak and renormalized solutions, we are requiring the equation to hold up to the boundary.

2.3. **On the equation** $\operatorname{div} \mathbf{v} = f$**.** Solutions to the following problem are vital to the upcoming convergence analysis:

$$\operatorname{div} \boldsymbol{v} = f \quad \text{in } \Omega, \qquad \boldsymbol{v} = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

If $f \in L^p(\Omega)$ with $\int_\Omega f \, dx = 0$, then a solution to (2.4) can be constructed through the Hodge decomposition

$$\boldsymbol{v} = \nabla s + \operatorname{curl} \xi,$$

where $s \in \boldsymbol{H}^2(\Omega)$ solves the Neumann Laplace problem, i.e.,

$$\Delta s = f \quad \text{in } \Omega, \qquad \nabla s \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

and $\xi \in \boldsymbol{H}^2(\Omega)$ is determined such that $\boldsymbol{v}|_{\partial\Omega} = 0$ (cf. [1]). Such a solution can be constructed using the Bogovskii solution operator [5]. Here, we define the solution operator $\mathcal{B}[\cdot] : L_0^p(\Omega) \to \boldsymbol{W}_0^{1,p}(\Omega)$ as one of the solutions to the problem

$$\operatorname{div} \mathcal{B}[\phi] = \phi \quad \text{in } \Omega, \qquad \mathcal{B}[\phi] = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

We shall need solutions $\boldsymbol{v}$ satisfying $\operatorname{curl} \boldsymbol{v} = 0$. Clearly, this is not compatible with the Dirichlet boundary condition. However, locally curl free solutions can be constructed using the operator $\mathcal{A}[\cdot] : L^p(\Omega) \to \boldsymbol{W}^{1,p}(\Omega)$,

$$\mathcal{A}[\phi] = \nabla \Delta^{-1}[\phi], \tag{2.6}$$

where Δ^{-1} is the inverse Neumann Laplace operator.

2.4. **Finite element spaces and some basic properties.** Let E_h denote a shape regular tetrahedral mesh of Ω. Let $\Gamma_h^I = \{\Gamma \in \Gamma_h : \Gamma \not\subset \partial\Omega\}$ denote the set of internal faces in E_h. We will approximate the density in the space of piecewise constants on E_h and denote this space by $Q_h(\Omega)$. For the approximation of the velocity we use the Crouzeix–Raviart element space [4]:

$$\boldsymbol{V}_h(\Omega) = \left\{\boldsymbol{v}_h \in L^2(\Omega) : \boldsymbol{v}_h|_E \in \mathbb{P}_1^N(E),\ \forall E \in E_h,\ \int_\Gamma [\![\boldsymbol{v}_h]\!]_\Gamma \ dS(x) = 0,\ \forall \Gamma \in \Gamma_h^I\right\}, \tag{2.7}$$

where $[\![\cdot]\!]_\Gamma$ denotes the jump across a face Γ. To incorporate the boundary condition, we let the degrees of freedom of $\boldsymbol{V}_h(\Omega)$ vanish at the boundary. Consequently, the finite element method is nonconforming in the sense that the velocity approximation space is not a subspace of the corresponding continuous space, $\boldsymbol{W}_0^{1,2}(\Omega)$.

We introduce the canonical interpolation operators

$$\Pi_h^V : \boldsymbol{W}_0^{1,2}(\Omega) \to \boldsymbol{V}_h(\Omega), \qquad \Pi_h^Q : L^2(\Omega) \to Q_h(\Omega),$$

defined by

$$\begin{aligned} \int_\Gamma \Pi_h^V \boldsymbol{v}_h \ dS(x) &= \int_\Gamma \boldsymbol{v}_h \ dS(x), \quad \forall \Gamma \in \Gamma_h, \\ \int_E \Pi_h^Q \phi \ dx &= \int_E \phi \ dx, \quad \forall E \in E_h. \end{aligned} \tag{2.8}$$

Then, by virtue of (2.8) and Stokes' theorem,

$$\operatorname{div}_h \Pi_h^V \boldsymbol{v} = \Pi_h^Q \operatorname{div} \boldsymbol{v}, \qquad \operatorname{curl}_h \Pi_h^V \boldsymbol{v} = \Pi_h^Q \operatorname{curl} \boldsymbol{v}, \tag{2.9}$$

for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega)$. Here, curl_h and div_h denote the curl and divergence operators, respectively, taken inside each element.

Now, (2.9) immediately gives

$$\operatorname{div}_h \Pi_h^V \mathcal{B}\left[q_h\right] = q_h, \qquad \forall q_h \in Q_h(\Omega) \cap \left\{\int_\Omega q_h \ dx = 0\right\},$$

and, away from the boundary,

$$\operatorname{div}_h \Pi_h^V \mathcal{A}\left[q_h\right] = q_h, \qquad \operatorname{curl}_h \Pi_h^V \mathcal{A}\left[q_h\right] = 0, \qquad \forall q_h \in Q_h(\Omega),$$

where $\mathcal{B}[\cdot]$ and $\mathcal{A}[\cdot]$ are defined in (2.5) and (2.6), respectively. Consequently, this configuration of elements enables us to construct discrete analogs of the continuous operators (2.5) and (2.6).

We associate to the space $\boldsymbol{V}_h(\Omega)$ the following semi–norm and norm:

$$\begin{aligned} |\boldsymbol{v}_h|_{\boldsymbol{V}_h}^2 &= \|\operatorname{curl}_h \boldsymbol{v}_h\|_{L^2(\Omega)}^2 + \|\operatorname{div}_h \boldsymbol{v}_h\|_{L^2(\Omega)}^2 \\ &\quad + \sum_{\Gamma \in \Gamma_h} h^{\epsilon-1}\left(\|[\![\boldsymbol{v}_h \cdot \nu]\!]_\Gamma\|_{L^2(\Gamma)}^2 + \|[\![\boldsymbol{v}_h \times \nu]\!]_\Gamma\|_{L^2(\Gamma)}^2\right), \\ \|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}^2 &= \|\boldsymbol{v}_h\|_{L^2(\Omega)}^2 + |\boldsymbol{v}_h|_{\boldsymbol{V}_h}^2. \end{aligned} \tag{2.10}$$

Let us now state some basic properties of the finite element spaces. We start by recalling from [3, 4] a few interpolation error estimates.

Lemma 2.8. *There exists a constant $C > 0$, depending only on the shape regularity of E_h and $|\Omega|$, such that for any $1 \le p < \infty$,*

$$\begin{aligned} &\|\Pi_h^Q \phi - \phi\|_{L^p(\Omega)} \le Ch\|\nabla\phi\|_{L^p(\Omega)}, \\ &\|\Pi_h^V \boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + h\|\nabla_h(\Pi_h^V \boldsymbol{v} - \boldsymbol{v})\|_{\boldsymbol{L}^p(\Omega)} \le ch^s\|\nabla^s \boldsymbol{v}\|_{\boldsymbol{L}^p(E)}, \quad s = 1, 2, \end{aligned}$$

for all $\phi \in W^{1,p}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{W}^{s,p}(E)$. Here, ∇_h is the gradient operator taken inside each element.

By scaling arguments, the trace theorem, and the Poincaré inequality, we obtain

Lemma 2.9. *For any $E \in E_h$ and $\phi \in W^{1,2}(E)$, we have the following inequalities:*

(1) *trace inequality,*

$$\|\phi\|_{L^2(\Gamma)} \leq c h_E^{-\frac{1}{2}} \left(\|\phi\|_{L^2(E)} + h_E \|\nabla \phi\|_{\boldsymbol{L}^2(E)} \right), \quad \forall \Gamma \in \Gamma_h \cap \partial E.$$

(2) *Poincaré inequality,*

$$\left\| \phi - \frac{1}{|E|} \int_E \phi \, dx \right\|_{L^2(E)} \leq C h_E \|\nabla \phi\|_{\boldsymbol{L}^2(E)}.$$

In both estimates, h_E is the diameter of the element E.

Lemma 2.10. *There exists a positive constant C, depending only on the shape regularity of E_h, such that for $1 \leq q, p \leq \infty$ and $r = 0, 1$,*

$$\|\phi_h\|_{W^{r,p}(E)} \leq C h^{-r+\min\{0, \frac{N}{p} - \frac{N}{q}\}} \|\phi_h\|_{L^q(E)},$$

for any $E \in E_h$ and all polynomial functions $\phi_h \in \mathbb{P}_k(E)$, $k = 0, 1, \ldots$.

Lemma 2.11. *Let $\{\boldsymbol{v}_h\}_{h>0}$ be a sequence in $\boldsymbol{V}_h(\Omega)$. Assume that there is a constant $C > 0$, independent of h, such that $\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \leq C$. Then there exists a function $\boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(\Omega)$ such that, by passing to a subsequence as $h \to 0$ if necessary,*

$$\boldsymbol{v}_h \rightharpoonup \boldsymbol{v} \text{ in } \boldsymbol{L}^2(\Omega), \quad \operatorname{curl}_h \boldsymbol{v}_h \rightharpoonup \operatorname{curl} \boldsymbol{v} \text{ in } \boldsymbol{L}^2(\Omega), \quad \operatorname{div}_h \boldsymbol{v}_h \rightharpoonup \operatorname{div} \boldsymbol{v} \text{ in } L^2(\Omega).$$

Proof. As $\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}$ is bounded independently of h, it follows that $\boldsymbol{v}_h \in_b \boldsymbol{L}^2(\Omega)$, $\operatorname{curl}_h \boldsymbol{v}_h \in_b \boldsymbol{L}^2(\Omega)$, and $\operatorname{div}_h \boldsymbol{v}_h \in_b L^2(\Omega)$. Thus, we have the existence of functions $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$, $\xi \in \boldsymbol{L}^2(\Omega)$, and $\zeta \in L^2(\Omega)$ such that, by passing to a subsequence if necessary,

$$\boldsymbol{v}_h \rightharpoonup \boldsymbol{v}, \quad \operatorname{curl}_h \boldsymbol{v}_h \rightharpoonup \xi, \quad \operatorname{div}_h \boldsymbol{v}_h \rightharpoonup \zeta.$$

Once we make the identifications $\xi = \operatorname{curl} \boldsymbol{v}$ and $\zeta = \operatorname{div} \boldsymbol{v}$, the proof is complete.

Fix any $\phi \in W_0^{1,2}(\Omega)$. An application of Green's theorem yields

$$\begin{aligned} \int_\Omega \operatorname{curl}_h \boldsymbol{v}_h \phi \, dx &= \sum_{E \in E_h} \int_E \boldsymbol{v}_h \operatorname{curl} \phi \, dx + \int_{\partial E} \phi (\boldsymbol{v}_h \times \nu) \, dS(x) \\ &= \int_\Omega \boldsymbol{v}_h \operatorname{curl} \phi \, dx + \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \phi \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x). \end{aligned}$$

By sending $h \to 0$ in the above identity, we discover

$$\int_\Omega \xi \phi \, dx = \int_\Omega \boldsymbol{v} \operatorname{curl} \phi \, dx + \lim_{h \to 0} \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \phi \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x). \tag{2.11}$$

Utilizing the bound

$$h^{\epsilon - 1} \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma^2 \, dS(x) \leq C, \tag{2.12}$$

cf. (2.10), and the second condition in (2.7), we control the last term of (2.11):

$$\begin{aligned} &\left| \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \phi \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x) \right| \\ &\quad = \left| \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma (\phi - \phi_\Gamma) \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x) \right| \end{aligned}$$

$$\leq h^{-\frac{\epsilon}{2}} \left(\sum_{\Gamma \in \Gamma_h^I} h^{\epsilon-1} \int_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma^2 \right)^{\frac{1}{2}} \left(\sum_{\Gamma \in \Gamma_h^I} h \int_\Gamma |\phi - \phi_\Gamma|^2 \, dS(x) \right)^{\frac{1}{2}},$$

where $\{\phi_\Gamma\}_{\Gamma \in \Gamma_h}$ is a given set of real numbers. For each $\Gamma \in \Gamma_h$, let us take $\phi_\Gamma := \frac{1}{|E|} \int_E \phi \, dx$, where E is arbitrarily fixed as one of the two elements sharing the edge Γ. Now, using Lemma 2.9 and (2.12), we deduce

$$\left| \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \phi \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \; dS(x) \right| \leq C h^{-\frac{\epsilon}{2}} h \|\nabla \phi\|_{\boldsymbol{L}^2(\Omega)}.$$

Hence,

$$\lim_{h \to} \left| \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \phi \, [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \; dS(x) \right| = 0.$$

By (2.11), this shows that

$$\int_\Omega \xi \phi \, dx = \int_\Omega \boldsymbol{v} \operatorname{curl} \phi \, dx,$$

and so our claim follows, i.e., $\xi = \operatorname{curl} \boldsymbol{v}$.

By almost identical arguments we find that $\zeta = \operatorname{div} \boldsymbol{v}$. □

The following lemma provides us with an estimate of the blow-up rate of $\nabla_h \boldsymbol{v}_h$, for any element $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$.

Lemma 2.12. *There exists a positive constant C, depending only on the shape regularity of E_h and the size of Ω, such that*

$$\|\nabla_h \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}^2 \leq C h^{-1-\frac{\epsilon}{2}} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \left(\sum_{\Gamma \in \Gamma_h^I} h^{\epsilon-1} \left\| [\![\boldsymbol{v}_h]\!]_\Gamma \right\|_{\boldsymbol{L}^2(\Gamma)}^2 \right)^{\frac{1}{2}},$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$.

Proof. By the linearity of $\boldsymbol{v}_h|_E$, $\Delta \boldsymbol{v}_h|_E = 0 \; \forall E \in E_h$. Using this we can apply Green's theorem to deduce the bound

$$\begin{aligned}
\|\nabla_h \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}^2 &= \sum_{E \in E_h} \int_E \nabla_h \boldsymbol{v}_h \cdot \nabla_h \boldsymbol{v}_h \, dx = \sum_{E \in E_h} \int_{\partial E} (\nabla \boldsymbol{v}_h \cdot \nu) \boldsymbol{v}_h \, dS(x) \\
&= \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\nabla \, [\![\boldsymbol{v}_h]\!]_\Gamma \cdot \nu \right) \boldsymbol{v}_h \, dS(x) \\
&\leq \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left| \nabla \, [\![\boldsymbol{v}_h]\!]_\Gamma \cdot \nu \right| |\boldsymbol{v}_h| \, dS(x) =: I.
\end{aligned}$$

To obtain the third equality we have used that the average of $\boldsymbol{v}_h$ is continuous across internal faces. Since $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$, we know that $\nabla \, [\![\boldsymbol{v}_h]\!]_\Gamma$ is constant for all internal faces $\Gamma \in \Gamma_h$. Moreover, there must exist a point $b_\Gamma \in \Gamma$, for every $\Gamma \in \Gamma_h^I$, such that $[\![\boldsymbol{v}_h(b_\Gamma)]\!]_\Gamma = 0$. By this and the Cauchy-Schwartz inequality, we deduce

$$\begin{aligned}
I &\leq C \sum_{\Gamma \in \Gamma_h^I} \frac{1}{h} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Gamma)} \| \, [\![\boldsymbol{v}_h]\!]_\Gamma \, \|_{\boldsymbol{L}^2(\Gamma)} \\
&\leq C h^{-1-\frac{\epsilon}{2}} \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \left(\sum_{\Gamma \in \Gamma_h^I} h^{\epsilon-1} \| \, [\![\boldsymbol{v}_h]\!]_\Gamma \, \|_{\boldsymbol{L}^2(\Gamma)}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The last inequality is achieved thanks to the trace inequality (1) in Lemma 2.9, together with Lemma 2.10. $\square$

Using the previous lemma, we can now establish a Poincaré inequality and a spatial compactness estimate.

Lemma 2.13. *There exists a positive constant C, depending only on the shape regularity of E_h and the size of Ω, such that for any $\xi \in \mathbb{R}^2$*

$$\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)} \leq C|\xi|^{\frac{1}{2}-\frac{\epsilon}{4}} |\boldsymbol{v}_h|_{\boldsymbol{V}_h(\Omega)}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega), \tag{2.13}$$

where $\Omega_\xi = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \xi\}$. Moreover,

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \leq C|\boldsymbol{v}_h|_{\boldsymbol{V}_h(\Omega)}, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega). \tag{2.14}$$

Proof. Fix $h > 0$, and select an arbitrary function $\boldsymbol{v}_h$ in $\boldsymbol{V}_h(\Omega)$. For any ξ we construct a new mesh G_h such that each $G \in G_h$ is a subset of one and only one element $E \in E_h$ (e.g., we can divide each element of $E \in E_h$ into a number of smaller elements). Moreover, we construct this new mesh G_h such that

$$C^{-1}|\xi| \leq h_G \leq C|\xi|, \qquad \forall G \in G_h, \tag{2.15}$$

where h_G denotes the diameter of the new element G and the constant C depends only on the shape–regularity of E_h.

Now, let $\boldsymbol{V}_{|\xi|}(\Omega)$ denote the Crouzeix–Raviart element space on G_h and denote by $\Pi^V_{|\xi|} : \boldsymbol{V}_h(\Omega) \to \boldsymbol{V}_{|\xi|}(\Omega)$ the canonical interpolation operator associated with $\boldsymbol{V}_{|\xi|}(\Omega)$.

Denote by $h_{|\xi|}$ the maximal element diameter in G_h. From standard properties of the Crouzeix–Raviart element [14], we have

$$\begin{aligned}
\|\Pi^V_{|\xi|}\boldsymbol{v}_h(x) - \Pi^V_{|\xi|}\boldsymbol{v}_h(x-\xi)\|^2_{\boldsymbol{L}^2(\Omega_\xi)} &\leq (h^2_{|\xi|} + |\xi|^2) \sum_{G \in G_h} \|\nabla \Pi^V_{|\xi|}\boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(G)} \\
&\leq (h^2_{|\xi|} + |\xi|^2)\|\nabla_h \boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}$$

where the second inequality follows from the properties of the operator $\Pi^V_{|\xi|}$.

Lemma 2.12 and the bounds (2.15) allow us to conclude the following estimate:

$$\begin{aligned}
&\|\Pi^V_{|\xi|}\boldsymbol{v}_h(x) - \Pi^V_{|\xi|}\boldsymbol{v}_h(x-\xi)\|^2_{\boldsymbol{L}^2(\Omega_\xi)} \\
&\quad \leq h_{|\xi|}^{-1-\frac{\epsilon}{2}}(h^2_{|\xi|} + |\xi|^2)\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \left(\sum_{\Gamma \in \Gamma^I_h} h^{\epsilon-1} \| \llbracket \boldsymbol{v}_h \rrbracket_\Gamma \|^2_{\boldsymbol{L}^2(\Gamma)} \right)^{\frac{1}{2}} \\
&\quad \leq C|\xi|^{1-\frac{\epsilon}{2}}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \left(\sum_{\Gamma \in \Gamma^I_h} h^{\epsilon-1} \| \llbracket \boldsymbol{v}_h \rrbracket_\Gamma \|^2_{\boldsymbol{L}^2(\Gamma)} \right)^{\frac{1}{2}}.
\end{aligned}$$

Keeping in mind that $\boldsymbol{v}_h|_E \in \boldsymbol{W}^{1,2}(E)$, $\forall E \in E_h$, we can apply Lemma 2.8 and the previous estimate to obtain

$$\begin{aligned}
&\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \xi)\|^2_{\boldsymbol{L}^2(\Omega_\xi)} \\
&\quad \leq 2\|\boldsymbol{v}_h - \Pi^V_{|\xi|}\boldsymbol{v}_h\|^2_{\boldsymbol{L}^2(\Omega_\xi)} + \|\Pi^V_{|\xi|}\boldsymbol{v}_h(x) - \Pi^V_{|\xi|}\boldsymbol{v}_h(x-\xi)\|^2_{\boldsymbol{L}^2(\Omega_\xi)} \\
&\quad \leq C|\xi|^{1-\frac{\epsilon}{2}}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \left(\sum_{\Gamma \in \Gamma^I_h} h^{\epsilon-1} \| \llbracket \boldsymbol{v}_h \rrbracket_\Gamma \|^2_{\boldsymbol{L}^2(\Gamma)} \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.16}$$

Next, denote by $\boldsymbol{v}_h^{\text{ext}}$ the extension of $\boldsymbol{v}_h$ by zero to all of $\mathbb{R}^N$. By the previous calculations, we conclude that $\boldsymbol{v}_h^{\text{ext}}$ satisfies (2.16) with Ω_ξ replaced by $\mathbb{R}^N$ (keep in mind that the jump terms are only summed over internal faces). Thus, we can fix $|\xi|$ large in (2.16) to discover that

$$\|\boldsymbol{v}_h^{\text{ext}}\|_{\boldsymbol{L}^2(\Omega)}^2 \leq C|\operatorname{diam}(\Omega)|^{1-\frac{\epsilon}{2}}\|\boldsymbol{v}_h^{\text{ext}}\|_{\boldsymbol{L}^2(\Omega)}\left(\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\|\left[\!\left[\boldsymbol{v}_h^{\text{ext}}\right]\!\right]_\Gamma\|_{\boldsymbol{L}^2(\Gamma)}^2\right)^{\frac{1}{2}},$$

and hence

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}^2 \leq C|\operatorname{diam}(\Omega)|^{1-\frac{\epsilon}{2}}\left(\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\|\left[\!\left[\boldsymbol{v}_h\right]\!\right]_\Gamma\|_{\boldsymbol{L}^2(\Gamma)}^2\right) \leq C|\operatorname{diam}(\Omega)|^{1-\frac{\epsilon}{2}}|\boldsymbol{v}_h|_{\boldsymbol{V}_h}^2,$$

which is (2.14).

Finally, setting (2.14) into (2.16) gives (2.13). □

We end this section with

Lemma 2.14. *There exists a constant $C>0$, which depends only on the shape regularity of E_h and the size of Ω, such that for any $\boldsymbol{v}_h\in\boldsymbol{V}_h(\Omega)$*

$$\begin{aligned}&\left|\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma\left[\!\left[\Pi_h^V\boldsymbol{w}\cdot\nu\right]\!\right]_\Gamma + [\![\boldsymbol{v}_h\times\nu]\!]_\Gamma\left[\!\left[\Pi_h^V\boldsymbol{w}\times\nu\right]\!\right]_\Gamma\, dS(x)\right|\\ &\qquad\leq Ch^{\frac{\epsilon}{2}}\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)},\qquad \forall\boldsymbol{w}\in\boldsymbol{W}_0^{1,2}(\Omega).\end{aligned}$$

Proof. Using the Hölder inequality,

$$\begin{aligned}&\left|\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma\left[\!\left[\Pi_h^V\boldsymbol{w}\cdot\nu\right]\!\right]_\Gamma\, dS(x)\right|\\
&\quad\leq h^{\frac{\epsilon}{2}}\left(\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma^2\, dS(x)\right)^{\frac{1}{2}}\left(\sum_{\Gamma\in\Gamma_h^I} h^{-1}\int_\Gamma \left[\!\left[\Pi_h^V\boldsymbol{w}\right]\!\right]_\Gamma^2\, dS(x)\right)^{\frac{1}{2}}\\
&\quad\leq h^{\frac{\epsilon}{2}}\left(\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma^2\, dS(x)\right)^{\frac{1}{2}}\\
&\qquad\qquad\times\left(\sum_{E\in E_h} h^{-1}\int_{\partial E}\left|\Pi_h^V\boldsymbol{w}-\boldsymbol{w}\right|^2\, dS(x)\right)^{\frac{1}{2}}.\end{aligned}$$

To obtain the last inequality, we have applied the calculation

$$\begin{aligned}\left[\!\left[\Pi_h^V\boldsymbol{w}\right]\!\right]_\Gamma^2 &= \left|(\Pi_h^V\boldsymbol{w})|_{\partial E^+}-\boldsymbol{w}+\boldsymbol{w}-(\Pi_h^V\boldsymbol{w})|_{\partial E^-}\right|^2\\ &\leq |(\Pi_h^V\boldsymbol{w})|_{\partial E^+}-\boldsymbol{w}|^2+|(\Pi_h^V\boldsymbol{w})|_{\partial E^-}-\boldsymbol{w}|^2,\end{aligned}$$

where E^+ and E^- are the two element sharing the face Γ.

By using (1) in Lemma 2.9, we further deduce that

$$\begin{aligned}&\left|\sum_{\Gamma\in\Gamma_h} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma\left[\!\left[\Pi_h^V\boldsymbol{w}\cdot\nu\right]\!\right]_\Gamma\, dS(x)\right|\\
&\quad\leq h^{\frac{\epsilon}{2}}\left(\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma^2\, dS(x)\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\times \left(\sum_{E \in E_h} h^{-2} \|\Pi_h^V \boldsymbol{w} - \boldsymbol{w}\|^2_{\boldsymbol{L}^2(E)} + \|\nabla(\Pi_h^V \boldsymbol{w} - \boldsymbol{w})\|^2_{\boldsymbol{L}^2(E)} \right)^{\frac{1}{2}} \\
&\leq h^{\frac{\epsilon}{2}} C \|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)},
\end{aligned}
$$

where the last inequality follows from Lemma 2.8.

By analogous calculations for the tangential jumps,

$$
\left| \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon-1} \int_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, [\![\Pi_h^V \boldsymbol{w} \times \nu]\!]_\Gamma \, dS(x) \right| \leq h^{\frac{\epsilon}{2}} C \|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}.
$$

This concludes the proof. □

3. Numerical method and main result

Given a time step $\Delta t > 0$, we discretize the time interval $[0, T]$ in terms of the points $t^m = m\Delta t$, $m = 0, \dots, M$, where it is assumed that $M\Delta t = T$. Regarding the spatial discretization, we let $\{E_h\}_h$ be a shape regular family of tetrahedral meshes of Ω, where h is the maximal diameter. It will be a standing assumption that h and Δt are related like $\Delta t = ch$ for some constant c. By shape regular we mean the existence of a constant $\kappa > 0$ such that every $E \in E_h$ contains a ball of radius $\lambda_E \geq \frac{h_E}{\kappa}$, where h_E is the diameter of E. Furthermore, we let Γ_h denote the set of faces in E_h. Throughout the paper, we will use "three dimensional" terminology (tetrahedron, face, etc.) when referring to both the three dimensional case and the two dimensional case (triangle, edge, etc).

On each element $E \in E_h$, we denote by $Q(E)$ the constants on E. The functions that are piecewise constant with respect to the elements of a mesh E_h are denoted by $Q_h(\Omega)$. We denote by $\boldsymbol{V}_h(\Omega)$ the Crouzeix–Raviart finite element space (2.7) formed on E_h. To incorporate the boundary condition, we let the degrees of freedom of $\boldsymbol{V}_h(\Omega$ vanish at the boundary:

$$
\int_\Gamma \boldsymbol{v}_h \, dS(x) = 0, \quad \forall \Gamma \in \Gamma_h \cap \partial\Omega, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega).
$$

We shall need to introduce some additional notation related to the discontinuous Galerkin method. Concerning the boundary ∂E of an element E, we write f_+ for the trace of the function f achieved from within the element E and f_- for the trace of f achieved from outside E. Concerning a face Γ that is shared between two elements E_- and E_+, we will write f_+ for the trace of f achieved from within E_+ and f_- for the trace of f achieved from within E_-. Here E_- and E_+ are defined such that ν points from E_- to E_+, where ν is fixed (throughout) as one of the two possible normal components on each face Γ. We also write $[\![f]\!]_\Gamma = f_+ - f_-$ for the jump of f across the face Γ, while forward time-differencing of f is denoted by $[\![f^m]\!] = f^{m+1} - f^m$ and $\partial_t^h f^m = \frac{[\![f^m]\!]}{\Delta t}$. The set of inner faces of Γ_h will be denoted by $\Gamma_h^I = \{\Gamma \in \Gamma_h; \Gamma \not\subset \partial\Omega\}$.

Definition 3.1 (Numerical scheme)**.** Let $\left\{\varrho_h^0(x)\right\}_{h>0}$ be a sequence (of piecewise constant functions) in $Q_h(\Omega)$ that satisfies $\varrho_h^0 > 0$ for each fixed $h > 0$ and $\varrho_h^0 \to \varrho_0$ a.e. in Ω and in $L^1(\Omega)$ as $h \to 0$. Set $\boldsymbol{f}_h(t, \cdot) = \boldsymbol{f}_h^m(\cdot) := \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \Pi_h^Q \boldsymbol{f}(s, \cdot) \, ds$, for $t \in (t_{m-1}, t_m)$, $m = 1, \dots, M$.

Determine functions

$$
(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad m = 1, \dots, M,
$$

such that for all $\phi_h \in Q_h(\Omega)$,

$$\int_\Omega \partial_t^h(\varrho_h^m)\phi_h \, dx = \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)_h^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)_h^- \right) [\![\phi_h]\!]_\Gamma \, dS(x). \tag{3.1}$$

and for all $\boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega)$,

$$\begin{aligned} \int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h^m \operatorname{curl}_h \boldsymbol{v}_h + [(\mu+\lambda)\operatorname{div}_h \boldsymbol{u}_h^m - p(\varrho_h^m)] \operatorname{div}_h \boldsymbol{v}_h \, dx & \\ + \mu \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon - 1} \int_\Gamma [\![\boldsymbol{u}_h^m \cdot \nu]\!]_\Gamma [\![\boldsymbol{v}_h \cdot \nu]\!]_\Gamma + [\![\boldsymbol{u}_h^m \times \nu]\!]_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x) & \\ = \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h \, dx, & \end{aligned} \tag{3.2}$$

for $m = 1, \dots, M$.

In (3.1), we have introduced the notation

$$(\boldsymbol{u}_h \cdot \nu)_h^\pm = \left(\frac{1}{|\Gamma|} \int_\Gamma \boldsymbol{u}_h \cdot \nu \, dS(x) \right)^\pm, \tag{3.3}$$

where $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Remark 3.2. Since the normal velocity components $(\boldsymbol{u}_h^m \cdot \nu)$ are discontinuous across element faces, the continuity method (3.1) approximates $(\varrho \boldsymbol{u} \cdot \nu)$ using instead the average normal velocity $\frac{1}{|\Gamma|}\int_\Gamma (\boldsymbol{u}_h^m \cdot \nu) \, dS(x)$, cf. (3.3), and traces of ϱ are taken in the upwind direction with respect to the average normal velocity.

We now make an observation that will simplify the subsequent analysis. Let $\mathcal{N}_h(\Omega)$ denote the lowest order div conforming Nedelec finite element space of the first kind [13, 11] on E_h. In two dimensions, $\mathcal{N}_h(\Omega)$ is the Raviart–Thomas space.

We will need the interpolation operator $\Pi_h^{\mathcal{N}} : \boldsymbol{V}_h(\Omega) \to \mathcal{N}_h(\Omega)$ defined by

$$\int_\Gamma (\Pi_h^{\mathcal{N}} \boldsymbol{v}_h) \cdot \nu \, dS(x) = \int_\Gamma \boldsymbol{v}_h \cdot \nu \, dS(x), \quad \forall \Gamma \in \Gamma_h.$$

Then, by definition, the interpolated velocity

$$\widetilde{\boldsymbol{u}_h^m} := \Pi_h^{\mathcal{N}} \boldsymbol{u}_h^m \tag{3.4}$$

satisfies

$$(\widetilde{\boldsymbol{u}_h^m} \cdot \nu)^\pm = \left(\widetilde{\boldsymbol{u}_h^m} \cdot \nu \right)_h^\pm = (\boldsymbol{u}_h^m \cdot \nu)_h^\pm, \tag{3.5}$$

where the first equality is valid since $\widetilde{\boldsymbol{u}_h^m} \cdot \nu$ is constant on each $\Gamma \in \Gamma_h$. Since both element spaces have piecewise constant divergence, a direct calculation yields

$$\operatorname{div} \widetilde{\boldsymbol{u}_h^m} = \operatorname{div}_h \boldsymbol{u}_h^m. \tag{3.6}$$

Now, setting (3.5) into the continuity method (3.1) leads to the relation

$$\int_\Omega \partial_t^h(\varrho_h^m)\phi_h \, dx = \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\widetilde{\boldsymbol{u}_h^m} \cdot \nu)^+ + \varrho_+^m (\widetilde{\boldsymbol{u}_h^m} \cdot \nu)^- \right) [\![\phi_h]\!]_\Gamma \, dS(x), \tag{3.7}$$

for all $\phi_h \in Q_h(\Omega)$. Hence, we can think of the pair $(\varrho_h^m, \widetilde{\boldsymbol{u}_h^m})$ as a solution to a continuity method in which $\mathcal{N}_h(\Omega)$ is used to approximate the velocity. In fact, (3.7) is the method examined in [11]. We will frequently utilize (3.7), instead of (3.1), to easily obtain properties of our continuity approximations.

For each fixed $h > 0$, the numerical solution $\{(\varrho_h^m, \boldsymbol{u}_h^m)\}_{m=0}^M$ is extended to the whole of $(0, T] \times \Omega$ by setting

$$(\varrho_h, \boldsymbol{u}_h)(t) = (\varrho_h^m, \boldsymbol{u}_h^m), \qquad t \in (t_{m-1}, t_m], \quad m = 1, \dots, M. \tag{3.8}$$

In addition, we set $\varrho_h(0) = \varrho_h^0$.

3.1. **Main result.** Our main result is that, passing if necessary to a subsequence, $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution. More precisely, we will prove

Theorem 3.3 (Convergence). *Suppose $\boldsymbol{f} \in \boldsymbol{L}^2((0,T)\times\Omega)$, and $\varrho_0 \in L^\gamma(\Omega)$ with $\gamma > 1$. Let $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.8) and Definition 3.1. Then, passing if necessary to a subsequence as $h \to 0$, $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$, $\varrho_h \boldsymbol{u}_h \rightharpoonup \varrho \boldsymbol{u}$ in the sense of distributions on $(0,T)\times\Omega$, and $\varrho_h \to \varrho$ a.e. in $(0,T)\times\Omega$, where the limit pair $(\varrho, \boldsymbol{u})$ is a weak solution as stated in Definition 2.4.*

This theorem will be a consequence of the results proved in Sections 4 and 5.

3.2. **The numerical method is well–defined.** We now turn to the existence of a solution to the discrete problem. However, we commence with the following easy lemma providing a positive lower bound for the density.

Lemma 3.4. *Fix any $m = 1, \dots, M$ and suppose $\varrho_h^{m-1} \in Q_h(\Omega)$, $\boldsymbol{u}_h^m \in \boldsymbol{V}_h(\Omega)$ are given bounded functions. Then the solution $\varrho_h^m \in Q_h(\Omega)$ of the discontinuous Galerkin method (3.1) satisfies*

$$\min_{x\in\Omega} \varrho_h^m \geq \min_{x\in\Omega} \varrho_h^{m-1} \left(\frac{1}{1 + \Delta t \|\operatorname{div}_h \boldsymbol{u}_h^m\|_{L^\infty(\Omega)}} \right).$$

Consequently, if $\varrho_h^{m-1}(\cdot) > 0$, then $\varrho_h^m(\cdot) > 0$.

Proof. Let $\widetilde{\boldsymbol{u}_h^m}$ be given by (3.4). Then, since $(\varrho_h^m, \widetilde{\boldsymbol{u}_h^m})$ satisfies (3.7), Lemma 4.1 in [11] can be applied. This concludes the proof. □

Lemma 3.5. *For each fixed $h > 0$, there exists a solution*

$$(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad \varrho_h^m(\cdot) > 0, \quad m = 1, \dots, M,$$

to the discrete problem posed in Definition 3.1.

Proof. As in the proof of [11, Lemma 4.2], the existence of a solution is established using a topological degree argument. By the arguments corresponding to those in [11, Lemma 4.2], one reduces the problem to proving existence of a solution $\boldsymbol{u}_h \in \boldsymbol{V}_h(\Omega)$ to the linear system:

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{v}_h) := & \int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h \operatorname{curl}_h \boldsymbol{v}_h + (\mu + \lambda) \operatorname{div}_h \boldsymbol{u}_h \operatorname{div}_h \boldsymbol{v}_h \, dx \\ & + \mu \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon - 1} \int_\Gamma [\![\boldsymbol{u}_h \cdot \nu]\!]_\Gamma [\![\boldsymbol{v}_h \cdot \nu]\!]_\Gamma + [\![\boldsymbol{u}_h \times \nu]\!]_\Gamma [\![\boldsymbol{v}_h \times \nu]\!]_\Gamma \, dS(x) \\ & = \int_\Omega \boldsymbol{g} \boldsymbol{v}_h \, dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega), \end{aligned} \tag{3.9}$$

where $\boldsymbol{g} \in \boldsymbol{L}^2(\Omega)$ is given.

Now, the bilinear form $a(\cdot,\cdot)$ is clearly bounded on the space $\boldsymbol{V}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$ equipped with the norm $\|\cdot\|_{\boldsymbol{V}_h}$. By an application of the Poincaré inequality (2.14), we also have the existence of a constant C, independent of h, such that

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq C \|\boldsymbol{u}_h\|_{\boldsymbol{V}_h}^2.$$

Hence, the bilinear form $a(\cdot,\cdot)$ is coercive on $\boldsymbol{V}_h$, and the existence of a function $\boldsymbol{u}_h \in \boldsymbol{V}_h(\Omega)$ satisfying (3.9) follows.

Since the remaining part of the topological degree argument is very similar to that found in [11, Lemma 4.2], we omit the details. □

4. Basic estimates

In this section we establish various a priori estimates for the discrete problem given in Definition 3.1, including a basic energy estimate and a higher integrability estimate for the density approximations.

We begin with a renormalized formulation of the continuity method (3.1).

Lemma 4.1 (Renormalized continuity method). *Fix any $m = 1, \dots, M$ and let $(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h \times \boldsymbol{V}_h$ satisfy the continuity method (3.1). Then $(\varrho_h^m, \boldsymbol{u}_h^m)$ also satisfies the renormalized formulation*

$$
\begin{aligned}
&\int_\Omega B(\varrho_h^m)\phi_h \, dx \\
&\qquad - \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(B(\varrho_-^m)(\boldsymbol{u}_h^m \cdot \nu)_h^+ + B(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)_h^- \right) [\![\phi_h]\!]_\Gamma \, dx \\
&\qquad + \Delta t \int_\Omega b(\varrho_h^m) \operatorname{div}_h \boldsymbol{u}_h^m \phi_h \, dx + \int_\Omega B''(\xi(\varrho_h^m, \varrho_h^{m-1})) \left[\!\left[\varrho_h^{m-1}\right]\!\right]^2 \phi_h \, dx \\
&\qquad + \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma B''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) [\![\varrho_h^m]\!]_\Gamma^2 (\phi_h)_- (\boldsymbol{u}_h^m \cdot \nu)_h^+ \\
&\qquad\qquad\qquad - B''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) [\![\varrho_h^m]\!]_\Gamma^2 (\phi_h)_+ (\boldsymbol{u}_h^m \cdot \nu)_h^- \, dS(x) \\
&\qquad = \int_\Omega B(\varrho_h^{m-1})\phi_h \, dx, \qquad \forall \phi_h \in Q_h(\Omega),
\end{aligned}
\tag{4.1}
$$

for any $B \in C[0,\infty) \cap C^2(0,\infty)$ with $B(0) = 0$ and $b(\varrho) := \varrho B'(\varrho) - B(\varrho)$. Given two positive real numbers a_1 and a_2, we use $\xi(a_1, a_2)$ and $\xi^\Gamma(a_1, a_2)$ to denote two corresponding numbers between a_1 and a_2 that arise from second order Taylor expansions utilized in the proof.

Proof. Recall the definition of $\widetilde{\boldsymbol{u}_h^m}$, cf. (3.4). By taking $B'(\varrho_h^m)\phi_h$ as test function in (3.7) and repeating the proof of Lemma 5.1 in [11], we obtain (4.1) with $\boldsymbol{u}_h^m$ replaced by $\widetilde{\boldsymbol{u}_h^m}$. In view of (3.5) and (3.6), this is identical to (4.1). □

Lemma 4.2 (Stability). *Let $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.8) and Definition 3.1. For $\varrho > 0$, set $P(\varrho) := \frac{a}{\gamma-1}\varrho^\gamma$. For any $m = 1, \dots, M$, we have*

$$
\begin{aligned}
\int_\Omega P(\varrho_h^m) \, dx + \frac{\mu}{2} \sum_{k=1}^m \Delta t \|\boldsymbol{u}_h^k\|_{\boldsymbol{V}_h(\Omega)}^2 + \mathcal{N}_{\textit{diffusion}}^m \\
\leq \int_\Omega P(\varrho_0) \, dx + C \sum_{k=1}^m \Delta t \|\boldsymbol{f}_h^k\|_{\boldsymbol{L}^2(\Omega)}^2,
\end{aligned}
\tag{4.2}
$$

where the numerical diffusion term $\mathcal{N}_{\textit{diffusion}}^m \geq 0$ takes the form

$$
\begin{aligned}
\mathcal{N}_{\textit{diffusion}}^m = &\sum_{k=1}^m \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\!\left[\varrho_h^{k-1}\right]\!\right]^2 \, dx \\
&+ \sum_{k=1}^m \sum_{\Gamma \in \Gamma_h^I} \Delta t \int_\Gamma P''(\varrho_\dagger^k) [\![\varrho_h^k]\!]_\Gamma^2 \left((\boldsymbol{u}_h^k \cdot \nu)_h^+ - (\boldsymbol{u}_h^k \cdot \nu)_h^- \right) \, dS(x).
\end{aligned}
$$

In particular, $\varrho_h \in_b L^\infty(0,T; L^\gamma(\Omega))$.

Proof. Since $P'(\rho)\rho - P(\rho) = p(\rho)$ and $\varrho_h > 0$, taking $\phi_h \equiv 1$ in (4.1) yields

$$\begin{aligned}\int_\Omega P(\varrho_h^k)\, dx + \Delta t \int_\Omega p(\varrho_h^k) \operatorname{div}_h \boldsymbol{u}_h^k\, dx + \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\!\left[\varrho_h^{k-1}\right]\!\right]^2 dx \\ + \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)_h^+ \\ - P''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)_h^-\, dS(x) = \int_\Omega P(\varrho^{k-1})\, dx.\end{aligned} \tag{4.3}$$

For $k = 1, \dots, M$ and $x \in \bigcup_{\Gamma \in \Gamma_h^I} \Gamma$, set

$$\varrho_\dagger^k(x) := \begin{cases} \max\{\varrho_+^k(x), \varrho_-^k(x)\}, & 1 < \gamma \le 2, \\ \min\{\varrho_+^k(x), \varrho_-^k(x)\}, & \gamma \ge 2, \end{cases}$$

and note that

$$\begin{aligned}\Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)_h^+ \\ - P''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)_h^-\, dS(x) \\ \ge \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\varrho_\dagger^k) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 \left((\boldsymbol{u}_h^k \cdot \nu)_h^+ - (\boldsymbol{u}_h^k \cdot \nu)_h^-\right) dS(x).\end{aligned} \tag{4.4}$$

Next, using $\boldsymbol{v}_h = \boldsymbol{u}_h^k$ as test function in (3.2), we obtain the estimate

$$\begin{aligned}\int_\Omega p(\varrho_h^k) \operatorname{div}_h \boldsymbol{u}_h^k\, dx = (\mu + \lambda) \|\operatorname{div}_h \boldsymbol{u}_h^k\|_{L^2(\Omega)}^2 + \mu \|\operatorname{curl}_h \boldsymbol{u}_h^k\|_{\boldsymbol{L}^2(\Omega)}^2 - \int_\Omega \boldsymbol{f}_h^k \boldsymbol{u}_h^k\, dx \\ + \mu \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon - 1} \int_\Gamma \left[\!\left[\boldsymbol{u}_h^k \cdot \nu\right]\!\right]_\Gamma^2 + \left[\!\left[\boldsymbol{u}_h^k \times \nu\right]\!\right]_\Gamma^2 dS(x) \\ \ge \mu \|\boldsymbol{u}_h^k\|_{\boldsymbol{V}_h}^2 - \int_\Omega \boldsymbol{f}_h^k \boldsymbol{u}_h^k\, dx, \quad k = 1, \dots, M.\end{aligned} \tag{4.5}$$

Applying (4.5) and (4.4) to (4.3) leads to the bound

$$\begin{aligned}\int_\Omega P(\varrho_h^k)\, dx + \mu \Delta t \|\boldsymbol{u}_h^k\|_{\boldsymbol{V}_h(\Omega)}^2 + \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\!\left[\varrho_h^{k-1}\right]\!\right]^2 dx \\ + \sum_{\Gamma \in \Gamma_h^I} \Delta t \int_\Gamma P''(\varrho_\dagger^k) \left[\!\left[\varrho_h^k\right]\!\right]_\Gamma^2 \left((\boldsymbol{u}_h^k \cdot \nu)_h^+ - (\boldsymbol{u}_h^k \cdot \nu)_h^-\right) dS(x) \\ \le \int_\Omega P(\varrho_h^{k-1})\, dx + \frac{1}{2\mu} \Delta t \int_\Omega |\boldsymbol{f}_h^k|^2 dx + \frac{\mu}{2} \Delta t \int_\Omega |\boldsymbol{u}_h^k|^2\, dx.\end{aligned}$$

Summing over $k = 1, \dots, M$ yields (4.2). □

Since the stability estimate only provides the bound $p(\varrho_h) \in_b L^\infty(0, T; L^1(\Omega))$, it is not clear that $p(\varrho_h)$ converges weakly to an integrable function. Hence, we shall next establish that the pressure is in fact uniformly bounded in $L^2(0, T; L^2(\Omega))$.

To increase the readability we introduce the notation

$$\langle \phi \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \phi\, dx.$$

Lemma 4.3 (Higher integrability on the pressure)**.** *Let $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.8) and Definition 3.1. Then*

$$p(\varrho_h) \in_b L^2((0, T) \times \Omega).$$

Proof. For $m = 1, \dots, M$, define $\boldsymbol{v}_h^m \in \boldsymbol{V}_h(\Omega)$ by

$$\boldsymbol{v}_h^m = \Pi_h^V \mathcal{B}\left[p(\varrho_h^m) - \langle p(\varrho_h^m)\rangle_\Omega\right],$$

where the operator $\mathcal{B}[\cdot]$ is defined in (2.5).

Since $\operatorname{div} \Pi_h^V = \Pi_h^Q \operatorname{div}$, we have the identity

$$\operatorname{div}_h \boldsymbol{v}_h^m = \Pi_h^Q \operatorname{div} \mathcal{B}\left[p(\varrho_h^m) - \langle p(\varrho_h^m)\rangle_\Omega\right] = p(\varrho_h^m) - \langle p(\varrho_h^m)\rangle_\Omega .$$

By using $\boldsymbol{v}_h^m$ as test function in the velocity method (3.2) and applying the previous identity, we obtain the relation

$$\begin{aligned}
\int_\Omega p(\varrho_h^m)^2 dx &= |\Omega| \langle p(\varrho_h^m)\rangle_\Omega^2 + (\lambda+\mu)\int_\Omega \operatorname{div}_h \boldsymbol{u}_h^m \operatorname{div}_h \boldsymbol{v}_h^m dx \\
&\quad + \int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h^m \operatorname{curl}_h \boldsymbol{v}_h^m - \boldsymbol{f}_h^m \boldsymbol{v}_h^m dx \\
&\quad + \mu \sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{u}_h^m\cdot\nu]\!]_\Gamma [\![\boldsymbol{v}_h^m\cdot\nu]\!]_\Gamma + [\![\boldsymbol{u}_h^m\times\nu]\!]_\Gamma [\![\boldsymbol{v}_h^m\times\nu]\!]_\Gamma \, dS(x).
\end{aligned}$$

Repeated applications of Hölder's inequality yields

$$\begin{aligned}
&\|p(\varrho_h^m)\|_{L^2(\Omega)}^2 \\
&\quad \le C\langle p(\varrho_h^m)\rangle_\Omega^2 + (\lambda+\mu)\|\operatorname{div}_h \boldsymbol{u}_h^m\|_{L^2(\Omega)}\|\operatorname{div}_h \boldsymbol{v}_h^m\|_{L^2(\Omega)} \\
&\qquad + \mu\|\operatorname{curl}_h \boldsymbol{u}_h^m\|_{\boldsymbol{L}^2(\Omega)}\|\operatorname{curl}_h \boldsymbol{v}_h^m\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{f}_h^m\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}_h^m\|_{\boldsymbol{L}^2(\Omega)} \\
&\qquad + \mu\left|\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_\Gamma [\![\boldsymbol{u}_h^m\cdot\nu]\!]_\Gamma [\![\boldsymbol{v}_h^m\cdot\nu]\!]_\Gamma + [\![\boldsymbol{u}_h^m\times\nu]\!]_\Gamma [\![\boldsymbol{v}_h^m\times\nu]\!]_\Gamma \, dS(x)\right|.
\end{aligned}$$

To bound the jump terms we apply Lemma 2.14:

$$\begin{aligned}
&\|p(\varrho_h^m)\|_{L^2(\Omega)}^2 \\
&\quad \le C\Big[\langle p(\varrho_h^m)\rangle_\Omega^2 \\
&\qquad\quad + \big((1+h^{\frac{\epsilon}{2}})\|\boldsymbol{u}_h^m\|_{\boldsymbol{V}_h(\Omega)} + \|\boldsymbol{f}_h^m\|_{\boldsymbol{L}^2(\Omega)}\big)\,\|\mathcal{B}\left[p(\varrho_h^m) - \langle p(\varrho_h^m)\rangle_\Omega\right]\|_{\boldsymbol{W}^{1,2}(\Omega)}\Big] \\
&\quad \le C\left[\langle p(\varrho_h^m)\rangle_\Omega^2 + \left((1+h^{\frac{\epsilon}{2}})\|\boldsymbol{u}_h^m\|_{\boldsymbol{V}_h(\Omega)} + \|\boldsymbol{f}_h^m\|_{\boldsymbol{L}^2(\Omega)}\right)\left(1+\|p(\varrho_h^m)\|_{L^2(\Omega)}\right)\right],
\end{aligned}$$

where the last inequality follows thanks to the estimate $\|\mathcal{B}[\phi]\|_{\boldsymbol{W}^{1,2}(\Omega)} \le C\|\phi\|_{L^2(\Omega)}$.

Finally, an application of Cauchy's inequality (with ϵ) yields

$$\|p(\varrho_h^m)\|_{L^2(\Omega)}^2 \le C\left(1 + \langle p(\varrho_h^m)\rangle_\Omega^2 + \|\boldsymbol{u}_h^m\|_{\boldsymbol{V}_h(\Omega)}^2 + \|\boldsymbol{f}_h^m\|_{\boldsymbol{L}^2(\Omega)}^2\right).$$

Finally, we multiply this inequality by Δt, sum over $m = 1, \dots, M$, and apply Lemma 4.2. This concludes the proof. □

5. Convergence

Let $\{(\varrho_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.8) and Definition 3.1. In this section we will prove that a subsequence of this sequence converges to a weak solution of the semi–stationary Stokes system, thereby proving Theorem 3.3.

In view of Section 4, we have the following h–independent bounds:

$$\begin{aligned}
&\varrho_h \in_{\mathrm{b}} L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \\
&\boldsymbol{u}_h \in_{\mathrm{b}} L^2(0,T;\boldsymbol{L}^2(\Omega)), \\
&\operatorname{div}_h \boldsymbol{u}_h \in_{\mathrm{b}} L^2(0,T;L^2(\Omega)), \\
&\operatorname{curl}_h \boldsymbol{u}_h \in_{\mathrm{b}} L^2(0,T;\boldsymbol{L}^2(\Omega)).
\end{aligned}$$

Consequently, we can assume that there exist limit functions $\varrho \in L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega)$ and $\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}_0^{1,2}(\Omega))$ such that

$$\begin{aligned} \varrho_h &\overset{h\to 0}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \\ \boldsymbol{u}_h &\overset{h\to 0}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)), \end{aligned} \tag{5.1}$$

and, by Lemma 2.11,

$$\begin{aligned} \operatorname{div}_h \boldsymbol{u}_h &\overset{h\to 0}{\rightharpoonup} \operatorname{div} \boldsymbol{u} \quad \text{in } L^2(0,T;L^2(\Omega)), \\ \operatorname{curl}_h \boldsymbol{u}_h &\overset{h\to 0}{\rightharpoonup} \operatorname{curl} \boldsymbol{u} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \end{aligned} \tag{5.2}$$

Moreover,

$$\varrho_h^\gamma \overset{h\to 0}{\rightharpoonup} \overline{\varrho^\gamma}, \qquad \varrho_h^{\gamma+1} \overset{h\to 0}{\rightharpoonup} \overline{\varrho^{\gamma+1}}, \qquad \varrho_h \log \varrho_h \overset{h\to 0}{\rightharpoonup} \overline{\varrho \log \varrho},$$

where each $\overset{h\to 0}{\rightharpoonup}$ signifies weak convergence in a suitable L^p space with $p > 1$.

Finally, ϱ_h, $\varrho_h \log \varrho_h$ converge respectively to ϱ, $\overline{\varrho \log \varrho}$ in $C([0,T];L^p_{\mathrm{weak}}(\Omega))$ for some $1 < p < \gamma$, cf. Lemma 2.2 and also [5, 12]. In particular, ϱ, $\varrho \log \varrho$, and $\overline{\varrho \log \varrho}$ belong to $C([0,T];L^p_{\mathrm{weak}}(\Omega))$.

5.1. **Density method.**

Lemma 5.1 (Convergence of $\varrho\boldsymbol{u}$). *Given* (5.1) *and* (5.2),

$$\varrho_h \boldsymbol{u}_h \overset{h\to 0}{\rightharpoonup} \varrho\boldsymbol{u} \quad \textit{in the sense of distributions on } (0,T)\times\Omega.$$

Proof. Denote by $\widetilde{\boldsymbol{u}_h}$ the function

$$\widetilde{\boldsymbol{u}_h}(t,\cdot) = \widetilde{\boldsymbol{u}_h^m}, \quad t \in (t^{m-1}, t^m], \ m = 1, \dots, M,$$

where $\widetilde{\boldsymbol{u}_h^m}$ is defined in (3.4). By standard properties of the Nedelec interpolation operator, $\|\widetilde{\boldsymbol{u}_h}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le C\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}$. This, (3.6), and Lemma 4.2 allow us to conclude that $\widetilde{\boldsymbol{u}_h} \in_{\mathrm{b}} L^2(0,T;\boldsymbol{W}^{\mathrm{div},2}(\Omega))$ and

$$\sum_{m=1}^{M} \sum_{\Gamma\in\Gamma_h^I} \Delta t \int_\Gamma P''(\varrho_\dagger^m)\, [\![\varrho_h^m]\!]_\Gamma^2 \left|\widetilde{\boldsymbol{u}_h^m}\cdot\nu\right| \, dS(x) \le C.$$

Using these bounds, we can apply to (3.7) the calculations leading to Lemma 5.6 in [11], resulting in the bound

$$\partial_t^h(\varrho_h) \in_{\mathrm{b}} L^1(0,T;W^{-1,1}(\Omega)). \tag{5.3}$$

At the same time, Lemma 2.13 tells us that

$$\|\boldsymbol{u}_h(t,x) - \boldsymbol{u}_h(t,x-\xi)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega_\xi))} \to 0 \quad \text{as } |\xi| \to 0, \text{ uniformly in } h. \tag{5.4}$$

In view of (5.3) and (5.4), an application of Lemma 2.3 concludes the proof. $\square$

Lemma 5.2 (Continuity equation). *The limit pair* $(\varrho, \boldsymbol{u})$ *constructed in* (5.1) *and* (5.2) *is a weak solution of the continuity equation* (1.1) *in the sense of Definition 2.4.*

Proof. Denote by $\widetilde{\boldsymbol{u}_h}$ the function

$$\widetilde{\boldsymbol{u}_h}(t,\cdot) = \widetilde{\boldsymbol{u}_h^m}, \quad t \in (t^{m-1}, t^m], \quad m = 1, \dots, M,$$

where $\widetilde{\boldsymbol{u}_h^m}$ is defined in (3.4).

Fix a test function $\phi \in C_0^\infty([0,T)\times\overline{\Omega})$ and introduce the piecewise constant projections $\phi_h := \Pi_h^Q \phi$, $\phi_h^m := \Pi_h^Q \phi^m$, and $\phi^m := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \phi(t,\cdot)\, dt$.

By using ϕ_h^m as test function in (3.7) and preforming the same calculations as in the proof of Lemma 6.4 in [11], we work out the identity

$$\int_0^T\int_\Omega \partial_t^h(\varrho_h)\phi_h \; dxdt = \int_0^T\int_\Omega \varrho_h\widetilde{\boldsymbol{u}_h}\nabla\phi \; dxdt + \omega(h), \tag{5.5}$$

where $|\omega(h)| \le Ch^{\frac{1}{2}}$ and

$$\int_0^T\int_\Omega \partial_t^h(\varrho_h)\phi_h \; dxdt \overset{h\to 0}{\longrightarrow} -\int_0^T\int_\Omega \varrho\phi_t \; dxdt - \int_\Omega \varrho_0\phi(0,x)\; dx,$$

where we have relied on (5.1) and the strong convergence $\varrho_h^0 \to \varrho^0$ a.e. in Ω.

Next,

$$\int_0^T\int_\Omega \varrho_h\widetilde{\boldsymbol{u}_h}\nabla\phi \; dxdt = \int_0^T\int_\Omega \varrho_h\boldsymbol{u}_h\nabla\phi + \varrho_h\left(\widetilde{\boldsymbol{u}_h} - \boldsymbol{u}_h\right)\nabla\phi \; dxdt.$$

In view of Lemma 5.1,

$$\int_0^T\int_\Omega \varrho_h\boldsymbol{u}_h\nabla\phi \; dxdt \overset{h\to 0}{\longrightarrow} \int_0^T\int_\Omega \varrho\boldsymbol{u}\nabla\phi \; dxdt.$$

By a standard error estimate for Π_h^W (cf. [13]),

$$\begin{aligned}&\left|\int_0^T\int_\Omega \varrho_h\left(\widetilde{\boldsymbol{u}_h} - \boldsymbol{u}_h\right)\nabla\phi \; dxdt\right| \\ &\qquad \le hC\|\varrho_h\|_{L^2(0,T;L^2(\Omega))}\|\nabla_h\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}\|\nabla\phi\|_{L^\infty(0,T;L^\infty(\Omega))} \le Ch^{\frac{2-\epsilon}{4}},\end{aligned}$$

where the final inequality follows from Lemmas 2.12, 4.2, and 4.3.

Summarizing, sending $h \to 0$ in (5.5) delivers the desired result (2.2). □

5.2. **Strong convergence of density approximations.** To establish the strong convergence of the density approximations ϱ_h, we will utilize a weak continuity property of the effective viscous flux: $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) = p(\varrho_h) - (\lambda+\mu)\operatorname{div}\boldsymbol{u}_h$.

To derive this property we exploit the div–curl structure of the velocity scheme (3.2) combined with the commutative properties (2.9) of $\boldsymbol{V}_h$. More specifically, in view of the commutative property (2.9), the function $\boldsymbol{v}_h = \Pi_h^V\nabla\Delta^{-1}\varrho_h$ satisfies $\operatorname{div}_h\boldsymbol{v}_h = \varrho_h$ and $\operatorname{curl}_h\boldsymbol{v}_h = 0$ on elements away from the boundary. The crucial point is that the curl part of the velocity method (3.2) vanishes when this $\boldsymbol{v}_h$ is utilized as a test function.

Lemma 5.3 (Discrete effective viscous flux)**.** *Given the weak convergences listed in* (5.1) *and* (5.2),

$$\lim_{h\to 0}\int_0^T\int_\Omega P_{\text{eff}}(\varrho_h,\boldsymbol{u}_h)\,\varrho_h\,\phi\psi\; dxds = \int_0^T\int_\Omega \overline{P_{\text{eff}}(\varrho,\boldsymbol{u})}\,\varrho\,\phi\psi\; dxds,$$

for all $\phi \in C_0^\infty(\Omega)$ and $\psi \in C^\infty(0,T)$.

Proof. Fix $\phi \in C_0^\infty(\Omega)$, $\psi \in C^\infty(0,T)$, and for each $h > 0$ introduce the test function

$$\boldsymbol{v}_h(\cdot,t) = \psi\Pi_h^V\left[\phi\mathcal{A}\left[\varrho_h - \varrho\right](\cdot,t)\right], \qquad t\in(0,T).$$

where the operator $\mathcal{A}[\cdot]$ is defined in (2.6).

By virtue of (2.9) and $\operatorname{curl}\mathcal{A}[\cdot] = 0$, we have the identities

$$\operatorname{div}_h\boldsymbol{v}_h = \psi\Pi_h^Q\left(\nabla\phi\mathcal{A}\left[\varrho_h-\varrho\right]\right) + \psi\Pi_h^Q\left(\phi(\varrho_h-\varrho)\right)$$

and

$$\operatorname{curl}_h\boldsymbol{v}_h = \psi\Pi_h^Q\left(\nabla\phi\times\mathcal{A}\left[\varrho_h-\varrho\right]\right).$$

For $m = 1, \dots, M$, set $\boldsymbol{v}_h^m := \frac{1}{\Delta t}\int_{t^{m-1}}^{t^m} \boldsymbol{v}_h(\cdot, s)\ ds$. Taking $\boldsymbol{v}_h^m$ as test function in the velocity method (3.2), utilizing the above identities, multiplying by Δt, and summing over m, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)(\varrho_h - \varrho)\phi\psi\ dxds \\
&= -\int_0^T\int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\nabla\phi\cdot\mathcal{A}\left[\varrho_h - \varrho\right]\psi + \boldsymbol{f}_h\left(\Pi_h^V\phi\mathcal{A}\left[\varrho_h - \varrho\right]\right)\psi\ dxds \\
&\qquad + \int_0^T\int_\Omega \mu\operatorname{curl}_h\boldsymbol{u}_h\left(\nabla\phi\times\mathcal{A}\left[\varrho_h - \varrho\right]\right)\psi\ dxds \\
&\qquad + \mu\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_0^T\psi\int_\Gamma [\![\boldsymbol{u}_h\cdot\nu]\!]_\Gamma [\![\boldsymbol{v}_h\cdot\nu]\!]_\Gamma + [\![\boldsymbol{u}_h\times\nu]\!]_\Gamma [\![\boldsymbol{v}_h\times\nu]\!]_\Gamma\ dS(x)ds.
\end{aligned} \tag{5.6}$$

In view of (5.3), the following h–independent bounds are immediate:

$$\begin{aligned}
&\partial_t^h\mathcal{A}\left[\varrho_h\right] = \mathcal{A}\left[\partial_t^h\varrho_h\right] \in_{\text{b}} L^1(0,T;W^{-1,1}(\Omega)), \\
&\mathcal{A}\left[\varrho_h\right] \in_{\text{b}} L^2(0,T;W^{1,2}(\Omega)).
\end{aligned} \tag{5.7}$$

Consequently, Lemma 2.3 can be applied with the result that $\left(\mathcal{A}\left[\varrho_h\right]\right)^2 \overset{h\to 0}{\rightharpoonup} \left(\mathcal{A}\left[\varrho\right]\right)^2$. Thus,

$$\mathcal{A}\left[\varrho_h - \varrho\right] \overset{h\to 0}{\to} 0, \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{5.8}$$

Now, using (5.8) together with (5.1) and (5.2), we send $h \to 0$ in (5.6) to obtain

$$\begin{aligned}
&\lim_{h\to 0}\int_0^T\int_\Omega P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)(\varrho_h - \varrho)\phi\psi\ dxds \\
&= \lim_{h\to 0}\mu\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_0^T\psi\int_\Gamma [\![\boldsymbol{u}_h\cdot\nu]\!]\left[\!\left[\Pi_h^V(\phi\mathcal{A}\left[\varrho_h - \varrho\right])\cdot\nu\right]\!\right] \\
&\qquad\qquad + [\![\boldsymbol{u}_h\times\nu]\!]\left[\!\left[\Pi_h^V(\phi\mathcal{A}\left[\varrho_h - \varrho\right])\times\nu\right]\!\right]\ dS(x)dt.
\end{aligned} \tag{5.9}$$

Lemma 2.14 yields

$$\begin{aligned}
&\left|\mu\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_0^T\psi\int_\Gamma [\![\boldsymbol{u}_h\cdot\nu]\!]_\Gamma\left[\!\left[\Pi_h^V(\phi\mathcal{A}\left[\varrho_h - \varrho\right])\cdot\nu\right]\!\right]_\Gamma\ dS(x)ds\right| \\
&\qquad + \left|\mu\sum_{\Gamma\in\Gamma_h^I} h^{\epsilon-1}\int_0^T\psi\int_\Gamma [\![\boldsymbol{u}_h\times\nu]\!]_\Gamma\left[\!\left[\Pi_h^V(\phi\mathcal{A}\left[\varrho_h - \varrho\right])\times\nu\right]\!\right]_\Gamma\ dS(x)ds\right| \\
&\leq h^{\frac{\epsilon}{2}}C\|\psi\|_{L^\infty(0,T)}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{V}_h(\Omega))}\|\nabla(\phi\mathcal{A}\left[\varrho_h - \varrho\right])\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \leq Ch^{\frac{\epsilon}{2}},
\end{aligned}$$

where the last inequality follows from (5.7) and Lemmas 4.2 and 4.3. Applying the previous bound to (5.9) yields the desired result. □

We can now infer the strong convergence of the density approximations.

Lemma 5.4 (Strong convergence of ϱ_h)**.** *Suppose that* (5.1)–(5.2) *holds. Then, passing to a subsequence as $h \to 0$ if necessary,*

$$\varrho_h \to \varrho \quad \textit{a.e. in } (0,T)\times\Omega.$$

Proof. In view of Lemma 5.2, the limit $(\varrho, \boldsymbol{u})$ is a weak solution of the continuity equation and hence, by Lemma 2.6, also a renormalized solution:

$$\left(\varrho\log\varrho\right)_t + \operatorname{div}\left(\left(\varrho\log\varrho\right)\boldsymbol{u}\right) = \varrho\operatorname{div}\boldsymbol{u} \quad \text{in the weak sense on } [0,T)\times\overline{\Omega}.$$

Since $t \mapsto \varrho \log \varrho$ is continuous with values in an appropriate Lebesgue space equipped with the weak topology, we can use this equation to obtain for any $t > 0$

$$\int_\Omega (\varrho \log \varrho)(t)\ dx - \int_\Omega \varrho_0 \log \varrho_0\ dx = -\int_0^t \int_\Omega \varrho \operatorname{div} \boldsymbol{u}\ dxds \tag{5.10}$$

Next, we specify $\phi_h \equiv 1$ as test function in the renormalized scheme (4.1), multiply by Δt, and sum the result over m. Making use of the convexity of $z \log z$, we conclude that for any $m = 1, \dots, M$

$$\int_\Omega \varrho_h^m \log \varrho_h^m\ dx - \int_\Omega \varrho_h^0 \log \varrho_h^0\ dx \leq -\sum_{k=1}^m \Delta t \int_\Omega \varrho_h^m \operatorname{div} \boldsymbol{u}_h^m\ dxdt. \tag{5.11}$$

In view of the convergences stated at the beginning of this section and strong convergence of the initial data, we can send $h \to 0$ in (5.11) to obtain

$$\int_\Omega \left(\overline{\varrho \log \varrho}\right)(t)\ dx - \int_\Omega \varrho_0 \log \varrho_0\ dx \leq -\int_0^t \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}}\ dxds. \tag{5.12}$$

Subtracting (5.10) from (5.12) gives

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho\right)(t)\ dx \leq -\int_0^t \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u}\ dxds,$$

for any $t \in (0, T)$. Lemma 5.3 tells us that

$$\int_0^t \int_\Omega \left(\overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u}\right) \phi\ dxds = \frac{a}{\mu + \lambda} \int_0^t \int_\Omega \left(\overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma}\varrho\right) \phi\ dxds \geq 0,$$

for all $\phi \in C_0^\infty(\Omega) \cap \{\phi \geq 0\}$, where the last inequality follows as in [5, 12], so the following relation holds:

$$\overline{\varrho \log \varrho} = \varrho \log \varrho \quad \text{a.e. in } (0, T) \times \Omega.$$

Now an application of Lemma 2.1 finishes the proof. □

5.3. **Velocity method.**

Lemma 5.5 (Velocity equation). *The limit pair $(\varrho, \boldsymbol{u})$ constructed in (5.1)–(5.2) is a weak solution to the velocity equation (1.2) in the sense of Definition 2.4.*

Proof. Fix $\boldsymbol{v} \in L^2(0, T; \boldsymbol{W}_0^{1,2}(\Omega))$, and set $\boldsymbol{v}_h = \Pi_h^V \boldsymbol{v}$ and $\boldsymbol{v}_h^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \boldsymbol{v}_h\ dt$.

Then, setting $\boldsymbol{v}_h^m$ as test function in the velocity method (3.2), multiplying with Δt, and summing over all $m = 1, \dots, M$, leads to the identity

$$\begin{aligned}
&\int_0^T \int_\Omega \mu \operatorname{curl}_h \boldsymbol{u}_h \operatorname{curl} \boldsymbol{v} + [(\mu + \lambda) \operatorname{div}_h \boldsymbol{u}_h - p(\varrho_h)] \operatorname{div} \boldsymbol{v}\ dxdt \\
&\qquad + \mu \sum_{\Gamma \in \Gamma_h^I} h^{\epsilon - 1} \int_0^T \int_\Gamma [\![\boldsymbol{u}_h \cdot \nu]\!]_\Gamma \left[\!\!\left[(\Pi_h^V \boldsymbol{v}) \cdot \nu\right]\!\!\right]_\Gamma \\
&\qquad\qquad + [\![\boldsymbol{u}_h \times \nu]\!]_\Gamma \left[\!\!\left[(\Pi_h^V \boldsymbol{v}) \times \nu\right]\!\!\right]_\Gamma\ dS(x)dt \\
&= \int_0^T \int_\Omega \boldsymbol{f}_h \Pi_h^V \boldsymbol{v}\ dxdt,
\end{aligned} \tag{5.13}$$

where we have also used (2.9). From Lemma 5.4 and (5.1), we have that $p(\varrho_h) \overset{h \to 0}{\to} p(\varrho)$ in $L^2(0, T; L^2(\Omega))$. Furthermore, Lemma 2.14 tells us that the jump terms converge to zero. Hence, we can send $h \to 0$ in (5.13) to obtain that the limit $(\varrho, \boldsymbol{u})$ constructed in (5.1)–(5.2) satisfies (2.3) for all test functions $\boldsymbol{v} \in L^2(0, T; \boldsymbol{W}_0^{1,2}(\Omega))$. □

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