Unique solutions to boundary value problems in the cold plasma model

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Abstract

The unique existence of a weak solution to the homogeneous closed Dirichlet problem is proven for a mixed elliptic-hyperbolic equation. Equations of this kind arise in models for electromagnetic wave propagation in cold plasma. A class of open boundary value problems for the equation is shown to possess strong solutions. *MSC2000*: 35M10, 35D05, 82D10.

Key words: Elliptic-hyperbolic equation, cold plasma model, closed boundary value problem, symmetric positive operator

1 Introduction

Boundary value problems for mixed elliptic-hyperbolic equations may be either *open* or *closed*. In the former case, data are prescribed on a proper subset of the boundary, whereas in the latter case, data are prescribed on the entire boundary. It is shown in Sec. 3 of [21] that if $\kappa = 1/2$, the closed Dirichlet problem is over-determined for the equation

$$\left(x - y^2\right) u_{xx} + u_{yy} + \kappa u_x = 0 \tag{1}$$

on a typical domain, where $u\left(x, y\right)$ is required to be twice-continuously differentiable on the domain. However, this equation arises in a qualitative model for electromagnetic wave propagation in an idealized cold plasma; physical reasoning suggests that the closed Dirichlet problem should be well-posed in a suitable function space, at least for some choice of lower order terms. See Sec. 1 of [21] for a discussion, in which the problem of formulating a closed Dirichlet-like problem that is well-posed in an appropriate sense is characterized as an "outstanding and significant problem for the cold plasma model."

Using methods introduced by Lupo, Morawetz, and Payne [17] for equations of Tricomi type, we show in Sec. 2 the weak existence of a unique solution to a homogeneous closed Dirichlet problem for the self-adjoint ($\kappa = 1$) form of eq.

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(1). This extends a result result [23] in which the existence of solutions having various degrees of smoothness was shown in certain cases to which uniqueness proofs did not seem to apply. At the same time, it extends the unique-existence arguments in [17] to an equation which is not of Tricomi type.

Another well known problem in elliptic-hyperbolic theory is the determination of natural conditions for boundary geometry; see the discussions in [1], [19], [20], [25], [26], and [27]. Heuristic approaches to determining boundary geometry tend to focus on physical [18] or geometric [24] analogies for the specific equation under study. In his theory of symmetric positive systems [6], Friedrichs proposed intrinsic mathematical criteria for the well-posedness, or *admissibility*, of boundary conditions. But Friedrichs' conditions are also tied to the specifics of the particular symmetric positive equation under study and are algebraic rather than explicitly geometric. In this note we require boundary arcs to be starlike with respect to an appropriate vector field. This approach to boundary geometry was introduced by Lupo and Payne [16], but algebraic conditions in certain very old results can be reinterpreted as the requirement of a starlike boundary; see, *e.g.*, [11]. Our results provide further evidence that domain boundaries which are starlike in this generalized sense are natural for elliptic-hyperbolic boundary value problems.

In Sec. 3 we investigate the solvability of open boundary value problems for a class of symmetric positive systems on star-shaped domains. The boundary conditions are *mixed* in the sense that a Dirichlet condition is placed on part of the boundary and a Neumann condition is placed on another part. However, our methods also apply to the case in which either a Dirichlet or a Neumann condition is imposed over the entire elliptic boundary. Because the boundary value problems considered in Sec. 3 are open, the results of that section may be less interesting than those of Sec. 2 in the physical context of the cold plasma model. But open boundary conditions can be expected to imply more smoothness on the part of solutions than is obtained from closed boundary conditions, and we show the existence of solutions which are strong in the sense of Friedrichs. Strong solutions to boundary value problems in the cold plasma model were also discussed in Sec. 3 of [23], but briefly and inadequately. Section 3 of this note revises and extends (to the open case of mixed and Neumann problems) the treatment of strong solutions in [23]. The existence question for weak solutions to open boundary value problems in the cold plasma model was considered in [22] and [37].

1.1 Remarks on the physical model

Plasma is the natural state of matter at temperatures on the order of 10,000 K or more. However, reasonably dense plasmas also exist at much lower temperatures, for example in interstellar media (see, *e.g.*, [4]). In the cold plasma model, the temperature is assumed to be zero in order to neglect the fluid properties of the medium, which is then treated as a linear dielectric. Somewhat surprisingly, this assumption is a useful first approximation to the products of tokamaks: low-density plasmas which are remarkably free of expected hightemperature phenomena such as collisions and wall effects; see the remarks in the introduction to [33] and the more detailed discussions in [35]. More generally still, the cold plasma model approximates the effects of small-amplitude electromagnetic waves, propagating with phase velocities which are sufficiently large in comparison to the thermal velocity of the particles.

In the case of wave propagation through an underlying static medium having axisymmetric geometry, equations of the form (1) model the tangency of a flux surface to a resonance surface. At the point of tangency, plasma heating might occur even in the cold plasma model. In two dimensions, flux surfaces (level sets of the magnetic flux function) can be represented by the lines $x = \text{const.}$, and a resonance surface (frequencies at which the field equations change from elliptic to hyperbolic type) by the curve $x = y^2$. It has been observed [35] that in such cases a plasma heating zone could lie at the origin of coordinates. This conjecture is supported by numerical [21] and classical [28] analysis which suggests that the origin is a singular point of eq. (1). For discussions of the physical context of the equation studied in this note, see Sec. V of [35] and Sec. 4 of [36], in which eq. (1) with $\kappa = 0$ is used as a qualitative model for erratic heating effects by lower hybrid waves in the plasma. See also [28], in which a model for electrostatic waves in a cold anisotropic plasma with a two-dimensional inhomogeneity yields, by a formal derivation, an equation for the field potential which is similar to (1); precisely, the equation derived in [28] is eq. (44) of Remark *iii)*, Sec. 3, below, with particular choices of $\sigma(y)$, κ_1, and κ_2. See Ch. 2 of [34] for a recent, general treatment of electromagnetic waves in cold plasma.

In the sequel we assume that Ω is a bounded connected domain of $\mathbb{R}^2$ having at least piecewise differentiable boundary with counterclockwise orientation; additional conditions will be placed on the domain where required.

2 Weak solutions to closed boundary value problems

Following Sec. 3 of [17] we define, for a given C^1 function $K(x,y)$, the space $L^2(\Omega;|K|)$ and its dual. These spaces consist, respectively, of functions u for which the norm

$$||u||_{L^2(\Omega;|K|)} = \left(\int_\Omega |K|u^2 dxdy\right)^{1/2}$$

is finite, and functions $u \in L^2(\Omega)$ for which the norm

$$||u||_{L^2(\Omega;|K|^{-1})} = \left(\int_\Omega |K|^{-1} u^2 dxdy\right)^{1/2}$$

is finite. Analogously, we define the space $H_0^1(\Omega;K)$ to be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_{H^1(\Omega;K)} = \left[\int\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right) dxdy\right]^{1/2}. \tag{2}$$

Using a weighted Poincaré inequality to absorb the zeroth-order term, we write the $H_0^1(\Omega;K)$-norm in the form

$$||u||_{H_0^1(\Omega;K)} = \left[\int\int_\Omega \left(|K|u_x^2 + u_y^2\right) \, dxdy \right]^{1/2}. \tag{3}$$

Various lower-order terms have been associated to eq. (1) in the literature on the cold plasma model, as such terms do not have explicit physical significance. We put the equation into the self-adjoint form

$$Lu \equiv \left[K\left(x,y\right)u_x\right]_x + u_{yy} = f\left(x,y\right) \tag{4}$$

for the type-change function $K\left(x,y\right) = x - y^2$; the inhomogeneous term $f\left(x,y\right)$ is assumed known.

In accordance with standard terminology, we will refer to the curve $K = 0$ on which eq. (4) changes type as the *sonic line.* This terminology is borrowed from fluid dynamics; in the context of the cold plasma model, the sonic transition occurs at a resonance frequency.

Following Lupo, Morawetz, and Payne [17], we define a *weak solution* of eq. (4) on Ω, with boundary condition

$$u(x,y) = 0 \, \forall \, (x,y) \in \partial\Omega, \tag{5}$$

to be a function $u \in H_0^1(\Omega;K)$ such that $\forall \xi \in H_0^1(\Omega;K)$ we have

$$\langle Lu, \xi \rangle \equiv -\int\int_\Omega \left(Ku_x\xi_x + u_y\xi_y\right) dxdy = \left(f, \xi\right),$$

where (,) denotes L^2 inner product. In this case the existence of a weak solution is equivalent to the existence of a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$||u_n - u||_{H_0^1(\Omega;K)} \to 0 \text{ and } ||Lu_n - f||_{H^{-1}(\Omega;K)} \to 0$$

as n tends to infinity.

Following Lupo and Payne (Sec. 2 of [16]), we consider a one-parameter family $\psi_\lambda\left(x,y\right)$ of inhomogeneous dilations given by

$$\psi_\lambda\left(x,y\right) = \left(\lambda^{-\alpha}x, \lambda^{-\beta}y\right),$$

where $\alpha, \beta, \lambda \in \mathbb{R}^+$, and the associated family of operators

$$\Psi_\lambda u = u \circ \psi_\lambda \equiv u_\lambda.$$

Denote by D the vector field

$$Du = \left[\frac{d}{d\lambda}u_\lambda\right]_{|\lambda=1} = -\alpha x\partial_x - \beta y\partial_y. \tag{6}$$

An open set $\Omega \subseteq \mathbb{R}^2$ is said to be *star-shaped* with respect to the flow of D if $\forall (x_0, y_0) \in \overline{\Omega}$ and each $t \in [0, \infty]$ we have $F_t(x_0, y_0) \subset \overline{\Omega}$, where

$$F_t(x_0, y_0) = (x(t), y(t)) = \left(x_0 e^{-\alpha t}, y_0 e^{-\beta t}\right).$$

If a domain is star-shaped with respect to a vector field D, then it is possible to "float" from any point of the domain to the origin along the flow lines of the vector field. If these flow lines are straight lines through the origin $(\alpha = \beta)$, then we recover the conventional notion of a star-shaped domain. By an appropriate translation, the origin can be replaced by any point (x_s, y_s) in the plane as a source of the flow. In that case we obtain a translated function $\tilde{F}_t$ for which

$$\lim_{t \to \infty} \tilde{F}_t(x_0, y_0) = (x_s, y_s) \;\; \forall (x_0, y_0) \in \overline{\Omega}.$$

Moreover, whenever a domain is star-shaped with respect to the flow of a vector field satisfying (6), the domain boundary will be *starlike* in the sense that

$$(\alpha x, \beta y) \cdot \hat{\mathbf{n}}(x, y) \geq 0,$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal vector on $\partial\Omega$. See Lemma 2.2 of [16]. In equivalent notation, given a vector field $V = -(b, c)$ and a boundary arc Γ which is starlike with respect to V, the inequality

$$bn_1 + cn_2 \geq 0 \tag{7}$$

is satisfied on Γ.

We employ an integral variant of the *abc* method, introduced by Didenko [5] and developed by Lupo and Payne [15]. Denote by v a C^1 solution to the boundary value problem

$$Hv = u \text{ in } \Omega \tag{8}$$

with v vanishing on $\partial\Omega \backslash \{0, 0\}$,

$$\lim_{(x,y) \to (0,0)} v(x, y) = 0, \tag{9}$$

and

$$Hv = av + bv_x + cv_y. \tag{10}$$

Assume that: Ω is star-shaped with respect to the flow of the vector field $V = -(b, c)$; $b = mx$ and $c = \mu y$; μ and m are positive constants and a is a negative constant. Arguing as in step 1 of the proof of Lemma 3.3, [17], which treats the harder case of a nondifferentiable coefficient, we conclude that v exists. We have the integral identities

$$(Iu, Lu) \equiv (v, Lu) = (v, LHv). \tag{11}$$

A good choice of the coefficients a, m, and μ on the right-hand side of this identity will allow us to derive an energy inequality, which will be used to prove weak existence via the Riesz Representation Theorem; see Ch. 2 of [2] for a general treatment of these kinds of arguments.

The following is a small but crucial extension of [23], Theorem 5.

Lemma 1. *Suppose that x is non-negative on Ω and that the origin of coordinates lies on $\partial\Omega$. Let Ω be star-shaped with respect to the flow of the vector field $V = -(b,c)$ for $b = mx$ and $c = \mu y$, where μ is a positive constant and m exceeds 3μ. Then for every $u \in C_0^\infty(\Omega)$ there exists a positive constant C for which*

$$||u||_{L^2(\Omega;|K|)} \leq C||Lu||_{H^{-1}(\Omega;K)},$$

where $K(x,y) = x - y^2$ and L satisfies (4).

Proof. Let v satisfy eqs. (8)-(10) on Ω for $a = -M$, where M is a positive number satisfying

$$M = \frac{m-3\mu}{2} - \delta$$

for some sufficiently small positive number δ. Integrate the integral identities (11) by parts, using Prop. 12 of [23] and the compact support of u. We have

$$\int\int_\Omega v \cdot LHv\,dxdy = \frac{1}{2}\oint_{\partial\Omega}\left(Kv_x^2 + v_y^2\right)(cdx - bdy)$$
$$+\int\int_\Omega \omega v^2 + \alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2 dxdy, \tag{12}$$

where $\omega = 0$,

$$\alpha = K\left(\frac{c_y - b_x}{2} - a\right) + \frac{1}{2}b + \frac{1}{2}K_y c$$
$$= \left(\frac{m}{2} - \mu - \delta\right)x + \delta y^2,$$
$$\beta = 0,$$

and

$$\gamma = -a - \frac{c_y}{2} + \frac{b_x}{2} = M - \frac{\mu - m}{2} = m - 2\mu - \delta > \mu - \delta.$$

On the *elliptic* region Ω^+, $K > 0$ and

$$\left(\frac{m}{2} - \mu - \delta\right)x > \left(\frac{\mu}{2} - \delta\right)x \geq \delta x$$

provided we choose δ so small that $\mu/4 \geq \delta$. Then on Ω^+,

$$\alpha \geq \delta\left(x + y^2\right) \geq \delta\left(x - y^2\right) = \delta K = \delta|K|.$$

On the *hyperbolic* region Ω^-, $K < 0$ and

$$\alpha = \left(\frac{m}{2} - \mu\right)x + \delta\left(y^2 - x\right) \geq \frac{\mu}{2}x + \delta(-K) \geq \delta|K|.$$

We find that if δ is sufficiently small relative to μ, then

$$(v, LHv) \geq \delta \int \int_{\Omega} \left(|K| v_x^2 + v_y^2 \right) dxdy. \tag{13}$$

The upper estimate is immediate, as

$$(v, LHv) = (v, Lu) \leq \|v\|_{H_0^1(\Omega;K)} \|Lu\|_{H^{-1}(\Omega;K)} . \tag{14}$$

Combining (13) and (14), we obtain

$$\|v\|_{H_0^1(\Omega;K)} \leq C \|Lu\|_{H^{-1}(\Omega;K)} . \tag{15}$$

The assertion of Lemma 1 now follows from (8) by the continuity of H as a map from $H_0^1(\Omega; K)$ into $L^2(\Omega; |K|)$. This completes the proof of Lemma 1.

Theorem 2. *Let Ω be star-shaped with respect to the flow of the vector field $-V = (mx, \mu y)$, where m and μ are defined as in Lemma 1. Suppose that x is nonnegative on Ω and that the origin of coordinates lies on $\partial\Omega$. Then for every $f \in L^2\left(\Omega; |K|^{-1}\right)$ there is a unique weak solution $u \in H_0^1(\Omega; K)$ to the Dirichlet problem (4), (5) where $K = x - y^2$.*

Proof. The proof follows the outline of the arguments in [17], Sec. 3. Defining a linear functional J_f by the formula

$$J_f(L\xi) = (f, \xi), \; \xi \in C_0^\infty(\Omega),$$

we estimate

$$|J_f(L\xi)| \leq ||f||_{L^2(\Omega;|K|^{-1})} ||\xi||_{L^2(\Omega;|K|)} \leq C ||f||_{L^2(\Omega;|K|^{-1})} ||L\xi||_{H^{-1}(\Omega;K)},$$

using Lemma 1. Thus J_f is a bounded linear functional on the subspace of $H^{-1}(\Omega; K)$ consisting of elements having the form $L\xi$ with $\xi \in C_0^\infty(\Omega)$. Extending J_f to the closure of this subspace by Hahn-Banach arguments, the Riesz Representation Theorem guarantees the existence of an element $u \in H_0^1(\Omega; K)$ for which

$$\langle u, L\xi \rangle = (f, \xi),$$

where $\xi \in H_0^1(\Omega; K)$. There exists a unique, continuous, self-adjoint extension $L : H_0^1(\Omega; K) \to H^{-1}(\Omega; K)$. Thus standard arguments imply that a sequence of smooth, compactly supported approximations u_n of $u \in H_0^1(\Omega; K)$ converges in norm to an element $\tilde{f}$ of $H^{-1}(\Omega; K)$. Taking the limit

$$\lim_{n\to\infty} \langle u - u_n, L\xi \rangle = \left(f - \tilde{f}, \xi \right),$$

we conclude that, because the left-hand side vanishes for all $\xi \in H_0^1(\Omega; K)$, the right-hand side must vanish as well. Taking the difference of two weak solutions, we find that this difference is zero in $H_0^1(\Omega; K)$ by the linearity of L and the weighted Poincaré inequality [17].

The unique-existence proofs of this section use similar estimates to those used in the proofs in Sec. 4 of [23] for existence alone. But the likelihood that the estimates would turn out to be similar appeared to be small on the basis of previous literature, and is rather surprising. In Sec. 5.1 of [23] it is shown that the estimates used to prove weak existence do not extend in an obvious way to proofs of uniqueness for the case $K = x$, in which the resonance curve is collinear with the flux line. Based on the physical discussion on p. 42 of [35], the collinear case would appear to be simpler than the case treated here, in which the two curves are tangent at an isolated point. But the cautionary example of [23], Sec. 5.1, which suggests the difficulty of modifying the weak-existence methods to prove uniqueness in the case $K = x$, happens to fail in the case $K = x - y^2$. As we have shown in this section, a modification of the weak existence estimates will in fact lead to a uniqueness proof for weak solutions to (4), (5) in the self-adjoint case.

3 Strong solutions to open boundary value problems

Consider a system of the form

$$L\mathbf{u} = \mathbf{f} \tag{16}$$

for an unknown vector

$$\mathbf{u} = (u_1(x,y), u_2(x,y)),$$

and a known vector

$$\mathbf{f} = (f_1(x,y), f_2(x,y)),$$

where $(x,y) \in \Omega \subset \mathbb{R}^2$. The operator L satisfies

$$(L\mathbf{u})_1 = K(x,y)\,u_{1x} + u_{2y} + \text{ zeroth-order terms,} \tag{17}$$

$$(L\mathbf{u})_2 = u_{1y} - u_{2x}. \tag{18}$$

As in the preceding section, $K(x,y)$ is continuously differentiable, negative on Ω^-, positive on Ω^+, and zero on a parabolic region (the sonic curve) separating the elliptic and hyperbolic regions. If $(f_1, f_2) = (f, 0)$, the components of the vector $\mathbf{u}$ are continuously differentiable, and $u_1 = u_x$, $u_2 = u_y$ for some twice-differentiable function $u(x,y)$, then the first-order system (16)-(18) reduces to a second-order scalar equation such as (4). Because the emphasis in this section is on the form of the boundary conditions, the presence or absence of zeroth-order terms will not affect the arguments provided the resulting system is symmetric positive.

We say that a vector $\mathbf{u} = (u_1, u_2)$ is in L^2 if each of its components is square-integrable. Such an object is a *strong solution* of an operator equation of the form (16), with given boundary conditions, if there exists a sequence $\mathbf{u}^\nu$

of continuously differentiable vectors, satisfying the boundary conditions, for which $\mathbf{u}^\nu$ converges to $\mathbf{u}$ in L^2 and $L\mathbf{u}^\nu$ converges to $\mathbf{f}$ in L^2. Strong solutions can be shown to be unique.

Sufficient conditions for a vector to be a strong solution were formulated by Friedrichs [6]. An operator L associated to an equation of the form

$$L\mathbf{u} = A^1\mathbf{u}_x + A^2\mathbf{u}_y + B\mathbf{u}, \tag{19}$$

where A^1, A^2, and B are matrices, is said to be *symmetric positive* if the matrices A^1 and A^2 are symmetric and the matrix

$$Q \equiv B^* - \frac{1}{2}\left(A^1_x + A^2_y\right)$$

is positive-definite, where B^* is the symmetrization of the matrix B :

$$B^* = \frac{1}{2}\left(B + B^T\right).$$

The differential equation associated to a symmetric positive operator is also said to be symmetric positive.

Boundary conditions for a symmetric positive equation can be given in terms of a matrix

$$\beta = n_1 A^1_{|\partial\Omega} + n_2 A^2_{|\partial\Omega}, \tag{20}$$

where (n_1, n_2) are the components of the outward-pointing normal vector on $\partial\Omega$. The boundary is assumed to be twice-continuously differentiable. Let $\mathcal{N}(\tilde{x}, \tilde{y})$, $(\tilde{x}, \tilde{y}) \in \partial\Omega$, be a linear subspace of a vector space $\mathcal{V}$, where $\mathbf{u} : \Omega \cup \partial\Omega \to \mathcal{V}$ and $\mathcal{N}(\tilde{x}, \tilde{y})$ depends smoothly on $\tilde{x}$ and $\tilde{y}$. A boundary condition $u \in \mathcal{N}$ is *admissible* if $\mathcal{N}$ is a maximal subspace of $\mathcal{V}$ and the quadratic form $(\mathbf{u}, \beta\mathbf{u})$ is non-negative on $\partial\Omega$.

A set of sufficient conditions for admissibility is the existence of a decomposition ([6], Sec. 5)

$$\beta = \beta_+ + \beta_-, \tag{21}$$

for which: the direct sum of the null spaces for β_+ and β_- spans the restriction of $\mathcal{V}$ to the boundary; the ranges $\mathfrak{R}_\pm$ of $\beta_\pm$ have only the vector $\mathbf{u} = 0$ in common; and the matrix $\mu = \beta_+ - \beta_-$ satisfies

$$\mu^* = \frac{\mu + \mu^T}{2} \geq 0. \tag{22}$$

These conditions imply that the boundary condition

$$\beta_-\mathbf{u} = 0 \text{ on } \partial\Omega \tag{23}$$

is admissible for eq. (16) and the boundary condition

$$\mathbf{w}^T\beta_+^T = 0 \text{ on } \partial\Omega \tag{24}$$

is admissible for the adjoint problem

$$L^* \mathbf{w} = \mathbf{g} \text{ in } \Omega.$$

The linearity of the operator L and the admissibility conditions on the matrices $\beta_\pm$ imply that both problems possess unique, strong solutions.

Boundary conditions are *semi-admissible* if they satisfy properties (22) and (23). A symmetric positive equation having semi-admissible boundary conditions possesses a unique weak solution in the ordinary sense: a vector $\mathbf{u} \in L^2(\Omega)$ such that

$$\int_\Omega (L^* \mathbf{w}) \cdot \mathbf{u} \, d\Omega = \int_\Omega \mathbf{w} \cdot \mathbf{f} \, d\Omega$$

for all vectors $\mathbf{w}$ having continuously differentiable components and satisfying (24).

Writing the higher-order terms of eqs. (17), (18) in the form

$$L\mathbf{u} = \begin{pmatrix} K(x,y) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y , \tag{25}$$

we will derive admissible boundary conditions for the system (16)-(18).

Slightly generalizing the type-change function of Sec. 2, we choose $K(x,y) = x - \sigma(y)$, where $\sigma(y) \geq 0$ is a continuously differentiable function of its argument satisfying [22]

$$\sigma(0) = \sigma'(0) = 0, \tag{26}$$

$$\sigma'(y) > 0 \, \forall y > 0, \tag{27}$$

and

$$\sigma'(y) < 0 \, \forall y < 0. \tag{28}$$

Let the operator L in (16) be given by

$$(L\mathbf{u})_1 = [x - \sigma(y)] \, u_{1x} + u_{2y} + \kappa_1 u_1 + \kappa_2 u_2, \tag{29}$$

$$(L\mathbf{u})_2 = u_{1y} - u_{2x}, \tag{30}$$

where κ_1 and κ_2 are constants.

By the *elliptic* portion $\partial\Omega^+$ of the boundary we mean points $(\tilde{x}, \tilde{y})$ of the domain boundary on which the type-change function $K(\tilde{x}, \tilde{y})$ is positive and by the *hyperbolic* portion $\partial\Omega^-$, boundary points for which the type-change function is negative. The *sonic* portion of the boundary consists of boundary points on which the type change function vanishes.

In this section we prove a revision and extension of [23], Theorem 9:

Theorem 3. *Let Ω be a bounded, connected domain of $\mathbb{R}^2$ having C^2 boundary $\partial\Omega$, oriented in a counterclockwise direction. Let $\partial\Omega_1^+$ be a (possibly empty and not necessarily proper) subset of $\partial\Omega^+$. Translate the vector field constructed in eq. (6) away from the origin. Let $\partial\Omega^+ \backslash \partial\Omega_1^+$ be starlike with respect to the* *vector field* $V^+ = -(b(x,y), c(x,y))$; *let* $\partial\Omega_1^+$ *be starlike with respect to the vector field* $V_1^+ = (b(x,y), c(x,y))$; *let* $\partial\Omega\backslash\partial\Omega^+$ *be starlike with respect to the vector field* $V^- = (b(x,y), -c(x,y))$. *Let* $b(x,y)$ *and* $c(x,y)$ *satisfy*

$$b^2 + c^2 K \neq 0 \tag{31}$$

on Ω, *and the inequalities:*

$$2b\kappa_1 - b_x K - b + c_y K - c\sigma'(y) > 0 \text{ in } \Omega; \tag{32}$$

$$\begin{aligned}&\left(2b\kappa_1 - b_x K - b + c_y K - c\sigma'(y)\right)\left(2c\kappa_2 + b_x - c_y\right)\\ &\qquad - \left(b\kappa_2 + c\kappa_1 - c_x K - c - b_y\right)^2 > 0 \text{ in } \Omega;\end{aligned} \tag{33}$$

$$K\left(bn_1 - cn_2\right)^2 + \left(cKn_1 + bn_2\right)^2 \leq 0 \text{ on } \partial\Omega\backslash\partial\Omega^+. \tag{34}$$

Let L *be given by (29), (30). Let the Dirichlet condition*

$$-u_1 n_2 + u_2 n_1 = 0 \tag{35}$$

be satisfied on $\partial\Omega^+\backslash\partial\Omega_1^+$ *and let the Neumann condition*

$$Ku_1 n_1 + u_2 n_2 = 0 \tag{36}$$

be satisfied on $\partial\Omega_1^+$. *Then eqs. (16), (29), (30) possess a strong solution on* Ω *for every* $\mathbf{f} \in L^2(\Omega)$.

Proof. Multiply both sides of eq. (16), (29), (30) by the matrix

$$E = \begin{pmatrix} b & -cK \\ c & b \end{pmatrix}. \tag{37}$$

Condition (31) implies that E is invertible on Ω, and conditions (32), (33) imply that the resulting system is symmetric positive.

Because $\partial\Omega^+\backslash\partial\Omega_1^+$ is starlike with respect to V^+, condition (7) is satisfied there. On $\partial\Omega_1^+$ we have, analogously,

$$bn_1 + cn_2 \leq 0 \tag{38}$$

and on $\partial\Omega\backslash\partial\Omega^+$,

$$-bn_1 + cn_2 \geq 0. \tag{39}$$

For all points $(\tilde{x}, \tilde{y}) \in \partial\Omega$, decompose the matrix

$$\beta\left(\tilde{x}, \tilde{y}\right) = \begin{pmatrix} K\left(bn_1 - cn_2\right) & cKn_1 + bn_2 \\ cKn_1 + bn_2 & -\left(bn_1 - cn_2\right) \end{pmatrix}$$

into a matrix sum having the form $\beta = \beta_+ + \beta_-$.

On $\partial\Omega^+\backslash\partial\Omega_1^+$, decompose β into the submatrices

$$\beta_+ = \begin{pmatrix} Kbn_1 & bn_2 \\ Kcn_1 & cn_2 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} -Kcn_2 & Kcn_1 \\ bn_2 & -bn_1 \end{pmatrix}.$$

Then $\beta_- \mathbf{u} = 0$ under boundary condition (35). We have

$$\mu^* = (bn_1 + cn_2) \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix},$$

so condition (7) implies that the Dirichlet condition (35) is semi-admissible on $\partial\Omega\backslash\partial\Omega_1^+$.

On $\partial\Omega_1^+$, choose

$$\beta_+ = \begin{pmatrix} -Kcn_2 & Kcn_1 \\ bn_2 & -bn_1 \end{pmatrix}$$

and

$$\beta_- = \begin{pmatrix} Kbn_1 & bn_2 \\ Kcn_1 & cn_2 \end{pmatrix}.$$

Then $\beta_- \mathbf{u} = 0$ under the Neumann boundary condition (36), and

$$\mu^* = -(bn_1 + cn_2) \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$$

is positive semi-definite under condition (38).

On $\partial\Omega\backslash\partial\Omega^+$, choose $\beta_+ = \beta$ and take β_- to be the zero matrix. Then $\mu = \mu^* = \beta$ and

$$\mu_{11} = K(bn_1 - cn_2).$$

Because μ_{11} is non-negative by (39), μ^* is positive semi-definite by inequality (34), and no conditions need be imposed outside the elliptic portion of the boundary. Semi-admissibility follows.

In fact, admissibility also follows, as we proceed to show.

On $\partial\Omega^+\backslash\partial\Omega_1^+$ the null space of β_- is composed of vectors satisfying the Dirichlet condition (35), which is imposed on that boundary arc. The null space of β_+ is composed of vectors satisfying the adjoint condition (36). On $\partial\Omega_1^+$, this relation is reversed. In order to show that the direct sum of these null spaces spans the two-dimensional space $\mathcal{V}_{|\partial\Omega^+}$, it is sufficient to show that the set

$$\left\{ \begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -Kn_1/n_2 \end{pmatrix} \right\}$$

is linearly independent there. Setting

$$c_1 \begin{pmatrix} 1 \\ n_2/n_1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -Kn_1/n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we find that $c_1 = -c_2$ and

$$-c_2 \left(\frac{n_2^2 + Kn_1^2}{n_1 n_2} \right) = 0. \tag{40}$$

Equation (40) can only be satisfied on the elliptic boundary if $c_2 = 0$, implying that $c_1 = 0$. Thus the direct sum of the null spaces of $\beta_\pm$ on $\partial\Omega^+$ is linearly independent and must span $\mathcal{V}$ over that portion of the boundary.

On $\partial\Omega\backslash\partial\Omega^+$, the null space of β_- contains every 2-vector and the null space of β_+ contains only the zero vector; so on that boundary arc, their direct sum spans $\mathcal{V}$.

On $\partial\Omega^+\backslash\partial\Omega_1^+$, the range $\mathfrak{R}_+$ of β_+ is the subset of the range $\mathfrak{R}$ of β for which

$$v_2 n_1 - v_1 n_2 = 0 \tag{41}$$

for $(v_1, v_2) \in \mathcal{V}$; the range $\mathfrak{R}_-$ of β_- is the subset of $\mathfrak{R}$ for which

$$K v_1 n_1 + v_2 n_2 = 0 \tag{42}$$

for $(v_1, v_2) \in \mathcal{V}$. Analogous assertions hold on $\partial\Omega_1^+$, in which the ranges of $\mathfrak{R}_+$ and $\mathfrak{R}_-$ are interchanged. Because if n_1 and n_2 are not simultaneously zero the system (41), (42) has only the trivial solution $v_2 = v_1 = 0$ on $\partial\Omega^+$, we conclude that $\mathfrak{R}_+ \cap \mathfrak{R}_- = \{0\}$ on $\partial\Omega^+$.

On $\partial\Omega\backslash\partial\Omega^+$, $\mathfrak{R}_- = \{0\}$, so $\mathfrak{R}_+ \cap \mathfrak{R}_- = \{0\}$ trivially.

The invertibility of E under condition (31) completes the proof of Theorem 3.

Remarks. *i*) Although the conditions of Theorem 3 are derived from purely mathematical considerations, mixed boundary value problems do arise in various contexts of plasma physics [12]. In any case, by taking $\partial\Omega_1^+$ to be either the empty set or all of $\partial\Omega^+$, Theorem 3 implies the existence of strong solutions for either the open Dirichlet problem or the open Neumann problem for the cold plasma model. Because only the open Dirichlet problem was considered in Theorem 9 of [23], Theorem 3 of this note extends that result to the open cases of the Neumann and mixed Dirichlet-Neumann problems.

ii) A misprint in eq. (45) of [23] has been corrected in eq. (29). In Theorem 3, condition (31) has been added to the list of hypotheses in Theorem 9 of [23], the redundant condition (57) removed, and an error in eq. (59) corrected by eq. (34) of the present note. The assumption that the boundary is piecewise smooth, which was default hypothesis in [23], seems to be too weak in general for strong solutions; see, however, [13], [14], [29], [30], and [31].

iii) Only conditions (32) and (33) have anything to do with the cold plasma model. Otherwise, Theorem 3 is about interpreting Friedrichs' theory in the context of starlike boundaries. For example, the argument leading to eq. (40) suggests that the Tricomi problem is strongly ill-posed under the hypotheses of the theorem, whatever the type-change function K. This is because in the Tricomi problem, data are given on both the elliptic boundary and a characteristic line; but on characteristic lines, K satisfies

$$K = -\frac{n_2^2}{n_1^2}. \tag{43}$$

Substituting this equation into eq. (40), we find that the equation is satisfied on characteristic lines without requiring the constants c_1 and c_2 to be zero.

However, the theorem is less restrictive if the operator in (16) is given by

$$(L\mathbf{u})_1 = [x - \sigma(y)]\, u_{1x} - u_{2y} + \kappa_1 u_1 + \kappa_2 u_2,$$
$$(L\mathbf{u})_2 = -u_{1y} + u_{2x}, \tag{44}$$

where, again, κ_1 and κ_2 are constants. This variant also arises in the cold plasma model (see [21] and [28]) and is analogous to the variant of the Tricomi equation $yu_{xx} - u_{yy} = 0$, studied in various contexts by Friedrichs [6], Katsanis [11], Sorokina [31], [32], and Didenko [5]. In that case, choose

$$E = \begin{pmatrix} b & cK \\ c & b \end{pmatrix}.$$

Obvious modifications of conditions (32) and (33) guarantee that the equation

$$EL\mathbf{u} = E\mathbf{f}$$

will be symmetric positive. Condition (31) must be replaced by the invertibility condition

$$b^2 - c^2 K \neq 0,$$

which is restrictive on the subdomain Ω^+ rather than on Ω^- as in (31). Most importantly, the discussion leading to Table 1 of [11] now applies, with only minor changes, and one can obtain a long list of possible starlike boundaries which result in strong solutions to suitably formulated problems of Dirichlet or Neumann type. In particular, one can formulate a Tricomi problem which is strongly well-posed.

The hypotheses of Theorem 3 are rather formal. We expect them to be harsh, as the known singularity at the origin should restrict the kind of smoothness results that we can prove. But the hypotheses do not seem to be vacuous, with the possible exception of condition (34). Thus we show that in fact condition (34) is always satisfied on the characteristic boundary.

Proposition 4. *Let Γ be a characteristic line for eq. (16), with the higher-order terms of the operator L satisfying (25). Then the left-hand side of inequality (34) is identically zero on Γ.*

Proof. We have, using eq. (43),

$$(cKn_1 + bn_2)^2 = c^2K^2n_1^2 + 2Kcbn_1n_2 + b^2n_2^2$$

$$= -c^2K^2\frac{n_2^2}{K} + 2Kcbn_1n_2 - b^2Kn_1^2 = -K\left(c^2n_2^2 - 2cbn_1n_2 + b^2n_1^2\right)$$

$$= -K\,(cn_2 - bn_1)^2 \,.$$

Substituting the extreme right-hand side of this equation into the second term of (34) completes the proof.

We conclude that condition (34) can be interpreted as the geometric requirement that the hyperbolic boundary be subcharacteristic.

On the basis of the observations in Remark *iii)*, we reformulate Theorem 3 for an arbitrary type-change function − that is, for a smooth function $K(x, y)$ which is positive on all points in a subset Ω^+ of Ω, negative on all points of a subset Ω^- of Ω, and zero on a smooth curve $\mathfrak{G} \in \Omega$ separating Ω^+ and Ω^-, where

$$\Omega = \Omega^+ \cup \Omega^- \cup \mathfrak{G}.$$

The proof of Theorem 3 will also prove:

Corollary 5. *Let Ω be a bounded, connected domain of $\mathbb{R}^2$ having C^2 boundary $\partial\Omega$, oriented in a counterclockwise direction. Let $\partial\Omega_1^+$ be a (possibly empty and not necessarily proper) subset of $\partial\Omega^+$. Suppose that EL is a symmetric positive operator, where L satisfies (25) (with the possible addition of lower-order terms) and E satisfies (37) with condition (31). Let $\partial\Omega^+\backslash\partial\Omega_1^+$ be starlike with respect to the vector field $V^+ = -(b(x,y), c(x,y))$; let $\partial\Omega_1^+$ be starlike with respect to the vector field $V_1^+ = (b(x,y), c(x,y))$; let $\partial\Omega\backslash\partial\Omega^+$ be starlike with respect to the vector field $V^- = (b(x,y), -c(x,y))$. Let the union $\partial\Omega\backslash\partial\Omega^+$ of the parabolic and hyperbolic boundaries be subcharacteristic in the sense of (34). Then the mixed boundary value problem given by eq. (16), with condition (35) satisfied on $\partial\Omega^+\backslash\partial\Omega_1^+$ and condition (36) satisfied on $\partial\Omega_1^+$, possesses a strong solution for every $\mathbf{f} \in L^2(\Omega)$.*

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