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ROUTING NUMBERS OF CYCLES, COMPLETE BIPARTITE GRAPHS, AND HYPERCUBES*

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Abstract. The routing number $rt(G)$ of a connected graph G is the minimum integer r so that every permutation of vertices can be routed in r steps by swapping the ends of disjoint edges. In this paper, we study the routing numbers of cycles, complete bipartite graphs, and hypercubes. We prove that $rt(C_n) = n - 1$ (for $n \geq 3$) and for $s \geq t$, $rt(K_{s,t}) = \lfloor \frac{3s}{2t} \rfloor + O(1)$. We also prove $n + 1 \leq rt(Q_n) \leq 2n - 2$ for $n \geq 3$. The lower bound $rt(Q_n) \geq n + 1$ was previously conjectured by Alon, Chung, and Graham [*SIAM J. Discrete Math.*, 7 (1994), pp. 513–530]. A variation, called fractional routing number, is also considered in this paper.

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1. Introduction.

The problem of routing permutations over graphs arose in different fields [2, 3, 6, 7], such as the study of communicating processes on networks, the data flow on parallel computation, and the analysis of routing algorithms on VLSI chips. This problem can be described as follows: Let $G = (V, E)$ be a connected graph with vertices $\{v_1, v_2 \dots v_n\}$ and π be a permutation on $[n]$. Initially, each vertex v_i of G is occupied by a "pebble." The pebble on v_i will be labeled as p_j if $\pi(i) = j$. Pebbles can be moved around by the following rule. At each step a disjoint collection of edges of G is selected, and the pebbles at each edge's two endpoints are interchanged. The goal is to move/route each pebble p_i to its destination v_i. Define $rt(G, \pi)$ to be the minimum number of steps to route the permutation π. Finally, define $rt(G)$ the routing number of G by

$$rt(G) = \max_{\pi} \; rt(G, \pi).$$

Determining the routing number of a graph is a quite difficult problem. For a few kinds of special graphs, the routing numbers are known in the literature [4]. For the path P_n ($n \geq 3$), $rt(P_n) = n$; for the complete graph K_n ($n \geq 3$), $rt(K_n) = 2$; for the complete bipartite graph $K_{n,n}$ ($n \geq 3$), $rt(K_{n,n}) = 4$; and for the star S_n ($n \geq 3$), $rt(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$.

Except for those exact results, some upper/lower bounds were obtained for certain graphs. A trivial lower bound is

$$rt(G) \geq D(G).$$

Here $D(G)$ is the diameter of G. If G has a connected spanning subgraph H, then

$$rt(G) \leq rt(H).$$

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In particular, we have

$$rt(G) \leq rt(T_n),$$

where T_n is a spanning tree of G. Alon, Chung, and Graham [1] proved

$$rt(T_n) \leq 2n,$$

and they conjectured that

$$rt(T_n) \leq \left\lfloor \frac{3(n-1)}{2} \right\rfloor,$$

and equality holds if and only if T_n is a star or P_4. This conjecture is still open. However, Zhang [8] proved that it is asymptotically correct by showing

$$rt(T_n) \leq \left\lfloor \frac{3n}{2} \right\rfloor + O(\log n).$$

Alon, Chung, and Graham gave an upper bound for the routing number of complete bipartite graph $K_{s,t}$, $s \geq t$,

$$rt(K_{s,t}) \leq 2 \left\lceil \frac{s}{t} \right\rceil + 2.$$

An important class of graphs is the hypercube Q_n, which is one of the most popular topological models in parallel computing. The previously best known bound for $rt(Q_n)$ is

$$n \leq rt(Q_n) \leq 2n - 1.$$

The lower bound comes from $rt(Q_n) \geq D(Q_n) = n$. The upper bound is a consequence of the following result on the Cartesian product of two graphs [1].

$$rt(G_1 \square G_2) \leq 2rt(G_1) + rt(G_2). \tag{1.1}$$

Alon, Chung, and Graham [1] conjectured $rt(Q_n) \geq n + 1$ for $n \geq 2$. Here we settle this conjecture by showing the following theorem.

THEOREM 1.1. *Let Q_n be the n-cube, then for $n \geq 3$*

$$n + 1 \leq rt(Q_n) \leq 2n - 2.$$

The improvements of both upper bound and lower bound are very limited. Alon, Chung, and Graham have a stronger conjecture $rt(Q_n) = n + O(1)$ [1]. Our result is just the first step toward this conjecture.

Furthermore, we have the following result on the cycle C_n.

THEOREM 1.2. *Let C_n be a cycle of length n, then*

$$rt(C_n) = n - 1.$$

We have the following theorem for the routing number of the complete bipartite graph $K_{s,t}$ with $s \geq t$.

THEOREM 1.3. *For the complete bipartite graph $K_{s,t}$ with $s \geq t \geq 1$,*

$$\left\lfloor \frac{3s}{2t} \right\rfloor - 1 \leq rt(K_{s,t}) \leq \left\lfloor \frac{3s}{2t} \right\rfloor + 7.$$

In other words, $rt(K_{s,t}) = \lfloor \frac{3s}{2t} \rfloor + O(1)$.

We will prove several lemmas on the lower bounds for the routing numbers of some graphs in section 2. The proofs of theorems are in section 3. A variation, called fractional routing number, is given in section 4. It deepens our understanding of various lower bounds for the routing numbers. In the last section, we conclude the paper by posing some problems.

2. Lower bounds for routing numbers. In this section, we will study lower bounds for routing numbers. Let C_n be a cycle on n vertices. The vertices are labeled as $v_1, v_2 \ldots, v_n$ in the cyclic order so that the edges are $e_i = v_i v_{i+1}$ for $i = 1, \ldots, n-1$ and $e_n = v_n v_1$.

LEMMA 2.1. *For the cycle C_n on n vertices, we have*

$$rt(C_n) \geq n - 1.$$

Proof. Let ρ be the cyclic rotation $(123 \cdots n)$ on $[n]$. We claim $rt(C_n, \rho) \geq n-1$.

Otherwise, suppose we can route ρ in at most $n-2$ steps. Let R be such a routing process. We say a pebble has a forward move if it is on v_i and moved to v_{i+1}, or it is on v_n and moved to v_1. By a backward move, we mean that a pebble is moved from v_{i+1} to v_i, or from v_1 to v_n. For $1 \leq i \leq n$, define $R^+(p_i)$ to be the number of forward moves of p_i and $R^-(p_i)$ to be the number of backward moves of p_i. For $1 \leq i \leq n$, we have

$$R^+(p_i) - R^-(p_i) \equiv 1 \mod n. \tag{2.1}$$

By assumption, this routing process takes at most $n-2$ moves. We have

$$0 \leq R^+(p_i) \leq n-2,$$

$$0 \leq R^-(p_i) \leq n-2,$$

$$-(n-2) \leq R^+(p_i) - R^-(p_i) \leq n-2. \tag{2.2}$$

Equations (2.1) and (2.2) together imply for $1 \leq i \leq n$,

$$R^+(p_i) - R^-(p_i) = 1. \tag{2.3}$$

Thus $\sum_{i=1}^{n} (R^+(p_i) - R^-(p_i)) = n$.

On the other hand, each swap is a forward move of one pebble but also a backward move of another pebble, so we have

$$\sum_{i=1}^{n} (R^+(p_i) - R^-(p_i)) = 0.$$

Contradiction. ∎

This idea can be generalized to obtain a lower bound for the routing number of the Cartesian product of two graphs. For two graphs $G = (V, E)$ and $G' = (V', E')$, the Cartesian product $G \square G'$ is the graph with vertex set $V \square V' = \{(v, v') | v \in V, v' \in V'\}$ and with $(u, u')(v, v')$ as an edge of $G \square G'$ if and only if either $u = v, u'v' \in E'$ or $u' = v', uv \in E$. A graph G is called an m-routing graph if there is a permutation σ such that the distance between each pebble p_i and its destination v_i is at least m.

LEMMA 2.2. *Let G be an m-routing graph, then*

$$rt(C_n \square G) \geq n + m - 1.$$

Proof. We can picture $C_n \square G$ as an array with each row spanning a copy of C_n and each column spanning a copy of G. Let π be the desired routing permutation such that each pebble at vertex (c, g) has the destination $(\rho(c), \sigma(g))$, where $\rho = (123 \cdots n)$ is the cyclic rotation of C_n and σ is the permutation of G in the definition of m-routing graph.

Just as in the proof of Lemma 2.1, there is a pebble moved at least $n - 1$ steps in horizontally. From $d(f, \sigma(f)) \geq m$, we need at least m vertical steps to route it. So totally we need $n + m - 1$ steps. ∎

3. Proof of theorems.

Proof of Theorem 1.1. Note that $Q_n = C_4 \square Q_{n-2}$ and Q_{n-2} is an $n-2$-routing graph. By Lemma 2.2, we have

$$rt(Q_n) \geq 4 + (n - 2) - 1 = n + 1.$$

The upper bound is due to a computational result (see the concluding remarks of the last section)

$$rt(Q_3) = 4.$$

This is done by an exhausting computer search. Applying (1.1), we have

$$\begin{aligned} rt(Q_n) &\leq rt(Q_{n-1}) + 2 \\ &\leq \cdots \\ &\leq rt(Q_3) + 2(n-3) \\ &= 4 + 2(n-3) \\ &= 2n - 2. \end{aligned}$$ ∎

The following algorithm is essentially the same as the *odd-even transposition sort* in parallel sorting networks (see [5] for a comprehensive survey). We rephrase it with our terminologies.

Denote the vertices of P_n by $v_1, \ldots, v_n$ in order and each edge $v_i v_{i+1}$ by e_i for $i = 1, \ldots, n-1$. We call e_i an odd edge if i is odd, and an even edge provided i is even. We interchange two pebbles on edge e_i if the pebble on v_i has an index greater than that of the pebble on v_{i+1}. In odd steps, we only exchange the pebbles on odd edges, while in even steps we exchange the pebbles on even edges. We exchange the pebbles on odd edges and even edges in every other steps.

This method routes any permutation of pebbles on P_n in at most n steps.

However, it is natural to ask, when do we need n steps to route a permutation of pebbles on P_n? Can it be done by less steps? The following lemma gives a sufficient condition of routing a permutation on P_n in at most $n - 1$ steps.

LEMMA 3.1. *Given a permutation π on $[n]$, if p_1 is on v_i and p_n is on v_j for some $i < j$, then $rt(P_n, \pi) \leq n - 1$.*

Proof. Let $S_i = p_{i_1} p_{i_2} \cdots p_{i_n}$ be the list of pebbles on $v_1, \ldots, v_n$ after we finished step i, where S_0 is just the initial distribution of pebbles when π is given. There will be $k + 1$ lists if we route π in k steps. For each S_i, $i \geq 1$, if p_1 is on v_ℓ and p_n is on

v_m, define T_{i-1} to be the following list.

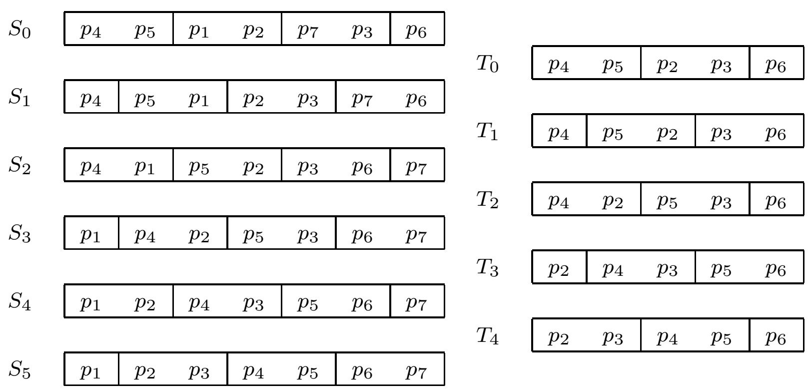
$$T_{i-1} = \underbrace{p_{(i-1)_1} \cdots p_{(i-1)_{\ell-1}}}_{\text{from } S_{i-1}} \underbrace{p_{i_{\ell+1}} \cdots p_{i_{m-1}}}_{\text{from } S_i} \underbrace{p_{(i-1)_{m+1}} \cdots p_{(i-1)_n}}_{\text{from } S_{i-1}} \text{ for } i = 1, \ldots, k.$$

In other words, we concatenate the first $\ell - 1$ pebbles and the last $n - m$ pebbles in S_{i-1} together to the middle pebbles $p_{i_{\ell+1}} \cdots p_{i_{m-1}}$ in S_i, forming a new list of pebbles.

For example, consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 1 & 2 & 7 & 3 & 6 \end{pmatrix}.$$

Note that p_1 is on v_3 and p_7 is on v_5. T_i's are shown in the following diagram.



Now we claim that

(1) Each T_i contains exactly the $n - 2$ pebbles $p_2, p_3, \ldots, p_{n-1}$.

(2) The lists $T_0, T_1, ..., T_{k-1}$ are corresponding to the process of routing $n-2$ pebbles on the path P_{n-2} using the odd-even transposition.

The following arguments we discuss about p_1 all work for p_n as well.

To show (1), notice that in the list S_i, the set $\{p_{i_1}, \ldots, p_{i_n}\} \backslash \{p_{i_\ell}, p_{i_m}\}$ contains exactly the pebbles $p_2, \ldots, p_{n-1}$ by our assumption. Since p_1 is placed on v_ℓ in S_i, at step i either we swap p_1 on $v_{\ell+1}$ and some pebble v_ℓ, or p_1 is already on v_ℓ and we do not swap it with the pebble on $v_{\ell+1}$. If a pebble is on v_j for some $j \leq \ell$, it cannot "jump" over v_ℓ at step i. Thus, $\{p_{i_1}, \ldots, p_{i_{\ell-1}}\}$ should be the same as $\{p_{(i-1)_1}, \ldots, p_{(i-1)_{\ell-1}}\}$. Hence T_i contains exactly the $n - 2$ pebbles $p_2, p_3, \ldots, p_{n-1}$.

For (2), if at step i p_1 is moved from $v_{\ell+1}$ to v_ℓ, it means we are exchanging pebbles on edges $v_{\ell-2}v_{\ell-1}$, $v_{\ell-4}v_{\ell-3}, \ldots.$ in S_{i-1}, which are $p_{(i-1)_{\ell-2}}p_{(i-1)_{\ell-1}}$, $p_{(i-1)_{\ell-4}}p_{(i-1)_{\ell-3}}, \ldots.$ Thus, at step $i+1$, we should exchange pebbles on $v_{\ell+1}v_{\ell+2}$, $v_{\ell+3}v_{\ell+4}, \ldots.$ in S_i, which are $p_{i_{\ell+1}}p_{i_{\ell+2}}$, $p_{i_{\ell+3}}p_{i_{\ell+4}}, \ldots$ So, from T_{i-1} to T_i, we perform the exchange of these pairs of pebbles $p_{(i-1)_{\ell-2}}p_{(i-1)_{\ell-1}}$, $p_{(i-1)_{\ell-4}}p_{(i-1)_{\ell-3}}, \ldots,$ and $p_{i_{\ell+1}}p_{i_{\ell+2}}$, $p_{i_{\ell+3}}p_{i_{\ell+4}}, \ldots.$ This is corresponding to the process of routing the pebbles $p_2, \ldots, p_{n-1}$.

Since it takes at most $n - 2$ steps to route any permutation on P_{n-2}, we have $k - 1 \leq n - 2$. Hence routing π on P_n takes at most $k \leq n - 1$ steps. $\square$

Since P_n is a spanning subgraph of C_n, we know $rt(C_n) \leq rt(P_n)$. Using the extra edge in C_n not in P_n, we are ready to prove our second theorem.

Proof of Theorem 1.2. The lower bound is proved by Lemma 2.1. It suffices to prove

$$rt(C_n) \le n - 1.$$

Given a permutation π, if p_1 is on v_i and p_n is on v_j with $i < j$, then we delete the edge e_n to get the path $v_1 v_2 \dots v_n$. According to Lemma 3.1, we can route π in $n-1$ steps. If not, let us cut another edge $e_k = v_k v_{k+1}$ and observe the pebbles p_k and p_{k+1}. If on the path $v_{k+1} v_{k+2} \dots v_n v_1 \dots v_k$, pebble p_k is on the right side of p_{k+1}, relabel the indices of pebbles and vertices by applying the permutation $\sigma : i \to i - k \mod n$. Let $q_i = p_{\sigma(i)}$ and $u_i = v_{\sigma(i)}$. Apply the odd-even transposition on q_i's and $u_1 \dots u_n$; we can route the pebbles in $n-1$ steps, but this is exactly the same as routing π. Thus, the only possible case that we might need n steps to route π is no matter what edge we cut to produce a path $v_{k+1} v_{k+2} \dots v_n v_1 \dots v_k$, p_k always appears on the left side of p_{k+1}.

Now assume that at the beginning each p_k is on v_{i_k}. Define d_k to be the length of the directed path $v_{i_k} v_{i_k+1} \dots v_k$ on C_n. By the previous conclusion, $d_k \ge d_{k+1}$ for all $k = 1, \dots n-1$, and $d_n \ge d_1$. Hence, $d_1 = d_2 = \dots = d_n = d$ for some $1 \le d \le n-1$. This tells us that π is sending i to $d + i \mod n$ and hence pebble p_{d+i} is on v_i; the indices are subject to modulo n. Therefore, the labeling, induced by π, of the pebbles on $v_1 v_2 ... v_n$ in order is $p_{d+1} p_{d+2} \dots p_n p_1 \dots p_d$. Let us call such kind of permutation a d-rotation.

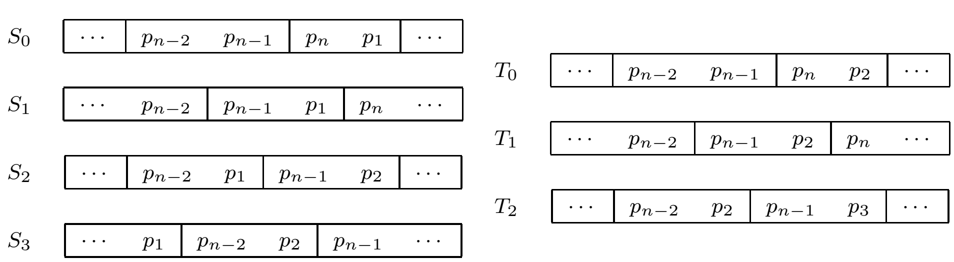
If we use the odd-even transposition to route π starting from swapping the pebbles on odd edges, then we cannot do any swap whenever $n - d$ is even. Let us modify the odd-even transposition in the following way in order to reduce one step.

For a d-rotation $\pi\colon i \to d + i \mod n$ for $1 \le i \le n$, we always start the odd-even transposition on edges of the same parity as $n - d$. Thus, the first step is to swap p_n and p_1 on the edge $v_{n-d} v_{n-d+1}$. After the first step, we swap pebbles on odd edges and even edges alternatively.

We show that the above modified odd-even transposition can route π in $n-1$ steps.

For $n = 3$, the initial distributions of pebbles are $p_1 p_2 p_3$, $p_2 p_3 p_1$, or $p_3 p_1 p_2$ if π is a d-rotation. One can easily verify that every pebble can reach its destination in two steps by using the modified odd-even transposition.

Suppose we now need k steps to route a rotation on P_n. Let $S_0, \dots, S_k$ be the lists of pebbles on $v_1, v_2, \dots, v_n$ as we defined in the proof of Lemma 3.1 after we applied the modified odd-even transposition on π. Again, if p_1 is on v_ℓ after step i, then concatenate the pebbles $p_{(i-1)_1} \cdots p_{(i-1)_{\ell-1}}$ to $p_{i_{\ell+1}} \cdots p_{i_n}$, forming a new list T_{i-1}. The new lists $T_0, \dots, T_{k-1}$ show the process of routing a rotation on P_{n-1}.



By induction hypothesis, we can show that $k - 1 \le n - 2$. Therefore, $k \le n - 1$. $\square$

Proof of Theorem 1.3. Denote the two node sets of $K_{s,t}$ by X and Y such that $|X| = s$ and $|Y| = t$. Given a permutation π, let A be the set of pebbles in X whose destinations are in Y and B be the set of pebbles in Y whose destinations are in X. We always have $|A| = |B|$. If both A and B are empty, do nothing. Otherwise, take one step to swap pebbles in A and B via a perfect matching between them. After this step, all pebbles in X will stay in X and all pebbles in Y will stay in Y.

Now route the pebbles in X first. The pebbles in X induce a permutation π' on X. Let $C_1 \circ C_2 \circ \cdots \circ C_m$ be the disjoint cycle decomposition of π'. When $C_1 \circ C_2 \circ \cdots \circ C_m$ is chosen, we pose a method to route the pebbles in X in $d+1$ steps. The number d will be determined later ($d = \lceil \frac{3s}{2t} \rceil$). In each step of routing we swap some pebbles in X with some pebbles in Y. We record the transpositions/swaps of each step in an array as shown below. In this array, $y_1, \ldots, y_t$ are the t vertices of Y; $x_{i,j}$ will be assigned a vertex in X. If a vertex x in X is assigned to $x_{i,j}$, it means that in step j we swap the pebbles on x and y_i. Note that we should avoid assigning x to both $x_{i_1,j}$ and $x_{i_2,j}$ for $i_1 \neq i_2$, since we cannot swap the pebble on x with the pebbles on y_{i_1} and y_{i_2} simultaneously at step j.

Step 1	Step 2	$\cdots$	Step d	Step $d+1$
$(x_{1,1}y_1)$ $(x_{2,1}y_2)$ $(x_{t,1}y_t)$	$(x_{1,2}y_1)$ $(x_{2,2}y_2)$ $(x_{t,2}y_t)$	$\cdots$ $\cdots$ $\cdots$	$(x_{1,d}y_1)$ $(x_{2,d}y_2)$ $(x_{t,d}y_t)$	$(x_{1,d+1}y_1)$ $(x_{2,d+1}y_2)$ $(x_{t,d+1}y_t)$

We arrange the entries of cycles $C_1, \ldots, C_m$ into the entries $x_{1,1}$, $x_{1,2}$, $x_{1,3} \ldots$, accordingly so that we can route the cycles by doing the indicated transpositions in each step. Start the procedure by the first row. For a cycle $C = (a_1 a_2 \cdots a_n)$, if $x_{i,j}$, $j \leq d$, is not assigned yet, then we assign a_1 to $x_{i,j}$, a_2 to $x_{i,j+1}$, and so on. If a_n is assigned to $x_{i,m}$, $m \leq d$, then we assign a_1 to $x_{i,m+1}$ again. If some a_k, $k < n$, is assigned to $x_{i,d}$, then we skip $x_{i,d+1}$ and go to the $(i+1)$th row. In addition, we have to assign a_k to $x_{i+1,1}$ again, namely, both $x_{i,d}$ and $x_{i+1,1}$ are a_k.

When $|C| \equiv 0 \bmod (d-1)$, using the above method leads to the following problem. The first a_1 is assigned to $x_{i_1,j}$ and the other a_1 is in another row but the same column $x_{i_2,j}$, $i_1 \leq i_2$. This is not allowed since the transpositions in the same column must be disjoint. For example, for $d = 6$ and $C = (a_1 a_2 \ldots a_{10})$, we have $|C| = 10$, which is divisible by $d - 1 = 5$. (See the following array.)

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7
			$(\mathbf{a_1 y_1})$	$(a_2 y_1)$	$(a_3 y_1)$	
$(a_3 y_2)$	$(a_4 y_2)$	$(a_5 y_2)$	$(a_6 y_2)$	$(a_7 y_2)$	$(a_8 y_2)$	
$(a_8 y_3)$	$(a_9 y_3)$	$(a_{10} y_3)$	$(\mathbf{a_1 y_3})$			

If this happens, we must have assigned a_{d-j+1} to $x_{i,d}$ and $x_{i+1,1}$ beforehand. We now solve this problem using $x_{i,d+1}$. We assign a_{d-j+1} to $x_{i,d}$, then a_{d-j+2} to $x_{i,d+1}$ and $x_{i+1,1}$. For the remaining a_i's, we still follow previous rules. Here is the solution of the previous example.

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7
			$(a_1 y_1)$	$(a_2 y_1)$	$(a_3 y_1)$	$(a_4 y_1)$
$(a_4 y_2)$	$(a_5 y_2)$	$(a_6 y_2)$	$(a_7 y_2)$	$(a_8 y_2)$	$(a_9 y_2)$	
$(a_9 y_3)$	$(a_{10} y_3)$	$(a_1 y_3)$				

Since all cycles are routed by this procedure, permutation π' is routed.

Let us determine how small d could be. Each entry a_i in the cycle $(a_1 a_2 \cdots a_n)$ appears either once or twice in the array. The entry a_1 must appear twice. If an entry a_k (not a_1 and a_n) in a cycle is assigned to $x_{i,d}$, then it is assigned to $x_{i+1,1}$ as well. The number of this kind of entry cannot exceed t. Hence we assign at most $|C_1| + \cdots + |C_m| + m + t$ entries to $t(d+1)$ entries $x_{i,j}$'s. Furthermore, $|C_1| + \cdots + |C_m| + m + t \le |X| + \frac{|X|}{2} + t \le \frac{3s}{2} + t$. So we have the inequality,

$$t(d+1) \ge \frac{3s}{2} + t.$$

This means that when $d = \lceil \frac{3s}{2t} \rceil$ the above procedure works. Hence the number of steps needed to route π' is no more than $\lceil \frac{3s}{2t} \rceil + 1$. Finally, we route the pebbles on Y, which have not been moved to their destinations. Select any t routed pebbles in X together with the pebbles in Y which induce a permutation on $K_{t,t}$. It can be routed in four steps (see [1]). Therefore $rt(K_{s,t}) \le 1 + \lceil \frac{3s}{2t} \rceil + 5 \le \lfloor \frac{3s}{2t} \rfloor + 7$.

Consider the permutation $\sigma = (x_1 x_2)(x_3 x_4) \cdots (x_{2\lfloor \frac{s}{2} \rfloor - 1} x_{2\lfloor \frac{s}{2} \rfloor})$. For each $(x_{2i-1} x_{2i})$, we need at least three transpositions to route it. (We need the pebbles on x_{2i-1} and x_{2i} leaving their original places and also entering their destinations. One can only save one move by moving the pebble $p_{x_{2i}}$ on x_{2i-1} out and moving the pebble $p_{x_{2i-1}}$ into x_{2i-1} at the same time.) Therefore, it takes at least $\lceil \frac{3\lfloor \frac{s}{2} \rfloor}{t} \rceil$ steps. Since $1 + \lceil \frac{3s-3}{2t} \rceil \ge 1 + \frac{3s-3}{2t} \ge \frac{3s}{2t}$, we have $1 + \lceil \frac{3s-3}{2t} \rceil \ge \lceil \frac{3s}{2t} \rceil$. Thus, $\lceil \frac{3\lfloor \frac{s}{2} \rfloor}{t} \rceil \ge \lceil \frac{3s-3}{2t} \rceil \ge \lceil \frac{3s}{2t} \rceil - 1 \ge \lfloor \frac{3s}{2t} \rfloor - 1$. This gives the lower bound for $rt(K_{s,t})$. ∎

4. Fractional routing numbers. We can relax the routing number $rt(G)$ to the fractional routing number $rt'(G)$ as follows. We assume that all pebbles have mass 1 and can be split into smaller pieces during the routing process. A piece from pebble p_i is said to have a type p_i. After reaching its destination, all pieces of type p_i can be assembled into the pebble p_i. The pieces can be exchanged through a fractional matching at one step. A fractional matching is a mapping

$$f\colon E(G) \to [0,1]$$

satisfying for any v, $\sum_{u \in \Gamma(v)} f(uv) \le 1$. Recall that $\Gamma(v)$ is the set of neighbors of v in G. For each edge uv, pieces of total mass $f(uv)$ at u can be exchanged with pieces of the same total mass $f(uv)$ at v. Given a permutation π, the pebble on v_i will be labeled as p_j if $\pi(i) = j$. The minimum number of steps to route each pebble p_i to v_i is denoted by $rt'(G, \pi)$. Finally, we define $rt'(G)$, the fractional routing number of G, by

$$rt'(G) = \max_{\pi} \; rt'(G, \pi).$$

Since every matching is a fractional matching, we have

$$rt'(G) \le rt(G). \tag{4.1}$$

Using this relation and the fact that $rt(K_n) = 2$, we can get $rt'(K_n) = 2$ for $n \ge 3$.

Many lower bounds for routing numbers are actually lower bounds for the fractional routing numbers. For example,

$$rt'(G) \ge D(G).$$

LEMMA 4.1. *Suppose C is a vertex-cut of a connected graph G, and the largest component after removing C has at most $\frac{n-|C|}{2}$ vertices. Then,*

$$rt'(G) \geq \frac{n}{|C|}. \tag{4.2}$$

Alon, Chung, and Graham [1] proved

$$rt(G) \geq \frac{2}{|C|} \min\{|A|, |B|\}, \tag{4.3}$$

where $A \cup B$ is a partition of $V(G) \setminus C$ so that there is no edge between A and B. The right hand side of inequality (4.3) reaches the maximum when $|A| = |B| = \frac{n-|C|}{2}$. In this case, inequality (4.3) gives $rt(G) \geq \frac{n}{|C|} - 1$. Lemma 4.1 gives a slightly better lower bound $rt'(G) \geq \frac{n}{|C|}$.

LEMMA 4.2. *Let $\cup_{i=1}^r S_i$ be a partition of a set S ($|S| \geq 2$) satisfying*

$$|S_i| \leq \frac{|S|}{2} \quad \textit{for all } i.$$

Then there is a permutation π on S so that for any $v \in S$, $\pi(v)$ and v are in different blocks.

Proof. Relabel the vertices if necessary. Without loss of generality, we can assume that the sets S_i consist of consecutively numbered vertices (e.g., $S_1 = \{1, 2, \dots, i_1\}$, $S_2 = \{i_1 + 1, i_1 + 2, \dots, i_2\}, \dots$, etc.) Consider the permutation

$$\pi\colon i \to i + \left\lfloor \frac{|S|}{2} \right\rfloor \mod |S|.$$

We claim i and $\pi(i)$ are not in the same block. If not, there is a block S_j containing both i and $\pi(i)$. Then S_j contains the interval $[i, \pi(i)]$ if $i < \pi(i)$, or the interval $[\pi(i), i]$ if $i > \pi(i)$. In either case, it contradicts $|S_i| \leq \frac{|S|}{2}$. ☐

Proof of Lemma 4.1. Suppose that the connected components of $G \setminus C$ are $C_1, C_2, \dots, C_r$. By Lemma 4.2, there exists a permutation σ on $C_1 \cup C_2 \cup \cdots \cup C_r$ so that for any vertex v, $\sigma(v)$ and v are always in different components. Extend this permutation σ over $V(G)$ so that the vertices in C are fixed points. Any path from v to $\sigma(v)$ must go through some vertex in C. The total mass of such pieces to get in and get out of C is at least $2(n - |C|)$. For any routing process of σ, at each step the total mass of such pieces that can get in and get out of C is at most $2|C|$. We also observe that at the first step and the last step, the amount of such pieces can get in and get out of C is at most $|C|$. Thus the number of steps is at least

$$\frac{2(n - |C|)}{2|C|} + 1 = \frac{n}{|C|}. \qquad ☐$$

THEOREM 4.3. *For any tree T on $n \geq 3$ vertices, we have*

$$rt'(T) \geq n.$$

Proof. For a vertex u, let $f(u)$ be the maximum size of connected components after deleting u from T. Let v be a vertex so that $f(v)$ reaches the minimum value. There are two cases.

Case 1. $f(v) \leq \frac{n-1}{2}$. Applying inequality (4.2) with $C = \{v\}$, we have $rt'(T) \geq n$.

Case 2. $f(v) \geq \frac{n}{2}$. Let u be the neighbor of v so that the component containing u is of size $f(v)$. Among the components of $T \setminus \{u\}$, the one containing v has at most $\frac{n}{2}$ vertices. All other components have at most $f(v) - 1$ vertices. Since $f(u) \geq f(v)$, we have $f(u) \leq \frac{n}{2}$. In particular, we have $f(v) = f(u) = \frac{n}{2}$, and n is even. So the edge uv cuts the tree T into two subtrees (denoted by T_u and T_v) of an equal size. Let π be a permutation of $V(T)$ that maps any vertex of $T_u \setminus \{u\}$ to a vertex of $T_v \setminus \{v\}$, $\pi(u) = v$, and $\pi(v) = u$.

Let k be the number of steps in the routing process. For $1 \leq i \leq k$, let a_i be the mass of pieces of types in T_v, which *first* goes through the edge uv at the ith step. At $(i+1)$th step, at least a_i mass at vertex u was from vertex v. By the definition of a_{i+1}, we have

$$a_{i+1} \leq 1 - a_i$$

for any $1 \leq i \leq k-1$. Note the first step and the last step can only route the piece of type v. We also have $a_1 + a_k \leq 1$.

$$\begin{aligned}\frac{n}{2} &\leq \sum_{i=1}^{k} a_i \\ &= \frac{1}{2}\left[(a_1 + a_k) + \sum_{i=1}^{k-1}(a_i + a_{i+1})\right] \\ &\leq \frac{1}{2}k.\end{aligned}$$

This implies $k \geq n$. ∎

COROLLARY 4.4. *For any tree T on $n \geq 3$ vertices, we have*

$$rt(T) \geq n.$$

For $n \geq 3$, we have $rt'(T_n) = n$. For the star S_n, it is known that $rt(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$. Here we show $rt'(S_n)$ is much smaller than $rt(S_n)$.

THEOREM 4.5. *For the star S_n on $n \geq 2$ vertices, we have*

$$rt'(S_n) = n.$$

Proof. Let c be the center of the star S_n and π be any permutation on $V(S_n)$. π can be written as the product of r disjoint circles $C_1 \circ C_2 \circ \cdots C_r$. We also assume the center c is in C_1. Let $\sigma = C_2 \circ \cdots C_r$. We will route pebbles in C_1 first.

Claim (a). The cycle C_1 can be routed in $|C_1| - 1$ steps.

Claim (a) is trivial for $|C_1| = 1$. Now we assume $|C_1| = s + 1$. Relabeling the vertices if needed, we assume $C_1 = (c, 1, 2, \ldots, s)$. First swap p_c and p_1, then p_1 and p_2, and so on. The last step is to swap p_{s-1} and p_s. Claim (a) is proved.

Claim (b). The permutation σ can be routed in $n - |C_1| + 1$ steps.

The vertices outside C_1 are labeled by $s+1, s+2, \ldots, n-1$. The pebbles on these vertices are divided into $n - s - 1$ pieces of equal mass $\frac{1}{n-s-1}$. At the first step, pick one piece at each vertex and swap it with a piece at the center. Now the center contains exactly $n-s-1$ pieces of all different types (other than type c). Each vertex outside C_1 has exactly one piece of type c. At the subsequent $n - s - 2$ steps, for $s + 1 \leq j \leq n - 1$, pick one piece of type j at the center and swap it with a piece of type $\sigma(j)$ at vertex j. The center always contains exactly $n-s-1$ pieces of all different types (other than type c). At the last step, pick a piece of type j at the center and swap it with the piece of type c at vertex j. The number of steps is $n-s=n-|C_1|+1$. The proof of Claim (b) is finished.

Thus for any permutation π, we have

$$rt'(S_n,\pi) \le |C_1|-1+n-|C_1|+1=n.$$

The upper bound $rt'(S_n) \le n$ is proved. The lower bound $rt'(S_n) \ge n$ is obtained by Theorem 4.3. ◻

One can prove the following lemma analogous to Lemma 2.1. Here we omit its proof.

LEMMA 4.6. *For the cycle C_n on n vertices, we have*

$$rt'(C_n) \ge n-1.$$

Thus for the cycles, we have the following theorem.

THEOREM 4.7. *For the cycle C_n on $n \ge 3$ vertices, we have*

$$rt'(C_n) = n-1.$$

Goddard (see [1]) showed $rt(K_{n,n}) = 4$. Here we determine $rt'(K_{n,n})$.

THEOREM 4.8. *For $n \ge 2$, we have*

$$rt'(K_{n,n}) = 3.$$

Proof. Denote the two vertex sets of $K_{n,n}$ by X and Y such that $|X|=|Y|=n$. Given a permutation π, if there are some pebbles in X whose destinations are in Y, then there will be the same number of pebbles in Y whose destinations are in X. The first step is to swap these pebbles so that all pebbles in X have destinations in X and also all pebbles in Y will stay in Y. Then we divide each pebble into n pieces of mass $\frac{1}{n}$. The fractional matching we used here is $f(xy)=\frac{1}{n}$ for any edge xy. The second step is to swap pairs of pieces through all edges. After this step, each vertex on X has n different pieces from Y, while each vertex on Y has n different pieces from X. At the last step, for each edge x_iy_j, swap the pair of pieces which belong to their destinations. Thus,

$$rt'(K_{3,3}) \le 3.$$

The lower bound comes from a special permutation σ. Let $x_1,x_2 \in X$ and $y_1,y_2 \in Y$. The induced graph on these four vertices is $K_{2,2}=C_4$. Let σ be the permutation of a rotation on these four vertices. Namely, for any $v \in X \cup Y$,

$$\sigma(v) = \begin{cases} y_1 & \text{if } v = x_1; \\ x_2 & \text{if } v = y_1; \\ y_2 & \text{if } v = x_2; \\ x_1 & \text{if } v = y_2; \\ v & \text{otherwise.} \end{cases}$$

We would like to show $rt'(K_{n,n},\sigma) \ge 3$. If a routing process routes a piece of type x_1 to a vertex u ($u \ne x_1,x_2,y_1,y_2$), then any walk from x_1 to y_1 passing through u needs at least three steps since $K_{n,n}$ is a bipartite graph. The same argument applies to the pieces of type x_2, y_1, y_2. Thus, we can assume that no piece of type x_1, y_1, x_2, y_2 is routed out of these four vertices. When we restrict to x_1, y_1, x_2, y_2, it induces a routing process on C_4. Since $rt'(C_4, \sigma|_{\{x_1,y_1,x_2,y_2\}}) = 3$, we need at least three steps. The proof is finished. ∎

Lemma 2.2 can also be generalized to the following lemma.

LEMMA 4.9. *Let G be an m-routing graph, then*

$$rt'(C_n \Box G) \geq n + m - 1.$$

In particular, we have

$$rt'(Q_n) \geq n + 1.$$

5. Concluding remarks.

- The routing number can be computed as follows. For an edge $e = v_i v_j \in E(G)$, let $P_e = (ij)$ be the swapping permutation of i and j. For a matching M, the move of swapping through the set of edges in M can be written as a permutation

$$P_M = \prod_{e \in M} P_e.$$

  Consider an auxiliary Cayley graph $H = (Perm_n, K)$, where $Perm_n$ is the permutation group and $K = \{P_M\colon \text{ for any matching } M\}$. For any permutations π and σ, the graph distance in H from σ to π in G is exactly $rt(G, \sigma^{-1}\pi)$. In particular, $rt(G)$ is the diameter of the graph H. So the value of $rt(G)$ does not depend on the labeling of vertices. This also suggests a simple algorithm for computing the $rt(G)$. Namely, construct the auxiliary graph H and apply the breadth-first-search algorithm to H. The worst-case running time of this algorithm is $O(|V(H)| + |E(H)|) = O((n!)^2)$. We implemented this algorithm and computed the routing number of any connected graph up to 8 vertices in this way. We found out that $rt(Q_3) = 4$. This approach fails miserably when we try to compute $rt(Q_4)$.

  Question: Is there an efficient algorithm to compute $rt(G, \pi)$?

- In the proof of Theorem 1.2, we prove that $rt(C_n, \pi) = n - 1$ if π is the rotation $(123 \cdots n)$ or its inverse.

  CONJECTURE 1. *For $n \geq 5$, if $rt(C_n, \pi) = n - 1$, then π is the rotation $(123 \cdots n)$ or its inverse. (Here we label the vertices of C_n in the standard way.)*

  The outputs of our program suggest this conjecture holds for $n = 5, 6, 7, 8$. For $n = 4$, there are four permutations $\pi = (1234), (4321), (13)$, and (24) giving $rt(C_4, \pi) = 3$. We didn't observe a similar pattern for the path P_n.

- Many lower bounds for routing numbers are actually the lower bounds for the corresponding fractional routing numbers. This observation motivates us to consider the fractional routing numbers. We proved $rt'(T) \geq n$, and the inequality holds for the paths and the stars. In many extremal situations, paths and stars are usually in the opposite direction. We conclude our paper with the following conjecture.

  CONJECTURE 2. *For any tree T on $n \geq 3$ vertices, $rt'(T) = n + O(1)$.*

  Maybe even $rt'(T) = n$ holds.

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