# **Directed Nowhere Dense Classes of Graphs**

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**Abstract** We introduce the concept of shallow directed minors and based on this a new classification of classes of directed graphs which is diametric to existing directed graph decompositions and width measures proposed in the literature.

We then study in depth one type of classes of directed graphs which we call *nowhere crownful*. The classes are very general as they include, on one hand, all classes of directed graphs whose underlying undirected class is nowhere dense, such as planar, bounded-genus, and H-minor-free graphs; and on the other hand, also contain classes of high edge density whose underlying class is not nowhere dense. Yet we are able to show that problems such as directed dominating set and many others become fixed-parameter tractable on nowhere crownful classes of directed graphs. This is of particular interest as these problems are not tractable on any existing digraph measure for sparse classes.

The algorithmic results are established via proving a structural equivalence of nowhere crownful classes and classes of graphs which are *directed uniformly quasi-wide*. This rather surprising result is inspired by [Nešetřil and Ossana de Mendez 2008] and yet a different and much more involved proof is needed, turning it into a particularly significant part of our contribution.

# 1 Introduction

Faced with the seeming intractability of problems such as variants of the dominating set problem, the independent set problem and many other problems naturally arising in applications, an intensively studied aspect of complexity theory is to explore the boundary of tractability of these problems by identifying specific classes of graphs on which they become tractable (in a parameterised setting, see Section 2). A central objective of this research is to identify structural properties of graphs or graph classes such that classes of this structure exhibit a rich algorithmic theory – i.e. many otherwise intractable graph problems become tractable – while at the same time these classes should be general enough so that graphs of this form do occur in applications.

To find such graph parameters, methods derived from structure theory for undirected graphs have proved to be extremely useful. Of particular importance in this context is the concept of *tree-width* (see e.g. [Die05]) developed by Robertson and Seymour as part of their celebrated graph minor project. Following the introduction of tree-width, a large number of generally intractable problems have been shown to become tractable on graph classes with a fixed upper bound on the tree-width. See [Bod05,Bod98,Bod97,Bod93] for surveys of tree-width related results.

Besides graph classes of bounded tree-width, many other structural parameters of graphs have been studied which allow for more efficient solutions of otherwise hard problems. Among the most important such parameters are planar graphs, or much more generally, graph classes excluding a fixed minor (see e.g. [Die05]). A relatively new addition to the family of graph parameters studied with algorithmic applications in mind are *nowhere dense* classes of graphs [NO08b] which will be of special importance for this paper.

The structural parameters discussed above all relate to undirected graphs. However, many models naturally occurring in computer science are directed. Given the enormous success width parameters had for problems defined on undirected graphs, it is natural to ask whether they can also be used to analyse the complexity of problems on directed graphs. While in principle it is possible to apply the structure theory for undirected graphs to directed graphs by ignoring the direction of edges, this implies a significant information loss. Hence, for computational problems which inherently apply to directed graphs, methods based on the structure theory for undirected graphs may not always be applicable.

Reed [Ree99] and Johnson, Robertson, Seymour and Thomas [JRST01] initiated the development of a decomposition theory for directed graphs with the aim of defining an analogue of the concept of undirected tree-width for directed graphs. Following their definition of a *directed tree-width*, several alternative notions have been introduced, for instance in [Saf05,Bar06,BDHK06,Obd06,HK08,KR09]. For each of these decompositions and associated width

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measures it has been shown that several problems become tractable if the width of digraphs with respect to these measures is bounded by a fixed constant. However, most examples of problems becoming tractable are either linkage problems, i.e. problems asking for the existence of certain pairwise disjoint paths in the digraph, or certain combinatorial games arising in verification. The only exception is bi-rank-width [KR09] where all problems definable in monadic second-order logic become tractable. However, this is a very special case as it is modelled after the concept of undirected clique-width and in fact bounded bi-rank-width implies bounded clique-width of the underlying undirected graphs. It is therefore a measure aimed to some extent at dense but homogeneous graphs whereas here we are concerned with sparse graphs in line with undirected graph structure theory such as excluded minors and nowhere dense classes of graphs.

Following these initial proposals for directed analogues of tree-width or other undirected graph measures, several papers investigated how rich the algorithmic theory of classes of digraphs of bounded width with respect to these measures is. Unfortunately, for many interesting problems other than those mentioned before, strong intractability results for these width measures were obtained showing that the algorithmic applicability of the existing directed width measures is very limited. See e.g. [KO08,KLM08,DGK09,GHK<sup>+</sup>] and references therein.

**Our contributions.** In this paper we define new and completely different width measures for directed graphs tailored towards algorithmic applications and which overcome the problems with existing measures mentioned above. The novelty and fundamental difference of our approach is that we will not consider acyclic digraphs as the algorithmically simplest class of digraphs, as it was done for previous proposals of directed width measures, and instead aim for width measures where the class of all acyclic graphs has unbounded width.

More specifically, on a conceptual level, we draw inspiration from the concept of nowhere dense classes of undirected graphs [NO08b] and introduce a new classification of classes of directed graphs based on the concept of shallow directed minors (see Section 3.1). In this way we introduce classes of directed graphs which we call *nowhere dense*, *somewhere dense*, *nowhere crownful* or of *directed bounded expansion* and show that these classes form a strict hierarchy into which all classes of directed graphs can be classified. We show various structural properties of these classes.

In the rest of the paper we then concentrate on classes of directed graphs which are nowhere crownful. As the main result in the first, conceptual part of the paper, we show that these classes can alternatively be characterised by a property called *uniformly quasi-wideness* (see Theorem 4.3). This characterisation in terms of wideness properties yields a new and different perspective of nowhere crownful classes which is algorithmically very useful. A similar characterisation for the undirected case was given in [NO10]. Unfortunately, their proof does not generalise to the directed graph setting and our proof of Theorem 4.3 requires completely new and different ideas.

This equivalence is indeed quite surprising when seen from the following perspective. The natural analogon to undirected nowhere dense classes seems to be our notion of directed nowhere dense, where tournaments are excluded as shallow minors. However, it turns out that in order to obtain quasi-wideness we need to exclude shallow *crowns*, a certain orientation of a subdivision of a clique. But even in this case, the proof of [NO10] breaks down at a somewhat unexpected place and can only be fixed via a much more delicate and, in a sense, fragile argument.

Nowhere crownful classes are incomparable to classes of digraphs of bounded width with respect to existing width measures. They can be very general. For instance, the class of planar directed graphs is nowhere crownful. More generally, if  $\mathcal{C}$  is a class of directed graphs such that the class  $\mathcal{C}^u$  of undirected graphs obtained from  $\mathcal{C}$  by forgetting the direction of edges is nowhere dense, e.g. excludes a fixed minor, then  $\mathcal{C}$  is nowhere crownful. However, there are simple examples for classes  $\mathcal{C}$  of directed graphs which are nowhere crownful but where the class  $\mathcal{C}^u$  is not nowhere dense. Hence, nowhere crownful classes can be much more general than classes of bounded directed tree-width or other width measures for directed graphs introduced so far. On the other hand, the class of acyclic directed graphs is not nowhere crownful but it has small width in all other digraph decompositions (except bi-rank-width) so that nowhere crownful classes are incomparable to classes defined by other width measures.

In the second, algorithmic part of the paper we then concentrate on algorithmic applications of the graph classes introduced in the first part. Our main algorithmic result is that using the alternative characterisation of nowhere crownful classes we can show that on these classes of digraphs problems such as directed (independent, etc.) dominating set, independent sets, and many other generally hard problems become tractable. This is particularly interesting as exactly these types of problems have been shown to be intractable on all existing directed width measures for sparse classes and therefore our definition provides the first structural property for sparse classes that can successfully be used in the analysis of domination problems on directed graphs. We finally study the connected dominating set problem and its directed analogues. In particular we show that the dominating outbranching problem is tractable on nowhere crownful classes.

Width measures for undirected graphs have proved to be extremely successful in tackling the complexity of hard algorithmic problems on undirected graphs with a huge number of applications. Developing a similar approach for directed graphs therefore has the potential for tremendous impact on the theory of hard problems on directed graphs and on the design of algorithms for solving these and must therefore be seen as a crucial aim.

The width measures introduced here allow for the first time to analyse the complexity of domination type problems on classes of directed graphs and we believe that the structural classification of directed graphs developed here may provide an interesting step towards establishing a structure theory for directed graphs more fruitful for algorithmic applications than the theory based on analogues of tree-width.

Organisation and results. The paper is organised as follows. We fix notation and introduce some basic concepts in Section 2. The concept of directed minors used in this paper is introduced in Section 3.1 where we establish basic properties of the minor relation. In Section 3.2 we establish our classification of classes of directed graphs by defining several measures for directed graphs of increasing complexity.

In the rest of the paper we will then concentrate on nowhere crownful classes of graphs. In Section 4, we provide an alternative characterisation of nowhere crownful classes based on the existence of scattered sets. We use this characterisation in Section 5 to show that many problems become tractable on classes of digraphs which are nowhere crownful. We conclude and state open problems in Section 6.

### **Preliminaries**

We write  $\mathbb{N}$  for the set of non-negative integers. If M is a set and  $k \in \mathbb{N}$  we write  $[M]^{\leq k}$  for the set of all subsets of M of cardinality at most k.  $[M]^k$ ,  $[M]^{\leq k}$  is defined analogously.

A digraph G = (V, E) is a pair where V is its set of vertices and  $E \subseteq V \times V$  is its set of edges. We often use V(G) and E(G) to refer to the set of vertices and edges of G, respectively, and write uv instead of (u, v) to denote an edge of G. In an undirected graph G = (V, E), we have  $E \subseteq [V]^2$  instead.

**Definition 2.1.** For a digraph G, we define its underlying undirected graph  $G^u$  to be an undirected graph on the same vertex set and with edge set  $\{\{u,v\}: uv \in E(G) \text{ or } vu \in E(G)\}$ . If C is a class of directed graphs, then its underlying undirected class is defined as  $C^u := \{G^u : G \in C\}$ .

Similarly, for an undirected graph G, we define its underlying bidirected graph  $G^{bd}$  as the directed graph on the same vertex set and with edge set  $\{(u,v),(v,u):\{u,v\}\in E(G)\}$  and define  $\mathcal{C}^{\bar{b}d}$  for a class of undirected graphs  $\mathcal{C}$ analogously.

A (directed) path of length k in a (di)graph G is a sequence  $v_1 \dots v_{k+1}$  of distinct vertices of G such that for each  $1 \le i \le k$ , there is an edge  $v_i v_{i+1}$  in E(G). If we allow and require  $v_1 = v_{k+1}$  we have a (directed) cycle instead. A directed acyclic graph (DAG) is a digraph that contains no directed cycles.

**Definition 2.2.** By  $N_d^+(v)$  we denote the d-outneighbourhood of v, i.e.

 $N_d^+(v) := \{ u \in V(G) : \text{ there is a directed path from } v \text{ to } u \text{ of length } \leq d \}.$ 

 $N_d^-(v)$  is defined analogously as the d-inneighbourhood of v. If  $X\subseteq V(G)$  then  $N_d^+(X):=\bigcup_{x\in X}N_d^+(x)$ , and  $N_d^-(X)$  is defined analogously. We skip the index d when it is

**Definition 2.3.** A directed bipartite graph is a directed graph  $G := (A \dot{\cup} B, E)$  whose vertex set is partitioned into two sets A and B and  $E \subseteq A \times B$ .

An *orientation* of an undirected graph G is a directed graph obtained from G by replacing every undirected edge by a directed edge. A clique of order n is the (up to isomorphism) complete undirected graph  $K_n$  on n vertices containing all possible edges. A tournament of order n is a digraph  $T_n$  that is an orientation of the clique  $K_n$ .

A subdivision of a (di)graph G is obtained by replacing some edges of G by paths; in case of digraphs, the paths must respect the directions of the replaced edges. The following class of graphs, which are acyclic orientations of subdivided cliques, will play a special role in this paper.

**Definition 2.4.** A crown of order q, for q > 0, is the graph  $S_q$  with

 $N_d^+(v)$ 

 $N_d^-(v)$ 

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- V(S_q) := \{v_1, \dots, v_q\} \dot{\cup} \{u_{i,j} : 1 \le i < j \le q\} and
- E(S_q) := \{(u_{i,j}, v_i), (u_{i,j}, v_j) : 1 \le i < j \le q\}.
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we call the vertices  $v_1, \ldots, v_n$  the principal vertices of the crown.

**Definition 2.5.** Let G be a digraph. A k-alternating path  $AP_k$  in G, for some  $k \geq 1$ , is an orientation of a path  $v_1 \dots v_{k+2}$  such that either all edges are directed towards their incident vertex with an odd index or all edges are directed towards their incident vertex with an even index.

The model of complexity we are using is parameterised complexity [DF98,FG06]. A problem of size n with parameter k is said to be fixed-parameter tractable (fpt), if it can be solved by an algorithm in time  $\mathcal{O}(f(k)n^{\mathcal{O}(1)})$ , for some computable function f. The class FPT is the set of all parameterised problems that are fixed-parameter tractable. The class XP is the set of all parameterised problems that can be solved by an algorithm in time  $\mathcal{O}(n^{f(k)})$ , for a computable function f. If the parameter of a problem is not explicitly specified, we assume the standard parameterization, that is, the parameter is the solution size. For example, the standard parameter of the dominating set problem is the size of a minimum dominating set in the given graph. Recall that  $D \subseteq V(G)$  is a d-dominating set if  $N_d^+(D) = V(G)$  and is just called a dominating set for d = 1.

# 3 A Classification of Directed Graph Classes

In this section we give a classification of directed graphs in terms of shallow directed minors. We first state our quite general definition for directed minors in terms of models and compare it with some other definitions in the literature. In Section 3.2, we define several properties of directed graph classes, show their relationship, and compare them to their undirected counterparts.

### 3.1 Directed Minors

For undirected graphs, we say that H is a minor of G, denoted as  $H \leq G$ , if H can be obtained from G by a series of vertex and edge deletions and edge contractions. This is equivalent to G containing a *model* of H. We define the minor relation for directed graphs in terms of models below.

**Definition 3.1.** A digraphs H has a directed model in a digraph G if there is a function  $\delta$  mapping vertices  $v \in V(H)$  of H to sub-graphs  $\delta(v) \subseteq G$  and edges  $e \in E(H)$  to edges  $\delta(e) \in E(G)$  such that

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- if v \neq u then \delta(v) \cap \delta(u) = \emptyset;
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- if e := uv then  $\delta(e)$  has its start point in  $\delta(u)$  and its end in  $\delta(v)$ .

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For v \in V(H) we set \operatorname{in}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e := uv \in E(H)} V(\delta(e)) and \operatorname{out}(\delta(v)) := V(\delta(v)) \cap \bigcup_{e := vw \in E(H)} V(\delta(e)) and require that
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- there is a directed path in  $\delta(v)$  from any  $u \in \operatorname{in}(\delta(v))$  to every  $u' \in \operatorname{out}(\delta(v))$ ;
- there is at least one source vertex  $s_v \in \delta(v)$  that reaches every element of  $\operatorname{out}(\delta(v))$ ;
- there is at least one sink vertex  $t_v \in \delta(v)$  that can be reached from every element of  $\operatorname{in}(\delta(v))$ .

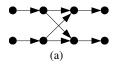
We write  $H \preceq^d G$  if H has a directed model in G and say H is a directed minor of G. We call the sets  $\delta(v)$  for  $v \in V(H)$  the branch-sets of the model.

It is obvious that this notion generalises the concept of undirected minors as follows.

**Lemma 3.2.** If G, H are undirected graphs, then  $H \preceq G \Leftrightarrow H^{bd} \preceq^d G^{bd}$ . If G, H are digraphs, then  $H \preceq^d G \Rightarrow H^u \preceq G^u$ .

For (di)graphs G, H, we say that H is a (directed) topological minor of G if G contains a subdivision of H as a subgraph and denote it by  $H \preccurlyeq^t G$ . As with undirected graphs, we have that  $H \preccurlyeq^t G \Rightarrow H \preccurlyeq^d G$  but the reverse is not true in general.

In the literature, another notion of directed minor has been considered [JRST01]:



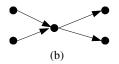


Figure 1. The digraph in (a) contains a model of (b) but does not contain it as a butterfly minor.

**Definition 3.3.** A butterfly contraction is the operation of contracting an edge e = uv where either u has outdegree 1 or v has indegree 1. A graph H is said to be a butterfly minor of G, i.e.  $H \preceq^b G$ , if it can be obtained from G by a series of vertex and edge deletions and butterfly contractions.

**Lemma 3.4.** For digraphs G and H, we have  $H \preceq^b G \Rightarrow H \preceq^d G$ , but the converse is not true in general.

*Proof.* We prove the claim by induction on the number of butterfly contractions necessary to obtain H from G. If no contractions are necessary, there is nothing to show. Otherwise, let e=uv be an edge of G that is to be butterfly contracted to a vertex z in G':=G/e. Consider any path in G' that passes through z via edges  $e_1$  and  $e_2$ , in this order. If  $e_1$  is incident to v in G and  $e_2$  is incident to u in G, then v would have indegree 2 and u would have outdegree 2 in G, making it impossible to butterfly contract e in G. Hence, every path in G' corresponds to a path in G by uncontracting e.

As H is a butterfly minor of G', we know by the induction hypothesis that there is a map  $\delta'$  that describes a model of H in G'. Let  $z_H \in V(H)$  be such that  $z \in \delta'(z_H)$ . Define  $\delta$  to be equal to  $\delta'$  except for  $\delta(z_H) := \delta'(z_H) \setminus \{z\} \cup \{u,v\}$ . Then  $\delta$  describes a valid model of H in G since all paths of G' are preserved when uncontracting e.

To see that the converse is not true in general, see Figure 1.

In the rest of the paper, models of directed bipartite graphs will play a special role. For these, we can take models to be of a particularly simple form. Recall that an in-branching is an orientation of a tree with all edges oriented towards the root; an out-branching is defined analogously.

**Lemma 3.5.** If H is a directed bipartite graph with  $H \preceq^d G$ , we can choose the branch-sets of the model of H in G to be in- or out-branchings. Furthermore, we have  $H \preceq^d G \Leftrightarrow H \preceq^b G$ .

*Proof.* The first claim follows immediately from the definition of a directed model since H consists solely of sources and sinks. As for the second claim, note that in order to obtain H from G as a butterfly minor, we can first delete all edges that are not in the model of H in G and then butterfly contract the in- and out-branchings of the model starting from the leaves.

If G is acyclic, we can detect fixed minors in polynomial time (more precisely, XP-time w.r.t. |H|). In order to prove this fact, we first show the following lemma.

**Lemma 3.6.** Let G be a DAG and let  $s_1, \ldots, s_k, t_1, \ldots, t_k$  be given vertices of G. Furthermore, let  $\mathcal{L} = \{(z_0 = 0, z_1], (z_1, z_2], \ldots, (z_{\ell-1}, z_{\ell} = k]\}$  be a set of  $\ell$  given intervals partitioning [1, k]. The goal is to find, for each  $1 \le i \le k$ , a path  $P_i$  between  $s_i$  and  $t_i$  such that if  $1 \le i < j \le k$  belong to different intervals of  $\mathcal{L}$ , then  $P_i$  and  $P_j$  are disjoint.

- (a) There is an algorithm running in time  $n^{\mathcal{O}(k)}$  that finds such paths if they exist.
- (b) If we are additionally given an integer r to bound the length that each path is allowed to have, then in time  $(nr)^{\mathcal{O}(k)} = n^{\mathcal{O}(k)}$  we can find the required paths or report that they do not exist.

*Proof.* Our ideas generalise that of Eppstein's algorithm [Epp95] (see also Perl and Shiloach [PS78]) for finding disjoint paths in DAGs. We first augment G by a vertex s and add edges  $(s, s_i)$  for all  $1 \le i \le k$ . Then we construct a new DAG D that contains a vertex for each tuple  $(v_1, \ldots, v_k) \in V(G)^k$ . Consider a topological order of the vertices of G and let f(v) denote the position of vertex v in this order. We add an edge from a tuple  $(v_1, \ldots, v_k)$  to a tuple  $(w_1, \ldots, w_k)$  in D if the following conditions apply for some  $w \in \{w_1, \ldots, w_k\}$  and  $d \in [1, \ell]$ :

- (i) for  $1 \le i \le k$ , if  $w_i = w$ , then there exists an edge  $(v_i, w_i) \in E(G)$ ;
- (ii) for  $1 \le i \le k$ , if  $w_i \ne w$ , then  $v_i = w_i$ , i.e. w is the only new vertex introduced;

- (iii) for  $i \notin (z_{d-1}, z_d]$ ,  $v_i = w_i$ , i.e. we may change only elements out of one interval of  $\mathcal{L}$  at a time;
- (iv) for  $1 \le i \le k$ ,  $f(w) > f(v_i)$ .

We claim that there is a one-to-one correspondence between the paths  $P_i$  that we are looking for in G and paths from  $(s, \ldots, s)$  to  $(t_1, \ldots, t_k)$  in D. Indeed, we can transform the latter to the former by taking the projection to the ith coordinate for each i; condition (iii) and (iv) above guarantee that paths from different intervals will be disjoint since w is required to be a vertex that has not appeared on the path so far and may be used by only one interval. In order to transform paths in G to a path in D, we consider the set of all vertices appearing on the paths  $P_1, \ldots, P_k$  and introduce them to our path in D in topological order. For further details, we refer to  $[Epp95]^3$ .

In order to prove (b), we additionally keep a table of size  $(r+1)^k$  at each node of D indexed by tuples  $(r_1, \ldots, r_k) \in \{0, \ldots, r\}^k$ . At a vertex  $v \in V(D)$ , an entry  $(r_1, \ldots, r_k)$  should indicate whether there exists a path P from  $(s, \ldots, s)$  to v in which the length of the path in G corresponding to the projection of the ith component of P has length  $r_i$ . If we traverse the vertices of D in topological order, these tables can be filled easily at each vertex using the tables of its predecessors, resulting in the claimed overall running time.

**Theorem 3.7.** Let G be a directed acyclic graph. There is a polynomial time algorithm to decide whether  $H \preceq^d G$  for any fixed digraph H.

Proof. Obviously, if H contains a cycle, we can reject. Otherwise, assume G has n vertices and H has h edges. We try to construct a map  $\delta$  that defines a model of H in G as follows. First, we guess the mapping of the edges of H, that is, for each  $e \in E(H)$ , we guess an edge  $e' \in E(G)$  as the value of  $\delta(e)$ . Furthermore, for each  $v \in V(H)$ , we guess vertices  $s_v, t_v \in V(G)$  that act as a source and a sink for  $\delta(v)$ , respectively. There are  $n^{\mathcal{O}(h)}$  possibilities for these choices and we can try all of them. This way, we have also partially guessed some vertices of  $\delta(v)$ , for  $v \in V(H)$ , namely, all vertices of  $\mathrm{in}(\delta(v))$ ,  $\mathrm{out}(\delta(v))$ , and its source and sink vertices. What remains is to connect these vertices by some paths as specified in Definition 3.1 such that the paths belonging to  $\delta(v)$  and  $\delta(w)$  for distinct  $v, w \in V(H)$  are disjoint. But this is exactly what we can check for using Lemma 3.6 in  $n^{\mathcal{O}(h^2)}$  time, which is polynomial for constant h (note that the number of paths we are looking for can be  $\mathcal{O}(h^2)$ ).

A further concept that is central to our work is that of *shallow* minors:

**Definition 3.8.** A digraph H is a (shallow) depth r minor of a digraph G, denoted as  $H \preccurlyeq^d_r G$ , if there exists a directed model of H in G in which the length of all the paths in the branch-sets of the model are bounded by r.

Hence, a depth 0 minor is simply a subgraph and any minor of G is a depth n minor. This is analogous to the definition of depth r minors in undirected graphs, denoted by  $H \preccurlyeq_r G$ .

**Theorem 3.9.** Let G, H be digraphs and r an integer. There is an algorithm that decides whether  $H \preccurlyeq^d_r G$  in time  $n^{\mathcal{O}(rh^2)}$ , where h denotes the number of edges of H. If G is acylcic, then the problem can be decided in time  $(nr)^{\mathcal{O}(h^2)} = n^{\mathcal{O}(h^2)}$ .

*Proof.* As in the proof of Lemma 3.7, we start by guessing the image of the edges of H in G and then check if the required paths to complete the model exist. But as there are at most  $n^r$  paths of length at most r starting at any given vertex of G, we can accomplish this in total time  $n^{\mathcal{O}(rh^2)}$ .

If G is a DAG, we can search for the required paths using Lemma 3.6 (b).

### 3.2 Classes of Digraphs Excluding Shallow Minors

In the realm of undirected graphs, classes of graphs excluding minors have been studied extensively, on one hand because some of these classes, like planar or bounded genus graphs, are very natural and important, and on the other hand due to their structural richness revealed by Robertson and Seymour's graph minor theory [RS]. More recently, Nešetřil and Ossana de Mendez [NO08b] offered a generalization of these classes based on the notion of shallow minors. They define a class of undirected graphs to be *somewhere dense* if there exists an integer r, such that the set of depth r minors of  $\mathcal C$  contains arbitrarily large cliques; otherwise, the class is called *nowhere dense*. It turns out that nowhere dense classes of graphs are the most general classes of (almost) sparse graphs that are structurally still very rich and allow for efficient (parameterised) algorithms [DK09,NO10].

<sup>&</sup>lt;sup>3</sup> the proof therein is for the all disjoint case but can be easily adapted.

In order to obtain an analogous theory for digraphs, we use the notion of shallow directed minors as introduced above. One of our design goals is to obtain proper generalizations of the undirected classes in the sense that is given in our Classification and Comparison Theorems 3.11 and 3.12 below. However, from an algorithmic point of view, we face the following important issue: every acyclic graph excludes any cycle as a minor, and every directed bipartite graph excludes a path of length 2 as a minor. But acyclic graphs are already algorithmically very hard for many problems; and directed bipartite graphs (together with a super-source) easily encode, for example, the general dominating set problem in undirected graphs, which is W[2]-hard. Hence it seems that even excluding an acylic graph will not facilitate finding efficient algorithms. Therefore, it appears that, algorithmically, it might be beneficial to additionally introduce and study classes of digraphs excluding a bipartite graph as a (shallow) minor.

## **Definition 3.10.** *Let* C *be a class of digraphs.*

- 1. C is directed somewhere dense if there is a radius  $r \ge 0$  so that the set of depth r minors of C contains arbitrarily large tournaments.
- 2. C is directed nowhere dense if for every r, there exists an n and an acyclic tournament  $T_n$  so that for all  $G \in \mathcal{C}$ ,
- 3. C is somewhere crownful if there is a radius  $r \ge 0$  so that every crown  $S_q$  occurs as a depth r minor of a graph  $G \in \mathcal{C}$ .
- 4.  $\mathcal{C}$  is nowhere crownful if for every r, there exists a q so that for all all  $G \in \mathcal{C}$ ,  $S_q \not\preccurlyeq^d_r G$ . 5.  $\mathcal{C}$  has directed bounded expansion if there is a function  $f: \mathbb{N} \to \mathbb{N}$  such that  $\nabla^d_r(G) \leq f(r)$  for all  $G \in \mathcal{C}$ , where  $\nabla^d_r(G) := \{\frac{|E(H)|}{|V(H)|}: H \preccurlyeq^d_r G\}$  (cf. [NO08a]).
- 6. C is crown-minor-free if there exists a q such that  $S_q \not \preccurlyeq^d G$  for any  $G \in \mathcal{C}$ .
- 7. C is alternating path-minor-free if there exists a k such that  $AP_k \not\preccurlyeq^d G$  for any  $G \in C$ .

#### Theorem 3.11 (Classification Theorem).

- (a) A class C is directed somewhere dense if and only if it is not directed nowhere dense, i.e. there exists a radius r so that every acyclic tournament occurs as a depth r minor of a graph  $G \in \mathcal{C}$ .
- (b) The property of being directed nowhere dense is more general than being nowhere crownful, which is in turn more general than being crown-minor-free, and the latter being in turn more general than being alternating-pathminor-free.
- (c) The property of being directed nowhere dense is also more general than being of directed bounded expansion, which is in turn more general than being alternating-path-minor-free. The property of being of directed bounded expansion is neither comparable to being crown-minor-free, nor to being nowhere crownful.
- *Proof.* (a) This follows from the fact that any tournament of order  $2^n$  contains all acyclic tournaments of order n as a subgraph. For, let T be a tournament of order  $2^n$  and T' an acyclic tournament of order n. Let v be a vertex of highest outdegree in T and let v' be the first vertex of T' in a topological order. Let  $T_v$  be the tournament induced by the out-neighbours of v in T. As the out-degree of v is at least  $2^{n-1}$ , we have by induction that there exists a copy of T'-v' in  $T_v$ . Now matching v with v' (note that v' can only have out-neighbours in T') and adding it to this copy gives us a copy of T' in T.

Claim (b) is immediate by definition.

(c) A class that is of directed bounded expansion cannot have arbitrarily dense graphs as depth r minors, in particular all acyclic tournaments, and hence any such class is directed nowhere dense. Also, as graphs of bounded treewidth and their minors are sparse, it follows from Theorem 3.14 that alternating-path-minor-free classes are of directed bounded expansion. In Theorem 3.13, we show that there exist crown-minor-free classes of digraphs that have high edge density – and are hence not of bounded expansion. Conversely, the set of all crowns and their minors (which are only their subgraphs) have edge density at most 2 and are hence of bounded expansion but clearly not nowhere crownful.

This theorem shows that our classes are robust; the next theorem shows that they properly generalise the respective undirected concepts and capture much larger classes of digraphs than what we would get by only considering their underlying undirected classes.

### Theorem 3.12 (Comparison Theorem).

- (a) If C is a class of undirected graphs, then C is somewhere dense/nowhere dense/of bounded expansion/H-minor-free if and only if C<sup>bd</sup> is directed somewhere dense/nowhere dense/of bounded expansion/crown-minor-free, respectively. Furthermore, C<sup>bd</sup> is directed somewhere dense if and only if it is somewhere crownful.
- (b) Let C be a class of digraphs. If the underlying undirected class  $C^u$  is nowhere dense/of bounded expansion/H-minor-free, then C is nowhere crownful/of directed bounded expansion/crown-minor-free, respectively.
- (c) The reverse of (b) is not true in general. There exist classes of digraphs that are crown-minor-free but whose underlying undirected class is somewhere dense.

*Proof.* (a,b) All claims are immediate by Lemma 3.2 and the fact that the underlying undirected graph of a crown can be contracted to a complete minor.

(c) Consider the class  $\mathcal{C} := \{S'_q | q \geq 1\}$  where  $S'_q$  is obtained by reversing the direction of all the edges of  $S_q$ . Since  $S'_q$  is a directed bipartite graph, it has no directed minors except for its subgraphs; but since the indegree of each vertex is at most 2, it does not contain any crown of order greater than 3 as a subgraph and is hence crown-minor-free. But  $\mathcal{C}^u$  obviously contains all cliques as minors and is hence somewhere dense.

The previous theorem already indicated that the concepts of directed graphs introduced here properly generalise their undirected counterparts. The following theorem gives further evidence of this, where we show that that crown-minor-free graphs can in fact be quite dense – something that classes of undirected graphs which are nowhere dense cannot be.

**Theorem 3.13.** For every  $\epsilon$ , there exists a q and an  $S_q$ -minor-free digraph that has edge density at least  $\Omega(n^{\frac{1}{2}-\epsilon})$  (whereas undirected nowhere dense classes have edge density at most  $n^{o(1)}$  [NO08b]).

*Proof.* We use a probabilistic construction to obtain a directed bipartite graph  $G=(A\dot{\cup}B,E)$  with |A|=|B|=n as follows. For a given density  $d:=d(\epsilon)$ , we choose d neighbours for every vertex  $a\in A$  uniformly at random. This way, our graph G has exactly dn edges and since it is a directed bipartite graph it has no directed minors except for its subgraphs. Now, let us count the expected number of  $S_q$ 's in this graph. For a set  $\beta=\{b_1,\ldots,b_q\}\subseteq B$  and an ordered sequence  $\alpha=(a_{12},\ldots,a_{(q-1)q})\in A^{\binom{q}{2}}$ , let  $X_\beta^\alpha$  be the indicator variable that  $\alpha\cup\beta$  forms an  $S_q$  such that  $a_{ij}$  is adjacent to  $b_i$  and  $b_j$ . By assuming d< n/2, we have

$$P[X_{\beta}^{\alpha} = 1] = \left(\frac{\binom{n}{d-2}}{\binom{n}{d}}\right)^{\binom{q}{2}} = \left(\frac{d(d-1)}{(n-d+2)(n-d+1)}\right)^{\binom{q}{2}} \le \left(\frac{2d}{n}\right)^{q(q-1)}$$

By linearity of expectation, and choosing  $d:=\left\lfloor\frac{n^{\frac{1}{2}-\epsilon}}{2}\right\rfloor$  and  $q>1+\frac{1}{\epsilon}$ , we obtain:

$$\begin{split} E[\sum_{\alpha,\beta} X_{\beta}^{\alpha}] &= \sum_{\alpha,\beta} E[X_{\beta}^{\alpha}] \leq \binom{n}{q} \binom{n}{\binom{q}{2}} \binom{q}{2}! \cdot (\frac{2d}{n})^{q(q-1)} \\ &\leq n^q \cdot n^{\frac{q(q-1)}{2}} \cdot \frac{(2d)^{q(q-1)}}{n^{q(q-1)}} = \frac{(2d)^{q(q-1)}}{n^{\frac{q(q-3)}{2}}} \\ &\leq n^{(\frac{1}{2} - \epsilon)q(q-1) - \frac{q(q-3)}{2}} = n^{q - \epsilon q(q-1)} < 1 \end{split}$$

Hence, there exists a graph G with edge density  $d = \Omega(n^{\frac{1}{2} - \epsilon})$  that contains no  $S_q$ .

One might notice that both the examples exhibited in the proofs of Theorems 3.12 (c) and 3.13 contain graphs with very long alternating path. One might think that alternating paths are a rather special case. But in fact, it is exactly the existence of such paths that make directed graphs complicated, as shown in the next theorem. The main idea is that if  $\mathcal{C}$  has large tree-width, then it contains large grid minors which in turn contain large alternating paths, independent of how they are oriented.

**Theorem 3.14.** If C is a class that is alternating path-minor-free then C (or, more precisely,  $C^u$ ) has bounded treewidth.

*Proof.* For  $l_1, l_2 \ge 1$ , the  $l_1 \times l_2$ -grid is the undirected graph with vertex set  $\{(i, j) : 1 \le i \le l_1, 1 \le j \le l_2\}$  and edge set  $\{(i, j), (i', j') : |i - i'| + |j - j'| = 1\}$ .

We claim that if H is an orientation of a  $2l \times 3$ -grid then H contains a subdivision of an l-alternating path beginning at the vertex (1,1) and terminating either at (2l,1) or (2l,3).

The claim is proved by induction on l. For l=1, let H be an orientation of a  $2 \times 3$ -grid. Consider the path  $P_1:=(1,1)(1,2)(1,3)(2,3)(2,2)(2,1)$ ; if it contains no alternations, then the path  $P_3:=(1,1)(1,2)(2,2)(2,3)$  must have at least one alternation.

Now let l>1 and consider the paths  $P_1$  and  $P_3$  described above. By the induction hypothesis, there exists a path  $P_1'$  starting at (3,1) and ending in  $(2l,z_1)$  containing at least l-1 alternations; by symmetry, there exists also a path  $P_3'$  starting at (3,3) and ending in  $(2l,z_3)$  containing at least l-1 alternations (where  $z_1,z_3\in\{1,3\}$ ). Now either the path  $P_1P_1'$  or the path  $P_3P_3'$  fulfill our claim.

Now suppose G is a graph of tree-width  $\operatorname{tw}(G) \ge f(2k)$ . Then, by the excluded grid theorem [RS86], G contains a  $2k \times 2k$ -grid as a minor and therefore a k-alternating path.<sup>4</sup>

**Corollary 3.15.** Let  $C_k$  be the class of  $AP_k$ -free digraphs. Then sub-graph isomorphism (and every other problem that is tractable on bounded treewidth graphs) is fixed-parameter tractable on  $C_k$ .

Note that the sub-graph isomorphism problem is of particular interest, as it is as hard on crown-minor-free classes as it is in the general undirected case: we can transform an undirected instance to a crown-minor-free instance simply by replacing every edge with an alternating path of length 2. The same is true for classes with directed bounded expansion and even bounded directed path- or tree-width.

# 4 An Alternative Characterisation of Nowhere Crownful Classes

In this section, we introduce the notion of directed uniformly quasi-wide classes as a generalization of undirected uniformly quasi-wide classes introduced by Dawar [Daw10]. Our main purpose is to prove Theorem 4.3 stating that this (rather algorithmic and seemingly unrelated) concept is exactly equivalent to the concept of nowhere crownful classes of digraphs. This is similar to the theorem of Nešetřil and Ossana de Mendez [NO10] showing that undirected nowhere dense classes are equivalent to uniformly quasi-wide classes. However, their proof does not generalise to the directed case, and hence we introduce significantly different ideas and a much more involved analysis to obtain this result.

**Definition 4.1.** Let G be a digraph and  $d \in \mathbb{N} \cup \{0\}$ . A set  $U \subseteq V(G)$  is d-scattered if there is no  $v \in V(G)$  and  $u_1 \neq u_2 \in U$  with  $u, u' \in N_d^+(v)$ .

Note that as  $N_0^+(v) = \{v\}$ , any subset of V(G) is 0-scattered.

**Definition 4.2.** A class C of directed graphs is uniformly quasi-wide if there are functions  $s: \mathbb{N} \to \mathbb{N}$  and  $N: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  such that for every  $G \in C$  and all  $d, m \in \mathbb{N}$  and  $W \subseteq V(G)$  with |W| > N(d, m) there is a set  $S \subseteq V(G)$  with  $|S| \le s(d)$  and  $U \subseteq W$  with |U| = m such that U is d-scattered in G - S. s, N are called the margin of C.

*If s and N are computable then we call C effectively uniformly quasi-wide.* 

Note that the class of reversed crowns described in the proof of Theorem 3.12 (c) is an example of a class that is directed uniformly quasi-wide but whose underlying undirected graph is not uniformly quasi-wide. The fact that the directed case properly generalises the undirected case also follows from Theorem 4.3 and the equivalence of undirected nowhere dense classes with undirected uniformly quasi-wide classes.

**Theorem 4.3.** A class C of digraphs is nowhere crownful if and only if it is directed uniformly quasi-wide.

The if direction is simple.

**Lemma 4.4.** If C is uniformly quasi-wide then it is nowhere crownful.

<sup>&</sup>lt;sup>4</sup> Note that by routing the path through a model, we can only get more alternations.

*Proof.* Let s,N be the margin of  $\mathcal{C}$ . Towards a contradiction, suppose there is r>0 such that for all q there is  $G_q\in\mathcal{C}$  with  $S_q\preccurlyeq^d_r G_q$ . Let  $s:=s(2r+1),\,N=N(2r+1,s+2)$ , and let  $G\in\mathcal{C}$  be such that  $S_{N+1}\preccurlyeq^d_r G$ . W.l.o.g. we assume  $N\geq s+2$ .

Then there are out-branchings  $\mathcal{A}:=\{A_1,\ldots,A_{\binom{N+1}{2}}\}$  and in-branchings  $\mathcal{B}=\{B_1,\ldots,B_{N+1}\}$  witnessing  $S_{N+1}\preccurlyeq^d_r G$ . Let W be the set of principal vertices of the in-branchings. As |W|>N there is a set  $S\subseteq V(G)$  with  $|S|\leq s$  and  $U\subseteq W$  with |U|=s+2 such that U is (2r+1)-scattered in G-S.

Let  $\mathcal{B}':=\{B'_1,\ldots,B'_t\}\subseteq\mathcal{B}$  be such that each  $B'_i\in\mathcal{B}'$  contains a vertex of U but no vertex of S; obviously, we have  $t\geq 2$ . Also, note that at most t-2 elements out of  $\mathcal{A}$  can be hit by S. But since  $\binom{t}{2}>t-2$  for  $t\geq 2$ , there must exist an  $A_k\in\mathcal{A}$  and  $B'_i,B'_j\in\mathcal{B}'$  none of which are hit by S. But then G-S contains a path of length at most 2r+1 from the root of  $A_k$  to the principal vertices of  $B'_i$  and  $B'_j$  contradicting the assumption that U is (2r+1)-scattered.  $\square$ 

In the remainder of this section we prove the converse. For this, we first introduce some auxiliary concepts.

**Definition 4.5.** (a) An r-controlled directed bipartite graph is a tuple  $(G, \beta, \lambda, \eta)$  such that

- $G := (A \dot{\cup} B, E)$  is a directed bipartite graph;
- $\beta: A \to [B]^{\leq 1}$  is a function assigning a base  $\beta(a) \in B$  to some vertices  $a \in A$ ;
- $\lambda: A \to \{0, \dots, r+1\}$  assigns a level to each vertex in A; and
- $\eta: E \to [A]^{\leq r+1}$  is a function that specifies for each  $e = ab \in E$  a subset  $\eta(e) \in [A]^{=\lambda(a)}$  s.t. if  $\eta(e) = \{a_0, \ldots, a_i\}$  then  $\beta(a_0) = \cdots = \beta(a_i) = b$  and  $\lambda(a_0) < \lambda(a_1) < \cdots < \lambda(a_i)$ ; furthermore, if  $\beta(a) = b$ , we have  $\lambda(a_i) < \lambda(a)$ .
- (b) A controlled crown of order q in an r-controlled directed bipartite graph  $(G, \beta, \lambda, \eta)$  is a crown  $S_q \subseteq G$  such that for all  $e \in E(S_q)$  we have  $\eta(e) \cap V(S_q) = \emptyset$ .

The main lemma we are going to prove about controlled directed bipartite graphs is the following:

**Lemma 4.6.** There exists a function  $F: \mathbb{N}^3 \to \mathbb{N}$  such that if  $(G := (A \dot{\cup} B, E), \beta, \lambda, \eta)$  is an r-controlled directed bipartite graph with  $|B| \geq F(r, p, q)$  then

- $-\exists S \in [A]^{\leq \binom{q}{2}}$  such that G-S contains a 1-scattered set of size p; or
- G contains a controlled crown of order q.

To see what this is good for, we first go on proving Theorem 4.3 assuming Lemma 4.6 and present the proof of Lemma 4.6 afterwards. Both of these proofs are based on several other intermediate lemmata.

**Lemma 4.7.** Let G be a directed graph,  $r \ge 0$ , and p, q > 0. Let I be an r-scattered set in G of size at least F(r, p, q) where F is the function defined in Lemma 4.6. Then  $S_q \preccurlyeq^d_r G$  or there is a set  $S \in [V(G)]^{\le \binom{q}{2}}$  s.t. G - S contains an (r+1)-scattered set of size p.

*Proof.* For  $u \in I$  let

$$P(u) := \{v : \text{there is a path of length} \le r \text{ from } v \text{ to } u \}.$$

By construction,  $P(v) \cap P(u) = \emptyset$  whenever  $u \neq v \in I$ .

We construct the following controlled directed bipartite graph  $(H:=(A\dot{\cup}B,E),\beta,\lambda,\eta)$ . Set B:=I. For each vertex  $v\in V(G)$ , if there are at least 2 distinct  $u,u'\in I$  reachable from v in G in at most r+1 steps, then we add  $\tilde{v}$  to A and initialise  $\lambda(\tilde{v}):=r+1$ . Then we do the following: for each pair  $\tilde{v}\in A$  and  $u\in B$  such that v can reach u in G in at most r+1 steps, we fix a shortest path  $v=v_i\dots v_0=u$  of length  $i\leq r+1$  in G. We add an edge  $\tilde{v}u$  to E, label e by  $\eta(e):=\{v_{i-1},\dots,v_0\}$ , and update  $\lambda(\tilde{v}):=\min\{i,\lambda(\tilde{v})\}$ . Note that in particular, we include edges  $e=\tilde{u}u$  for all  $u\in I$  and label it by  $\eta(e)=\emptyset$ . If  $v\in P(u)$  for some  $u\in I$ , then we define  $\beta(\tilde{v}):=\{u\}$  and otherwise  $\beta(\tilde{v}):=\emptyset$ . Note that by construction, (i) v can be in at most one set P(u); (ii)  $\beta(\tilde{v})=\emptyset\Leftrightarrow\lambda(\tilde{v})=r+1$ ; and (iii) our construction of  $\beta$  and  $\eta$  is fully conform to Definition 4.5 (a).

By Lemma 4.6, either there is a set  $\tilde{S} \subseteq A$  of size at most  $\binom{q}{2}$  and a set  $I' \subseteq B$  of size p which is 1-scattered in H - S, or there is a controlled crown  $S_q := (A' \cup B', E') \subseteq H$  of order q contained in H.

In the first case, let  $S \subseteq V(G)$  be the vertices in G corresponding to  $\tilde{S}$ . We claim that  $I' \subseteq I$  is (r+1)-scattered in G-S. For, suppose there were v,u,u' with  $v \in V(G) \setminus S$  and  $u,u' \in I'$  such that both u,u' are reachable from v in at most v+1 steps. Then  $\tilde{v} \in A$  and there are edges from  $\tilde{v}$  to u and u' in u, contradicting the fact that u' was 1-scattered in u in u

So suppose  $S_q:=(A'\cup B',E')\subseteq H$ . We claim that  $S_q\preccurlyeq^d_r G$  as follows. For  $\tilde{a}\in A'$  let  $T_a$  be the tree consisting only of a. For  $b\in B'$  let  $T_b$  be a spanning in-branching of  $\bigcup_{e=ab\in E'}\eta(e)$ , and note that  $T_b\subseteq P(b)$  clearly exists. By Definition 4.5 (b),  $A'\cap \eta(e)=\emptyset$  whenever  $e\in E'$  and therefore  $T_a\cap T_b=\emptyset$  for each  $\tilde{a}\in A'$  and  $b\in B'$ . Also, if  $\tilde{a}b\in E'$  then a has an edge to  $\eta(e)$  and hence  $T_a$  has an edge to  $T_b$ . This shows that  $\{T_a,T_b:\tilde{a}\in A',b\in B'\}$  are the branch sets of an  $S_q$ -minor of  $T_a$  of depth  $T_a$ .

We would like to emphasise that Lemma 4.7 works in particular for r = 0, i.e. when I is an arbitrary set in G which is large enough. Now we obtain our main theorem by repeated application of the above.

**Lemma 4.8.** If C is nowhere crownful then it is uniformly quasi-wide.

Proof. Let  $f: \mathbb{N} \cup \{0\} \to \mathbb{N}$  be the function such that  $S_{f(r)} \not \preccurlyeq^d_r G$  for all  $G \in \mathcal{C}$ . We construct the margins s(r) and N(r,m) witnessing that  $\mathcal{C}$  is uniformly quasi-wide. Let F be the function from Lemma 4.6. We define  $\tilde{N}(r,m,r) := m$ , and  $\tilde{N}(r,m,i) := F(r,\tilde{N}(r,m,i+1),f(i))$ . We let  $N(r,m) := \tilde{N}(r,m,0)$  and  $s(r) := \sum_{i=0}^{r-1} {f(i) \choose 2}$ . Let W be a set of size at least N(r,m) in  $G \in \mathcal{C}$ .

We let  $I_0 := W$ ,  $Z_0 := \emptyset$ . Assuming  $I_i$  and  $Z_i$  are given and that  $I_i$  is i-scattered in  $G - Z_i$  with  $|I_i| \ge \tilde{N}(r,m,i)$ , we construct  $I_{i+1}$  and  $Z_{i+1}$  as follows. By Lemma 4.7, we know that either  $S_{f(i)} \preccurlyeq_i^d G - Z_i \preccurlyeq_i^d G$  or there exists a set  $Z_i'$  of size at most  $\binom{f(i)}{2}$  and  $I_{i+1} \subseteq I_i$  of size  $\tilde{N}(r,m,i+1)$  such that  $I_{i+1}$  is (i+1)-scatterd in  $G - Z_i - Z_i'$ . Since the former cannot happen in our nowhere crownful class, the latter must be the case, and our claim is proved by defining  $Z_{i+1} := Z_i \cup Z_i'$  and letting  $S := Z_r$  and  $U := I_r$  be the output of the algorithm.

This finishes the proof of Theorem 4.3 based on Lemma 4.6. We proceed by proving the latter in a bottom-up fashion. Let us first state the following simple observation:

**Lemma 4.9.** If  $S_q = (A' \dot{\cup} B', E')$  is a crown in an r-controlled directed bipartite graph  $(G, \beta, \lambda, \eta)$  such that for all  $a \in A'$ ,  $\beta(a) \cap B' = \emptyset$ , then  $S_q$  is a controlled crown.

*Proof.* Suppose there is a  $v \in A'$  and  $e = ab \in E'$  such that  $v \in \eta(e)$ . But then we know by definition that  $\beta(v) = b \in B'$ , a contradiction.

Our plan is to apply several Ramsey-type arguments – which we state from an algorithmic viewpoint – to either find a large 1-scattered set, a controlled crown by way of Lemma 4.9, or a controlled crown in which all vertices of A have the same level according to  $\lambda$ . The latter restriction is crucial to our proof and will be applied when we invoke Lemma 4.10, a very useful auxiliary lemma stated below.

**Lemma 4.10.** There is a computable function  $f: \mathbb{N} \to \mathbb{N}$  such that if  $G:=K_{f(n)}$  and  $\gamma: E(G) \to [V(G)]^{\leq 1}$  s.t.  $\gamma(e) \cap e = \emptyset$  for all  $e \in E(G)$  then there is  $H \cong K_n \subseteq G$  such that  $\gamma(e) \cap V(H) = \emptyset$  for all  $e \in E(H)$ .

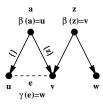
*Proof.* Let R(n) denote the  $n^{\text{th}}$  Ramsey number, i.e. the least integer m such that if we 2-color the edges of  $K_m$ , there always exists a monochromatic  $K_n$ . We define the function f(n) recursively as f(1) := 1 and  $f(n+1) := 1 + R(2 \cdot f(n))$ .

The lemma is proved by induction on n. For n=1 there is nothing to prove as any  $K_1$  contains a  $K_1$  as required. Suppose the statement is proved for n and let  $G\cong K_{f(n+1)}$  and  $\gamma$  be as required. Choose a vertex  $v\in V(G)$  and colour all edges e in  $G_v:=G-v$  by v if  $\gamma(e)=\{v\}$  and by  $\overline{v}$  otherwise. By Ramsey's theorem, as  $|G_v|=R(2\cdot f(n))$  either there is a set  $X\subseteq V(G_v)$  of size  $2\cdot f(n)$  such that all edges between elements of X are coloured v or there is such a set where all edges are coloured  $\overline{v}$ . In the first case, any  $X'\in [X]^{n+1}$  induces the required graph Y0 in Y2.

So suppose X induces a sub-graph where all edges are labelled  $\overline{v}$ . We construct a set X' as follows. Initially, set  $X' := \emptyset$ . While  $X \neq \emptyset$ , choose a vertex  $u \in X$ . Add u to X' and remove u and  $\gamma(\{v,u\})$  from X. As we are removing at most 2 elements of X in each step, we get a set X' of size at least f(n) with the property that for each  $u \in X'$ ,  $\gamma(\{v,u\}) \cap X' = \emptyset$ .

Let G' := G[X'] and  $\gamma'(e) := \gamma(e)$  for all  $e \in E[G']$ . As  $|G'| \ge f(n)$ , by the induction hypothesis, G' contains a subgraph  $H' \cong K_n$  such that  $\gamma(e) \cap V(H') = \emptyset$  for all  $e \in E(H')$ . Hence,  $H := G[V(H') \cup \{v\}]$  is the required subgraph of G isomorphic to  $K_{n+1}$  with  $\gamma(e) \cap V(H) = \emptyset$  for all  $e \in E(H)$ .

Next, we state and prove Lemma 4.11. An interesting point about this lemma is that it somewhat surprisingly includes two unrelated applications of Lemma 4.10, and in a sense, bears the delicate part that makes our overall proof work correctly.



**Figure 2.** Illustration of how to define  $\gamma$  in case (ii) of the proof of Lemma 4.11.

**Lemma 4.11.** Let  $(G := (A \dot{\cup} B, E), \beta, \lambda, \eta)$  be an r-controlled directed bipartite graph such that

- (i)  $\lambda$  is a constant function equal to  $c \in \{0, \dots, r+1\}$ ;
- (ii) any two vertices in B have a common neighbour in A; and
- (iii) every vertex in A has at most n successors.

If  $|B| \geq g(q,n) := (2n)^{2f(q)}$ , where f is as in Lemma 4.10, then G contains a controlled crown of order q.

*Proof.* For  $u, v \in B$ , let us say that u and v have a *red connection* if there is an  $a \in A$  with  $u, v \in N^+(a)$  such that  $\beta(a) \neq u$  and  $\beta(a) \neq v$ . If u and v are both neighbors to a vertex  $u' \in A$  with  $\beta(u') = u$  or  $\beta(u') = v$ , we say u and v have a *yellow connection*. Note that u and v can have both red and yellow connections. If the vertices u, u do not have any other neighbours, we say the connection is *pure*.

Let  $D_0 := B$  and let  $R_0 := Y_0 := \emptyset$ . Suppose  $D_i$ ,  $R_i$ , and  $Y_i$  have been defined maintaining the invariants that

- any vertex v in  $R_i$  has a pure red connection to any vertex in  $R_i \cup D_i \setminus \{v\}$ ; and
- any vertex v in  $Y_i$  has a pure yellow connection to any vertex in  $Y_i \cup D_i \setminus \{v\}$ .

Choose an arbitrary vertex  $v_{i+1} \in D_i$ , remove it from  $D_i$ , and set  $D_R = D_Y = \emptyset$ . For  $j = 1 \dots 2 \cdot (2n)^{2f(q) - (i+1)}$  do the following. Choose  $u_j \in D_i$  and  $a \in A$  with  $\{v_{i+1}, u_j\} \subseteq N^+(a)$ . If a induces a red connection between  $v_{i+1}$  and  $u_j$ , put  $u_j$  in  $D_R$ , otherwise put  $u_j$  in  $D_Y$ . Delete  $N^+(a)$  from  $D_i$ .

As no  $a \in A$  has more than n successors, this process can complete successfully. At this time, either  $D_R$  or  $D_Y$  have at least  $(2n)^{2f(q)-(i+1)}$  elements. If  $|D_R| \ge |D_Y|$ , define  $D_{i+1} := D_R$ ,  $R_{i+1} := R_i \cup \{v_{i+1}\}$ , and  $Y_{i+1} := Y_i$ ; otherwise let  $D_{i+1} := D_Y$ ,  $Y_{i+1} := Y_i \cup \{v_{i+1}\}$ , and  $R_{i+1} := R_i$ . After 2f(q) iterations, either  $R_{2f(q)}$  or  $Y_{2f(q)}$  contain at least f(q) vertices.

Case (i),  $|R_{2f(q)}| \geq f(q)$ : We construct a complete undirected graph  $G_R := (V_R, E_R)$  with  $V_R := R_{2f(q)}$ . For vertices  $u, v \in V_R$  and  $e = uv \in E_R$ , let  $a \in A$  be the vertex inducing the pure red connection between u and v, and define  $\gamma(e) := \beta(a)$ . As  $\beta(a) \neq u$  and  $\beta(a) \neq v$ , we have that G and  $\gamma$  fulfil the requirements of Lemma 4.10, and we obtain a clique  $H_R \subseteq G_R$  of size q such that  $\gamma(e) \cap V(H_R) = \emptyset$  for all  $e \in E(H_R)$ . This translates to a controlled crown of order q in G by way of Lemma 4.9.

Case (ii),  $|Y_{2f(q)}| \geq f(q)$ : We construct a complete undirected graph  $G_Y := (V_Y, E_Y)$  with  $V_Y := Y_{2f(q)}$ . As shown in Figure 2, for vertices  $u, v \in V_Y$  and  $e = uv \in E_Y$ , let  $a \in A$  be the vertex inducing the pure yellow connection between u and v, and w.l.o.g. assume  $\beta(a) = u$ . Let  $e_1 = au$  and  $e_2 = av$ . Note that for all  $z' \in \eta(e_1)$ , we know that  $\lambda(z') < \lambda(a) = c$ , and hence  $z' \notin A$ . In  $\eta(e_2)$ , there exists at most one z with  $\lambda(z) = c$ . If no such z exists or z does not occur as a pure yellow connection between vertices of  $Y_{2f(q)}$ , define  $\gamma(e) := \emptyset$ . Otherwise, we know by Definition 4.5 (a) that  $\beta(z) = v$ , and hence z induces a pure yellow connection between v and some vertex  $v \in Y_{2f(q)}$ . If v = u, then  $v \in V_{2f(q)}$  is redundant and we delete it from  $v \in V_{2f(q)}$  and set  $v \in V_{2f(q)}$  be for all  $v \in V_{2f(q)}$ . If  $v \in V_{2f(q)}$  is redundant and we delete it from  $v \in V_{2f(q)}$  and set  $v \in V_{2f(q)}$  be for all  $v \in V_{2f(q)}$ . If  $v \in V_{2f(q)}$  is redundant and we delete it from  $v \in V_{2f(q)}$  and set  $v \in V_{2f(q)}$  be for all  $v \in V_{2f(q)}$  in the  $v \in V_{2f(q)}$  such a vertex  $v \in V_{2f(q)}$  of size  $v \in V_{2f(q)}$  of size  $v \in V_{2f(q)}$  in the count in the crown induced by  $v \in V_{2f(q)}$  in  $v \in V_{2f(q)}$  induces a controlled crown in  $v \in V_{2f(q)}$  in  $v \in V_$ 

**Lemma 4.12.** Let  $f: \mathbb{N}^4 \to \mathbb{N}$  be defined as  $f(r, p, q, n) := (r+3)^{p+(r+2)\cdot g(q,n)}$ , where g(q, n) is the function defined in Lemma 4.11. Let  $(G:=(A, \dot{\cup}B, E), \beta, \lambda, \eta)$  be an r-controlled directed bipartite graph. For all  $r \geq 0$  and p, q, n > 0, if |B| > f(r, p, q, n) then

- 1. A contains a vertex with n + 1 successors; or
- 2. B contains a 1-scattered set of size p; or
- 3. G contains a controlled crown of order q.

*Proof.* If there is a vertex  $a \in A$  of degree at least n+1 we are done. So suppose all vertices  $a \in A$  have degree at most n. We are going to define a sequence of sets  $B_i, I_i, D_i^0, \dots, D_i^{r+1}$  with the invariant that

- $|B_i| \ge (r+3)^{p+(r+2)\cdot g(q,n)-i}$ ,
- no two  $v \neq v' \in I_i$  and no  $v \in I$  and  $u \in B_i$  have a common predecessor in A,
- for  $j=0,\ldots,r+1$ , any two  $v,v'\in D_i^j$  and any  $v\in D_i^j$  and  $u\in B_i$  have a common predecessor in  $a\in A$  with

Set  $I_0=D_0^0=\cdots=D_0^{r+1}:=\emptyset$  and  $B_0:=B$  which clearly satisfies the invariant. Now suppose  $i< p+(r+2)\cdot g(q,n)$  and  $B_i,I_i,D_i^0,\ldots,D_i^{r+1}$  have been defined. Choose  $v\in B_i$ . For  $j=0,\ldots,r+1$ , define  $A^j:=\{a\in N^-(v):\lambda(a)=j\}$  and  $B^j:=N^+(A_j)\cap B_i$ . Let  $B':=B_i\setminus\bigcup_j B^j$ .

- If  $|B'| \ge \frac{1}{r+3}|B_i|$  then  $B_{i+1} := B'$ ,  $I_{i+1} := I_i \cup \{v\}$ , and  $D_{i+1}^j := D_i$ . Clearly, this maintains the invariant. Otherwise, there must exist a t s.t.  $|B^t| \ge \frac{1}{r+3}|B_i|$ ; define  $B_{i+1} := B^t$ ,  $I_{i+1} := I_i$ ,  $D_{i+1}^t := D_i^t \cup \{v\}$ , and  $D_{i+1}^j := D_i^j$  for all  $j \neq t$ . Again, the invariant is maintained.

Now if for some i we have  $|I_i| = p$ , then we have found a 1-scattered set of size p. Otherwise, there must exist a ts.t.  $|D_{p+(r+2)\cdot q(q,n)}^t| \ge g(q,n)$ . Applying Lemma 4.11 to this set results in a controlled crown of order q.

Finally, we are ready to prove Lemma 4.6, concluding this section.

Proof (of Lemma 4.6). Let  $\tilde{F}(r,p,q,0) := q$  and  $\tilde{F}(r,p,q,t) := f(r,p,q,\tilde{F}(r,p,q,t-1))$ , where f is as in Lemma 4.12. Define  $F(r, p, q) := \tilde{F}(r, p, q, \binom{q}{2})$  and assume  $|B| \ge F(r, p, q)$ .

We are going to construct a sequence  $S_i \subseteq A, B_i \subseteq B$  as follows, where we will maintain the invariant that  $|B_i| \geq \tilde{F}(r, p, q, \binom{q}{2} - i)$  and that for all  $v \in S_i$ ,  $B_i \subseteq N^+(v)$  and  $\beta(v) \cap B_i = \emptyset$ .

Set  $S_0 := \emptyset$  and  $B_0 := B$ . Now suppose  $S_i, B_i$  have been defined. If  $A \setminus S_i$  contains a vertex a such that  $|N^+(v)\cap B_i|>\tilde{F}(r,p,q,\binom{q}{2}-i-1)$  then set  $S_{i+1}:=\{v\}\cup S_i$  and  $B_{i+1}:=(B_i\cap N^+(v))\setminus\beta(v)$ . Clearly, this maintains the invariant.

If we can continue this process until we obtain a set  $S := S_{\binom{q}{2}}$ , then S together with any subset of  $B_{\binom{q}{2}}$  of size q contain an  $H \cong S_q$ , which is in fact a controlled crown by Lemma 4.9.

Otherwise, if at some point  $A \setminus S_i$  does not contain a vertex a such that  $|N^+(v) \cap B_i| > \tilde{F}(r, p, q, \binom{q}{2} - i - 1)$ , then, applying Lemma 4.12 to  $(G' := G[A \cup B_i \setminus S_i], \beta', \lambda, \eta)$ , where  $\beta'(v) := \emptyset$  if  $\beta(v) \notin B_i$  and  $\beta'(v) := \beta(v)$ otherwise, we obtain either a controlled crown of order q or a 1-scattered set  $I \subseteq B_i$  of size p in G'. But then, setting  $S := S_i$  we see that I is 1-scattered in G - S.

### **Algorithmic Results**

The aim of this section is to show that problems such as directed dominating sets, directed independent dominating set (Kernel problem), and many variations are fixed-parameter tractable on any class of directed graphs which is nowhere crownful. Domination problems on nowhere dense classes of undirected graphs have been studied in [DK09] and some of the ideas developed there can be adapted to the directed setting.

As a first step towards algorithmic applications we show that we can efficiently compute scattered sets in graph classes which are effectively uniformly quasi-wide.

**Lemma 5.1.** Let C be effectively uniformly quasi-wide with margin s, N. The following problem is fixed-parameter tractable.

```
Input: G \in \mathcal{C}, d, m \in \mathbb{N}, W \subseteq V(G) such that |W| \ge N(d, m)
Parameter: d + m
  Problem: Compute S \subseteq V(G), |S| \le s(d) and U \subseteq W, |U| = m
             such that U is d-scattered in G - S
```

*Proof.* To compute U, we first choose an arbitrary subset  $U' \subseteq W$  of size exactly N(d, m). By definition of uniformly quasi-wideness, we are guaranteed to find a set S and a set  $U \subseteq U'$  as required. But |U'| depends only on the parameter and hence we can simply traverse through all subsets U of U' of size at least m. For each such set U we compute  $C:=\bigcup_{u\neq u'\in U}N_d^-(u)\cap N_d^-(u'). \text{ Clearly, } U\setminus C \text{ is } d\text{-scattered in } G-C \text{ and } C \text{ can be computed in time } \mathcal{O}(|U|^2\cdot |G|).$ Hence, if  $|U \setminus C| \ge m$  and  $|C| \le s(d)$  we return U and S := C. П We briefly recall the definition of the algorithmic problems we refer to below. As defined in Section 2, a *d*-dominating set is a set  $D \subseteq V(G)$  such that  $N_d^+(D) = V(G)$  and is just called a *dominating set* for d = 1. An independent dominating set is a dominating set D which is itself independent, i.e.  $(u, v) \notin E(G)$  for all  $u, v \in D$ . The corresponding decision problems are defined as the problem, given a digraph G and a number K, to decide whether G contains an (independent or unrestricted) G-dominating set of size G. The parameter is G-dominating set of size G-dominating set of s

Recall that an important variant of the undirected dominating set problem is the connected dominating set problem, where we are asked to find a dominating set D of size k such that D induces a connected subgraph. There are various natural translations of this problem to the directed case: we can require the dominating set to induce a strongly connected subgraph or we can simply require it to induce an out-branching, i.e. a directed tree whose edges are oriented away from a root. The second variation, which we call *dominating out-branching*, still captures the idea that information can flow from the root to all vertices in the dominating set.

We now establish our main algorithmic result of the paper where we show that many covering problems become fixed-parameter tractable on classes of directed graphs which are nowhere crownful.

**Theorem 5.2.** Let C be a class of directed graphs which is nowhere crownful. Then the directed (independent or unrestricted) dominating set problem, the dominating out-branching problem, and the independent set problem as well as their distance-d-versions are fixed-parameter tractable on C.

We present the proof of Theorem 5.2 for the cases of independent dominating set, d-dominating set, and dominating out-branching. This captures the essence of the ideas involved in designing such algorithms for directed uniformly quasi-wide classes. The other problems and some variations such as Red-Blue domination or Roman domination follow similarly. See [Ces06] for detailed definitions of these problems.

Proof (of Theorem 5.2 for the case of independent dominating set). Let  $G \in \mathcal{C}$  and k be given. We solve a slightly more general version, where we are additionally given a set  $Y \subseteq V(G)$  and require that the solution belongs to  $V(G) \setminus Y$ . If  $|V(G)| \leq N(1, k+1)$ , then we solve the problem by exhaustively searching for all possible sets of k vertices.

Otherwise, we apply Lemma 5.1 to obtain a set  $U \subseteq V(G)$  of size k+1 and a set S of at most s(1) vertices such that U is 1-scattered in G-S. As U is 1-scattered and of size k+1, it follows that any dominating set must contain at least one vertex from S. Hence, it suffices that for each  $v \in S \setminus Y$ , we recursively solve the problem on  $G' := G - s - N^+(s)$  and  $Y' := Y \cup N^-(s)$  with k' := k-1 and report a positive answer in case one is found. This results in a recursion tree of depth at most k and degree at most s(1) which are both bounded in terms of our parameters.

Next, we consider the d-dominating set problem. The following lemmata are proved for the undirected version in [DK09] and the proofs work exactly as they are for the directed version as well.

**Lemma 5.3.** The following problem is fixed-parameter tractable.

```
Input: a directed graph G,W\subseteq V(G) and k,d\in\mathbb{N} Parameter: |W|,k,d Problem: Does there exist a set X\in [V(G)]^{\leq k} such that X d-dominates W?
```

**Lemma 5.4.** Let C be effectively uniformly quasi-wide with margin s, N. The following problem is fixed-parameter tractable. Furthermore, such a vertex w always exists.

```
\begin{array}{c} \textit{Input: } G \in \mathcal{C}, k, d \in \mathbb{N}, W \subseteq V(G) \text{ such that } |W| > N(\overline{d, (k+2)(d+1)^{s(d)}}) \\ \textit{Parameter: } k+d \\ \textit{Problem: } \text{Compute a vertex } w \in W \text{ such that for any set } X \in [V(G)]^{\leq k}, X \text{ $d$-dominates } W \text{ if and only if } X \text{ $d$-dominates } W \setminus \{w\}. \end{array}
```

In order to solve the d-dominating set problem, we start with W = V(G) and apply Lemma 5.4 until the size of W becomes bounded by  $N(d, (k+2)(d+1)^{s(d)})$ . Then we can apply Lemma 5.3.

Finally, let us consider the dominating out-branching problem. We first need the following lemma, which is proved by using the fixed-parameter tractability of *directed Steiner tree* in general graphs.

**Lemma 5.5.** The following problem is fixed-parameter tractable.

```
Input: a directed graph G, u_1, \ldots, u_t \in V(G) and W \subseteq V(G), j \in \mathbb{N}
Parameter: |W|, t+j
Problem: Do there exist distinct vertices v_1, \ldots, v_j such that G[v_1, \ldots, v_j, u_1, \ldots, u_t] is an out-branching and W \subseteq N^+(v_1, \ldots, v_j)?
```

*Proof.* Suppose there are distinct vertices  $v_1, \ldots, v_j$  such that  $G[v_1, \ldots, v_j, u_1, \ldots, u_t]$  is an out-branching and  $W \subseteq N^+(v_1, \ldots, v_j)$ . Then we can partition W into (possibly empty) sets  $W_1, \ldots, W_j$  so that  $W_i \subseteq N^+(v_i)$ . Let

$$X_i := \{ x \in V(G) : W_i \subseteq N^+(x) \};$$

then  $v_i \in X_i$  for all i. Hence, to decide whether such vertices exists it suffices to compute all possible partitions of W into sets  $W_1, \ldots, W_j$  (some of which may be empty), for each partition the sets  $X_i$ , and then check if an out-branching of size at most t+j exists that includes  $u_1, \ldots, u_t$  and at least one element out of each  $X_i$ .

But this problem can easily be transformed to a *directed Steiner tree* instance: for each  $X_i$  add a veretex  $x_i$  and for each  $v \in X_i$  a directed edge  $vx_i$ . Now, the goal is to find a directed Steiner tree, i.e. an out-branching, of size at most t+2j on the terminal set  $\{u_1,\ldots,u_t,x_1,\ldots,x_j\}$ . Since the directed Steiner tree problem is fixed-parameter tractable (see e.g. Guo et al. [GNS09]) and the size of W is part of the parameter, all steps can be erformed in fpt time.  $\square$ 

Proof (of Theorem 5.2 for the case of dominating out-branching). Let  $G \in \mathcal{C}$  and k be given. We will recursively solve a slightly more general problem where the input consists of a graph G, a set  $W \subseteq V(G)$ , a number j and vertices  $u_1, \ldots, u_t \in V(G)$ . The problem is to decide whether there are j vertices  $v_1, \ldots, v_j$  such that  $G[u_1, \ldots, u_t, v_1, \ldots, v_j]$  is an out-branching and  $\{v_1, \ldots, v_j\}$  dominate W. The parameter is t+j.

Solving the problem for W := V(G), t := 0 and j := k solves the dominating out-branching problem.

If  $|W| \leq N(1,t+j+1)$ , then we can apply Lemma 5.5. Otherwise, let  $S \subseteq V(G)$  and  $A \subseteq W$  be such that |A| > t+j and A is 1-scattered in G-S. This implies that every dominating set must contain a vertex of S. For each  $u_{t+1} \in S$  we call the algorithm recursively on  $G, u_1, \ldots, u_{t+1}, W \setminus N^+(u_{t+1}), j-1$  to check whether  $u_1, \ldots, u_{t+1}$  can be extended to a dominating out-branching. This results in a recursion tree of depth at most k and degree at most s(1) which are both bounded in terms of our parameters.

To adapt this proof idea to a solution of the strongly connected dominating set problem we would need to solve the directed strongly connected Steiner subgraph problem, i.e. the problem, given vertices  $t_1, \ldots, t_p$  in a digraph G and a number k, to decide whether there are k vertices  $v_1, \ldots, v_k$  so that  $t_1, \ldots, t_p, v_1, \ldots, v_k$  induce a strongly connected sub-graph. This problem is known *not* to be fixed-parameter tractable (with parameter k+p) on general directed graphs but it might become fpt on nowhere crownful classes of graphs. We leave this for future research.

# 6 Conclusion

In this paper we have given a classification of classes of directed graphs that is fundamentally different from the existing proposals for directed width measures based on tree-width. We have seen that even for the very general concept of nowhere crownful classes of directed graphs, "covering" problems such as the (independent) dominating set problem become fixed-parameter tractable, problems that are not tractable on classes of bounded width for any other width measure (except bi-rank width, see the introduction).

We believe that our proposal here may lead to a new and promising structural theory for directed graphs with algorithmic applications in mind. However, this clearly is only a first step and many questions remain open.

In particular, linkage problems become (XP-)tractable on classes of directed graphs of bounded directed tree-width [JRST01] but we do not presently know whether they are tractable on any of our classes. But from this it follows that if we consider classes of directed graphs which have bounded directed tree-width and at the same time are nowhere crownful, then on such classes we can solve a wide range of problems efficiently, both linkage and covering problems. Clearly, this definition is extremely technical and as such not very useful. But it shows that there are structural concepts for directed graphs which do allow for a very rich algorithmic theory and we believe it would be worth studying these concepts in more detail.

Additionally, from a structural point of view, the perspective presented in our work opens a number of intriguing problems to look at. For example, a slight adaption of Theorem 3.14 shows that if a class of digraphs has unbounded undirected tree-width, then it contains arbitrarily large alternating cycles, i.e. an orientation of a cycle that contains

a long alternating path minor. In terms of the directed minor relation, this results in an *anti-chain* in the class. One question is, if we exclude such anti-chains, does a minor-closed class become *well quasi-ordered*? Since such a class must have bounded tree-width, it seems plausible that the answer might be positive.

Another structural question relates to *colourings* of our classes. It is easily seen that classes of directed bounded expansion can be coloured by a constant number of colours. However, could it be possible that more powerful colourings are possible, such as acyclic colourings or colourings that induce some other bounded width measure (cf.[NO08a])? This seems to be a quite difficult question as known methods for undirected graphs do not generalise. However, it might be that at least on the intersection of classes of directed bounded expansion and crown-minor-free, one can obtain some results.

### References

- [Bar06] János Barát. Directed path-width and monotonicity in digraph searching. *Graphs and Combinatorics*, 22(2):161–172, 2006.
- [BDHK06] D. Berwanger, A. Dawar, P. Hunter, and S. Kreutzer. Dag-width and parity games. In *Symp. on Theoretical Aspects of Computer Science (STACS)*, 2006.
- [Bod93] Hans L. Bodlaender. A tourist guide through treewidth. Acta Cybern., 11(1-2):1-22, 1993.
- [Bod97] H. L. Bodlaender. Treewidth: Algorithmic techniques and results. In Proc. of Mathematical Foundations of Computer Science (MFCS), volume 1295 of Lecture Notes in Computer Science, pages 19–36, 1997.
- [Bod98] H. Bodlaender. A partial k-aboretum of graphs with bounded tree-width. *Theoretical Computer Science*, 209:1 45, 1998.
- [Bod05] Hans L. Bodlaender. Discovering treewidth. In SOFSEM, pages 1–16, 2005.
- [Ces06] Marco Cesati. Compendium of parameterized problems. Technical report, University of Rome, 2006.
- [Daw10] Anuj Dawar. Homomorphism preservation on quasi-wide classes. J. Comput. Syst. Sci., 76(5):324–332, 2010.
- [DF98] R. Downey and M. Fellows. *Parameterized Complexity*. Springer, 1998.
- [DGK09] Peter Dankelmann, Gregory Gutin, and Eun Jung Kim. On complexity of minimum leaf out-branching problem. *Discrete Applied Mathematics*, 157(13):3000–3004, 2009.
- [Die05] R. Diestel. *Graph Theory*. Springer-Verlag, 3rd edition, 2005.
- [DK09] A. Dawar and S. Kreutzer. Domination problems in nowhere-dense classes of graphs. In *Foundations of software technology and theoretical computer science (FSTTCS)*, 2009.
- [Epp95] David Eppstein. Finding common ancestors and disjoint paths in DAGs. Technical Report 95-52, University of California, Irvine, CA, 1995.
- [FG06] J. Flum and M. Grohe. Parameterized Complexity Theory. Springer, 2006. ISBN 3-54-029952-1.
- [GHK<sup>+</sup>] Robert Ganian, Petr Hliněný, Joachim Kneis, Alexander Langer, Jan Obržálek, and Peter Rossmanith. On digraph width measures in parameterized algorithmics. available on arxiv.org.
- [GNS09] Jiong Guo, Rolf Niedermeier, and Ondrej Suchý. Parameterized complexity of arc-weighted directed steiner problems. In Yingfei Dong, Ding-Zhu Du, and Oscar H. Ibarra, editors, *ISAAC*, volume 5878 of *Lecture Notes in Computer Science*, pages 544–553. Springer, 2009.
- [HK08] P. Hunter and S. Kreutzer. Digraph measures: Kelly decompositions, games, and ordering. *Theoretical Computer Science (TCS)*, 399(3), 2008.
- [JRST01] Thor Johnson, Neil Robertson, Paul D. Seymour, and Robin Thomas. Directed tree-width. *J. Comb. Theory, Ser. B*, 82(1):138–154, 2001.
- [KLM08] Georgia Kaouri, Michael Lampis, and Valia Mitsou. On the algorithmic effectiveness of digraph decompositions and complexity measures. Submitted to and rejected from ICALP 2008. It is the ICALP version that the pdf link points to., 2008.
- [KO08] S. Kreutzer and S. Ordyniak. Digraph decompositions and monotonocity in digraph searching). In *34th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, 2008.
- [KR09] Mamadou Moustapha Kanté and Michaël Rao. Directed rank-width and displit decomposition. In *Workshop on Graph-Theoretical Concepts in Computer Science (WG)*, pages 214–225, 2009.
- [NO08a] J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion I–III. *European Journal of Combinat-orics*, 29, 2008. Series of 3 papers appearing in volumes (3) and (4).
- [NO08b] Jaroslav Nešetřil and Patrice Ossona de Mendez. On nowhere dense graphs. *European Journal of Combinatorics*, 2008. submitted.
- [NO10] J. Nešetřil and P. Ossona de Mendez. First order properties on nowhere dense structures. *Journal of Symbolic Logic*, 75(3):868–887, 2010.
- [Obd06] Jan Obdrzálek. Dag-width: connectivity measure for directed graphs. In *Symp. on Discrete Algorithms (SODA)*, pages 814–821, 2006.
- [PS78] Y. Perl and Y. Shiloach. Finding two disjoint paths between two pairs of vertices in a graph. J. ACM, 1(25):1–9, 1978.

- [Ree99] B. Reed. Introducing directed tree-width. *Electronic Notes in Discrete Mathematics*, 3:222 229, 1999.
- [RS86] N. Robertson and P. D. Seymour. Graph minors V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.
- [RS] N. Robertson and P.D. Seymour. Graph minors I XXIII, 1982 –. Appearing in Journal of Combinatorial Theory, Series B since 1982.
- [Saf05] M. A. Safari. D-width: A more natural measure for directed tree width. In *Proc. of Mathematical Foundations of Computer Science (MFCS)*, number 3618 in Lecture Notes in Computer Science, pages 745 756, 2005.