The Space Complexity of 2-Dimensional Approximate Range Counting and Combinatorial Discrepancy*

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Abstract

We study the problem of 2-dimensional orthogonal range counting with additive error. Given a set P of n points drawn from an $n \times n$ grid and an error parameter ε, the goal is to build a data structure, such that for any orthogonal range R, it can return the number of points in $P \cap R$ with additive error εn. A well-known solution for this problem is the *ε-approximation*, which is a subset $A \subseteq P$ that can estimate the number of points in $P \cap R$ with the number of points in $A \cap R$. It is known that an ε-approximation of size $O(\frac{1}{\varepsilon}\log^{2.5}\frac{1}{\varepsilon})$ exists for any P with respect to orthogonal ranges, and the best lower bound is $\Omega(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$.

The ε-approximation is a rather restricted data structure, as we are not allowed to store any information other than the coordinates of the points in P. In this paper, we explore what can be achieved without any restriction on the data structure. We first describe a simple data structure that uses $O(\frac{1}{\varepsilon}(\log^2\frac{1}{\varepsilon} + \log n))$ bits and answers queries with error εn. We then prove a lower bound that any data structure that answers queries with error εn must use $\Omega(\frac{1}{\varepsilon}(\log^2\frac{1}{\varepsilon} + \log n))$ bits. Our lower bound is information-theoretic: We show that there is a collection of $2^{\Omega(n\log n)}$ point sets with large *union combinatorial discrepancy*, and thus are hard to distinguish unless we use $\Omega(n\log n)$ bits.

1 Introduction

Range counting is one of the most fundamental problems in computational geometry and data structures. Given n points in d dimensions, the goal is to preprocess the points into a data structure, such that the number of points in any query range can be returned. Range counting has been studied intensively, and a lot of work has focused on the space-query time tradeoff or the update-query tradeoff of the data structure. We refer the reader to the survey by Agarwal and Erickson [2] for these results. In this paper, we look at the problem from a data summarization/compression point of view: What is the minimum amount of space that is needed to encode all the range counts approximately? Approximation is necessary here, since otherwise we will have to remember the entire the point set. It is also easy to see that relative approximation will not help either, as it requires us to differentiate between empty ranges and those containing only one point. Thus, we aim at an absolute error guarantee. As we will be dealing with bit-level space complexity, it is convenient to focus on an integer grid. More formally, we are given a set of n points P drawn from

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an $n \times n$ grid and an error parameter ε . The goal is to build a data structure, such that for any orthogonal range R, the data structure can return the number of points in $P \cap R$ with additive error εn.

We should mention that there is another notion of approximate range counting that approximates the range, i.e., points near the boundary of the range may or may not be counted [5]. Such an approximation notion clearly precludes any sublinear-space data structure as well.

1.1 Background and related results

ε-approximations. Summarizing point sets while preserving range counts (approximately) is a fundamental problem with applications in numerical integration, statistics, and data mining, among many others. The classical solution is to use the *ε-approximation* from discrepancy theory. Consider a range space $(P, \mathcal{R})$, where P is a finite point set of size n. A subset $A \subseteq P$ is called an ε-approximation of $(P, \mathcal{R})$ if

$$\max_{R \in \mathcal{R}} \left| \frac{|R \cap A|}{|A|} - \frac{|R \cap P|}{|P|} \right| \leq \varepsilon.$$

This means that we can approximate $|R \cap P|$ by counting the number of points in $R \cap A$ and scaling back, with error at most εn.

Finding ε-approximations of small size for various geometric range spaces has been a central research topic in computational geometry. Please see the books by Matousek [19] and Chazelle [9] for a comprehensive coverage on this topic. Here we only review the most relevant results, i.e., when the range space is the set of all orthogonal rectangles in 2 dimensions, which we denote as $\mathcal{R}_2$. This question dates back to Beck [7], who showed that there are ε-approximations of size $O(\frac{1}{\varepsilon} \log^4 \frac{1}{\varepsilon})$ for any point set P. This was later improved to $O\left(\frac{1}{\varepsilon} \log^{2.5} \frac{1}{\varepsilon}\right)$ by Srinivasan [25]. These were not constructive due to the use of a non-constructive coloring with combinatorial discrepancy $O(\log^{2.5} n)$ for orthogonal rectangles. Recently, Bansal [6] and Lovett et al. [17] proposed algorithms to construct such a coloring, and therefore has made these results constructive. On the lower bound side, it is known that there are point sets that require ε-approximations of size $\Omega(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ [7].

Combinatorial discrepancy. Given a range space $(P, \mathcal{R})$ and a coloring function $\chi : P \to \{-1, +1\}$, we define the *discrepancy* of a range $R \in \mathcal{R}$ under χ to be

$$\chi(P \cap R) = \sum_{p \in P \cap R} \chi(p).$$

The discrepancy of the range space $(P, \mathcal{R})$ is defined as

$$\operatorname{disc}(P, \mathcal{R}) = \min_{\chi} \max_{R \in \mathcal{R}} |\chi(P \cap R)|,$$

namely, we are looking at the coloring that minimizes the color difference of any range in $\mathcal{R}$. This kind of discrepancy is called *combinatorial discrepancy* or sometimes *red-blue discrepancy*. Taking the maximum over all point sets of size n, we say that the combinatorial discrepancy of $\mathcal{R}$ is $\operatorname{disc}(n, \mathcal{R}) = \max_{|P|=n} \operatorname{disc}(P, \mathcal{R})$.

There is a close relationship between combinatorial discrepancy and ε-approximations, as observed by Beck [7]. For orthogonal ranges, the relationship is particularly simple: The combinatorial discrepancy is at most $t(n)$ if and only if there is an ε-approximation of size $O(\frac{1}{\varepsilon}t(\frac{1}{\varepsilon}))$. In fact, all the aforementioned results on ε-approximations follow from the corresponding results on combinatorial discrepancy. So the current upper bound on the combinatorial discrepancy of $\mathcal{R}_2$ is $O(\log^{2.5} n)$ [25]. The lower bound is $\Omega(\log n)$ [7], which follows from the Lebesgue discrepancy lower bound (see below). Closing the $\Theta(\log^{1.5} n)$ gap between the upper and the lower bound remains a major open problem in discrepancy theory. For orthogonal ranges in $d \geq 3$ dimensions, the current best upper bound is $O(\log^{d+1/2} n)$ by Larsen [16], while the lower bound is $\Omega((\log n)^{d-1})$, which is recently proved by Matoušek and Nikolov [18].

Lebesgue discrepancy. Suppose the points of P are in the unit square $[0,1)^2$. The Lebesgue discrepancy of $(P, \mathcal{R})$ is defined to be

$$D(P, \mathcal{R}) = \sup_{R \in \mathcal{R}} \left| |P \cap R| - \left| R \cap [0,1)^2 \right| \right|.$$

The Lebesgue discrepancy describes how uniformly the point set P is distributed in $[0,1)^2$. Taking the infimum over all point sets of size n, we say that the Lebesgue discrepancy of $\mathcal{R}$ is $D(n, \mathcal{R}) = \inf_{|P|=n} D(P, \mathcal{R})$.

The Lebesgue discrepancy for $\mathcal{R}_2$ is known to be $\Theta(\log n)$. The lower bound is due to Schmidt [23], while there are many point sets (e.g., the Van der Corput sets [26] and the b-ary nets [24]) that are proved to have $O(\log n)$ Lebesgue discrepancy. It is well known that the combinatorial discrepancy of a range space cannot be lower than its Lebesgue discrepancy, so this also gives the $\Omega(\log n)$ lower bound on the combinatorial discrepancy of $\mathcal{R}_2$ mentioned above.

ε-nets. For a range space $(P, \mathcal{R})$, a subset $A \subseteq P$ is called an ε-net of P if for any range $R \in \mathcal{R}$ that satisfies $|P \cap R| \geq \varepsilon n$, there is at least 1 point in $A \cap R$. Note that an ε-approximation is an ε-net, but the converse may not be true.

For a range space $(P, \mathcal{A})$, Haussler and Welzl [14] show that if the range space has finite VC-dimension d, there exists an ε-net of size $O(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon})$. For $\mathcal{R}_2$, the current best construction is due to Aronv, Ezra and Sharir [3], which has size $O(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon})$. A recent result by Pach and Tardos [20] shows that this bound is essentially optimal. For more results on ε-nets, please refer to the book by Matoušek [19]. In this paper, our data structure will be based an *ε-net* for $\mathcal{R}_2$.

Approximate range counting data structures. The ε-approximation is a rather restricted data structure, as we are not allowed to store any information other than the coordinates of a subset of points in P. In this paper, we explore what can be achieved without any restriction on the data structure. In 1 dimension, there is nothing better: An ε-approximation has size $O(\frac{1}{\varepsilon})$, which takes $O(\frac{1}{\varepsilon}\log n)$ bits. On the other hand, simply consider the case where the n points are divided into groups of size εn, where all points in each group have the same location. There are $n^{1/\varepsilon}$ such point sets and the data structure has to differentiate all of them. Thus $\log(n^{1/\varepsilon}) = \frac{1}{\varepsilon}\log n$ is a lower bound on the number of bits used by the data structure.

Finally, we remark that there are also other work on approximate range counting with various error measure, such as relative ε-approximation [13], relative error data structure [1, 4], and absolute error model [5]. These error measures are different from ours, and it is not clear if these problems admit sublinear space solutions.

1.2 Our results

This paper settle the following problem: How many bits do we need to encode all the orthogonal range counts with additive error εn for a point set on the plane? We first show that if we are allowed to store extra information other than the coordinates of the points, then there is a data structure that uses $O(\frac{1}{\varepsilon}(\log^2\frac{1}{\varepsilon}+\log n))$ bits. This is a $\Theta(\log^{1.5}\frac{1}{\varepsilon})$ improvement from ε-approximations.

The majority of the paper is the proof of a matching lower bound: We show that for $\varepsilon \geq c\log n/n$ for some constant c, any data structure that answers queries with error εn must use $\Omega(\frac{1}{\varepsilon}(\log^2\frac{1}{\varepsilon}+\log n))$ bits. In particular, if we set $\varepsilon = c\log n/n$, then any data structure that answers queries with error εn must use $\Omega(n\log n)$ bits, which implies that that answering queries with error $O(\log n)$ is as hard as answering the queries exactly.

The core of our lower bound proof is the construction of a collection $\mathcal{P}^*$ of $2^{\Omega(n\log n)}$ point sets with large *union combinatorial discrepancy*. More precisely, we show that the union of any two point sets in $\mathcal{P}^*$ has high combinatorial discrepancy, i.e., at least $c\log n$. Then, for any two point sets $P_1, P_2 \in \mathcal{P}^*$, if $\mathrm{disc}(P_1 \cup P_2, \mathcal{R}_2) \geq c\log n$, that means for any coloring χ on $P_1 \cup P_2$, there must exist a rectangle R such that $|\chi(R)| \geq c\log n$. Consider the coloring χ where $\chi(p) = 1$ if $p \in P_1$ and $\chi(p) = -1$ if $p \in P_2$. Then there exists a rectangle R such that $|\chi(R)| = ||R\cap P_1| - |R \cap P_2|| \geq c\log n$. This implies that a data structure that answers queries with error $\frac{c}{2}\log n$ have to distinguish P_1 and P_2. Thus, to distinguish all the $2^{\Omega(n\log n)}$ point sets in $\mathcal{P}^*$, the data structure has to use at least $\Omega(n\log n)$ bits, which is a tight lower bound for $\varepsilon = n/\log n$. We will show how the combinatorial discrepancy bound implies tight lower bound for arbitrary ε in Section 3.

While point sets with low Lebesgue discrepancy or high combinatorial discrepancy have been extensively studied, here we have constructed a large collection of point sets in which the pairwise union has high combinatorial discrepancy. This particular aspect appears to be novel, and our construction could be useful in proving other space lower bounds. It may also have applications in situations where we need a "diverse" collection of (pseudo) random point sets.

2 Upper Bound

In this section, we build a data structure that supports approximate range counting queries. Given a set of n points on an $n \times n$ grid, our data structure uses $O(\frac{1}{\varepsilon}(\log^2\frac{1}{\varepsilon}+\log n)$ bits and answers an orthogonal range counting query with error εn. We note that it is sufficient to only consider two-sided ranges, since an 4-sided range counting query can be expressed as a linear combination of four two-sided range counting queries by the inclusion-exclusion principle. A two-sided range is specified by a rectangle of the form $[0,x) \times [0,y)$, where (x,y) is called the *query point*.

The data structure. Our data structure is an approximate variant of Chazelle's linear-space version of the range tree, originally for exact orthogonal range counting [10]. Consider a set P of n points on an $n \times n$ grid. We divide P into the left point set P_L and the right point set P_R by the median of the x-coordinates. We will recursively build a data structure for P_L and P_R. Let B be a parameter to be determined later. Let $Q(P)$ denote the $\frac{n}{B}$ quantiles of the y-coordinates of P. Note that the i-th quantile is the y-coordinate in P with exactly iB points below it. We use indices $[\frac{n}{B}] = 1, \ldots, \frac{n}{B}$ to represent $Q(P)$, where i denote the i-th quantile. We don't explicitly store the y-values or even the indices of $Q(P)$. Instead, for each index i in $Q(P)$ with coordinate y, we store a pointer to the successor of y in $Q(P_L)$. Note that these $\frac{n}{B}$ pointers form a monotone increasing sequence of $\frac{n}{B}$ indices in $[\frac{n}{2B}]$, and can be encoded in $O(\frac{n}{B})$ bits. Similarly, we store the successor pointers from $Q(P)$ to $Q(P_R)$ with $O(\frac{n}{B})$ bits. It follows that the space in bits satisfies recursion $S(n) = 2S(\frac{n}{2}) + O(\frac{n}{B})$, with base case $S(B) = 0$. The recurrence solves to $S(n) = O(\frac{n}{B}\log\frac{n}{B})$. Finally, we explicitly store the $\frac{1}{\varepsilon}$ quantiles $Q_0(P)$ for the y-coordinates of P with $O(\frac{1}{\varepsilon}\log n)$ bits.

Given a query $q = (q.x, q.y)$. For simplicity, we assume $q.y$ is in $Q_0(P)$. If not, we can use the successor of $q.y$ in $Q_0(P)$ as an estimation with additive error at most εn to the final count. If q is in P_L, we follow the pointer to find the successor of $q.y$ in $Q.L$, and the recurse the problem in P_L. If q is in P_r, we first follow the pointer to get the successor of $q.y$ in $Q(P_L)$. This gives an approximate count for $P_L \cap q$ with additive error B. We then follow the pointer to get the successor of $q.y$ in $Q(P_R)$, and recurse the problem in P_R. Note that rounding $q.y$ with the successor in P_R or R_L causes additive error B, and using the approximate count for $P_L \cap q$ also causes additive error B. Thus, the overall additive error satisfies $E(n) = E(\frac{n}{2}) + 2B$, with base case $E(B) = B$. The recurrence solves to $E(n) = O(B \log \frac{n}{B})$, and we can then set $B = \varepsilon n / \log \frac{1}{\varepsilon}$ to make $E(n) = O(\varepsilon n)$. It follows that $S(n) = O(\frac{1}{\varepsilon} \log^2 \frac{1}{\varepsilon})$, and thus total space usage is $O(\frac{1}{\varepsilon}(\log^2 \frac{1}{\varepsilon} + \log n))$ bits. The query time can also be made $O(\log \frac{1}{\varepsilon})$, if we use succinct rank-select structures to encode the pointers, as in Chazelle's method.

Theorem 2.1. *Given a set of n points drawn from an $n \times n$ grid, there is a data structure that uses $O(\frac{1}{\varepsilon}(\log^2 \frac{1}{\varepsilon} + \log n))$ bits and answers orthogonal range counting query with additive error εn.*

3 Lower Bound

In this section, we prove a lower bound that matches the upper bound in Theorem 2.1.

Theorem 3.1. *Consider a set of n points drawn from an $n \times n$ grid. A data structure that answers orthogonal range counting query with additive error εn for any point set must use $\Omega(\frac{1}{\varepsilon}(\log^2 \frac{1}{\varepsilon} + \log n))$ bits.*

To prove Theorem 3.1, we need the following theorem on union discrepancy.

Theorem 3.2. *Let $\mathcal{P}$ denote the collection of all n-point sets drawn from an $n \times n$ grid. There exists a constant c and a sub-collection $\mathcal{P}^* \subseteq \mathcal{P}$ of size $2^{\Omega(n \log n)}$, such that for any two point sets $P_1, P_2 \in \mathcal{P}^*$, their union discrepancy* $\mathrm{disc}(P_1 \cup P_2, \mathcal{R}_2) \geq c \log n$.

We first show how Theorem 3.2 implies Theorem 3.1.

of Theorem 3.1. We only need to prove the $\Omega(\frac{1}{\varepsilon} \log^2 \frac{1}{\varepsilon})$ lower bound. Suppose we group the points into $N = \frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$ fat points, each of size $\varepsilon n / \log \frac{1}{\varepsilon}$. By Theorem 3.2, there is a collection $\mathcal{P}^*$ of $2^{\Omega(N \log N)}$ fat point sets, such that for any two fat point sets $P_1, P_2 \in \mathcal{P}^*$, there exists a rectangle R such that the number of fat points in $R \cap P_1$ and $R \cap P_2$ differs by at least $\geq c \log N$. Since each fat points corresponds to $\varepsilon n / \log \frac{1}{\varepsilon}$ points, it follows that the counts of $P_1 \cap R$ and $P_2 \cap R$ differs by at least

$$\frac{\varepsilon n}{\log \frac{1}{\varepsilon}} \cdot c \log N = \frac{\varepsilon n}{\log \frac{1}{\varepsilon}} \cdot c \log \left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \right) \geq c \varepsilon n.$$

Therefore, a data structure that answers queries with error $\frac{c}{2} \varepsilon n$ have to distinguish P_1 and P_2. Thus, to distinguish all the $2^{\Omega(N \log N)}$ point sets in $\mathcal{P}^*$, the data structure has to use at least $\Omega(N \log N) = \Omega(\frac{1}{\varepsilon} \log^2 \frac{1}{\varepsilon})$ bits. □

In the rest of this section, we will focus on proving Theorem 3.2. To derive the sub-collection $\mathcal{P}^*$ in Theorem 3.2, we begin by looking into a collection of point sets called *binary nets*. Binary nets are a special type of point sets under a more general concept called (t, m, s)-*nets*, which are

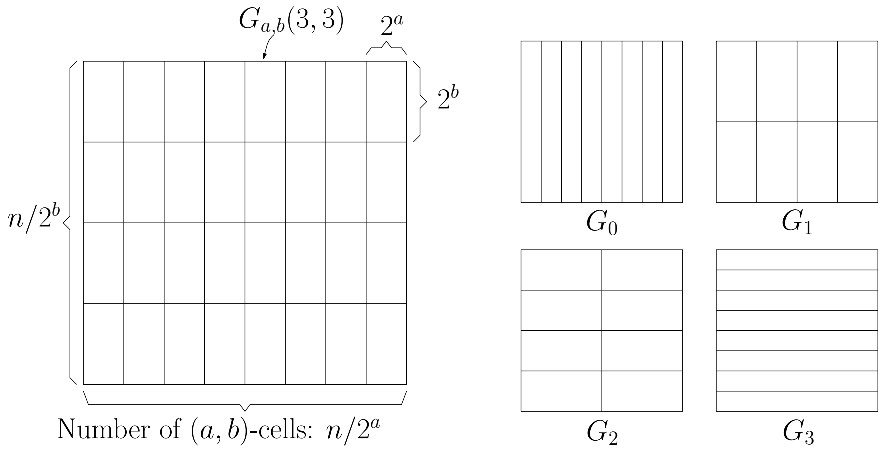

Figure 1: Illustrations of (a, b)-cells and canonical cells.

introduced in [19] as an example of point sets with low Lebesgue discrepancy. See the survey by Clayman et al. [11] or the book by Hellekalek et al. [15] for more results on (t, m, s)-nets. In this paper we will show that binary nets have two other nice properties: 1) A binary net has high combinatorial discrepancy, i.e., $\Omega(\log n)$; 2) there is a bit vector representation for every binary net, which allows us to extract a sub-collection by constructing a subset of bit vectors. In the following sections, we will define binary nets, and formalize these two properties.

3.1 Definitions

For ease of the presentation, we assume that the $n \times n$ grid is embedded in the unit square $[0, 1)^2$. We partition $[0, 1)^2$ into $n \times n$ squares, each of size $\frac{1}{n^2}$. We assume the grid points are placed at the mass centers of the n^2 squares, that is, each grid point has coordinates $(\frac{i}{n} + \frac{1}{2n}, \frac{j}{n} + \frac{1}{2n})$, for $i, j \in [n]$, where $[n]$ denote the set of all integers in $[0, n)$. For the sake of simplicity, we define the grid point (i, j) to be the grid point with coordinates $(\frac{i}{n} + \frac{1}{2n}, \frac{j}{n} + \frac{1}{2n})$, and we do not distinguish a grid point and the square it resides in.

Now we introduce the concepts of (a, b)-*cell* and k-*canonical* cell.

Definition 3.1. *A* (a, b)-cell *at position* (i, j) *is the rectangle* $[\frac{i2^a}{n}, \frac{(i+1)2^a}{n}) \times [\frac{j2^b}{n}, \frac{(j+1)2^b}{n})$. *We use* $G_{a,b}(i, j)$ *to denote the* (a, b)*-cell at position* (i, j)*, and* $G_{a,b}$ *to denote the set of all* (a, b)*-cells.*

Definition 3.2. *A* k-canonical cell *at position* (i, j) *is a* $(k, \log n - k)$*-cell with coordinates* (i, j)*. We use* $G_k(i, j)$*, to denote the* k*-canonical cell at position* (i, j)*, and* G_k *to denote the set of all* k*-canonical cells.*

Figure 1 is the illustration of (a, b)-cells and canonical cells. Note that the position (i, j) for a (a, b)-cell takes value in $[n/2^a] \times [n/2^b]$. In particular, we call $G_0(i, 0)$ the i-th column and $G_{\log n}(0, j)$ the j-th row. Note that for a fixed k, G_k partitions the grid $[0, 1)^2$ into n rectangles. Based on the definition of k-canonical cells, we define the binary nets:

Definition 3.3. *A point set* P *is called a binary net if for any* $k \in [\log n]$*,* P *has exactly one point in each* k*-canonical cell.*

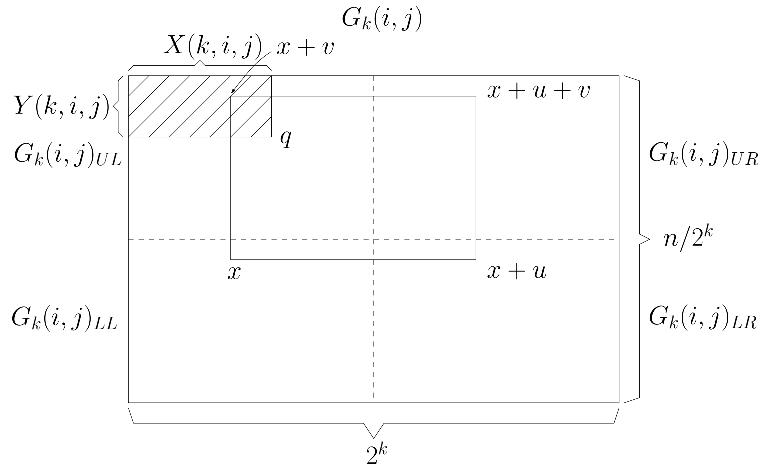

Figure 2: Illustration of the corner volume and the four analogous points. The area in shadow represents the corner volume $V_P(k,i,j)$.

Let $\mathcal{P}_0$ denote the collection of binary nets. In other word, $\mathcal{P}_0$ is the set

$$\{P \mid |P \cap G_k(i,j)| = 1, k \in [\log n], i \in [n/2^k], j \in [2^k]\}.$$

It is known that the point sets in $\mathcal{P}_0$ have Lebesgue discrepancy $O(\log n)$; below we show that they also have $\Omega(\log n)$ combinatorial discrepancy. However, the union of two point sets in $\mathcal{P}_0$ could have combinatorial discrepancy as low as $O(1)$. Thus we need to carefully extract a subset from $\mathcal{P}_0$ with high pairwise union discrepancy.

3.2 Combinatorial Discrepancy and Corner Volume

In this section, we focus on proving the following theorem, which shows that the combinatorial discrepancy of a binary net is large.

Theorem 3.3. *For any point set $P \in \mathcal{P}_0$, we have* $\mathrm{disc}(P, \mathcal{R}_2) = \Omega(\log n)$.

Strictly speaking, Theorem 3.2 does not depend on Theorem 3.3, but this theorem gives us some insights on the binary nets. Moreover, a key lemma to proving Theorem 3.2 (Lemma 3.3) shares essentially the same proof with Theorem 3.3. To prove Theorem 3.3, we need the following definition of *corner volume*:

Definition 3.4. *Consider a point set $P \in \mathcal{P}_0$ and a k-canonical cell $G_k(i,j)$. Let q be the point of P in $G_k(i,j)$. We define the* corner volume *$V_P(k,i,j)$ to be the volume of the orthogonal rectangle defined by q and its nearest corner of $G_k(i,j)$. We use S_P to denote the summation of the corner volumes over all possible triples (k,i,j), that is,*

$$S_P = \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} V_P(k,i,j).$$

See Figure 2 for the illustration of corner volumes. A key insight of our lower bound proof is the following lemma, which relates the combinatorial discrepancy of P with its corner volume sum S_P.

Lemma 3.1. *There exists a constant c, such that for any point set $P \in \mathcal{P}_0$ with corner volume sum*

$$S_P \geq c \log n,$$

we have $\mathrm{disc}(P, \mathcal{R}_2) = \Omega(\log n)$.

The proof of Lemma 3.1 makes use of the Roth's orthogonal function method [21], which is widely used for proving lower bounds for Lebesgue discrepancy (see [9, 19]).

Proof. Consider a binary net $P \in \mathcal{P}_0$ that satisfies $S_P \geq c \log n$, where c is constant to be determined later. Given any coloring $\chi : P \to \{-1, +1\}$ and a point $x = (x_1, x_2) \in [0, 1)^2$, the combinatorial discrepancy $D(x)$ at a point x is defined to be

$$D(x) = \sum_{p \in P \cap [0, x_1) \times [0, x_2)} \chi(p).$$

If we can prove $\sup_{x \in [0,n)^2} |D(x)| = \Omega(\log n)$, the lemma will follow.

For $k \in [\log n]$, we define normalized wavelet functions f_k as follow: for each k-canonical cell $G_k(i, j)$, let q denote the point contained in it. We subdivide $G_k(i, j)$ into four equal-size quadrants, and use $G_k(i, j)_{UR}$, $G_k(i, j)_{UL}$, $G_k(i, j)_{LR}$, $G_k(i, j)_{LL}$ to denote the upper right, upper left, lower right and lower left quadrants, respectively (See Figure 2). Set $f_k(x) = \chi(q)$ over quadrants $G_k(i, j)_{UR}$ and $G_k(i, j)_{LL}$, and $f_k(x) = -\chi(q)$ over the other two quadrants. To truly reveal the power of these wavelet functions, we define a more general class of functions called *checkered* functions.

Definition 3.1. *We say a function $f : [0, 1)^2 \to \mathbb{R}$ is* (a, b)-checkered *if for each (a, b)-cell, there exists a color $C \in \{-1, +1\}$ such that f is equal to C over $G_{a,b}(i, j)_{UR}$ and $G_{a,b}(i, j)_{LL}$ and $-C$ over the other two quadrants.*

Note that our definition of checkered function is slight different from the one used in [9]. It is easy to see the wavelet function f_k is $(k, \log n - k)$-checkered, and the integration of a (a, b)-checkered function over an (a, b)-cell is 0. The following lemma states that the checkered property is "closed" under multiplication.

Fact 3.1. *If f is (a_1, b_1)-checkered and g is (a_2, b_2) checkered, where $a_1 < a_2$ and $b_1 > b_2$, then fg is (a_1, b_2)-checkered.*

For a proof, consider an (a_1, b_2)-cell $G_{a_1,b_2}(i, j)$. We observe that this cell is defined by the intersection of an (a_1, b_1)-cell and an (a_2, b_2)-cell, and we use $G_{a,b}(i_1, j_1)$ and $G_{a_2,b_2}(i_2, j_2)$ to denote these two cells, respectively. Therefore the four quadrants of $G_{a_1,b_2}(i, j)$ are defined by the intersections of two neighboring quadrants of $G_{a_1,b_1}(i_1, j_1)$ and two neighboring quadrants of $G_{a_2,b_2}(i_2, j_2)$. Without loss of generality, we assume the four quadrants are defined by the intersections of the two upper quadrants of $G_{a_1,b_1}(i_1, j_1)$ and two left quadrants of $G_{a_2,b_2}(i_2, j_2)$ (see Figure 3). Since f is (a_1, b_1)-checkered and g is (a_2, b_2) checkered, we can assume f equal to C_1 and $-C_1$ over $G_{a_1,b_1}(i_1, j_1)_{UR}$ and $G_{a_2,b_2}(i_2, j_2)$, and g equal to C_2 and $-C_2$ over $G_{a_2,b_2}(i_2, j_2)_{UL}$ and $G_{a_2,b_2}(i_2, j_2)_{LL}$, respectively. It follows that the fg is equal to C_1C_2 over $G_{a_1,b_2}(i, j)_{UL}$ and $G_{a_1,b_2}(i, j)_{LR}$, and $-C_1C_2$ over $G_{a_1,b_2}(i, j)_{UR}$ and $G_{a_1,b_2}(i, j)_{LL}$. Thus fg is an (a_1, b_2)-checkered function.

A direct corollary from Fact 3.1 is that the wavelet functions are *generalized orthogonal*:

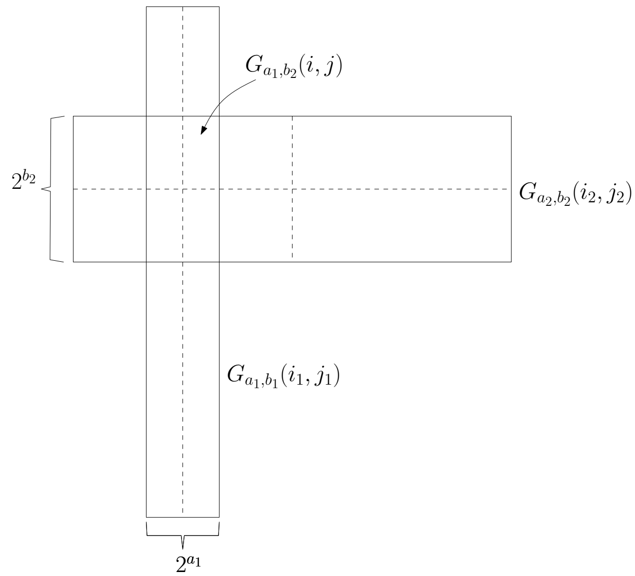

Figure 3: Illustration of the intersection of two cells

Corollary 3.1. *For $0 \le k_1 < \cdots < k_l \le \log n$, the function $f_{k_1}(x)\cdots f_{k_l}(x)$ is a $(k_1, \log n - k_l)$-checkered. As a consequence, we have*

$$\int_{[0,1)^2} f_{k_1}(x)\cdots f_{k_l}(x)dx = 0.$$

In the remaining of the paper we assume the range of the integration is $[0,1)^2$ and the variable of integration is dx when not specified. We define the Riesz product

$$G(x) = -1 + \prod_{k=0}^{\log n}(\gamma f_k(x) + 1),$$

where γ is some constant to be determined later. By the inequality

$$\left|\int GD\right| \le \int |GD| \le \sup_{x\in[0,1)^2} |D| \cdot \int |G|,$$

we can lower-bound the combinatorial discrepancy of P as follows:

$$\sup_{x\in[0,1)^2} |D| \ge \left|\int GD\right| \Big/ \int |G|. \tag{3.1}$$

For the denominator $\int |G|$, we have

$$\begin{aligned}\int |G| = \int \left| -1 + \prod_{k=0}^{\log n}(\gamma f_k + 1)\right| &\le 1 + \sum_{l=0}^{\log n}\gamma^l \sum_{0\le k_1<\ldots<k_l\le \log n}\int f_{k_1}\cdots f_{k_l} \\ &= 2 + \sum_{l=1}^{\log n}\gamma^l \sum_{0\le k_1<\ldots<k_l\le \log n}\int f_{k_1}\cdots f_{k_l} = 2. \qquad (3.2)\end{aligned}$$

The last equation is due to Corollary 3.1. The numerator $\left|\int G(x)D(x)dx\right|$ can be expressed as follow:

$$\begin{aligned}
\left|\int GD\right| &= \left|\int \left(-1+\prod_{k=0}^{\log n}(\gamma f_k+1)\right)\cdot D\right| \\
&= \left|\int \left(\gamma\sum_{k=0}^{\log n} f_k + \sum_{l=2}^{\log n}\gamma^l \sum_{0\le k_1<\ldots<k_l\le \log n} f_{k_1}\cdots f_{k_l}\right)\cdot D\right| \\
&\ge \gamma\left|\sum_{k=0}^{\log n}\int f_k D\right| - \sum_{l=2}^{\log n}\gamma^l\left|\sum_{0\le k_1<\ldots<k_l\le\log n}\int f_{k_1}\cdots f_{k_l} D\right|. \qquad (3.3)
\end{aligned}$$

In order to estimate $\int f_k D$, we consider the integration of a single product $f_k(x)D(x)$ over a k-canonical cell $G_k(i,j)$. Recall that there is exactly one point of P that lies in $G_k(i,j)$. We use q to denote this point in P, and $\chi(q)$ denote its color. Define horizontal vector $u = (\frac{2^{k-1}}{n}, 0)$ and vertical vector $v = (0, \frac{1}{2^{k+1}})$. Then for any point $x \in G_k(i,j)_{LL}$, points $x+u$, $x+v$ and $x+u+v$ are the analogous points in quadrants $G_k(i,j)_{LR}$, $G_k(i,j)_{UL}$ and $G_k(i,j)_{UR}$ of x, respectively (see Figure 2). The four analogous points defines an orthogonal rectangle. We use R_x to denote the orthogonal rectangle, and function $R(x)$ to denote the indicator function of point q and R_x, that is, $R(x) = 1$ if $q \in R_x$ and $R(x) = 0$ if otherwise. We can express the integral as

$$\begin{aligned}
\int_{G_k(i,j)} f_k(x)D(x)dx &= \int_{G_k(i,j)_{LL}} \chi(q)\left(D(x)-D(x+u)-D(x+v)+D(x+u+v)\right)dx \\
&= \int_{G_k(i,j)_{LL}} \chi(q)\cdot\chi(q)R(x)dx = \int_{G_k(i,j)_{LL}} R(x)dx.
\end{aligned}$$

The second equation is because $(D(x)-D(x+u)-D(x+v)+D(x+u+v))$ only counts points inside R_x, which can only be q, or nothing otherwise. Observe that $R(x) = 1$ if and only if one of x's analogous points lies inside the rectangle defined by q and its nearest corner (see Figure 2), so we have

$$\int_{G_k(i,j)} f_k D = \int_{G_k(i,j)_{LL}} R = V_P(k,i,j). \qquad (3.4)$$

Now we can compute the first term in (3.3):

$$\begin{aligned}
\gamma\left|\sum_{k=0}^{\log n}\int f_k D\right| &= \gamma\left|\sum_{k=0}^{\log n}\sum_{i=0}^{n/2^k-1}\sum_{j=0}^{2^k-1}\int_{G_k(i,j)} f_k D\right| = \gamma\left|\sum_{k=0}^{\log n}\sum_{i=0}^{n/2^k-1}\sum_{j=0}^{2^k-1} V_P(k,i,j)\right| \\
&= \gamma S_P \ge c\gamma\log n. \qquad (3.5)
\end{aligned}$$

For the second term in (3.3), consider a $(k_1, \log n - k_l)$-cell $G_{k_1,\log n-k_l}$. Note that P intersects $G_{k_1,\log n-k_l}(i,j)$ with at most 1 point. By Fact 3.1, function $f_{k_1}\cdots f_{k_l}$ is $(k_1, \log n - k_l)$-checkered, so following similar arguments in the proof of equation (3.4), we can show that the integral $\left|\int_{G_{k_1,\log n-k_l}(i,j)} f_{k_1}\cdots f_{k_l} D\right|$ is 0 if $P\cap G_{k_1,\log n-k_l}=\emptyset$ and otherwise equal to the corner volume of $G_{k_1,k_l}(i,j)$. In the latter case, we can relax the corner volume to the volume of $G_{k_1,\log n-k_l}(i,j)$, that is, $\frac{1}{2^{k_l-k_1}n}$. Thus we can estimate the integral as follows:

$$\left|\int_{G_{k_1,\log n-k_l}(i,j)} f_{k_1}\cdots f_{k_l} D\right| \le \frac{1}{2^{k_l-k_1}n}.$$

Since there are n non-empty $(k_1,\log n-k_l)$-cells, we have

$$\left|\int f_{k_1}\cdots f_{k_l} D\right| \le n\cdot\frac{1}{2^{k_l-k_1}n}=\frac{1}{2^{k_l-k_1}}.$$

Now we can estimate the second term in (3.3):

$$\begin{aligned}\sum_{l=2}^{\log n}\gamma^l\left|\sum_{0\le k_1<\ldots<k_l\le\log n}\int f_{k_1}\cdots f_{k_l} D\right| &\le \sum_{l=2}^{\log n}\gamma^l\sum_{0\le k_1<\ldots<k_l\le\log n}\frac{1}{2^{k_l-k_1}}\\ &=\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\sum_{k_l-k_1=w}\frac{1}{2^w}\binom{w-1}{l-2}. \end{aligned}\tag{3.6}$$

For the last equation we replace k_l-k_1 with a new index w and use the fact that there are $\binom{w-1}{l-2}$ ways to choose $k_2,\ldots,k_{l-1}$ in an interval of length w. Note that for a fixed w, there are $\log n+1-w$ possible values for k_1, so

$$\begin{aligned}\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\sum_{k_l-k_1=w}\frac{1}{2^w}\binom{w-1}{l-2} &=\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\frac{\log n+1-w}{2^w}\binom{w-1}{l-2}\\ &\le\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\frac{\log n}{2^w}\binom{w-1}{l-2}\\ &=\log n\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\frac{1}{2^w}\binom{w-1}{l-2}. \end{aligned}\tag{3.7}$$

By inverting the order of the summation,

$$\begin{aligned}\log n\sum_{l=2}^{\log n}\gamma^l\sum_{w=l-1}^{\log n+1}\sum_{k_l-k_1=w}\frac{1}{2^w}\binom{w-1}{l-2} =&\gamma^2\log n\sum_{w=1}^{\log n+1}\frac{1}{2^w}\sum_{l=2}^{w+1}\binom{w-1}{l-2}\gamma^{l-2}\\ =&\gamma^2\log n\sum_{w=1}^{\log n+1}\frac{1}{2^w}(1+\gamma)^{w-1}\\ =&2\gamma^2\log n\sum_{w=1}^{\log n+1}\left(\frac{1+\gamma}{2}\right)^{w-1}\le\frac{2\gamma^2}{1-\gamma}\log n. \end{aligned}\tag{3.8}$$

So from (3.5), (3.6), (3.7) and (3.8) we have

$$\left|\int GD\right|\ge c\gamma\log n-\frac{2\gamma^2}{1-\gamma}\log n.$$

Setting γ small enough while combining with (3.1) and (3.2) completes the proof. $\square$

Now we can give a proof to Theorem 3.3. By Lemma 3.1, we only need to show that the corner volume sum of any point set $P \in \mathcal{P}_0$ is large. Fix k and consider a k-canonical cell $G_k(i,j)$. Let q denote the point in $P \cap G_k(i,j)$. We define the corner x-distance of $G_k(i,j)$ to be the difference between the x-coordinate of q and that of its nearest corner of $G_k(i,j)$. The corner y-distance is defined in similar manner. See Figure 2. We use $X(k,i,j)$ and $Y(k,i,j)$ to denote the corner x-distance and corner y-distance, respectively. Note that the corner volume $V_P(k,i,j)$ is the product of $X(k,i,j)$ and $Y(k,i,j)$. The following fact holds for the x-distances of canonical cells in a column:

Fact 3.2. *Fix k and i, we have $\{X(k,i,j) \mid j \in [2^k]\} = \{\frac{j}{n} + \frac{1}{2n}, \frac{j}{n} + \frac{1}{2n} \mid j \in [2^{k-1}]\}$, where both are taken as multisets.*

For a proof, note that the k-canonical cell $G_k(i,j)$ is intersecting with 2^k columns: $G_0(i2^k,0),\ldots,G_0((i+1)2^k-1,0)$. There are 2^k points in $G_k(i,0),\ldots,G_k(i,2^k-1)$, and they must reside in different columns. Therefore there is exactly one point in the each of the 2^k columns, and their corner x-distances span from $\frac{1}{2n}$ to $\frac{2^{k-1}-1}{n} + \frac{1}{2n}$, and each value is hit exactly twice. Similarly, we have

Fact 3.3. *Fix k and j, we have $\{X(k,i,j) \mid i \in [n/2^k]\} = \{\frac{i}{n} + \frac{1}{2n}, \frac{i}{n} + \frac{1}{2n} \mid i \in [n/2^{k+1}]\}$, where both are taken as multisets.*

Now consider the product of $X(k,i,j)$ and $Y(k,i,j)$ over all (i,j) for a fixed k:

$$
\begin{aligned}
\prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} V_P(k,i,j) &= \prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} X(k,i,j)Y(k,i,j) \\
&= \prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} X(k,i,j) \cdot \prod_{j=0}^{2^k-1}\prod_{i=0}^{n/2^k-1} Y(k,i,j) \\
&= \prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^{k-1}-1} (\frac{j}{n}+\frac{1}{2n})^2 \cdot \prod_{j=0}^{2^k-1}\prod_{i=0}^{n/2^{k+1}-1} (\frac{i}{n}+\frac{1}{2n})^2.
\end{aligned}
$$

The last equation is due to Fact 3.2 and Fact 3.3. By relaxing $\frac{i}{n} + \frac{1}{2n}$ and $\frac{j}{n} + \frac{1}{2n}$ to $\frac{i+1}{2n}$ and $\frac{j+1}{2n}$, we have

$$
\begin{aligned}
\prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} V_P(k,i,j) &\geq \prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^{k-1}-1} \left(\frac{i+1}{2n}\right)^2 \cdot \prod_{j=0}^{2^k-1}\prod_{i=0}^{n/2^{k+1}-1} \left(\frac{j+1}{2n}\right)^2 \\
&= \frac{1}{n^n}\prod_{i=0}^{n/2^k-1} \left(\frac{(2^{k-1})!}{2^{2^{k-1}}}\right)^2 \cdot \prod_{j=0}^{2^k-1} \left(\frac{(n/2^{k+1})!}{2^{n/2^{k+1}}}\right)^2.
\end{aligned}
$$

By the inequality $x! \geq (x/e)^x$,

$$\begin{aligned}
\prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} V(k,i,j) \geq & \frac{1}{n^{2n}} \prod_{i=0}^{n/2^k-1}\left(\left(\frac{2^{k-1}}{2e}\right)^{2^{k-1}}\right)^2 \cdot \prod_{j=0}^{2^k-1}\left(\left(\frac{n/2^{k+1}}{2e}\right)^{n/2^{k+1}}\right)^2 \\
= & \frac{1}{n^{2n}} \prod_{i=0}^{n/2^k-1}\left(\frac{2^{k-1}}{2e}\right)^{2^k} \cdot \prod_{j=0}^{2^k-1}\left(\frac{n/2^{k+1}}{2e}\right)^{n/2^k} \\
= & \frac{1}{n^{2n}} \left(\frac{2^{k-1}}{2e}\right)^{2^k \cdot n/2^k} \cdot \left(\frac{n/2^{k+1}}{2e}\right)^{n/2^k \cdot 2^k} \\
= & \frac{1}{n^{2n}} \left(\frac{2^k}{4e}\right)^n \cdot \left(\frac{n/2^k}{4e}\right)^n = \left(\frac{1}{16en}\right)^n.
\end{aligned}$$

Using the inequality of geometric means,

$$\sum_{i=0}^{n/2^k-1}\sum_{j=0}^{2^k-1} V_P(k,i,j) \geq n \cdot \left(\prod_{i=0}^{n/2^k-1}\prod_{j=0}^{2^k-1} V_P(k,i,j)\right)^{1/n} \geq n \cdot \frac{1}{16en} = \frac{1}{16e}.$$

So the corner volume sum $S_P = \sum_{k=0}^{\log n}\sum_{i=0}^{n/2^k-1}\sum_{j=0}^{2^k-1} V(k,i,j)$ is lower bounded by $\log n/16e$, and by Lemma 3.1, Theorem 3.3 follows.

3.3 A bit vector representation for $\mathcal{P}_0$

Another nice property of $\mathcal{P}_0$ is that we can derive the exact number of point sets in it. The following lemma is from the book [12]. We sketch the proof here, as it also provides a bit vector presentation of each binary net, which is essential in our lower bound proof.

Lemma 3.2 ([12])**.** *The number of point sets in $\mathcal{P}_0$ is $2^{\frac{1}{2}n\log n}$.*

Proof. It is equivalent to prove that the number of possible ways to place n points on the $n \times n$ grid such that any k-canonical cell $G_k(i,j)$ has exactly 1 point is $2^{\frac{1}{2}n\log n}$. Our proof proceeds by an induction on n. Let $\mathcal{P}_0(n)$ denote the collection of binary nets of size n in a $n \times n$ grid.

Observe that the line $y = n/2$ divides the grid $[0,1)^2$ into two rectangles: the upper grid $[0,1) \times [\frac{1}{2},1)$ and the lower grid $[0,1) \times [0,\frac{1}{2})$. For i even, let R_i denote the rectangle defined by the union of i-th and $(i+1)$-th columns $G_0(i,0)$ and $G_0(i+1,0)$. Note that the line $y = n/2$ divides R_i into $G_1(\frac{i}{2},0)$ and $G_1(\frac{i}{2},1)$, and therefore defines four quadrants. By the definition of $\mathcal{P}_0$, for any point set $P \in \mathcal{P}_0$, the two points in $G_0(i,0)$ and $G_1(i+1,0)$ must either reside in the lower left and upper right quadrants or in the lower right and upper left quadrants. There are in total $n/2$ even i's, so the number of the possible choices is $2^{n/2}$. See Figure 4. Note that after determining which half the point in each column resides in, the problem is divided into two sub-problems: counting the number of possible ways to place $n/2$ points in the upper grid and the lower grid. It is easy to show that each sub-problem is identical to the problem of counting the number of point sets in $\mathcal{P}_0(n/2)$, so we have the following recursion:

$$|\mathcal{P}_0(n)| = 2^{\frac{n}{2}} \cdot |\mathcal{P}_0(n/2)|^2.$$

Solving this recursion with $\mathcal{P}_0(1) = 1$ yields that $|\mathcal{P}_0(n)| = 2^{\frac{1}{2}n\log n}$. $\square$

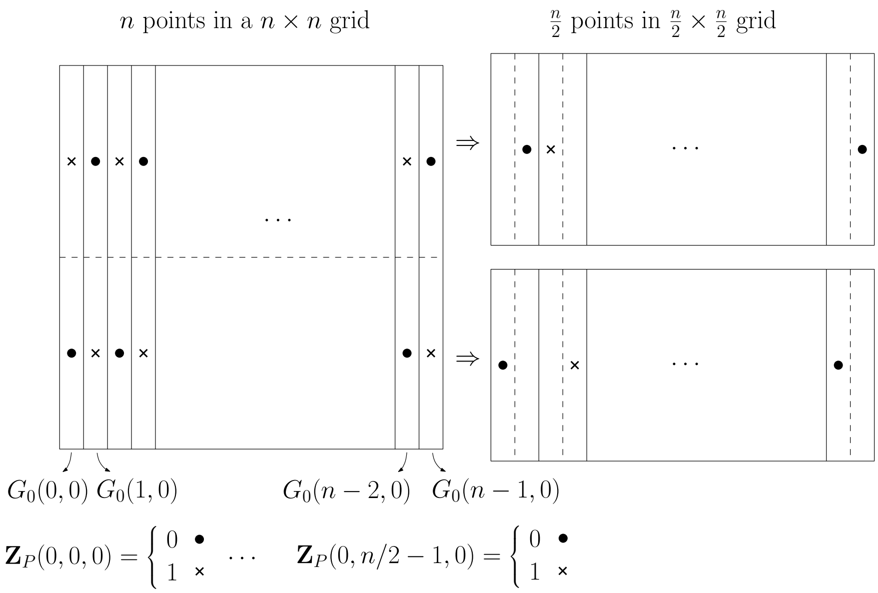

Figure 4: Illustration of the partition vector of G_0.

A critical observation is that the proof of Lemma 3.2 actually reveals a bit vector representation for each of the point sets in $\mathcal{P}_0$, which will allow us to refine the collection $\mathcal{P}_0$. To see this, we define the *partition vector* $\mathbf{Z}_P$ for a point set $P \in \mathcal{P}_0$ as follows. For any $(k,i,j) \in [\log n] \times [n/2^{k+1}] \times [2^k]$, consider the k-canonical cells $G_k(2i,j)$ and $G_k(2i+1,j)$ and $(k+1)$-canonical cells $G_{k+1}(i,2j)$ and $G_{k+1}(i,2j+1)$. The two k-canonical cells overlap with the two $(k+1)$-canonical cells, which defines four quadrants. By the definition of binary nets, there are two points in P contained in these quadrants. We define $\mathbf{Z}_P(k,i,j) = 0$ if the two points are in the lower left and upper right quadrants and $\mathbf{Z}_P(k,i,j) = 1$ if they are in the lower right and upper left quadrants. See Figure 4. We say the k-canonical cells $G_k(2i,j)$ and $G_k(2i+1,j)$ is associated with bit $\mathbf{Z}_P(k,i,j)$. Note that we use the triple (k,i,j) as the index into $\mathbf{Z}_P$ for the ease of presentation; we can assume that the bits in $\mathbf{Z}_P$ are stored in for example the lexicographic order of (k,i,j). Since the number of triples (k,i,j) is $\frac{1}{2}n\log n$, the total number of bits in $\mathbf{Z}_P$ is $\frac{1}{2}n\log n$. Let $\mathcal{Z}_0 = \{0,1\}^{\frac{1}{2}n\log n}$ denote the set of all possible partition vector $\mathbf{Z}_P$'s. By the proof of Lemma 3.2, there is a bijection between $\mathcal{Z}_0$ and $\mathcal{P}_0$.

3.4 Combinatorial discrepancy and corner volume distance

Although we have proved that binary nets have large combinatorial discrepancy, it does not yet lead us to Theorem 3.2. In this section, we will refine $\mathcal{P}_0$, the collection of all binary nets, to derive a collection $\mathcal{P}^*$, such that the union of any two point sets in $\mathcal{P}^*$ has large combinatorial discrepancy. In order to characterize the combinatorial discrepancy of the union of two point sets, we will need the following definition of *corner volume distance*.

Definition 3.5. *For two point sets $P_1, P_2 \in \mathcal{P}_0$, the* corner volume distance *of P_1 and P_2 is the summation of $|V_{P_1}(k,i,j) - V_{P_2}(k,i,j)|$, over all (k,i,j). In other words, let $\Delta(P_1,P_2)$ denote the* *corner volume distance of P_1 and P_2, then*

$$\Delta(P_1, P_2) = \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k,i,j) - V_{P_2}(k,i,j)| .$$

The following lemma relates the combinatorial discrepancy of the union of two point sets with their corner volume distance:

Lemma 3.3. *Let $\mathcal{P}^*$ be a subset of $\mathcal{P}_0$. If there exists a constant c, such that for any two point sets $P_1, P_2 \in \mathcal{P}_0$, that their corner volume distance satisfies $\Delta(P_1, P_2) \geq c \log n$, then* $\mathrm{disc}(P_1 \cup P_2, \mathcal{R}_2) = \Omega(\log n)$.

Proof. The proof follows the same framework as the proof for Lemma 3.1. Note that there are exactly two points of $P_1 \cup P_2$ in each k-canonical cell $G_k(i,j)$, and we use q_1, q_2 denote the two points from P_1 and P_2, respectively. We will set $f_k(x) = C$ for quadrants $G_k(i,j)_{UR}$ and $G_k(i,j)_{LL}$ and $f_k(x) = -C$ for the other two quadrants, where C is determined as follows:

$$C = \begin{cases} \chi(q_1) & \text{if } V_{P_1}(k,i,j) \geq V_{P_2}(k,i,j); \\ \chi(q_2) & \text{if } V_{P_1}(k,i,j) < V_{P_2}(k,i,j). \end{cases}$$

Let $D(x)$ be the combinatorial discrepancy at x over $P_1 \cup P_2$. By equation (3.5) in the proof of Lemma 3.1, we get

$$\int_{G_k(i,j)} f_k D = \begin{cases} (V_{P_1}(k,i,j) + V_{P_2}(k,i,j)) & \text{if } \chi(q_1) = \chi(q_2); \\ |V_{P_1}(k,i,j) - V_{P_2}(k,i,j)| & \text{if } \chi(q_1) \neq \chi(q_2). \end{cases}$$

In either case,

$$\int_{G_k(i,j)} f_k D \geq |V_{P_1}(k,i,j) - V_{P_2}(k,i,j)| .$$

And the rest of the proof follows the same argument in the proof of Lemma 3.1. □

Here we briefly explain the high level idea for proving Theorem 3.2. By Lemma 3.3, it is sufficient to find a sub-collection $\mathcal{P}^* \subseteq \mathcal{P}_0$, such that for any two point sets in $\mathcal{P}^*$, their corner volume distance is large. We will choose a subset $\mathcal{Z}_1 \subseteq \mathcal{Z}_0$, and project each vector in $\mathcal{Z}_1$ down to a slightly shorter bit vector $\mathbf{T}$. The collection $\mathcal{T}$ of all resulted bit vector $\mathbf{T}$'s induces a sub-collection $\mathcal{P}_1 \subseteq \mathcal{P}_0$, and each $\mathbf{T}$ represents a point set in $\mathcal{P}_1$. Then we prove that for any two point sets $P_1, P_2 \in \mathcal{P}_1$, there is a linear dependence between the corner volume distance $\Delta(P_1, P_2)$ and the Hamming distance of their bit vector representations $\mathbf{T}_{P_1}$ and $\mathbf{T}_{P_2}$. Finally, we show that there is a large sub-collection of $\mathcal{T}$ with large pair-wise Hamming distances, and this sub-collection induces a collection of point sets $\mathcal{P}^* \in \mathcal{P}_1$ in which the union of any two point sets has large combinatorial discrepancy.

We focus on an $(k+6, \log n - k)$-cell $G_{k+6, \log n - k}(i,j)$, for $k \in \{0, 6, 12, \ldots, \log n - 6\}$. Note that $G_{k+6, \log n - k}(i,j)$ only contains $(k+l)$-canonical cells for $l \in [7]$. Let $F_{k,i,j}(l)$ denote the set of all $(k+l)$-canonical cells in $G_{k+6, \log n - k}(i,j)$, which can be listed as

$$F_{k,i,j}(l) = \{G_{k+l}(2^{6-l} i + s, 2^l j + t) \mid s \in [2^{6-l}], t \in [2^l]\}.$$

Note that $|F_{k,i,j}(l)| = 64$ for each $l \in [7]$. Let $Z_{k,i,j}(l)$ denote the set of indices of bits in the partition vector that are associated with the some $(k+l)$-canonical cells in $G_{k+6,\log n-k}(i,j)$, for $l \in [6]$, i.e.,

$$Z_{k,i,j}(l) = \{(k+l, 2^{5-l}i + s, 2^l j + t) \mid s \in [2^{5-l}], t \in [2^l]\}.$$

Define $Z_{k,i,j}$ to be the union of the $Z_{k,i,j}(l)$'s. Since there are 32 bits in $Z_{k,i,j}(l)$ for each $l \in [6]$, the total number of bits in $Z_{k,i,j}$ is 192 (here we use the indices in $Z_{k,i,j}$ to denote their corresponding bits in the partition vector of P, with a slightly abuse of notation). The following fact shows the $Z_{k,i,j}$'s partition all the $\frac{1}{2}n\log n$ bits:

Fact 3.4. *The number of $Z_{k,i,j}$'s is $\frac{1}{384}n\log n$; For different (k,i,j) and (k',i',j'), $Z_{k,i,j} \cap Z_{k',i',j'} = \emptyset$.*

The proof of the above claims are fairly straightforward: The number of different $Z_{k,i,j}$'s is equal to the number of different $G_{k+6,\log n-k}(i,j)$'s. For a fixed k, the number of different $(k+6, \log n-k)$-cells is $n/64$, and the number of different k's is $\log n/6$, so the total number of different $Z_{k,i,j}$'s is $\frac{1}{384}n\log n$. For the second claim, we consider the following two cases: If $k = k'$, we have$(i,j) \neq (i',j')$. This implies that the two $(k, \log n - k + 6)$-cells are disjoint, therefore the bits associated with the canonical cells inside them are disjoint. For $k \neq k'$, observe that we choose k and k' from $\{0, 6, \dots, \log n - 6\}$, and $Z_{k,i,j}$ and $\mathbf{Z}_{k',i',j'}$ only contain bits associated with $(k+l)$-canonical cells and $(k'+l')$-canonical cells, respectively, for $l, l' \in [6]$, so $Z_{k,i,j}(l)$ and $Z_{k'i'j'}(l')$ are disjoint, for $l, l' \in [6]$.

The reason we group the bits in the partition vector into small subsets is that we can view each subset $Z_{k,i,j}$ as a partition vector of the cell $G_{k+6,\log n-k}(i,j)$, which allows us to manipulate the positions of the points inside it. More precisely, we can view $G_{k+6,\log n-k}(i,j)$ as a 64×64 grid, with each grid cell being a $(k, \log n - k - 6)$-cell in the original $[0,1)^2$ grid. Moreover, a $(k+l)$-canonical cell contained in $G_{k+6,\log n-k}(i,j)$ corresponds to a l-canonical cell in the 64×64 grid. Note that there are 64 points in this grid, and the bits in $Z_{k,i,j}$ correspond to the partition vector of this 64-point set. Now consider a $(k+3)$-canonical cell $G_{k+3}(8i, 8j)$, which corresponds to the lower left 8×8 grid in $G_{k+6,\log n-k}(i,j)$. For each point set $P \in \mathcal{P}_0$, there is exactly one point in $G_{k+3}(8i,8j)$, and the bits in $Z_{k,i,j}$ encode the position of the point on the 8×8 grid. Suppose s_1 and s_2 are two bit vectors of length 192, such that when the bits in $Z_{k,i,j}$ are assigned as s_1 (denoted $Z_{k,i,j} = s_1$), the point in $G_{k+3}(8i,8j)$ resides in the upper left grid cell,; and when $Z_{k,i,j} = s_2$, it resides in the grid cell to the upper left of the center of $G_{k+3}(8i,8j)$ (see Figure 5). Note that by this definition, the corner volume distance of this two point is at least $n/8$. Meanwhile, since there are no constraints on the other 63 points in $G_{k+6,\log n-k}(i,j)$, it is easy to show that such assignments s_1 and s_2 indeed exist.

By restricting the assignments of $Z_{k,i,j}$ to $\{s_1, s_2\}$, we have created a subset $\mathcal{Z}_1$ of $\mathcal{Z}_0 = \{0,1\}^{\frac{1}{2}n\log n}$:

$$\mathcal{Z}_1 = \{\mathbf{Z} \mid Z_{k,i,j} = s_1 \text{ or } s_2, k \in \{0, 6, \dots, \log n - 6\}, i \in [n/2^{k+6}], j \in [2^k]\}.$$

Let $\mathcal{P}_1$ denote the sub-collection of $\mathcal{P}_0$ that $\mathcal{Z}_1$ encode. By Fact 3.4, the number of $Z_{k,i,j}$'s is $\frac{1}{384}n\log n$, so $|\mathcal{P}_1| = 2^{\frac{1}{384}n\log n}$. Define a bit vector $\mathbf{T}$ of length $\frac{1}{384}n\log n$, such that $\mathbf{T}(k,i,j) = 0$ if $Z_{k,i,j} = s_1$ and $\mathbf{T}(k,i,j) = 1$ if $\mathbf{Z}_{k,i,,j} = s_2$, then a bit vector $\mathbf{T}$ encodes a bit vector $\mathbf{Z} \in \mathcal{Z}_1$, and therefore encodes a point set in $\mathcal{P}_1$. Let $\mathcal{T} = \{0,1\}^{\frac{1}{384}n\log n}$ denote the collection of all bit vectors $\mathbf{T}$. Then there is a bijection between $\mathcal{T}$ and $\mathcal{P}_1$, and $|\mathcal{T}| = |\mathcal{P}_1| = 2^{\frac{1}{384}n\log n}$.

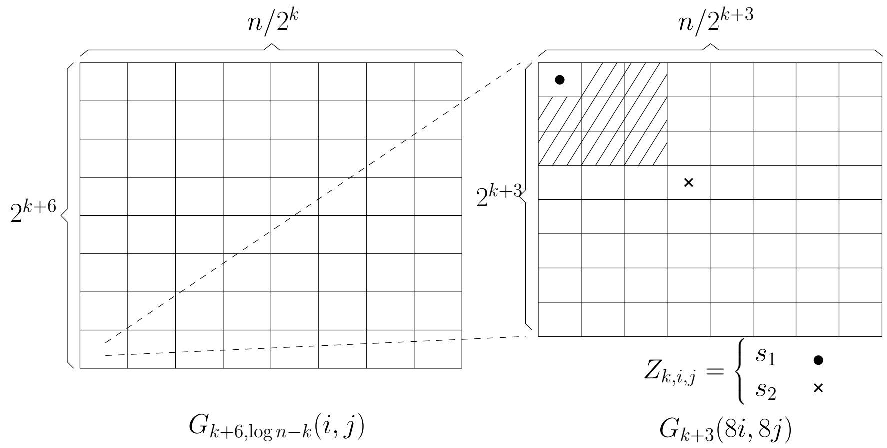

Figure 5: Illustration of the 64×64 grid. The volume of each cell in $G_{k+3}(8i, 8j)$ is $n/64$. The cells in shadow represent the corner volume difference of s_1 and s_2.

Consider two point sets P_1 and P_2 in $\mathcal{P}_1$. Let $\mathbf{T}_{P_1}$ and $\mathbf{T}_{P_2}$ denote the bit vector that encode these two point sets, respectively. The following lemma relates the corner volume distance of P_1 and P_2 with the Hamming distance between $\mathbf{T}_{P_1}$ and $\mathbf{T}_{P_2}$.

Lemma 3.4. *Suppose there exists a constant c, such that for any $P_1, P_2 \in \mathcal{P}_1$, the Hamming distance $H(\mathbf{T}_{P_1}, \mathbf{T}_{P_2}) \geq cn \log n$, then the corner volume distance between P_1 and P_2, $\Delta(P_1, P_2)$, is $\Omega(n^2 \log n)$.*

Proof. We make the following relaxation on $\Delta(P_1, P_2)$:

$$\begin{aligned}
\Delta(P_1, P_2) &= \sum_{k=0}^{\log n} \sum_{i=0}^{n/2^k-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k,i,j) - V_{P_1}(k,i,j)| \\
&\geq \sum_{k \in \{0,6,\ldots,\log n-6\}} \sum_{i=0}^{n/2^{k+6}-1} \sum_{j=0}^{2^k-1} |V_{P_1}(k+3,8i,8j) - V_{P_1}(k+3,8i,8j)| \,.
\end{aligned}$$

Now consider the bits $\mathbf{T}_{P_1}(k,i,j)$ and $\mathbf{T}_{P_2}(k,i,j)$. If $\mathbf{T}_{P_1}(k,i,j) \neq \mathbf{T}_{P_2}(k,i,j)$, then by the choice of s_1 and s_2 we have $|V_{P_1}(k+3, 8i, 8j - V_{P_2}(k+3, 8i, 8j)| \geq n/8$. So the corner volume distance $\Delta(P_1, P_2)$ is lower bounded by the Hamming distance $H(\mathbf{T}_{P_1}, \mathbf{T}_{P_2})$ multiplied by $n/8$, and the lemma follows. ☐

The following lemma (probably folklore; we provide a proof here for completeness) states that there is a large subset of $\mathcal{T}$, in which the vectors are well separated in terms of Hamming distance.

Lemma 3.5. *Let $N = \frac{1}{384} n \log n$. There is a subset $\mathcal{T}^* \subseteq \mathcal{T} = \{0,1\}^N$ of size $2^{\frac{1}{16}N}$, such that for any $\mathbf{T}_1 \neq \mathbf{T}_2 \in \mathcal{T}^*$, the Hamming distance $H(\mathbf{T}_1, \mathbf{T}_2) \geq \frac{1}{4}N$.*

Proof. We embed $\mathcal{T}$ into a graph (V, E). Each node in V represents a vector $\mathbf{T} \in \mathcal{T}$, and there is edge between two nodes $\mathbf{T}_1$ and $\mathbf{T}_2$ if and only if $H(\mathbf{T}_1, \mathbf{T}_2) < \frac{1}{4}N$. By this embedding, it is equivalent to prove that there is an independent set of size $2^{\frac{1}{16}N}$ in (V, E).

Fix a vector $\mathbf{T} \in \mathcal{T}$, and consider a random vector $\mathbf{T}'$ uniformly drawn from $\mathcal{T}$. It is easy to see that the Hamming distance $H(\mathbf{T}, \mathbf{T}')$ follows binomial distribution. By Chernoff bound

$$\Pr[H(\mathbf{T}, \mathbf{T}') < \frac{1}{4}N] \leq e^{-\frac{1}{16}N} \leq 2^{-\frac{1}{16}N}.$$

This implies that the probability that there is an edge between $\mathbf{T}$ and $\mathbf{T}'$ is at most $2^{-\frac{1}{16}N}$. By the fact that $\mathbf{T}'$ is uniformly chosen from $\mathcal{T}$, it follows that the degree of $\mathbf{T}$ is at most $d = 2^N \cdot 2^{-\frac{1}{16}N} = 2^{\frac{15}{16}N}$. Since a graph with maximum degree d must have an independent set of size at least $|V|/d$, there must be an independent set of size at least $2^{\frac{1}{16}N}$. $\square$

Let $\mathcal{P}^*$ denote the collection of point sets encoded by $\mathcal{T}^*$. By Lemma 3.5, $|\mathcal{P}^*| \geq 2^{\frac{1}{16}N} = 2^{\frac{1}{6144}n \log n}$. From Lemma 3.3 and 3.4 we know that for any two point sets $P_1 \neq P_2 \in \mathcal{P}^*$, the combinatorial discrepancy of the union of P_1 and P_2 is $\Omega(\log n)$. This completes the proof of Theorem 3.2.

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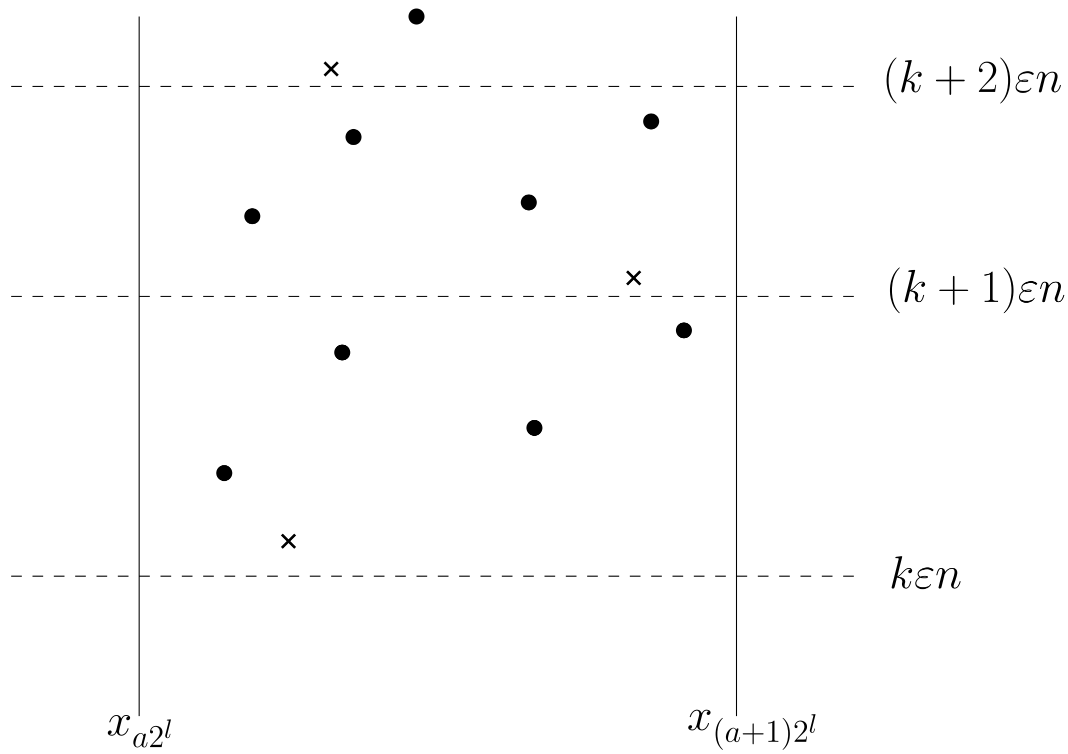

$(k+2)\varepsilon n$
$(k+1)\varepsilon n$
$k\varepsilon n$
x_{a2^l}
$x_{(a+1)2^l}$