# Optimal Dynamic Sequence Representations * 

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#### Abstract

We describe a data structure that supports access, rank and select queries, as well as symbol insertions and deletions, on a string $S[1, n]$ over alphabet $[1 . . \sigma]$ in time $O(\lg n / \lg \lg n)$, which is optimal even on binary sequences and in the amortized sense. Our time is worst-case for the queries and amortized for the updates. This complexity is better than the best previous ones by a $\Theta(1+\lg \sigma / \lg \lg n)$ factor. We also design a variant where times are worst-case, yet rank and updates take $O(\lg n)$ time. Our structure uses $n H_{0}(S)+o(n \lg \sigma)+O(\sigma \lg n)$ bits, where $H_{0}(S)$ is the zero-order entropy of $S$. Finally, we pursue various extensions and applications of the result.


## 1 Introduction

String representations supporting rank and select queries are fundamental in many data structures, including full-text indexes [GGV03, FMMN07, GMR06, permutations GMR06, BGNN10, inverted indexes BFLN08, BGNN10, graphs CN10, document retrieval indexes VM07, labeled trees GMR06, BHMR11, XML indexes GHSV07, FLMM09, binary relations BHMR11, and many more. The problem is to encode a string $S[1, n]$ over alphabet $\Sigma=[1 . . \sigma]$ so as to support the following queries:

$$
\begin{aligned}
\operatorname{rank}_{a}(S, i) & =\text { number of occurrences of } a \in \Sigma \text { in } S[1, i] \text {, for } 1 \leq i \leq n . \\
\operatorname{select}_{a}(S, i) & =\text { position in } S \text { of the } i \text {-th occurrence of } a \in \Sigma \text {, for } 1 \leq i \leq \operatorname{rank}_{a}(S, n) . \\
\operatorname{access}(S, i) & =S[i] .
\end{aligned}
$$

There exist various static representations of $S$ (i.e., $S$ cannot change) that support these operations [GGV03, GMR06, FMMN07, BGNN10, BN12]. The most recent one [BN12] shows a lower bound of $\Omega\left(\lg \frac{\lg \sigma}{\lg w}\right)$ time for operation rank on a RAM machine with $w$-bit words, using any $O\left(n \lg ^{O(1)} n\right)$ space. It also provides a matching upper bound that in addition achieves almost constant time for select and access, using compressed space. Thus the problem is essentially closed.

However, various applications need dynamism, that is, the ability to insert and delete symbols in $S$. A lower bound for this case, in order to support operations rank, insertions and deletions, even for bitmaps $(\sigma=2)$ and in the amortized sense, is $\Omega(\lg n / \lg \lg n)$ [FS89]. On the other hand the best known upper bound HM10, NS10 is $O((1+\lg \sigma / \lg n) \lg n / \lg \lg n)$, that is, a factor

[^0]$\Theta(\lg \sigma / \lg \lg n)$ away from the lower bound for alphabets larger than polylogarithmic. Their space is $n H_{0}(S)+o(n \lg \sigma)$, where $H_{0}(S)=\sum_{a \in[1 . . \sigma]}\left(n_{a} / n\right) \lg \left(n / n_{a}\right) \leq \lg \sigma$ is the zero-order entropy of $S$, where $n_{a}$ is the number of occurrences of $a$ in $S$.

In this paper we close this gap by providing an optimal-time dynamic representation of sequences. Our representation takes $O(\lg n / \lg \lg n)$ time for all the operations, worst-case for the three queries and amortized for insertions and deletions. We present a second variant achieving worst-case bounds for all the operations, $O(\lg n / \lg \lg n)$ for select and access and $O(\lg n)$ for rank, insertions and deletions. The space is also $n H_{0}(S)+o(n \lg \sigma)$. This $O(\lg n)$ is still faster than previous work for $\lg \sigma=\Omega\left((\lg \lg n)^{2}\right)$. This gets much closer to closing this problem under the dynamic scenario as well.

We then show how to handle general alphabets, such as $\Sigma=\mathbb{R}$, or $\Sigma=\Gamma^{*}$ for a symbol alphabet $\Gamma$, in optimal time. For example, in the comparison model for $\Sigma=\mathbb{R}$, the time is $O(\lg \sigma+\lg n / \lg \lg n)$, where $\sigma$ is the number of distinct symbols that appear in $S$; in the case $\Sigma=\Gamma^{*}$ for general $\Gamma$, the time is $O(|a|+\lg \gamma+\lg n / \lg \lg n)$, where $|a|$ is the length of the involved symbol (a string) and $\gamma$ the number of distinct symbols of $\Gamma$ that appear in the elements of $S$. Previous dynamic solutions have assumed that the alphabet [1.. $\sigma$ ] was static.

At the end we describe several applications where our result offers improved time/space tradeoffs. These include compressed indexes for dynamic text collections, construction of the BurrowsWheeler transform [BW94] and static compressed text indexes within compressed space, as well as compressed representations of dynamic binary relations, directed graphs, and inverted indexes.

We start with an overview of the state of the art, putting our solution in context, in Section 2, We review the wavelet tree data structure [GGV03], which is at the core of our solution (and most previous ones) in Section 3, In Section 4 we describe the core of our amortized solution, deferring to Section 5 the management of deletions and its relation with a split-find data structure needed for rank and inserts. Section 6 deals with the changes in $\lg n$ and how we obtain times independent of $\sigma$, and concludes with Theorem 1 , our result on uncompressed sequences. Then Section 7 shows how to improve the data encoding to obtain compressed space in Theorem [2, and Section 8 how to obtain worst-case times, Theorem 3. Finally, Section 10 describes some extensions and several applications of our results. We conclude in Section 11. An important technical part of the paper, describing the structure of blocks that handle subsequences of polylogarithmic size, is deferred to Section 9 to avoid distractions.

## 2 Related Work

With one exception GHSV07, all the previous work on dynamic sequences build on the wavelet tree structure GGV03. The wavelet tree decomposes $S$ hierarchically. In a first level, it separates larger from smaller symbols, by marking in a bitmap which symbols of $S$ were larger and which were smaller. The two subsequences of $S$ are recursively separated. The $\lg \sigma$ levels of bitmaps describe $S$, and access, rank and select operations on $S$ are carried out via $\lg \sigma$ rank and select operations on the bitmaps (see Section 3 for more details).

In the static case, rank and select operations on bitmaps take constant time, and therefore access, rank and select on $S$ takes $O(\lg \sigma)$ time GGV03. This can be reduced to $O(1+\lg \sigma / \lg \lg n)$ by using multiary wavelet trees [FMMN07]. These separate the symbols into $\rho=o(\lg n)$ ranges, and instead of a bitmap store a sequence over an alphabet of size $\rho$.

Insertions and deletions in $S$ can also be carried out by inserting and deleting bits from $\lg \sigma$
bitmaps. However, the operations on dynamic bitmaps are bound to be slower. Fredman and Saks [FS89] show that $\Omega(\lg n / \lg \lg n)$ time is necessary, even in the amortized sense, to support rank, insert and delete operations on a bitmap. By using dynamic bitmap solutions HSS03, CHL04, BB04, CHLS07, HSS11 on the wavelet tree levels, one immediately obtains a dynamic sequence representation, where the space and the time of the dynamic bitmaps solution is multiplied by $\lg \sigma$ (the sum of the zero-order entropies of the bitmaps also adds up to $n H_{0}(S)$ GGV03]). With this combination one can obtain times as good as $O(\lg \sigma \lg n / \lg \lg n)$ (and $n \lg \sigma+o(n \lg \sigma)$ bits) CHLS07] and spaces as good as $O\left(n H_{0}(S)\right)$ (and $O(\lg \sigma \lg n)$ time) BB04.

Mäkinen and Navarro [MN06, MN08 made the above combination explicit and obtained compressed bitmaps with logarithmic-time operations, yielding $O(\lg \sigma \lg n)$ time for all the sequence operations and the best compressed space until then, $n H_{0}(S)+o(n \lg \sigma)$ bits. They also obtained $O((1+\lg \sigma / \lg \lg n) \lg n)$ query time, but with an update time of $O\left(\lg \sigma \lg ^{1+\varepsilon} n\right)$, for any constant $0<\varepsilon<1$. This was achieved by replacing binary with multiary wavelet trees, and obtaining $O(\lg n)$ query time for the operations on sequences over a small alphabet of size $o(\lg n)$.

Lee and Park [LP07, [PP09] pursued this path further, obtaining the $O(1+\lg \sigma / \lg \lg n) \lg n)$ time for queries and update operations, yet the space was not compressed, $n \lg \sigma+o(n \lg \sigma)$, and update times were amortized. Shortly after, González and Navarro GN08, GN09] obtained the best of both worlds, making all the times worst-case and compressing the space again to $n H_{0}(S)+o(n \lg \sigma)$ bits. Both solutions managed to solve all query and update operations in $O(\lg n)$ time on sequences over small alphabets of size $o(\lg n)$.

Finally, almost simultaneously, He and Munro [HM10] and Navarro and Sadakane [NS10] obtained the currently best result, $O((1+\lg \sigma / \lg \lg n) \lg n / \lg \lg n)$ time, still within the same compressed space. They did so by improving the times of the dynamic sequences on small alphabets to $O(\lg n / \lg \lg n)$, which is optimal even on bitmaps and in the amortized sense FS89.

As mentioned, the solution by Gupta et al. GHSV07] deviates from this path and is a general framework for using any static data structure and periodically rebuilding it. By using it over a given representation GMR06, it achieves $O(\lg \lg n)$ query time and $O\left(n^{\varepsilon}\right)$ amortized update time. It would probably achieve compressed space if combined with more recent static data structures BGNN10]. This shows that query times can be significantly smaller if one allows for much higher update times. In this paper, however, we focus in achieving similar times for all the operations.

Table 1 gives more details on previous and our new results. Wavelet trees can also be used to model $n \times n$ grids of points, in which case $\sigma=n$. Bose et al. BHMM09] used a wavelet-tree-like structure to solve range counting in optimal static time $O(\lg n / \lg \lg n)$, using operations slightly more complex than rank on the wavelet tree levels. It is conceivable that this can be turned into an $O\left((\lg n / \lg \lg n)^{2}\right)$ time algorithm using dynamic sequences on the wavelet tree levels. On the other hand, $\Omega\left((\lg n / \lg \lg n)^{2}\right)$ is a lower bound for dynamic range counting in two dimensions [Pat07]. This seems to suggest that it is unlikely to obtain better results than those previously known for dynamic wavelet trees.

In this paper we show that this dead-end can be broken by abandoning the implicit assumption that, to provide access, rank and select on $S$, we must provide rank and select on the bitmaps (or sequences over $[1 . . \rho]$ ). We show that all what is needed is to track positions of $S$ downwards and upwards along the wavelet tree. It turns out that this tracking can be done in constant time per level, breaking the $\Theta(\lg n / \lg \lg n)$ per-level barrier.

As a result, we obtain the optimal time complexity $O(\lg n / \lg \lg n)$ for all the queries (worst-case) and update operations (amortized), independently of the alphabet size. This is $\Theta(1+\lg \sigma / \lg \lg n)$

Table 1: History of results on managing dynamic sequences $S[1, n]$ over alphabet [1.. $\sigma$ ], assuming $\sigma=o(n / \lg n)$ to simplify. Some results HSS03, BB04, CHL04 were presented only for binary sequences and the result we give is obtained by using them in combination with wavelet trees. Column W/A tells whether the update times are (W)orst-case or (A)mortized.

| Source | Space (bits) | Query time | Update time | W/A |
| :---: | :---: | :---: | :---: | :---: |
| [FS89] |  | $\Omega(\lg n / \lg \lg n)$ for rank and indels |  | A |
| HSS03, HSS11] | $n \lg \sigma+O\left(n \lg \sigma(\lg \lg n)^{2} / \lg n\right)$ | $O(\lg \sigma \lg n / \lg \lg n)$ | $O\left(\lg \sigma(\lg n / \lg \lg n)^{2}\right)$ | A |
| CHL04 | $O(n \lg \sigma)$ | $O(\lg \sigma \lg n)$ | $O(\lg \sigma \lg n)$ | W |
| BB04 | $O\left(n H_{0}(S)+\lg n\right)$ | $O(\lg \sigma \lg n)$ | $O(\lg \sigma \lg n)$ | W |
| MN06 MN08 | $n H_{0}(S)+O(n \lg \sigma / \sqrt{\lg n})$ | $O(\lg \sigma \lg n)$ | $O(\lg \sigma \lg n)$ | W |
|  | $n H_{0}(S)+O\left(n \lg \sigma / \lg ^{1 / 2-\varepsilon} n\right)$ | $O\left(\left(1+\frac{1}{\varepsilon} \lg \sigma / \lg \lg n\right) \lg n\right)$ | $O\left(\frac{1}{\varepsilon} \lg \sigma \lg ^{1+\varepsilon} n\right)$ | W |
| CHLS07 | $O(n \lg \sigma)$ | $O(\lg \sigma \lg n / \lg \lg n)$ | $O(\lg \sigma \lg n / \lg \lg n)$ | W |
| GHSV07] | $n \lg \sigma+O(n \lg \sigma / \lg \lg \sigma)$ | $O\left(\frac{1}{\varepsilon} \lg \lg n+\lg \lg \sigma\right)$ | $O\left(\frac{1}{\varepsilon} n^{\varepsilon}\right)$ | A |
| LP07, LP09] | $n \lg \sigma+O(n \lg \sigma / \sqrt{\lg n})+O(n)$ | $O((1+\lg \sigma / \lg \lg n) \lg n)$ | $O((1+\lg \sigma / \lg \lg n) \lg n)$ | A |
| GN08 GN09 | $n H_{0}(S)+O(n \lg \sigma / \sqrt{\lg n})$ | $O((1+\lg \sigma / \lg \lg n) \lg n)$ | $O((1+\lg \sigma / \lg \lg n) \lg n)$ | W |
| HM10 | $n H_{0}(S)+O(n \lg \sigma / \sqrt{\lg n})$ | $O((1+\lg \sigma / \lg \lg n) \lg n / \lg \lg n)$ | $O((1+\lg \sigma / \lg \lg n) \lg n / \lg \lg n)$ | W |
| NS10] | $n H_{0}(S)+O\left(n \lg \sigma /\left(\varepsilon \lg ^{1-\varepsilon} n\right)\right)$ | $O\left(\left(1+\frac{1}{\varepsilon} \lg \sigma / \lg \lg n\right) \lg n / \lg \lg n\right)$ | $O\left(\left(1+\frac{1}{\varepsilon} \lg \sigma / \lg \lg n\right) \lg n / \lg \lg n\right)$ | W |
| Ours | $n H_{0}(S)+O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)$ | $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$ | $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$ | A |
| Ours | $n H_{0}(S)+O\left(n H_{0}(S) / \lg \lg n\right)+O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)$ | $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right), O(\lg n)$ for rank | $O(\lg n)$ | W |

times faster than what was believed to be the "ultimate" solution. Our space is $n H_{0}(S)+o(n \lg \sigma)$ bits, similar to previous solutions. We develop, alternatively, a data structure achieving worst-case time for all the operations, yet this raises to $O(\lg n)$ for rank, insertions and deletions.

Among the many applications of this result, it is worth mentioning that any dynamic sequence representation supporting rank and insertions in $O(t(n))$ amortized time can be used to compute the Burrows-Wheeler transform (BWT) [BW94] of a sequence $S[1, n]$ in worst-case time $O(n t(n))$. Thus our results allow us to build the BWT in $O(n \lg n / \lg \lg n)$ time and compressed space. The best existing space-time tradeoffs are by Okanohara and Sadakane OS09, who achieve optimal $O(n)$ time within $O(n \lg \sigma \lg \lg \sigma)$ bits, and Hon et al. HSS09, who achieve $O(n \lg \lg \sigma)$ time with $O(n \lg \sigma)$ bits. Kärkkäinen [Kär07] had obtained before $O(n \lg n+n v)$ time and $O(n \lg n / \sqrt{v})$ extra bits for a parameter $v$. Using less space allows us to improve BWT-based compressors (like BziP2) by allowing them to cut the sequence into larger blocks, given a fixed amount of main memory for the compressor. Many other results will be mentioned in Section 10 .

## 3 The Wavelet Tree

Let $S$ be a string over alphabet $\Sigma=[1 . . \sigma]$. We associate each $a \in \Sigma$ to a leaf $v_{a}$ of a full balanced binary tree $\mathcal{T}$. The essential idea of the wavelet tree structure GGV03] is the representation of elements from a string $S$ by bit sequences stored in the nodes of tree $\mathcal{T}$. We associate a subsequence $S(v)$ of $S$ with every node $v$ of $\mathcal{T}$. For the root $v_{r}, S\left(v_{r}\right)=S$. In general, $S(v)$ consists of all the occurrences of symbols $a \in \Sigma_{v}$ in $S$, where $\Sigma_{v}=\left\{a \in \Sigma, v_{a}\right.$ descends from $\left.v\right\}$. The wavelet tree does not store $S(v)$ explicitly, but just a bit vector $B(v)$. We set $B(v)[i]=t$ if the $i$-th element of $S(v)$ also belongs to $S\left(v_{t}\right)$, where $v_{t}$ is the $t$-th child of $v$ (the left child corresponds to $t=0$ and the right to $t=1$ ). This data structure (i.e., $\mathcal{T}$ and bit vectors $B(v)$ ) is called a wavelet tree. Since $\mathcal{T}$ has $O(\sigma)$ nodes and $\lceil\lg \sigma\rceil$ levels, and the bitmaps at each level add up to length $n$, it requires $n\lceil\lg \sigma\rceil+O(\sigma \lg n)$ bits of space. If the bitmaps $B(v)$ are compressed to $|B(v)| H_{0}(B(v)$ bits, the total size adds up to $n H_{0}(S)+O(\sigma \lg n)$ bits [GGV03]. Various surveys on wavelet trees are available, for example [NM07, Nav12].

For any symbol $S[i]$ and every internal node $v$ such that $S[i] \in \Sigma_{v}$, there is exactly one bit $b_{v}$ in $B(v)$ that indicates in which child of $v$ the leaf $v_{S[i]}$ is stored. We will say that such $b_{v}$ encodes $S[i]$ in $B(v)$; we will also say that bit $b_{v}$ from $B(v)$ corresponds to a bit $b_{u}$ from $B(u)$ if both $b_{v}$ and $b_{u}$ encode the same symbol $S[i]$ in two nodes $v$ and $u$. Identifying the positions of bits that encode the same symbol plays a crucial role in wavelet trees. Other, more complex, operations rely on the ability to navigate in the tree and keep track of bits that encode the same symbol.

The wavelet tree encodes $S$, in the sense that it allows us to extract any $S[i]$. To implement $\operatorname{access}(S, i)$ we traverse a path from the root to the leaf $v_{S[i]}$. In each visited node we read the bit $b_{v}$ that encodes $S[i]$ and proceed to the corresponding bit in the $b_{v}$-th child of $v$. Upon arriving to a leaf $v_{a}$ we answer $\operatorname{access}(S, i)=a$.

The wavelet tree also implements operations rank and select. To compute select ${ }_{a}(S, i)$, we start at the position of $S\left(v_{a}\right)[i]$ and identify the corresponding bit $b_{v}$ in the parent $v$ of $v_{a}$. We continue this process to the root until reaching a position $B\left(v_{r}\right)[j]$. Then the answer is select ${ }_{a}(S, i)=j$. Finally, to compute $\operatorname{rank}_{a}(S, i)$, we traverse the wavelet tree from $B\left(v_{r}\right)[i]$ to the leaf $v_{a}$. At each element $b_{v}$ of each node $v$ in the path, we identify the last bit $b^{\prime}$ that precedes $b_{v}$ and encodes an $a$. Then we move to the bit $b_{u}$ corresponding to $b^{\prime}$ in the $b^{\prime}$-th child of $v$. Upon arriving at position $B\left(v_{a}\right)[j]$, the answer is $\operatorname{rank}(S, i)=j$.

The standard method used in wavelet trees for identifying the corresponding bits is to maintain rank/select data structures on the bit vectors $B(v)$. Let $B(v)[e]=t$; we can find the offset of the corresponding bit in the child of $v$ by answering a query $\operatorname{rank}_{t}(B(v), e)$. If $v$ is the $t$-th child of a node $u$, we can find the offset of the corresponding bit in $u$ by answering a query $\operatorname{select}_{t}(B(u), e)$. Finally, the more complicated process of finding $b^{\prime}$ needed for $\operatorname{rank}_{a}(S, i)$ is easily solved using binary rank: if $b_{v}$ is at position $e$ in $B(v)$, then without need of finding $b^{\prime}$ we know that its corresponding position in the $b^{\prime}$-th child of $v$ is $\operatorname{rank}_{b^{\prime}}(B(v), e)$. This approach leads to $O(\lg \sigma)$ query times in the static case because rank/select queries on a bit vector $B(v)$ can be answered in constant time and $|B(v)|+o(|B(v)|)$ bits of space (Mun96, Cla96, and even using $|B(v)| H_{0}(B(v))+o(|B(v)|)$ bits RRR07. However, we need $\Omega(\lg n / \lg \lg n)$ time to support rank/select and updates on a bit vector [FS89], which multiplies the operation times in the dynamic case.

An improvement (for both static and dynamic wavelet trees) can be achieved by increasing the fan-out of the wavelet tree to $\rho=\Theta\left(\lg ^{\varepsilon} n\right)$ for a constant $0<\varepsilon<1$ : as before, $B(v)[e]=t$ if the $e$-th element of $S(v)$ also belongs to $S\left(v_{t}\right)$ for the $t$-th child $v_{t}$ of $v$. This enables us to reduce the height of the wavelet trees and the query time by a $\Theta(\lg \lg n)$ factor, because the rank/select times over alphabet [1.. $\rho$ ] is still constant in the static case FMMN07] and $O(\lg n / \lg \lg n)$ in the dynamic case HM10, NS10]. However, it seems that further improvements that are based on dynamic rank/select queries in every node are not possible.

In this paper we use a different approach to identifying the corresponding elements. We partition sequences $B(v)$ into blocks, which are stored in compact list structures $L(v)$. Pointers from selected positions in $L(v)$ to the structure $L(u)$ in children nodes $u$ (and vice versa) enable us to navigate between nodes of the wavelet tree in constant time. We extend the idea to multiary wavelet trees. While similar techniques have been used in some geometric data structures [Nek11, Ble08, applying them on compressed data structures where the bit budget is severely limited is much more challenging. We describe our new solution next.

## 4 Basic Structure

We start by describing the main components of our modified wavelet tree. Then, we show how our structure supports $\operatorname{access}(S, i)$ and $\operatorname{select}_{a}(S, i)$. In the third part of this section we describe additional structures that enable us to answer $\operatorname{rank}_{a}(S, i)$. Finally, we show how to support updates.

### 4.1 Structure

We assume that the wavelet tree $\mathcal{T}$ has node degree $\rho=\Theta\left(\lg ^{\varepsilon} n\right)$. We divide sequences $B(v)$ into blocks and store those blocks in a doubly-linked list $L(v)$. Each block $G_{j}(v)$ contains $\Theta\left(\lg ^{3} n / \lg \rho\right)$ consecutive elements from $B(v)$, except the last, which can be smaller. That is, $\left|G_{j}(v)\right|=O\left(\lg ^{3} n\right)$ if measured in bits. For each $G_{j}(v)$ we maintain a data structure $R_{j}(v)$ that supports access, rank and select queries on elements of $G_{j}(v)$. Since a block contains a poly-logarithmic number of elements over an alphabet of size $\rho$, we can answer those queries in $O(1)$ time using $O\left(\left|G_{j}(v)\right| / \lg ^{1-\varepsilon} n\right)$ additional bits (see Section 9 for details).

A pointer to an element $B(v)[e]$ consists of two parts: a unique id of the block $G_{j}(v)$ that contains the offset $e$ and the index of $e$ in $G_{j}(v)$. Such a pair (block id, local index) will be called the position of $e$ in $v$.

We maintain pointers between selected corresponding elements in $L(v)$ and its children. If an
element $B(v)[e]=t$ is stored in a block $G_{j}(v)$ and $B(v)\left[e^{\prime}\right] \neq t$ for all $e^{\prime}<e$ in $G_{j}(v)$ (i.e., $B(v)[e]$ is the first occurrence of $t$ in its block), then we store a pointer from $e$ to the offset $e_{t}$ of the corresponding element $B\left(v_{t}\right)\left[e_{t}\right]$ in $L\left(v_{t}\right)$, where $v_{t}$ is the $t$-th child of $v$. Pointers are bidirectional, that is, we also store a pointer from $e_{t}$ to $e$. In addition, if $B(v)[e]$ is the first offset in its block and $B(u)\left[e^{\prime}\right]$ corresponds to $B(v)[e]$ in the parent $u$ of $v$, then we store a pointer from $e$ to $e^{\prime}$ and, by bidirectionality, from $e^{\prime}$ to $e$. All these pointers will be called inter-node pointers. We describe how they are implemented later in this section.

It is easy to see that the number of inter-node pointers from $e$ in $L(v)$ to $e_{t}$ in $L\left(v_{t}\right)$, for any fixed $t$, is $\Theta(g(v))$, where $g(v)$ is the number of blocks in $L(v)$. Hence, the total number of pointers that point down from a node $v$ is bounded by $O(g(v) \rho)$. Additionally, there are $O(g(v))$ pointers up to the parent of $v$. Thus, the total number of pointers in the wavelet tree equals $O\left(\sum_{v \in \mathcal{T}} g(v) \rho\right)=O\left(n \lg \sigma / \lg ^{3-\varepsilon} n+\sigma \lg ^{\varepsilon} n\right)$. Note that the term $\sigma \lg ^{\varepsilon} n$ is only necessary to account for nodes that have just one block (with $o\left(\lg ^{3} n\right)$ bits). Since the children of those nodes must also have just one block, we avoid storing their pointers, as we know that all point to the same block and their index inside the block can be found with constant-time rank/select operations inside the block from where pointers leave. Their upwards pointer to their parent, if the parent has more than one block, can be represented and charged to the space of the parent. This yields the cleaner expression $O\left(n \lg \sigma / \lg ^{3-\varepsilon} n\right)$ for the number of pointers.

The pointers from a block $G_{j}(v)$ are stored in a data structure $F_{j}(v)$. Using $F_{j}(v)$ we can find, for any offset $e$ in $G_{j}(v)$ and any $1 \leq t \leq \rho$, the last $e^{\prime} \leq e$ in $G_{j}(v)$ such that there is a pointer from $e^{\prime}$ to an offset $e_{t}^{\prime}$ in $L\left(v_{t}\right)$. We describe in Section 9 how $F_{j}(v)$ implements the queries and updates in constant time.

For the root node $v_{r}$, we store a dynamic searchable partial-sum data structure $K\left(v_{r}\right)$ that contains the number of positions in each block of $L\left(v_{r}\right)$. Using $K\left(v_{r}\right)$, we can find the block $G_{j}\left(v_{r}\right)$ that contains the $i$-th element of $S\left(v_{r}\right)=S$ (query search on the partial sums), as well as the number of elements in all the blocks that precede a given block $G_{j}\left(v_{r}\right)$ (operation sum on the partial sums). Both operations can be supported in $O(\lg n / \lg \lg n)$ time and linear space [NS10, Lem. 1]. The same data structures $K\left(v_{a}\right)$ are also stored in the leaves $v_{a}$ of $\mathcal{T}$. Since $g\left(v_{r}\right)=O\left(n \lg \rho / \lg ^{3} n\right)$, and also $\sum_{a \in \Sigma} g\left(v_{a}\right)=O\left(n \lg \rho / \lg ^{3} n\right)$, we store $O\left(n \lg \lg n / \lg ^{3} n\right)$ elements in the partial sums $K\left(v_{r}\right)$ and $K\left(v_{a}\right)$, for an overall size of $O\left(n \lg \lg n / \lg ^{2} n\right)$ bits.

We observe that we do not store a sequence $B\left(v_{a}\right)$ in a leaf node $v_{a}$, only in internal nodes. Nevertheless, we divide the (implicit) sequence $B\left(v_{a}\right)$ into blocks and store the number of positions in each block in $K\left(v_{a}\right)$; we maintain $K\left(v_{a}\right)$ only if $L\left(v_{a}\right)$ consists of more than one block. Moreover we store inter-node pointers from the parent of $v_{a}$ to $v_{a}$ and vice versa. Pointers in a leaf are maintained using the same rules of any other node.

For future reference, we provide the list of secondary data structures in Table 2,

### 4.2 Access and Select Queries

Assume the position of an element $B(v)[e]=t$ in $L(v)$ is known, and let $i_{v}$ be the index of offset $e$ in its block $G_{j}(v)$. Then the position of the corresponding offset $e_{t}$ in $L\left(v_{t}\right)$ is computed as follows. Using $F_{j}(v)$, we find the index $i_{v}^{\prime}$ of the largest $e^{\prime} \leq e$ in $G_{j}(v)$ such that there is a pointer from $e^{\prime}$ to some $e_{t}^{\prime}$ in $L\left(v_{t}\right)$. Due to our construction, such $e^{\prime}$ must exist (it may be $e$ itself). Let $G_{\ell}\left(v_{t}\right)$ denote the block that contains $e_{t}^{\prime}$, and let $i_{t}^{\prime}$ be the index of $e_{t}^{\prime}$ in $G_{\ell}\left(v_{t}\right)$. Due to our rules to define pointers, $e_{t}$ also belongs to $G_{\ell}\left(v_{t}\right)$, since if it belonged to another block $G_{m}\left(v_{t}\right)$ the upward pointer from $G_{m}\left(v_{t}\right)$ would point between $e^{\prime}$ and $e$, and since pointers are bidirectional, this would

Table 2: Structures inside any node $v$ of the wavelet tree $\mathcal{T}$, or only in the root node $v_{r}$ and the leaves $v_{a}$. The third column gives the extra space in bits, on top of the data, for the whole structure; here $|d a t a|$ is $n \lg \sigma$ in the uncompressed case and $n H_{0}(S)$ in the compressed case.

| Structure | Meaning | Extra space in bits |
| :--- | :--- | :---: |
| $L(v)$ | List of blocks storing $B(v)$ | $O\left(n \lg \sigma / \lg ^{2} n+\sigma \lg n\right)$ |
| $G_{j}(v)$ | $j$-th block of list $L(v)$ | $O(\|d a t a\| \lg \lg n / \lg n+n \lg \sigma / \lg n+\sigma \lg n)$ |
| $R_{j}(v)$ | Supports rank/select/access inside $G_{j}(v)$ | $O\left(\|d a t a\| / \lg ^{1-\varepsilon} n\right)$ |
| $F_{j}(v)$ | Pointers leaving from $G_{j}(v)$ | $O\left(n \lg \sigma / \lg ^{2-\varepsilon} n\right)$ |
| $H_{j}(v)$ | Pointers arriving at $G_{j}(v)$ | $O\left(n \lg \sigma / \lg ^{2} n\right)$ |
| $P_{t}(v)$ | Predecessor in $L(v)$ containing symbol $t$ | $O\left(n \lg \sigma / \lg ^{2-\varepsilon} n\right)$ |
| $K(v)$ | Partial sums on block lengths for $v_{r}$ and $v_{a}$ | $O\left(n \lg \lg n / \lg ^{2} n\right)$ |
| $D_{j}(v)$ | Deleted elements in $G_{j}(v)$, for $v_{r}$ and $v_{a}$ | $O\left(n(\lg \lg n)^{2} / \lg n\right)$ |
| $D E L$ | Global list of deleted elements in $\bar{S}$ | $O\left(n / \lg n+n \lg \sigma / \lg ^{2} n\right)$ |

contradict the definition of $e^{\prime}$. Furthermore, let $r_{v}=\operatorname{rank}_{t}\left(G_{j}(v), i_{v}\right)$ and $r_{v}^{\prime}=\operatorname{rank}_{t}\left(G_{j}(v), i_{v}^{\prime}\right)$. Then the index of $e_{t}$ is $i_{t}^{\prime}+\left(r_{v}-r_{v}^{\prime}\right)$. Thus we can find the position of $e_{t}$ in $O(1)$ time if the position of $B(v)[e]=t$ is known.

Analogously, assume we know a position $B\left(v_{t}\right)\left[e_{t}\right]$ at $G_{j}\left(v_{t}\right)$ and want to find the position of the corresponding offset $e$ in its parent node $v$. Using $F_{j}\left(v_{t}\right)$ we find the last $e_{t}^{\prime} \leq e_{t}$ in $G_{j}\left(v_{t}\right)$ that has a pointer to its parent, which exists by construction (it can be the upward pointer from the first index in $G_{j}\left(v_{t}\right)$ or the reverse of some pointer from $v$ to $\left.v_{t}\right)$. Let $e_{t}^{\prime}$ point to $e^{\prime}$, with index $i_{v}^{\prime}$ in a block $G_{\ell}(v)$. Then, by our construction, $e$ is also in $G_{\ell}(v)$, since if it belonged to a different block $G_{m}(v)$, then the first occurrence of $t$ in $G_{m}(v)$ would point between $e_{t}^{\prime}$ and $e_{t}$, and its bidirectional version would contradict the definition of $e_{t}^{\prime}$. Furthermore, let $i_{t}^{\prime}$ and $i_{t}$ be the indexes of $e_{t}^{\prime}$ and $e_{t}$ in $G_{j}\left(v_{t}\right)$, respectively. Then the index of $e$ is $\operatorname{select}_{t}\left(G_{\ell}(v), \operatorname{rank}_{t}\left(G_{\ell}(v), i_{v}^{\prime}\right)+\left(i_{t}-i_{t}^{\prime}\right)\right)$.

To solve $\operatorname{access}(S, i)$, we visit the nodes $v_{0}=v_{r}, v_{1} \ldots v_{h}=v_{a}$, where $h=\lg _{\rho} \sigma$ is the height of $\mathcal{T}, v_{k}$ is the $t_{k}$-th child of $v_{k-1}$ and $B\left(v_{k-1}\right)\left[e_{k-1}\right]=t_{k}$ encodes $S[i]$. We do not find out the offsets $e_{1}, \ldots, e_{h}$, but just their positions. The position of $e_{0}=i$ is found in $O(\lg n / \lg \lg n)$ time using the partial-sums structure $K\left(v_{r}\right)$. If the position of $e_{k-1}$ is known, we can find that of $e_{k}$ in $O(1)$ time, as explained above. When a leaf node $v_{h}=v_{a}$ is reached, we know that $S[i]=a$.

To solve $\operatorname{select}_{a}(S, i)$, we set $e_{h}=i$ and identify its position in the list $L\left(v_{a}\right)$ of the leaf $v_{a}$, using structure $K\left(v_{a}\right)$. Then we traverse the path $v_{h}, v_{h-1}, \ldots, v_{0}=v_{r}$ where $v_{k-1}$ is the parent of $v_{k}$, until the root node is reached. In every node $v_{k}$, we find the position of $e_{k-1}$ in $L\left(v_{k-1}\right)$ that corresponds to $e_{k}$ as explained above. Finally, we compute the number of elements that precede $e_{0}$ in $L\left(v_{r}\right)$ using structure $K\left(v_{r}\right)$.

Thus access and select require $O\left(\lg _{\rho} \sigma+\lg n / \lg \lg n\right)=O((\lg \sigma+\lg n) / \lg \lg n)$ worst-case time.

### 4.3 Rank Queries

We need some additional data structures for the efficient support of rank queries. In every node $v$ such that $L(v)$ consists of more than one block, we store a data structure $P(v)$. Using $P(v)$ we can find, for any $1 \leq t \leq \rho$ and for any block $G_{j}(v)$, the last block $G_{\ell}(v)$ that precedes $G_{j}(v)$ and
contains an element $B(v)[e]=t . P(v)$ consists of $\rho$ predecessor data structures $P_{t}(v)$ for $1 \leq t \leq \rho$. We describe in Section 5 a way to support these predecessor queries in constant time in our scenario.

Let the position of offset $e$ be the $i$-th element in a block $G_{j}(v) . P(v)$ enables us to find the position of the last $e^{\prime} \leq e$ such that $B(v)\left[e^{\prime}\right]=t$. First, we use $R_{j}(v)$ to compute $r=\operatorname{rank}_{t}\left(G_{j}(v), i\right)$. If $r>0$, then $e^{\prime}$ belongs to the same block as $e$ and its index in the block $G_{j}(v)$ is $\operatorname{select}_{t}\left(G_{j}(v), r\right)$. Otherwise, we use $P_{t}(v)$ to find the last block $G_{\ell}(v)$ that precedes $G_{j}(v)$ and contains an element $B(v)\left[e^{\prime}\right]=t$. We then find the last such element in $G_{\ell}(v)$ using $R_{\ell}(v)$.

Now we are ready to describe the procedure to answer $\operatorname{rank}_{a}(S, i)$. The symbol $a$ is represented as a concatenation of symbols $t_{0} \circ t_{1} \circ \ldots \circ t_{h}$, where each $t_{k}$ is between 1 and $\rho$. We traverse the path from the root $v_{r}=v_{0}$ to the leaf $v_{a}=v_{h}$. We find the position of $e_{0}=i$ in $v_{r}$ using the data structure $K\left(v_{r}\right)$. In each node $v_{k}, 0 \leq k<h$, we identify the position of the last element $B\left(v_{k}\right)\left[e_{k}^{\prime}\right]=t_{k}$ that precedes $e_{k}$, using $P_{t_{k}}\left(v_{k}\right)$. Then we find the offset $e_{k+1}$ in the list $L\left(v_{k+1}\right)$ that corresponds to $e_{k}^{\prime}$.

When our procedure reaches the leaf node $v_{h}$, the element $B\left(v_{h}\right)\left[e_{h}\right]$ encodes the last symbol $a$ that precedes $S[i]$. We know the position of offset $e_{h}$, say index $i_{h}$ in its block $G_{\ell}\left(v_{h}\right)$. Then we find the number $r$ of elements in all the blocks that precede $G_{\ell}\left(v_{h}\right)$ using $K\left(v_{h}\right)$. Finally, $\operatorname{rank}_{a}(S, i)=r+i_{h}$.

Since structures $P_{t}$ answer queries in constant time, the overall time for rank is $O\left(\lg _{\rho} \sigma+\right.$ $\lg n / \lg \lg n)=O((\lg \sigma+\lg n) / \lg \lg n)$.

### 4.4 Updates

Now we describe how inter-node pointers are implemented. We say that an element of $L(u)$ is pointed if there is a pointer to its offset. Unfortunately, we cannot store the local index of a pointed element in the pointer: when a new element is inserted into a block, the indexes of all the elements that follow it are incremented by 1 . Since a block can contain $\Theta\left(\lg ^{3} n / \lg \rho\right)$ pointed elements, we would have to update that many pointers after each insertion and deletion.

Therefore we resort to the following two-level scheme. Each pointed element in a block is assigned a unique id. When a new element is inserted, we assign it the id max_id +1 , where $m a x_{-} i d$ is the maximum id value used so far. We also maintain a data structure $H_{j}(v)$ for each block $G_{j}(v)$ that enables us to find the position of a pointed element if its id in $G_{j}(v)$ is known. Implementation of $H_{j}(v)$ is based on standard word RAM techniques and a table that contains ids of the pointed elements; details are given in Section 9 .

We describe now how to insert a new symbol $a$ into $S$ at position $i$. Let $e_{0}, e_{1}, \ldots, e_{h}$ be the offsets of the elements that will encode $a=t_{0} \circ \ldots \circ t_{h}$ in $v_{r}=v_{0}, v_{1}, \ldots, v_{h}=v_{a}$. We can find the position of $e_{0}=i$ in $L\left(v_{r}\right)$ in $O(\lg n / \lg \lg n)$ time using $K\left(v_{r}\right)$, and insert $t_{0}$ at that position, $B\left(v_{r}\right)\left[e_{0}\right]=t_{0}$. Now, given the position of $e_{k}$, in $L\left(v_{k}\right)$, where $B\left(v_{k}\right)\left[e_{k}\right]=t_{k}$ has just been inserted, we find the position of the last $e_{k}^{\prime}<e_{k}$ such that $B\left(v_{k}\right)\left[e_{k}^{\prime}\right]=t_{k}$, in the same way as for rank queries. Once we know the position of $e_{k}^{\prime}$ in $L\left(v_{k}\right)$, we find the position of $e_{k+1}^{\prime \prime}$ in $L\left(v_{k+1}\right)$ that corresponds to $e_{k}^{\prime}$. The element $t_{k+1}$ must then be inserted into $L\left(v_{k+1}\right)$ immediately after $e_{k+1}^{\prime \prime}$, at position $e_{k+1}=e_{k+1}^{\prime \prime}+1$.

The insertion of a new element $B\left(v_{k}\right)\left[e_{k}\right]=t$ into a block $G_{j}\left(v_{k}\right)$ is handled by structure $R_{j}\left(v_{k}\right)$ and the memory manager of the block. We must also update structures $F_{j}\left(v_{k}\right)$ and $H_{j}\left(v_{k}\right)$ to keep the correct alignments, and possibly to create and destroy a constant number inter-node pointers to maintain our invariants. Also, since pointers are bidirectional, a constant number of inter-node pointers in the parent and children of node $v_{k}$ may be updated. All those changes can be done in
$O(1)$ time; see Section 9 for the details. Insertions may also require updating structures $P_{t}\left(v_{k}\right)$, which require $O(1)$ amortized time, see Section 5. Finally, if $v_{k}$ is the root node or a leaf, we also update $K\left(v_{k}\right)$. This update is only by $\pm 1$, so it requires just $O(\lg n / \lg \lg n)$ time [NS10, Lem. 1].

If the number of elements in $G_{j}\left(v_{k}\right)$ exceeds $2 \lg ^{3} n$, we split $G_{j}\left(v_{k}\right)$ evenly into two blocks, $G_{j_{1}}\left(v_{k}\right)$ and $G_{j_{2}}\left(v_{k}\right)$. Then, we rebuild the data structures $R, F$ and $H$ for the two new blocks. Note that there are inter-node pointers to $G_{j}\left(v_{k}\right)$ that now could become dangling pointers, but all those can be known from $F_{j}\left(v_{k}\right)$, since pointers are bidirectional, and updated to point to the right places in $G_{j_{1}}\left(v_{k}\right)$ or $G_{j_{2}}\left(v_{k}\right)$. Finally, if $v_{k}$ is the root or a leaf, then $K\left(v_{k}\right)$ is updated, meaning that we replace an existing element by two.

The total cost of splitting a block is dominated by that of building the new data structures $R$, $F$ and $H$. These are easily built in $O\left(\lg ^{3} n / \lg \rho\right)$ time. Since we split a block $G_{j}(v)$ at most once per sequence of $\Theta\left(\lg ^{3} n\right)$ insertions in $G_{j}(v)$, the amortized cost incurred by splitting a block is $o(1)$. Therefore the total cost of an insertion in $L(v)$ is $O(1)$. The insertion of a new symbol leads to $O\left(\lg _{\rho} \sigma\right)$ insertions into lists $L(v)$.

Our partial-sums structures $K\left(v_{r}\right)$ and $K\left(v_{a}\right)$ do not support updates with large values. Inserting a new value for $G_{j_{2}}\left(v_{k}\right)$ and moving part of the value of $G_{j}\left(v_{k}\right)$ to $G_{j_{2}}\left(v_{k}\right)$ can be done in $O\left(\lg ^{3} n / \lg \lg n\right)$ time by subtracting $O(\lg n)$ units from the value for $G_{j}(v)$ until it becomes $\left|G_{j_{1}}\left(v_{k}\right)\right|$, then inserting a number after it with value zero and increasing it by $O(\lg n)$ units until it becomes $\left|G_{j_{2}}\left(v_{k}\right)\right|$. Each such increment/decrement and insertion takes $O(\lg n / \lg \lg n)$ time NS10, Lem. 1] and we carry it out $O\left(\lg ^{2} n / \lg \rho\right)$ times. Still this total cost amortizes to $o(1)$ per operation.

Hence, the total cost of an insertion is $O\left(\lg _{\rho} \sigma+\lg n / \lg \lg n\right)=O((\lg \sigma+\lg n) / \lg \lg n)$.
We describe next how deletions are handled, where we also describe the data structure $P(v)$.

## 5 Lazy Deletions and Data Structure $P(v)$

We do not process deletions immediately, but in lazy form: we do not maintain exactly $S$ but a supersequence $\bar{S}$ of it. When a symbol $S[i]=a$ is deleted from $S$, we retain it in $\bar{S}$ but take a notice that $S[i]=a$ is deleted. When the number of deleted symbols exceeds a certain threshold, we expunge from the data structure all the elements marked as deleted. We define $\bar{B}(v)$ and the list $\bar{L}(v)$ for the sequence $\bar{S}$ in the same way as $B(v)$ and $L(v)$ are defined for $S$.

Since elements of $\bar{L}(v)$ are never removed, we can implement $P(v)$ as an insertion-only data structure. For any $t, 1 \leq t \leq \rho$, we store information about all the blocks of a node $v$ in a data structure $P_{t}(v) . P_{t}(v)$ contains one element for each block $G_{j}(v)$ and is implemented as an incremental split-find data structure that supports insertions and splitting in $O(1)$ amortized time and queries in $O(1)$ worst-case time [A84]. The splitting positions in $P_{t}(v)$ are the blocks $G_{j}(v)$ that contain an occurrence of $t$, so the operation "find" in $P_{t}(v)$ allows us to locate, for any $G_{j}(v)$, the last block preceding $G_{j}(v)$ that contains an occurrence of $t$.

The insertion of a symbol $t$ in $\bar{L}(v)$ may induce a new split in $P_{t}(v)$. Furthermore, overflows in a block $G_{j}(v)$, which convert it into two blocks $G_{j_{1}}(v)$ and $G_{j_{2}}(v)$, induce insertions in $P_{t}(v)$. Note that an overflow in $G_{j}(v)$ triggers $\rho$ insertions in the $P_{t}(v)$ structures, but this $O(\rho)$ time amortizes to $o(1)$ because overflows occur every $\Theta\left(\lg ^{3} n / \lg \rho\right)$ operations.

Structures $P_{t}(v)$ do not support "unsplitting" nor removals. The replacement of $G_{j}(v)$ by $G_{j_{1}}(v)$ and $G_{j_{2}}(v)$ is implemented as leaving in $P_{t}(v)$ the element corresponding to $G_{j}(v)$ and inserting one corresponding to either $G_{j_{1}}(v)$ or $G_{j_{2}}(v)$. If $G_{j}(v)$ contained $t$, then at least one of $G_{j_{1}}(v)$ and $G_{j_{2}}(v)$ contain $t$, and the other can be inserted as a new element (plus possibly a split, if it also
contains $t$ ).
We need some additional data structures to support lazy deletions. A data structure $\bar{K}(v)$ stores the number of non-deleted elements in each block of $\bar{L}(v)$ and supports partial-sum queries. We will maintain $\bar{K}(v)$ in the root of the wavelet tree and in all leaf nodes. Moreover, we maintain a data structure $D_{j}(v)$ for every block $G_{j}(v)$, where $v$ is either the root or a leaf node. $D_{j}(v)$ can be used to count the number of deleted and non-deleted elements before the $i$-th element in a block $G_{j}(v)$ for any query index $i$, as well as to find the index in $G_{j}(v)$ of the $i$-th non-deleted element. The implementation of $D_{j}(v)$ is described in Section 9. We can use $\bar{K}(v)$ and $D_{j}(v)$ to find the index $\bar{i}$ in $\bar{L}(v)$ where the $i$-th non-deleted element occurs, and to count the number of non-deleted elements that occur before the index $\bar{i}$ in $\bar{L}(v)$.

We also store a global list $D E L$ that contains, in any order, all the deleted symbols that have not yet been expunged from the wavelet tree. For any symbol $\bar{S}[i]$ in the list $D E L$ we store a pointer to the offset $e$ in $\bar{L}\left(v_{r}\right)$ that encodes $\bar{S}[i]$. Pointers in list $D E L$ are implemented in the same way as inter-node pointers.

### 5.1 Queries

Queries are answered very similarly to Section 4. The main idea is that we can essentially ignore deleted elements except at the root and at the leaves.
$\operatorname{access}(S, i):$ Exactly as in Section 3 , except that $e_{0}$ encodes the $i$-th non-deleted element in $\bar{L}\left(v_{r}\right)$, and is found using $\bar{K}\left(v_{r}\right)$ and $D_{j}\left(v_{r}\right)$.
$\operatorname{select}_{a}(S, i)$ : We find the position of the offset $e_{h}$ of the $i$-th non-deleted element in $\bar{L}\left(v_{h}\right)$, where $v_{h}=v_{a}$, using $\bar{K}\left(v_{a}\right)$ and some $D_{j}\left(v_{a}\right)$. Then we move up in the tree exactly as in Section 4, When the root node $v_{0}=v_{r}$ is reached, we count the number of non-deleted elements that precede offset $e_{0}$ using $\bar{K}\left(v_{r}\right)$.
$\operatorname{rank}_{a}(S, i)$ : We find the position of the offset $e_{0}$ of the $i$-th non-deleted element in $\bar{L}\left(v_{r}\right)$. Let $v_{k}, t_{k}$ be defined as in Section 4. In every node $v_{k}$, we find the last offset $e_{k}^{\prime} \leq e_{k}$ such that $\bar{B}\left(v_{k}\right)\left[e_{k}^{\prime}\right]=t_{k}$. Note that this element may be a deleted one, but it still drives us to the correct position in $\bar{L}\left(v_{k+1}\right)$. We proceed exactly as in Section 4 until we arrive at a leaf $v_{h}=v_{a}$. At this point, we count the number of non-deleted elements that precede offset $e_{h}$ using $\bar{K}\left(v_{a}\right)$ and $D_{j}\left(v_{a}\right)$.

### 5.2 Updates

Insertions are carried out just as in Section 4. The only difference is that we also update the data structure $D_{j}\left(v_{k}\right)$ when an element $B\left(v_{k}\right)\left[e_{k}\right]$ that encodes the inserted symbol $a$ is added to a block $G_{j}\left(v_{k}\right)$. When a symbol $S[i]=a$ is deleted, we append it to the list $D E L$ of deleted symbols. Then we visit each block $G_{j}\left(v_{k}\right)$ containing the element $B\left(v_{k}\right)\left[e_{k}\right]$ that encodes $S[i]$ and update the data structures $D_{j}\left(v_{k}\right)$. Finally, $\bar{K}\left(v_{r}\right)$ and $\bar{K}\left(v_{a}\right)$ are also updated. This takes in total $O\left(\lg _{\rho} \sigma+\lg n / \lg \lg n\right)$ time.

When the number of symbols in the list $D E L$ reaches $n / \lg ^{2} n$, we perform a cleaning procedure and get rid of all the deleted elements. Therefore $D E L$ never requires more than $O(n / \lg n)$ bits, and the overhead due to storing deleted symbols is $O\left(n \lg \sigma / \lg ^{2} n\right)$ bits.

Let $B\left(v_{k}\right)\left[e_{k}\right], 0 \leq k \leq h$, be the sequence of elements that encode a symbol $\bar{S}[i] \in D E L$. The method for tracking the elements $B\left(v_{k}\right)\left[e_{k}\right]$, removing them from their blocks $G_{j}\left(v_{k}\right)$, and updating the block structures, is symmetric to the insertion procedure described in Section 4. In this case we do not need the predecessor queries to track the symbol to delete, as the procedure is similar to that for accessing $S[i]$. When the size of a block $G_{j}\left(v_{k}\right)$ falls below $\left(\mathrm{lg}^{3} n\right) / 2$ and it is not the last block of $L\left(v_{k}\right)$, we merge it with $G_{j+1}\left(v_{k}\right)$, and then split the result if its size exceeds $2 \lg ^{3} n$. This retains $O(1)$ amortized time per deletion in any node $v_{k}$, including the updates to $K\left(v_{k}\right)$ structures, and this adds up to $O((\lg \sigma+\lg n) / \lg \lg n)$ amortized time per deleted symbol.

Once all the pointers in $D E L$ are processed, we rebuild from scratch the structures $P(v)$ for all nodes $v$. The total size of all the $P(v)$ structures is $O\left(\rho n \lg \sigma / \lg ^{3} n\right)$ elements. Since a data structure for incremental split-find is constructed in linear time, all the $P(v)$ s are rebuilt in $O\left(n \lg \sigma / \lg ^{3-\varepsilon} n\right)$ time. Hence the amortized time to rebuild the $P(v)$ s is $O\left(\lg \sigma / \lg ^{1-\varepsilon} n\right)$, which does not affect the amortized time $O((\lg \sigma+\lg n) / \lg \lg n)$ to carry out the effective deletions.

## 6 Changes in $\lg n$ and Alphabet Independence

Note that our structures depend on the value of $w=\lg n$, so they should be rebuilt when $\lg n$ changes. We use $w=\lceil\lg n\rceil$ as a fixed value and rebuild the structure from scratch when $n$ reaches another power of two (more precisely, we use words of $w=\lceil\lg n\rceil$ bits until $\lceil\lg n\rceil$ increases by 1 or decreases by 2 , and only then update $w$ and rebuild). These reconstructions do not affect the amortized complexities, and the slightly larger words waste an $O(1 / \lg n)$ extra space factor in the redundancy.

We take advantage of using a fixed $w$ value to get rid of the alphabet dependence. If $\lg \sigma \leq w$, our time complexities are the optimal $O(\lg n / \lg \lg n)$. However, if $\sigma$ is larger, this means that not all the alphabet symbols can appear in the current sequence (which contains at most $n \leq 2^{w}<\sigma$ distinct symbols). Therefore, in this case we create the wavelet tree for an alphabet of size $s=2^{w}$, not $\sigma$ (this wavelet tree is created when $w$ changes). We also set up a mapping array $S N[1, \sigma]$ that will tell to which value in $[1 . . s]$ is a symbol mapped, and a reverse mapping $N S[1, s]$ that tells to which original symbol in $[1 . . \sigma]$ does a mapped symbol correspond. Both $S N$ and $N S$ are initialized in constant time Meh84, Section III.8.1] and require $O(\sigma \lg n+n \lg \sigma)$ bits of space. Since this is used only when $\sigma>n$, the space is $O(\sigma \lg n)$.

Upon operations $\operatorname{rank}_{a}(S, i)$ and $\operatorname{select}_{a}(S, j)$, the symbol $a$ is mapped using $S N$ (the answer is obvious if $a$ does not appear in $S N$ ) in constant time. The answer of operation access $(S, i)$ is mapped using $N S$ in constant time as well. Upon insertion of $a$, we also map $a$ using $S N$. If not present in $S N$, we find a free slot $N S[i]$ (we maintain a list of free slots) and assign $N S[i]=a$ and $S N[a]=i$. When the last occurrence of a symbol $a$ is deleted we return its slot to the free list and unitialize its entry in $S N$. In this way, when $\lg \sigma>\lg n$, we can support all the operations in time $O(\lg s / \lg \lg s)=O(\lg n / \lg \lg n)$.

We are ready to state a first version of our result, not yet compressing the sequence. In Section 9 it is seen that the time for the operations is $O(1 / \varepsilon)$. Since the height of the wavelet tree is $\lg _{\rho} \min (\sigma, s)=O\left(\frac{1}{\varepsilon} \lg n / \lg \lg n\right)$, then we have $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$ time for all the operations.

As for the space, we show in Section 9 how to manage the data in blocks $G_{j}(v)$ so that all the elements stored in lists $L(v)$ use $n \lg \sigma$ bits, plus the overhead $O(n \lg \sigma \lg \lg n / \lg n+\sigma \lg n)$ of the data organization and the memory manager. The internal structures $R_{j}(v)$ add up to $O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)$ extra bits. Since there are $O\left(n \lg \sigma / \lg ^{3} n+\sigma\right)$ blocks overall, all the pointers
between blocks of the same lists add up to $O\left(n \lg \sigma / \lg ^{2} n+\sigma \lg n\right)$ bits. All the data structures $K(v)$ add up to $O\left(n \lg \rho / \lg ^{2} n\right)$ bits. We have shown that there are $O\left(n \lg \sigma / \lg ^{3-\varepsilon} n\right)$ inter-node pointers, hence all inter-node pointers (i.e., $F_{j}$ and $H_{j}$ structures) use $O\left(n \lg \sigma / \lg ^{2-\varepsilon} n\right)$ bits. Structures $P_{t}(v)$ use $O\left(n \lg \sigma / \lg ^{2-\varepsilon} n\right)$ bits as they have $\rho$ integers per block, and $D E L$ takes $O(n / \lg n)$ bits plus the overhead of $O\left(n \lg \sigma / \lg ^{2} n\right)$ of keeping deleted elements. The overall space is then $n \lg \sigma+O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)+O(\sigma \lg n)$ bits. (Note that when $\sigma>n$ we use an alphabet of size $O(n)$, but then still we need the $S N$ mapping, that takes $O(\sigma \lg n)$ bits.) This gives our first result.

Theorem 1 A dynamic string $S[1, n]$ over alphabet $[1 . . \sigma]$ can be stored in a structure using $n \lg \sigma+$ $O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)+O(\sigma \lg n)$ bits, for any $0<\varepsilon<1$, and supporting queries access, rank and select in time $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$. Insertions and deletions of symbols are supported in $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$ amortized time.

## 7 Compressed Space

We now compress the space of the data structure to zero-order entropy ( $n H_{0}(S)$ plus redundancy). We show how a different encoding of the bits within the blocks reduces the $n \lg \sigma$ to $n H_{0}(S)$ in the space without affecting the time complexities.

Raman et al. RRR07 describe an encoding for a bitmap $B[1, n]$ that occupies $n H_{0}(B)+$ $O(n \lg \lg n / \lg n)$ bits of space. It consists of cutting the bitmap into chunks of length $b=(\lg n) / 2$ and encoding each chunk $i$ as a pair $\left(c_{i}, o_{i}\right): c_{i}$ is the class, which indicates how many 1 s are there in the chunk, and $o_{i}$ is the offset, which is the index of this particular chunk within its class. The $c_{i}$ components add up to $O(n \lg \lg n / \lg n)$ bits, whereas the $o_{i}$ components add up to $n H_{0}(B)$. Navarro and Sadakane [NS10, Sec. 8] describe a technique to maintain a dynamic bitmap in this format. They allow the chunk length $b$ to vary, so they encode triples ( $b_{i}, c_{i}, o_{i}$ ) maintaining the invariant that $b_{i}+b_{i+1}>b$ for any $i$. They show that this retains the same space, and that each update affects $O(1)$ chunks.

We extend this encoding to handle an alphabet [1.. $\rho$ ] [FMMN07], so that $b=\left(\lg _{\rho} n\right) / 2$ symbols, and each chunk is encoded as a tuple $\left(b_{i}, c_{i}^{1}, \ldots, c_{i}^{\rho}, o_{i}\right)$ where $c_{i}^{t}$ counts the occurrences of $t$ in the block. The classes $\left(b_{i}, c_{i}^{1}, \ldots, c_{i}^{\rho}\right)$ use $O(\rho n \lg \lg n / \lg n)$ bits, and the offsets still add up to $n H_{0}(B)$. Blocks are encoded/decoded in $O(1)$ time, as the class takes $O(\rho \lg \lg n)=o(\lg n)$ bits and the block encoding requires at most $O(\lg n)$ bits. In Section 9 we show how using compressed chunks does not affect their handling inside blocks.

The sum of the local entropies of the chunks, across the whole $L(v)$, adds up to $n H_{0}\left(B_{v}\right)$, and these add up to $n H_{0}(S)$ [GGV03]. The redundancy over the entropy is $O(\rho \lg \lg n)$ bits per miniblock, adding up to $O\left(n H_{0}(S) \lg \lg n / \lg ^{1-\varepsilon} n\right)$ bits, and we have also a fixed redundancy of $O\left(n \lg \sigma / \lg n+n(\lg \lg n)^{2} / \lg n+\sigma \lg n\right)$, according to Section 9. The fact that we store $\bar{S}$ instead of $S$, with up to $O\left(n / \lg ^{2} n\right)$ spurious symbols, can increase $n H_{0}(S)$ up to $n H_{0}(\bar{S}) \leq n H_{0}(S)+$ $O(n / \lg n)$ bits. Thus we get the following result, for any desired $0<\varepsilon<1$.

Theorem 2 A dynamic string $S[1, n]$ over alphabet $[1 . . \sigma]$ can be stored in a structure using $n H_{0}(S)+$ $O\left(n H_{0}(S) / \lg ^{1-\varepsilon} n+n \lg \sigma / \lg n+n(\lg \lg n)^{2} / \lg n+\sigma \lg n\right)=n H_{0}(S)+O\left(n \lg \sigma / \lg ^{1-\varepsilon} n+\sigma \lg n\right)$ bits, for any $0<\varepsilon<1$, and supporting queries access, rank and select in time $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$. Insertions and deletions of symbols are supported in $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$ amortized time.

## 8 Worst-Case Complexities

While in previous sections we have obtained optimal time and compressed space, the time for the update operations is amortized. In this section we derive worst-case time complexities, at the price of losing the time optimality, which will now become logarithmic for some operations. Along the rest of the section we remove the various sources of amortization in our solution.

### 8.1 Block Splits and Merges

Our amortized solution splits overflowing blocks and rebuilds the two new blocks from scratch (Section 4.4). Similarly, it merges underflowing blocks (as a part of the cleaning of the global $D E L$ list in Section 5.2). This gives good amortized times but in the worst case the cost is $\Omega\left(\lg ^{3} n / \lg \lg n\right)$.

We use a technique [GN09] that avoids global rebuildings. A block is called dense if it contains at least $\lg ^{3} n$ bits, and sparse otherwise. While sparse blocks of any size (larger than zero) are allowed, we maintain the invariant that no two consecutive sparse blocks may exist. This retains the fact that there are $O\left(n \lg \sigma / \lg ^{3} n+\sigma\right)$ blocks in the data structure. The maximum size of a block will be $2 \lg ^{3} n$ bits. When a block overflows due to an insertion, we move its last element to the beginning of the next block. If the next block would also overflow, then we are entitled to create a new sparse block between both dense blocks, containing only that element. Analogously, when a deletion converts a dense block into sparse (i.e., it falls below length $\lg ^{3} n$ ), we check if both neighbors are dense. If they are, the current block can become sparse. If, instead, there is a sparse neighbor, we move its first/last element into the current block to avoid it becoming sparse. If this makes that sparse neighbor block become of size zero, we remove it.

Therefore, we only create and destroy empty blocks, and move a constant number of elements to neighboring blocks. This can be done in constant worst-case time. It also simplifies the operations on the partial-sum data structures $K(v)$, since now only updates by $\pm 1$ and insertions/deletions of elements with value zero are necessary, and these are carried out in $O(\lg n / \lg \lg n)$ worst case time [NS10, Lem. 1]. Recall that $\lg n$ is fixed in each instance of our data structure, so the definition of sparse and dense is static.

### 8.2 Split-Find Data Structure and Lazy Deletions

The split-find data structure [IA84] we used in Section 5 to implement the $P_{t}$ structures has constant amortized insertion time. We replace it by another one [Mor03, Thm 4.1] achieving $O(\lg \lg n)$ worstcase time. Their structure handles a list of colored elements (list nodes), where each element can have $O(1)$ colors (each color is a positive integer bounded by $O\left(\log ^{\varepsilon} n\right)$ for a constant $0<\varepsilon<1$ ). We will only use list nodes with 0 or 1 color. The operations of interest to us are: creating a new list node without colors, assigning or removing a color to/from a list node, and finding the last list node preceding a given node and having some given color. Node deletions are not supported. The number of list nodes must be smaller than a certain upper bound $n^{\prime}$, and the operations cost $O\left(\lg \lg n^{\prime}\right)$. In our case, since $\lg n$ is fixed, we can use $n^{\prime}=2^{w}=O(n)$ as the upper bound.

We use $\rho$ colors, one per symbol in the sequences. Each time we create a block, we add a new uncolored node to the list, with a bidirectional pointer to the block. Each time we insert a symbol $t \in[1 . . \rho]$ for the first time in a block, we add a new node colored $t$ to the list, right after the uncolored element that represents the block, and also set a bidirectional pointer between this node and the block.

We cannot use the lazy deletions mechanism of Section 5, as it gives only good amortized complexity. We carry out the deletions immediately in the blocks, as said in Section 8.1. Each time the last occurrence of a symbol $t \in[1 . . \rho]$ is deleted from a block, we remove the color from the corresponding list node (if the symbol reappears later, we reuse the same node and color it, instead of creating a new one).

Therefore, finding the last block where a symbol $t$ appears, as needed by the rank query and for insertions, corresponds to finding the last list node colored $t$ and preceding the uncolored node that represents the current block.

Since list nodes cannot be deleted, when a block disappears its (uncolored) list nodes are left without an associated block. This does not alter the result of queries, but there is the risk of maintaining too many useless nodes. We permanently run an incremental list "copying" process, traversing the current list of blocks and inserting the corresponding nodes into a new list. This new list is also updated, together with the current list, on operations concerning the blocks already copied. When the new list is ready it becomes the current list and the previous list is incrementally deleted. In $O\left(n \rho \lg \rho / \lg ^{3} n\right)$ steps we have copied the current list; by this time the number of useless nodes is at most $O\left(n \rho \lg \rho / \lg ^{3} n\right)$ and just poses $O\left(n \lg \lg n / \lg ^{2-\varepsilon} n\right)$ bits of space overhead.

Note that blocks must manage the sets of up to $\rho$ pointers to their colored nodes. This is easily handled in constant time with the same techniques used for structure $F_{j}(v)$ in Section 9 ,

Since the colored list data structure requires $O(\lg \lg n)$ time, operations rank and insert take worst-case time $O\left(\frac{1}{\varepsilon} \lg n\right)$, whereas access, select and delete still stay in $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$.

### 8.3 Changes in $\lg n$

As an alternative to reconstructing the whole structure when $n$ doubles or halves, Mäkinen and Navarro MN08] describe a way to handle this problem without affecting the space nor the time complexities, in a worst-case scenario. The sequence is cut into a prefix, a middle part, and a suffix. The middle part uses a fixed value $\lceil\lg n\rceil$, the prefix uses $\lceil\lg n\rceil-1$ and the suffix uses $\lceil\lg n\rceil+1$. Insertions and deletions trigger slight expansions and contractions in this separation, so that when $n$ doubles all the sequence is in the suffix part, and when $n$ halves all the sequence is in the prefix part, and we smoothly move to a new value of $\lg n$. This means that the value of $\lg n$ is fixed for any instance of our data structure. Operations access, rank and select, as well as insertions and deletions, are easily adapted to handle this split string.

Actually, to have sufficient time to build universal tables of size $O\left(n^{\alpha}\right)$ for $0<\alpha<1$, the solution [MN08] maintains the sequence split into five, not three, parts. This gives also sufficient time to build any universal table we need to handle block operations in constant time, as well as to build the wavelet tree structures of the new partitions.

### 8.4 Memory Management Inside Blocks

The EAs of Lemma 1 (Section 9) have amortized times to grow and shrink. Converting those to worst-case time requires a constant space overhead factor. While this is acceptable for the EAs of structures $T b l$ in Section 9, they raise the overall space to $O\left(n H_{0}(S)\right)$ bits if used to maintain the main data. Instead, we get rid completely of the EA mechanism to maintain the data, and use a single large memory area for all the miniblocks of Section 9, using Munro's technique Mun86.

The problem of using a single memory area is that the pointers to the miniblocks require $\Theta(\lg n)$ bits, which is excessive because miniblocks are also of $\Theta(\lg n)$ bits. Instead, we use slightly
larger miniblocks, of $\Theta(\lg n \lg \lg n)$ bits. This makes the overhead due to pointers to miniblocks $O\left(\left|G_{j}(v)\right| / \lg \lg n\right)$, adding up to additional $O\left(n H_{0}(S) / \lg \lg n+n \lg \sigma / \lg ^{1-\varepsilon} n\right)=o(n \lg \sigma)$ bits.

The price of using larger miniblocks is that now the operations on blocks are not anymore constant time because they need to traverse a miniblock, which takes time $O(\lg \lg n)$. We can still retain constant time for the query operations, by considering logical miniblocks of $\Theta(\lg n)$ bits, which are stored in physical areas of $\Theta(\lg \lg n)$ miniblocks. However, update operations like insert and delete must shift all the data in the miniblock area and possibly relocate it in the memory manager, plus updating pointers to all the logical miniblocks displaced or relocated. This costs $O(\lg \lg n)$ time per insertion and deletion. This completes our result.

Theorem 3 A dynamic string $S[1, n]$ over alphabet $[1 . . \sigma]$ can be stored in a structure using $n H_{0}(S)+$ $O\left(n H_{0}(S) / \lg \lg n\right)+O\left(n \lg \sigma / \lg ^{1-\varepsilon} n\right)+O(\sigma \lg n)=n H_{0}(S)+o(n \lg \sigma)+O(\sigma \lg n)$ bits, for any constant $0<\varepsilon<1$, and supporting queries access and select in worst-case time $O\left(\frac{1}{\varepsilon^{2}} \lg n / \lg \lg n\right)$, and query rank, insertions and deletions in worst-case time $O\left(\frac{1}{\varepsilon} \lg n\right)$.

## 9 Data Structures for Handling Blocks

We describe the way the data is stored in blocks $G_{j}(v)$, as well as the way the various structures inside blocks operate. All the data structures are based on the same idea: We maintain a tree with node degree $\lg ^{\delta} n$ and leaves that contain $O(\lg n)$ bits. Since elements within a block can be addressed with $O(\lg \lg n)$ bits, each internal node and each leaf fits into one machine word. Moreover, we can support searching and basic operations in each node in constant time.

### 9.1 Data Organization

The block data is physically stored as a sequence of miniblocks of $\Theta\left(\lg _{\rho} n\right)$ symbols, or $\Theta(\lg n)$ bits. Thus there are $O\left(\lg ^{2} n\right)$ miniblocks in a block. These miniblocks will be the leaves of a $\tau$-ary tree $T$, for $\tau=\Theta\left(\lg ^{\delta} n\right)$ and some constant $0<\delta<1$. The height of this tree is constant, $O(1 / \delta)$. Each node of $T$ stores $\tau$ counters telling the number of symbols stored at the leaves that descend from each child. This requires just $O(\tau \lg \lg n)=o(\lg n)$ bits. To access any position of $G_{j}(v)$, we descend in $T$, using the counters to determine the correct child. When we arrive at a leaf, we know the local offset of the desired symbol within the leaf, and can access it directly. Since the counters fit in less than a machine word, a small universal table gives the correct child in constant time, therefore we have $O(1)$ time access to any symbol (actually to any $\Theta$ ( $\left.\lg _{\rho} n\right)$ consecutive symbols).

Upon insertions or deletions, we arrive at the correct leaf, insert or delete the symbol (in constant time because the leaf contains $\Theta(\lg n)$ bits overall), and update the counters in the path from the root (in constant time as they have $o(\lg n)$ bits). The leaves may have $\lg n$ to $2 \lg n$ bits. Splits/merges upon overflows/underflows are handled as usual, and can be solved in a constant number of $O(1)$-time operations ( $T$ operates as a B-tree; internal nodes may have $\tau$ to $2 \tau$ children).

The space overhead due to the nodes of $T$ is $O\left(\left|G_{j}(v)\right| \lg \lg n / \lg n\right)$ bits, where we measure $\left|G_{j}(v)\right|$ in bits, not symbols. The factor $\tau$ disappears because each leaf of $T$ has $\tau$ miniblocks.

We consider now the space used by the data itself. In order not to waste space, the miniblock leaves are stored using a memory management structure by Munro Mun86]. For our case, it allows us to allocate, free, and access miniblocks of length up to $2 \lg n$ in constant time. Its space waste, given that our pointers are internal to blocks and require $O(\lg \lg n)$ bits, is $O(\lg \lg n)$ per allocated miniblock, which adds up to $O\left(\left|G_{j}(v)\right| \lg \lg n / \lg n\right)$, plus a global redundancy of
$O\left(\lg ^{2} n\right)$ bits. If we used one allocation structure per block, handling its miniblocks, the global redundancy of $O\left(\lg ^{2} n\right)$ bits per block would add $O\left(n \lg \sigma / \lg n+\sigma \lg ^{2} n\right)$ bits overall. This is reduced to $O\left(n \lg \sigma / \lg ^{2} n+\sigma \lg n\right)$ by using one allocation structure per group of $\lg n$ blocks. This reduces the overhead of the structures and the address space is still of size $O\left(\lg ^{4} n\right)$, so pointers can still be of length $O(\lg \lg n)$.

Each allocation structure uses a memory area of fixed-size cells (inside which the variable-length miniblocks are stored) that grows or shrinks at the end as miniblocks are created or destroyed. A structure to store those memory areas with fixed-size cells and allowing them to grow and shrink is the extendible array (EA) RR03. We need to handle a set of $O\left(n \lg \sigma / \lg ^{4} n+\sigma / \lg n\right)$ EAs, what is called a collection of extendible arrays. It supports accessing any cell of any EA, letting any EA grow or shrink by one cell, and create and destroy EAs. The following lemma, simplified from the original [RR03, Lemma 1], and using words of $\lg n$ bits, is useful.

Lemma $1 A$ collection of a EAs of total size $s$ bits can be represented using $s+O(a \lg n+\sqrt{s a \lg n})$ bits of space, so that the operations of creation of an empty EA and access take constant worst-case time, whereas grow/shrink take constant amortized time. An EA of s' bits can be destroyed in time $O\left(s^{\prime} / \lg n\right)$.

In our case $a=O\left(n \lg \sigma / \lg ^{4} n+\sigma / \lg n\right)$ and $s=O(n \lg \sigma)$, so the space overhead posed by the EAs is $O\left(n \lg \sigma / \lg ^{3} n+\sigma+n \lg \sigma / \lg ^{3 / 2} n+\sqrt{n \sigma \lg \sigma}\right)=O(n \lg \sigma / \lg n+\sigma \lg n)$.

When we store the miniblocks in compressed form, in Section 7, they could use as little as $O\left(\lg ^{\varepsilon} n \lg \lg n\right)$ bits, and thus we could store up to $\Theta\left(\lg ^{1-\varepsilon} n / \lg \lg n\right)$ miniblocks in a single leaf of $T$. This can still can be handled in constant time using (more complicated) universal tables MN08, and the counters and pointers of $O(\lg \lg n)$ bits are still large enough.

### 9.2 Structure $R_{j}(v)$

To support rank and select we enrich $T$ with further information per node. We store $\rho$ counters with the number of occurrences of each symbol in the subtree of each child. The node size becomes $O(\tau \rho \lg \lg n)=O\left(\lg ^{\varepsilon+\delta} n \lg \lg n\right)=o(\lg n)$ as long as $\varepsilon+\delta<1$. This adds up to $O\left(\left|G_{j}(v)\right| \rho \lg \lg n / \lg n\right)$ bits because the leaves of $T$ handle $\tau$ miniblocks.

With this information on the nodes we can easily solve rank and select in constant time, by descending on $T$ and determining the correct child (and accumulating data on the leftward children) in $O(1)$ time using universal tables. Nodes can also be updated in constant time even upon splits and merges, since all the counters can be recomputed in $O(1)$ time.

### 9.3 Structure $F_{j}(v)$

This structure stores all the inter-node pointers leaving from block $G_{j}(v)$, to its parent and to any of the $\rho$ children of node $v$.

The structure is a tree $T_{f}$ very similar in spirit to $T$. The pointers are stored at the leaves of $T_{f}$, in increasing order of their source position inside $G_{j}(v)$. The pointers stored are inter-node, and thus require $\Theta(\lg n)$ bits. Thus we store a constant number of pointers per leaf of $T_{f}$. For each pointer we store the position in $G_{j}(v)$ holding the pointer (relative to the starting position of the leaf node inside $G_{j}(v)$ ) and the target position (as an absolute pointer to another $G_{\ell}(u)$ ). The internal nodes, of arity $\tau$, maintain information on the number of positions of $G_{j}(v)$ covered by each child, and the number of pointers of each kind ( $1+\rho$ counters) stored in the subtree of each
child. This requires $O(\tau \rho \lg \lg n)=o(\lg n)$ bits, as before. To find the last position before $i$ holding a pointer of a certain kind (parent or $t$-th wavelet tree child, for any $1 \leq t \leq \rho$ ), we traverse $T_{f}$ from the root looking for position $i$. At each node $x$, it might be that the child $y$ where we have to enter holds pointers of that kind, or not. If it does, then we first enter into child $y$. If we return with an answer, we recursively return it. If we return with no answer, or there are no pointers of the desired kind below $y$, we enter into the last sibling to the left of $y$ that holds a pointer of the desired kind, and switch to a different mode where we simply go down the tree looking for the rightmost child with a pointer of the desired kind. It is not hard to see that this procedure visits $O(1 / \delta)$ nodes, and thus it is constant-time because all the computations inside nodes can be done in $O(1)$ time with universal tables. When we arrive at the leaf, we scan for the desired pointer in constant time.

The tree $T_{f}$ must be updated when a symbol $t$ is inserted before any other occurrence of $t$ in $G_{j}(v)$, when a symbol is inserted at the first position of $G_{j}(v)$ and, similarly, when symbols are deleted from $G_{j}(v)$. The needed queries are easily answered with tree $T$. Moreover, due to the bidirectionality, we must also update $T_{f}$ when pointers to $G_{j}(v)$ are created from the parent or a child of $v$, or when they are deleted. Those updates work just like on the tree $T . T_{f}$ is also updated upon insertions and deletions of symbols, even if they do not change pointers, to maintain the positions up to date. In this case we traverse $T_{f}$ looking for the position of the update, change the offsets stored at the leaf, and update the subtree sizes stored at the nodes.

### 9.4 Structure $H_{j}(v)$

This structure manages the inter-node pointers that point inside $G_{j}(v)$. As explained in Section4.4, we give a handle to the outside nodes, that does not change over time, and $H_{j}(v)$ translates handles to positions in $G_{j}(v)$.

We store a tree $T_{h}$ that is just like $T_{f}$, where the incoming pointers are stored. $T_{h}$ is simpler, however, because at each node we only need to store the number of positions covered by the subtree of each child. It must also be possible to traverse $T_{h}$ from a leaf to the root.

In addition, we manage a table $T b l$ so that $T b l[h]$ points to the leaf of $T_{h}$ where the pointer corresponding to handle $h$ is stored. $T b l$ is also managed as a tree similar to $T_{f}$, with pointers sorted by id, where a constant number of ids $h$ are stored at the leaves together with their pointers to the leaves of $T_{h}$ (note that there are $O\left(\lg ^{3} n / \lg \rho\right)$ ids at most, so we need $O(\lg \lg n)$ bits for both ids and their pointers to $T_{h}$ ). Each internal node in $T b l$ maintains the maximum id stored at its leaves and the number of ids stored at its leaves. Thus one can in constant time find the pointer to $T_{h}$ corresponding to a given id, and also find the smallest unused id when a fresh one is needed (by looking for the first leaf of $T b l$ where the maximum id is larger than the number of ids).

At the leaves of $T_{h}$ we store, for each pointer, a backpointer to the corresponding leaf of Tbl and the position in $G_{j}(v)$ (in relative form). Given a handle $h$, we find using $T b l$ the corresponding position in the leaf of $T_{h}$, and move upwards up to the root of $T_{h}$, adding to the leaf offset the number of positions covered by the leftward children of each node. At the end we have obtained the position in constant time.

When pointers to $G_{j}(v)$ are created or destroyed, we insert or remove pointers in $T_{h}$. This requires traversing it top-down to find the appropriate leaf position and returning back to the root updating offsets. Backpointers to $T b l$ are used to adjust a constant number of positions in the leaf of $T_{h}$. We must also update $T_{h}$ upon symbol insertions and deletions in $G_{j}(v)$, to maintain the positions up to date. When a leaf splits or merges, we also update the pointers from a constant
number of positions in $T b l$, found with the backpointers. Similarly, the insertion and deletion of pointers from outside require updating $T b l$, and the backpointers from $T_{h}$ are maintained up to date using the pointers from $T b l$ to $T_{h}$.

Tbl may contain up to $\Theta\left(\lg ^{3} n / \lg \rho\right)$ pointers of $O(\lg \lg n)$ bits, which can be significant for some blocks. However, across the whole structure there can be only $O\left(\rho n \lg \sigma / \lg ^{3} n\right)$ pointers, adding up to $s=O\left(\rho n \lg \sigma \lg \lg n / \lg ^{3} n\right)$ bits, spread across $a=O\left(n \lg \sigma / \lg ^{3} n\right)$ tables Tbl. Using again Lemma 1, a collection of EAs poses an overhead of $O\left(n \lg \sigma / \lg ^{2} n\right)$ bits.

### 9.5 Structure $D_{j}(v)$ and the Final Result

Structure $D_{j}(v)$ is implemented as a tree $T_{d}$ analogous to $T$, storing at each node the number of positions and the number of non-deleted positions below each child. It requires $O\left(\left|G_{j}(v)\right| \lg \lg n / \lg n\right)$ bits. Since these are stored only for the root $v_{r}$ and the leaves $v_{a}$ of $\mathcal{T}$, its space adds up to $O(n \lg \rho \lg \lg n / \lg n)=O\left(n(\lg \lg n)^{2} / \lg n\right)$ bits.

While the raw data adds up to $n \lg \sigma$ bits, the space overhead adds up to $O\left(n \lg \sigma \lg ^{\varepsilon} n \lg \lg n / \lg n\right)$ for all the pointers plus $O(n \lg \sigma / \lg n+\sigma \lg n)$ for the memory management overhead. We can use, say, $\delta=\varepsilon$ and then have $O(1 / \varepsilon)$ time and $O\left(n \lg \sigma / \lg ^{1-\varepsilon} n+\sigma \lg n\right)$ bits for any $0<\varepsilon<1$ (renaming $2 \varepsilon$ as $\varepsilon$ ). However, when the data is compressed (Section [7), the sum of all the $\left|G_{j}(v)\right|$ terms in the space is $n H_{0}(S)+O\left(n \lg \sigma \lg \lg n / \lg ^{\varepsilon} n\right)$. This makes the space overhead related to the memory management and of $R_{j}(v)$ structures add up to $O\left(n H_{0}(S) / \lg ^{1-\varepsilon} n+n \lg \sigma / \lg n+\sigma \lg n\right)$ bits.

## 10 Extensions and Applications

We first describe an extension of our results to handling general alphabets, and then various applications of the original and the extended results.

### 10.1 Handling General Alphabets

Our time results do not depend on the alphabet size $\sigma$, yet our space does, in a way that ensures that $\sigma$ gives no problems as long as $\sigma=o(n)$ (so $\sigma \lg n=o(n \lg \sigma)$ ).

Let us now consider the case where the alphabet $\Sigma$ is much larger than the effective alphabet of the string, that is, the set of symbols that actually appear in $S$ at a given point in time. Let us now use $s \leq n$ to denote the effective alphabet size. Our aim is to maintain the space within $n H_{0}(S)+o(n \lg s)+O(s \lg n)$ bits, even when the symbols come from a large universe $\Sigma=[1 . .|\Sigma|]$, or even from a general ordered universe such as $\Sigma=\mathbb{R}$ or $\Sigma=\Gamma^{*}$ (i.e., $\Sigma$ are words over another alphabet $\Gamma$ ).

Our mappings $S N$ and $N S$ of Section 6 give a simple way to handle a sequence over an unbounded ordered alphabet. By changing $S N$ to a custom structure to search $\Sigma$, and storing elements of $\Sigma$ in array $N S$, we obtain the following results, using respectively Theorems 2 and 3.

Theorem 4 A dynamic string $S[1, n]$ over a general alphabet $\Sigma$ can be stored in a structure using $n H_{0}(S)+o(n \lg s)+O(s \lg n)+\mathcal{S}(s)$ bits and supporting queries access, rank and select in time $O(\mathcal{T}(s)+\lg n / \lg \lg n)$. Insertions and deletions of symbols are supported in $O(\mathcal{U}(s)+\lg n / \lg \lg n)$ amortized time. Here $s \leq n$ is the number of distinct symbols of $\Sigma$ occurring in $S, \mathcal{S}(s)$ is the number of bits used by a dynamic data structure to search over $s$ elements in $\Sigma$ plus to refer to $s$
elements in $\Sigma, \mathcal{T}(s)$ is the worst-case time to search for an element among $s$ of them in $\Sigma$, and $\mathcal{U}(s)$ is the amortized time to insert/delete symbols of $\Sigma$ in the structure.

Theorem 5 A dynamic string $S[1, n]$ over a general alphabet $\Sigma$ can be stored in a structure using $n H_{0}(S)+o(n \lg s)+O(s \lg n)+\mathcal{S}(s)$ bits and supporting queries access and select in time $O(\mathcal{T}(s)+$ $\lg n / \lg \lg n)$ and rank in time $O(\mathcal{T}(s)+\lg n)$. Insertions and deletions of symbols are supported in $O(\mathcal{U}(s)+\lg n)$ time. Here $s \leq n$ is the number of distinct symbols of $\Sigma$ occurring in $S, \mathcal{S}(s)$ is the number of bits used by a dynamic data structure to search over $s$ elements in $\Sigma$ plus to refer to $s$ elements in $\Sigma, \mathcal{T}(s)$ is the time to search for an element among $s$ of them in $\Sigma$, and $\mathcal{U}(s)$ is the time to insert/delete symbols of $\Sigma$ in the structure. All times are worst-case

Using general and dynamic alphabets had not been achieved in previous dynamic sequence data structures, because the wavelet has a static shape (and changing it is costly). These results open the door to using these solutions in various scenarios where alphabet dynamism is essential. We examine a few interesting particular cases:

- We can handle a sequence of arbitrary real numbers in the comparison model, by using a balanced tree for the alphabet data structure. If $\Sigma=\mathbb{R}$ we have $O(\lg s+\lg n / \lg \lg n)$ times using Theorem 4 and $O(\lg n)$ worst-case times using Theorem 5. Those complexities are optimal in the comparison model.
- We can handle a sequence of strings, that is, $\Sigma=\Gamma^{*}$ on a general alphabet $\Gamma$. Here we can store the effective set of strings in a data structure by Franceschini and Grossi [FG04], so that operations involving a string $a$ take $O(|a|+\lg \gamma+\lg n / \lg \lg n)$, where $\gamma$ is the number of symbols of $\Gamma$ actually in use. With Theorem 5 we obtain worst-case times $O(|a|+\lg \gamma+\lg n)$.
- If $\Sigma=[1 . .|\Sigma|]$ is a large integer range, we can obtain time $O(\lg \lg |\Sigma|+\lg n / \lg \lg n)$, or worstcase times $O(\lg \lg |\Sigma|+\lg n)$, and the space increases by $O(s \lg |\Sigma|)$ bits, by using y-fast tries Wil83] to handle the alphabet.
- Another important particular case is when we maintain a contiguous effective alphabet [1..s], and only insert new symbols $\sigma+1$. This is the case where the symbol identities themselves are not important. In this case there is no time penalty for letting the alphabet grow dynamically.


### 10.2 Dynamic Sequence Collections

A landmark application of dynamic sequences, stressed out in several papers along time CHL04, MN06, CHLS07, MN06, LP07, MN08, GN08, LP09, GN09, HM10, NS10, is to maintain a collection $\mathcal{C}$ of texts, where one can carry out indexed pattern matching, as well as inserting and deleting texts from the collection. Plugging in our new representation we can significantly improve the time and space of previous work, with an amortized and with a worst-case update time, respectively.

Theorem 6 There exists a data structure for handling a collection $\mathcal{C}$ of texts over an alphabet [1.. $\sigma$ ] within size $n H_{h}(\mathcal{C})+o(n \lg \sigma)+O\left(\sigma^{h+1} \lg n+m \lg n\right)$ bits, simultaneously for all $h$. Here $n$ is the length of the concatenation of $m$ texts, $\mathcal{C}=T_{1} \circ T_{2} \cdots \circ T_{m}$, and we assume that the alphabet size is $\sigma=o(n)$. The structure supports counting of the occurrences of a pattern $P$ in $O(|P| \lg n / \lg \lg n)$ time. After counting, any occurrence can be located in time $O\left(\lg { }_{\sigma} n \lg n\right)$. Any substring of length $\ell$ from any $T$ in the collection can be displayed in time $O\left(\left(\ell / \lg \lg n+\lg _{\sigma} n\right) \lg n\right)$. Inserting or
deleting a text $T$ takes $O(\lg n+|T| \lg n / \lg \lg n)$ amortized time. For $0 \leq h \leq\left(\alpha \lg _{\sigma} n\right)-1$, for any constant $0<\alpha<1$, the space simplifies to $n H_{h}(\mathcal{C})+o(n \lg \sigma)+O(m \lg n)$ bits.

Theorem 7 There exists a data structure for handling a collection $\mathcal{C}$ of texts over an alphabet $[1 . . \sigma]$ within size $n H_{h}(\mathcal{C})+o(n \lg \sigma)+O\left(\sigma^{h+1} \lg n+m \lg n\right)$ bits, simultaneously for all $h$. Here $n$ is the length of the concatenation of $m$ texts, $\mathcal{C}=T_{1} \circ T_{2} \cdots \circ T_{m}$, and we assume that the alphabet size is $\sigma=o(n)$. The structure supports counting of the occurrences of a pattern $P$ in $O(|P| \lg n)$ time. After counting, any occurrence can be located in time $O\left(\lg _{\sigma} n \lg n \lg \lg n\right)$. Any substring of length $\ell$ from any $T$ in the collection can be displayed in time $O\left(\left(\ell+\lg _{\sigma} n \lg \lg n\right) \lg n\right)$. Inserting or deleting a text $T$ takes $O(|T| \lg n)$ time. For $0 \leq h \leq\left(\alpha \lg _{\sigma} n\right)-1$, for any constant $0<\alpha<1$, the space simplifies to $n H_{h}(\mathcal{C})+o(n \lg \sigma)+O(m \lg n)$ bits.

The theorems refer to $H_{h}(\mathcal{C})$, the $h$-th order empirical entropy of sequence $\mathcal{C}$ Man01. This is a lower bound to any semistatic statistical compressor that encodes each symbol as a function of the $h$ preceding symbols in the sequence, and it holds $H_{h}(\mathcal{C}) \leq H_{h-1}(\mathcal{C}) \leq H_{0}(\mathcal{C}) \leq \lg \sigma$ for any $h>0$. To offer search capabilities, the Burrows-Wheeler Transform (BWT) BW94 of $\mathcal{C}, \mathcal{C}^{b w t}$, is represented, not $\mathcal{C}$; then access and rank operations on $\mathcal{C}^{\text {bwt }}$ are used to support pattern searches and text extractions. Kärkkäinen and Puglisi [KP11] showed that, if $\mathcal{C}^{b w t}$ is split into superblocks of size $\Theta\left(\sigma \lg ^{2} n\right)$, and a zero-order compressed representation is used for each superblock, the total bits are $n H_{h}(\mathcal{C})+o(n)$.

We use their partitioning, and Theorems 2 or 3 to represent each superblock. For Theorem 6, the superblock sizes are easily maintained upon insertions and deletions of symbols, by splitting and merging superblocks and rebuilding the structures involved, without affecting the amortized time per operation. They KP11 also need to manage a table storing the rank of each symbol up to the beginning of each superblock. This is arranged, in the dynamic scenario, with $\sigma$ partial sum data structures containing $O\left(n /\left(\sigma \lg ^{2} n\right)\right)$ elements each, plus another one storing the superblock lengths. This adds $O(n / \lg n)$ bits and $O(\lg n / \lg \lg n)$ time per operation [NS10, Lem. 1]. Upon blocks splits and merges, we use the same techniques used for $K$ structures described in Section 4.4,

For Theorem 7 we use the smooth block size management algorithm described in Section 8.1 for the superblocks, which guarantees worst-case times and the same space redundancy. Then partial-sum data structures are used without problems.

Finally, the locating and displaying overheads are obtained by marking one element out of $\lg _{\sigma} n \lg \lg n$, so that the space overhead of $o(n \lg \sigma)$ is maintained. Other simpler data structures used in previous work MN08, such as mappings from document identifiers to their position in $\mathcal{C}^{\text {bwt }}$ and the samplings of the suffix array, can easily be replaced by $O(\lg n / \lg \lg n)$ time partial-sums data structures and simpler structures to maintain dictionaries of values [NS10, Lem. 1].

### 10.3 Burrows-Wheeler Transform

Another application of dynamic sequences is to build the BWT of a text $T, T^{b w t}$, within compressed space, by starting from an empty sequence and inserting each new character, $T[n], T[n-1], \ldots$, $T[1]$, at the proper positions. Equivalently, this corresponds to initializing an empty collection and then inserting a single text $T$ using Theorem 6. The result is also stated as the compressed construction of a static FM-index FMMN07, a compressed index that consists essentially of a (static) wavelet tree of $T^{b w t}$. Our new representation improves upon the best previous result on compressed space NS10.

Theorem 8 The Alphabet-Friendly FM-index FMMN07], as well as the BWT [BW94], of a text $T[1, n]$ over an alphabet of size $\sigma$, can be built using $n H_{h}(T)+o(n \lg \sigma)$ bits, simultaneously for all $1 \leq h \leq\left(\alpha \lg _{\sigma} n\right)-1$ and any constant $0<\alpha<1$, in time $O(n \lg n / \lg \lg n)$. It can also be built within the same time and $n H_{0}(T)+o(n \lg \sigma)+O(\sigma \lg n)$ bits, for any alphabet size $\sigma$.

We are using Theorem 6 for the case $h>0$, and Theorem 2 to obtain a less alphabet-restrictive result for $h=0$ (in this case, we do not split the text into superblocks of $O\left(\sigma \lg ^{2} n\right)$ symbols, but just use a single sequence). Note that, although insertion times are amortized in those theorems, this result is worst-case because we compute the sum of all the insertion times.

This is the first time that $o(n \lg n)$ time complexity is obtained within compressed space. Other space-conscious results that achieve better time complexity (but more space) are Okanohara and Sadakane [OS09], who achieved optimal $O(n)$ time within $O\left(n \lg \sigma \lg \lg _{\sigma} n\right)$ bits, and Hon et al. [HSS09], who achieved $O(n \lg \lg \sigma)$ time and $O(n \lg \sigma)$ bits. Older results, like Kärkkäinen's Kär07, are superseded.

### 10.4 Binary Relations

Barbay et al. BGMR07] show how to represent a binary relation of $t$ pairs relating $n$ "objects" with $\sigma$ "labels" by means of a string of $t$ symbols over alphabet [1.. $\sigma$ ] plus a bitmap of length $t+n$. The idea is to traverse the matrix, say, object-wise, and write down in a string the labels of the pairs found. Meanwhile we append a 1 to the bitmap each time we find a pair and a 0 each time we move to the next object. Then queries like: find the objects related to a label, find the labels related to an object, and tell whether an object and a label are related, are answered via access, rank and select operations on the string and the bitmap.

A limitation in the past to make this representation dynamic was that creating or removing labels implied changing the alphabet of the string. Now we can use Theorem 4 to obtain a fully dynamic representation. We illustrate the case where labels and objects are contiguous values in integer intervals $[1 . . \sigma]$ and $[1 . . n]$, respectively. We note that the structure on the sequence of labels is so fast that the bitmap, which is longer, dominates the times.

Theorem 9 A dynamic binary relation consisting of $t$ pairs relating $n$ objects with $\sigma$ labels can support the operations of counting and listing the objects related to a given label, counting and listing the labels related to a given object, and telling whether an object and a label are related, all in time $O(\lg (n+t) / \lg \lg (n+t))$ per delivered datum. Pairs, objects and labels can also be added and deleted in amortized time $O(\lg (n+t) / \lg \lg (n+t))$. The space required is $t H+o(t \lg \sigma)+n \lg n+\sigma \lg \sigma+$ $O(t+n+\sigma \lg t)$ bits, where $H=\sum_{1 \leq i \leq \sigma}\left(t_{i} / t\right) \lg \left(t / t_{i}\right) \leq \lg \sigma$, where $t_{i}$ is the number of objects related to label i. Only labels and objects with no related pairs can be deleted.

Theorem 10 A dynamic binary relation consisting of $t$ pairs relating $n$ objects with $\sigma$ labels can support the operations of counting and listing the objects related to a given label, counting and listing the labels related to a given object, and telling whether an object and a label are related, all in time $O(\lg (n+t))$ per delivered datum. Pairs, objects and labels can also be added and deleted in time $O(\lg (n+t))$. The space required is $t H+o(t \lg \sigma)+n \lg n+\sigma \lg \sigma+O(t+n+\sigma \lg t)$ bits, where $H=\sum_{1 \leq i \leq \sigma}\left(t_{i} / t\right) \lg \left(t / t_{i}\right) \leq \lg \sigma$, where $t_{i}$ is the number of objects related to label $i$. Only labels and objects with no related pairs can be deleted.

The careful reader may notice that we have uniformized the times of all the operations for simplicity, yet some can be slightly faster. For example, listing the $m$ labels related to a given object requires only $O(\lg (n+t) / \lg \lg (n+t)+m \lg t / \lg \lg t)$ time. Also, obviously, we can exchange labels and objects if desired.

### 10.5 Directed Graphs

A particularly interesting and general binary relation is a directed graph with $n$ nodes and $e$ edges. Our binary relation representation allows one to navigate a directed graph in forward and backward direction, and modify it, within about the space needed by a classical adjacency list representation, and even less.

Theorem 11 A dynamic directed graph consisting of $n$ nodes and e edges can support the operations of counting and listing the neighbors pointed from a node, counting and listing the reverse neighbors pointing to a node, and telling whether there is a link from one node to another, all in time $O(\lg (n+e) / \lg \lg (n+e))$ per delivered datum. Nodes and edges can be added and deleted in amortized time $O(\lg (n+e) / \lg \lg (n+e))$. The space used is $e H+o(e \lg n)+n \lg n+O(e+n \lg e)$ bits, where $H=\sum_{1 \leq i \leq n}\left(e_{i} / e\right) \lg \left(e / e_{i}\right) \leq \lg n$ and $e_{i}$ is the outdegree of node $i$.

Theorem 12 A dynamic directed graph consisting of $n$ nodes and e edges can support the operations of counting and listing the neighbors pointed from a node, counting and listing the reverse neighbors pointing to a node, and telling whether there is a link from one node to another, all in time $O(\lg (n+e))$ per delivered datum. Nodes and edges can be added and deleted in time $O(\lg (n+e))$. The space used is eH $+o(e \lg n)+n \lg n+O(e+n \lg e)$ bits, where $H=\sum_{1 \leq i \leq n}\left(e_{i} / e\right) \lg \left(e / e_{i}\right) \leq \lg n$ and $e_{i}$ is the outdegree of node $i$.

Note also that we can change "outdegree" by "indegree" in the theorem by representing the transposed graph, as operations are symmetric. Our ability to handle dynamic alphabets is essential here to allow node insertions and deletions in the graph.

### 10.6 Inverted Indexes

Finally, we consider an application where the symbols are strings. Take a text $T$ as a sequence of $n$ words, which are strings over a set of letters $\Gamma$. The alphabet $\Gamma$ is integer and fixed, of size $\gamma$. The alphabet of $T$ is $\Sigma=\Gamma^{*}$, and its effective alphabet is called the vocabulary $V$ of $T$, of size $|V|=\sigma$. A positional inverted index is a data structure that, given a word $w \in V$, returns the positions in $T$ where $w$ appears [BYR11.

A well-known way to simulate a positional inverted index within no extra space on top of the compressed text is to use a compressed sequence representation for $T$ (over alphabet $\Sigma$ ), so that operation select ${ }_{w}(T, i)$ simulates access to the $i$ th position of the list of word $w$, whereas access to the original $T$ is provided via access $(T, i)$. Operation rank can be used to emulate various inverted index algorithms, particularly for intersections [BN09]. The space is the zero-order entropy of the text seen as a sequence of words, which is very competitive in practice [BYR11]. Our new technique permits modifying the underlying text, that is, it simulates a dynamic inverted index. For this sake we use Theorem 4 and compact tries to handle a vocabulary over a fixed alphabet.

Theorem 13 A text of $n$ words with a vocabulary of $\sigma$ words and total length $\nu$ over a fixed alphabet $\Gamma$ of size $\gamma$ can be represented within $n H_{0}(T)+o(n \lg \sigma)+O(\nu \lg \gamma+\sigma \lg n)$ bits of space, where $H_{0}(T)$ is the word-wise entropy of $T$. The representation outputs any word $T[i]=w$ given $i$, finds the position of the ith occurrence of any word $w$, and tells the number of occurrences of any word $w$ up to position $i$, all in time $O(|w|+\lg n / \lg \lg n)$. A word $w$ can be inserted or deleted at any position in $T$ in amortized time $O(|w|+\lg n / \lg \lg n)$.

Theorem 14 A text of $n$ words with a vocabulary of $\sigma$ words and total length $\nu$ over a fixed alphabet $\Gamma$ of size $\gamma$ can be represented within $n H_{0}(T)+o(n \lg \sigma)+O(\nu \lg \gamma+\sigma \lg n)$ bits of space, where $H_{0}(T)$ is the word-wise entropy of $T$. The representation outputs any word $T[i]=w$ given $i$, and finds the position of the ith occurrence of any word $w$, in time $O(|w|+\lg n / \lg \lg n)$. It tells the number of occurrences of any word $w$ up to position $i$, and supports the insertion or deletion of any word $w$ in $T$, in time $O(|w|+\lg n)$.

We remark that $\sigma$ and $\nu$ are assumed to be $O\left(n^{\alpha}\right)$ for some $0<\alpha<1$ in information retrieval models [BYR11. Under this assumption the space is just $n H_{0}(T)+o(n \lg \sigma)$.

Another kind of inverted index, a non-positional one, relates each word with the documents where it appears (not to the exact positions). This can be seen as a direct application of our binary relation representation [BCN10], and our dynamization theorems apply to it as well.

## 11 Conclusions and Further Challenges

We have obtained $O(\lg n / \lg \lg n)$ time for all the operations that handle a dynamic sequence on an arbitrary alphabet [1.. $\sigma$ ], matching lower bounds that apply to binary alphabets [FS89], and using zero-order compressed space. Our structure is faster than the best previous work HM10, NS10, by a factor of $\Theta(\lg \sigma / \lg \lg n)$ when the alphabet is larger than polylogarithmic. The query times are worst-case, yet the update times are amortized. We also show that it is possible to obtain worstcase for all the operations, although times for rank and updates raises to $O(\lg n)$. We also show how to handle general and infinite alphabets. Our result can be applied to a number of problems and improve previous upper bounds on those; we have described several ones.

We remark that the lower bounds [FS89] are valid also for amortized times, so our amortized solution is optimal, yet it is not known whether our worst-case solution is optimal. Thus the main remaining challenge is whether it is possible to attain the optimal $O(\lg n / \lg \lg n)$ worst-case time for all the operations.

Another interesting challenge is to support a stronger set of update operations, such as block edits, concatenations and splits in the sequences. Navarro and Sadakane [NS10 support those operations within time $O\left(\sigma \lg ^{1+\varepsilon} n\right)$. While it seems feasible to achieve, in our structure, $O(\sigma \lg n)$ time by using blocks of $\Theta\left(\lg ^{2} n\right)$ bits, the main hurdle is the difficulty of mimicking the same splits and concatenations on the list maintenance data structures we use [IA84, Mor03].

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