# A Constant Factor Approximation Algorithm for Reordering Buffer Management 

Noa Avigdor-Elgrabli ${ }^{\S} \quad$ Yuval Rabani ${ }^{〔}$


#### Abstract

In the reordering buffer management problem (RBM) a sequence of $n$ colored items enters a buffer with limited capacity $k$. When the buffer is full, one item is removed to the output sequence, making room for the next input item. This step is repeated until the input sequence is exhausted and the buffer is empty. The objective is to find a sequence of removals that minimizes the total number of color changes in the output sequence. The problem formalizes numerous applications in computer and production systems, and is known to be NP-hard.

We give the first constant factor approximation guarantee for RBM. Our algorithm is based on an intricate "rounding" of the solution to an LP relaxation for RBM, so it also establishes a constant upper bound on the integrality gap of this relaxation. Our results improve upon the best previous bound of $O(\sqrt{\log k})$ of Adamaszek et al. (STOC 2011) that used different methods and gave an online algorithm. Our constant factor approximation beats the super-constant lower bounds on the competitive ratio given by Adamaszek et al. This is the first demonstration of an offline algorithm for RBM that is provably better than any online algorithm.


[^0]
## 1 Introduction

Problem statement and motivation. In the reordering buffer management problem (RBM) a sequence of $n$ items of colors $c(1), c(2), \ldots, c(n)$ (taken from a finite set of colors $C$ ) enters a buffer with capacity $k \in \mathbb{N}$. When the buffer is full, one item is removed, making room for the next input item. This step is repeated until the input sequence is exhausted and the buffer is empty. Thus, the buffer can be used to permute the input sequence in a limited way. In the permuted output sequence, we are interested in the number of times there is a color change between adjacent positions. Out of all feasible solutions, the objective is to find a sequence of removals that batches items of the same color and minimizes the total number of color changes in the output sequence.

Introduced in [19], this elegant model formalizes a wide scope of resource management problems in production engineering, logistics, computer systems, network optimization, and information retrieval (see, e.g., [19, 8, [17, 14]). For example, one of the motivating examples of [19] is batching cars by color in the paint shop of a car manufacturing plant to minimize the consumption of paint solvent used to wash spray guns each time the paint color is changed between two consecutive cars. Naturally, the buffer capacity is limited by physical space constraints, and the incoming stream of cars is dictated by the schedule of the assembly line. (This particular application is part of the ROADEF Challenge 2005 of the French Operations Research Society [10], see also [14].) Generally, in computer systems and production systems buffers are often prepended to subsystems to facilitate better control of their input (see [20, 18]), so understanding how to optimize buffer utilization is a fundamental and important problem.

Our results. We give the first constant factor approximation guarantee for RBM, improving on the best previous bound of $O(\sqrt{\log k})$ [2]. Our algorithm is based on "rounding" the solution to a linear programming relaxation that we recently proposed [6]. Thus, our work also establishes an $O(1)$ upper bound on the integrality gap of this relaxation, improving upon the best previous bound of $O(\sqrt{\log k})$ (which is not explicit in [2], but can be derived from their work). Most previous work on RBM (including the above-mentioned [6, [2]) discussed online algorithms. There are recent lower bounds on the competitive ratio of $\Omega(\sqrt{\log k / \log \log k})$ for deterministic algorithms, and $\Omega(\log \log k)$ for randomized algorithms (against the oblivious adversary) [2]. Thus, our (deterministic) algorithm shows, for the first time, that an efficient offline RBM algorithm can beat any online algorithm. We note that in some applications, e.g. the paint shop sequencing problem mentioned above, the setting enables an offline computing of a good solution. Moreover, proving strong upper bounds on the integrality gap of a natural linear programming relaxation seems to be one of the major stumbling blocks on the way to design randomized online algorithms that beat the deterministic lower bound.

Our algorithm for transforming a fractional solution into an integer one, without increasing the cost by more than a constant factor, works in phases. A phase starts at the time reached by the previous phase. A phase has a time horizon target, which is the time that the fractional solution increases its cost by some small constant factor. The goal of a phase is to reach the target by evicting a constant number of colors. This goal might be impossible to achieve. In such a case, we use an intricate charging scheme that chooses colors to evict and charges their eviction to the past fractional cost of other colors. The main difficulty in the analysis is to prove that the conditions under which the simple strategy fails to reach the target imply that the charging scheme can be used to bridge the gap. Our proofs involve illuminating observations on the structure of fractional RBM solutions.

Previous work. As mentioned above, RBM was introduced in [19], who gave an $O\left(\log ^{2} k\right)$-competitive online algorithm for the problem. The guarantee was improved through a sequence of papers [12, 6, 2], culminating in the $O(\sqrt{\log k})$ bound of [2]. This was also the best known approximation guarantee for RBM prior to our work. While the algorithms evolved gradually, each paper uses completely different tools of analysis. The $O(\log k / \log \log k)$-competitive analysis of [6] applies a dual fitting argument, using the same relaxation that we use in this paper. The later and better result of [2] does not use linear programming. However, their proof can be
modified to show that the $O(\sqrt{\log k})$ bound on the competitive ratio also holds when competing against a fractional adversary, and therefore the integrality gap of the [6] relaxation is $O(\sqrt{\log k})$. As mentioned above, in [2] lower bounds on the competitive ratio of $\Omega(\sqrt{\log k / \log \log k})$ and $\Omega(\log \log k)$, respectively, were established for deterministic and randomized online algorithms, respectively.

Beyond the implications of the online setting, not much was known about the offline case prior to our work. Recent work shows that the problem is NP-hard [9, 5]. Allowing resource augmentation, [9] give, for every $\epsilon>0$, an $O(1 / \epsilon)$-approximation algorithm for RBM with the caveat that the approximation algorithm is allowed to use a buffer of size $(2+\epsilon) \cdot k$. This strengthens a similar result implicit in [12], proving this for their online algorithm, but only for $\epsilon=2$. The paper [12] also shows that the optimum for a buffer of size $k$ can be at most a factor of $O(\log k)$ larger than the optimum for a buffer of size $4 k$. In [1], a matching lower bound of $\Omega(\log k)$ was established, so the above resource augmentation arguments cannot yield constant factor approximation guarantees for RBM.

There are simple constant factor approximation algorithms for the complement objective of maximizing the number of adjacent pairs with no color change in the output sequence [16, 7] (the constants are 20 and 9 , respectively). The minimization version that we consider here seems more adequate for the applications in mind, and it also seems more challenging. Clearly, if we expect successful batching into relatively long monochromatic sequences, then guarantees on the complement maximization objective do not guarantee good performance in terms of the minimization objective.

For some applications, it is suitable to use more general cost functions to measure the cost of color changes in the output sequence. In particular, non-uniform costs, where the cost of switching to a color depends on the color, were discussed in [12, 6, 2]. Metric costs, where the cost of switching between colors is determined by a metric on the colors, were discussed for the line metric in [15, 13] and for general metrics in [11]. None of these models are known to have constant factor polynomial time approximation algorithms.

## 2 The Algorithm

Consider an instance $\mathcal{I}$ of RBM that is given by the buffer size $k$ and by a sequence of $n$ items of colors $c(1), c(2), \ldots, c(n)$. Let $C$ denote the set of colors that appear in the sequence. Our algorithm solves a linear programming relaxation for $\mathcal{I}$, and then uses the fractional optimal solution to derive an integer solution whose cost is at most a constant factor greater than the fractional solution we started with. We use a time indexed relaxation that was first defined in our previous work [6], where it was used in a dual fitting analysis of an online algorithm for the problem. For completeness, we define the relaxation and motivate it here.

We use $0-1$ variables $x_{i, j}$, for $i=1,2, \ldots, n$ and $j=\max \{k+1, i\}, \ldots, k+n$. An assignment $x_{i, j}=1$ indicates that the $i$-th input item is removed from the buffer at time $j$. The reordering buffer management problem can be expressed as an integer linear program IP on these variables. We require the following notation. For every input item $i$, let last $(i)$ denote the last input item of color $c(i)$, and for $i \neq \operatorname{last}(i)$ let $n(i)$ denote the next input item of color $c(i)$. For notational convenience, we put $n(i)=k+n+2$ for all $i=\operatorname{last}(i)$. Then, IP is:

$$
\begin{array}{rcl}
\operatorname{minimize} & \sum_{i=1}^{n} \sum_{j=\max \{i, k+1\}}^{n(i)-2} x_{i, j} & \\
\text { s.t. } & \sum_{j=\max \{i, k+1\}}^{k+N} x_{i, j}=1 & \forall i \\
& \sum_{i=1}^{j} x_{i, j}=1 & \forall j \\
& x_{n(i), j}-x_{i, j-1} \geq 0 & \forall i \neq \operatorname{last}(i) ; \forall j \geq n(i) \\
& x_{i, j} \in\{0,1\} & \forall i ; \forall j \geq i . \tag{4}
\end{array}
$$

The constraints (1) guarantee that each item is eventually removed from the buffer. The constraints (2) guarantee that at each time slot one item is removed from the buffer. The constraints (3) prevent the solution from switching colors while there are still items of the current color in the buffer. These constraints are needed to guarantee that the linear objective function measures the cost of the solution correctly. Notice that the objective function simply counts the number of items that are removed from the buffer before the next item of the same color is encountered in the input. Without constraints (3), we could avoid paying for color changes by keeping in the buffer just the last encountered item of a color $c$ until the next item of this color is reached. We denote the optimal value of IP by $z_{\text {IP }}$.

Proposition 2.1. The value of an optimal solution for $\mathcal{I}$ is exactly $z_{\mathrm{IP}}$.
Proof Sketch. The obvious correspondence between RBM output sequences and feasible IP solutions matches RBM solutions and IP solutions with the same cost.

A linear programming relaxation LP is derived by relaxing the constraints (4) to

$$
\begin{equation*}
x_{i, j} \geq 0 \quad \forall i ; \forall j \geq i \tag{5}
\end{equation*}
$$

We denote the value of the relaxation at a feasible solution $x$ by $z(x)$, and the optimal value by $z_{\mathrm{LP}}$. Given a feasible fractional solution $x$ of LP (i.e., $x$ satisfying constraints (1), (2), (3), and (5)) and a time step $j$, we can think of $x$ as defining a fractional packing of input items into the buffer at time $j$. I.e., every input item $i \leq j$ is in the buffer with weight $w_{i}^{j}=w(x)_{i}^{j}$ where $w_{i}^{j}=1-\sum_{j^{\prime}=i}^{j} x_{i, j^{\prime}}$. For notational convenience, we define $w_{i}^{i-1}=1$. Notice that $w$ is a function of $x$; we usually omit $x$ from the notation. Also notice that due to constraints (1), $w_{i}^{j} \in[0,1]$.

Proposition 2.2. If $j \leq n$, then $\sum_{i \leq j} w_{i}^{j}=k$, and if $j>n$ then $\sum_{i \leq j} w_{i}^{j}=k+n-j$.
Proof Sketch. By constraints (2), for every $k+1 \leq j \leq k+n$, it holds that $\sum_{i \leq j} x_{i, j}=1$. Therefore, if $j \leq n$,

$$
\sum_{i \leq j} w_{i}^{j}=\sum_{i \leq j}\left(1-\sum_{j^{\prime}=i}^{j} x_{i, j^{\prime}}\right)=j-\sum_{j^{\prime}=k+1}^{j} \sum_{i \leq j^{\prime}} x_{i, j^{\prime}}=j-(j-k)=k .
$$

A similar argument shows the case of $j>n$.
Proposition 2.3. For every $i \neq \operatorname{last}(i)$, for every $j \geq n(i)-1, w_{i}^{j} \leq w_{n(i)}^{j}$.
Proof Sketch. If at some point $j \geq n(i)-1$ we have $w_{i}^{j}>w_{n(i)}^{j}$, then $\sum_{j^{\prime}>j} x_{i, j^{\prime}}=w_{i}^{j}>w_{n(i)}^{j} \geq w_{n(i)}^{j+1}=$ $\sum_{j^{\prime}>j+1} x_{n(i), j^{\prime}}$, so at some point $j^{\prime}>j+1, x_{n(i), j^{\prime}}-x_{i, j^{\prime}-1}<0$, violating constraints (3).

Our main result is the following theorem.
Theorem 2.4. There is a constant $\alpha>1$ and a (deterministic) polynomial time algorithm which given a feasible fractional solution $x$ of LP computes a feasible $0-1$ solution $\bar{x}$ of IP such that $z(\bar{x}) \leq \alpha \cdot z(x)$.

Corollary 2.5. There is a (deterministic) polynomial time $\alpha$-approximation algorithm for reordering buffer management.

Proof. Compute an optimal solution $x^{*}$ of LP with cost $z\left(x^{*}\right)=z_{\mathrm{LP}} \leq z_{\mathrm{IP}}$. Use Theorem 2.4 to compute a $0-1$ solution $\bar{x}^{*}$ with cost $z\left(\bar{x}^{*}\right) \leq \alpha z_{\mathrm{LP}} \leq \alpha z_{\mathrm{IP}}$. The corollary follows from Proposition 2.1.

We now describe the rounding algorithm of Theorem 2.4. The algorithm works in phases. Each phase evicts one or more colors from the buffer. To evict a color, the algorithm removes the items of this color from the buffer until the buffer contains no such item. We refer to the eviction of one color from the buffer as a step. The algorithm uses constants $\delta_{1}, \delta_{2}, \delta_{3} \in(0,1)$, and $\gamma=\gamma\left(\delta_{1}, \delta_{2}\right)>1$, to be defined later. In order to describe the algorithm, we need the following definition:

Definition 2.6. For $q=1,2, \ldots,\left\lfloor z(x) / \delta_{3}\right\rfloor$,

$$
t_{q}=\min \left\{t: \sum_{j=k+1}^{t} \sum_{i \leq j} y_{i, j} \geq q \cdot \delta_{3}\right\}
$$

where $y_{i, j}=x_{i, j}$ if $n(i)>j+1$, and $y_{i, j}=0$ otherwise.
In other words, $t_{q}$ denotes the earliest time at which the cost of the fractional solution $x$ increases to at least $q \cdot \delta_{3}$. The goal of phase $q$ is to reach $t_{q}$. (If the last $t_{q}$ was already reached, the goal of the last phase is to reach the end of the output sequence.) Each phase includes three types of steps. There are steps that are charged against the past $x$-cost of the items removed by that step. There are steps that are paid for by charging the past $x$-cost of other items in the buffer. Finally, there are a constant number of steps that cannot be charged to the past $x$-cost, so they are charged to the increase in $x$-cost that sets $t_{q}$.

For the second type of steps, we will use a charging scheme to determine the choice of colors to evict, and also to analyze the algorithm. For each item $i$ in the buffer at time $j$ we maintain an index $\tau_{i, j}$ which is the last time before $j$ that $i$ was charged. Initially, $\tau_{i, i}=i-1$. At time $j$, if $i$ was not charged at time $j-1$, we put $\tau_{i, j}=\tau_{i, j-1}$. Otherwise, we put $\tau_{i, j}=j-1$. (The charge is implied by the sequence of charging times.) For each time step $j$, for each item $i$ in the buffer of the algorithm at time $j$, define $d_{i}^{j}=w_{i}^{\tau_{i, j}}-w_{i}^{j}$. This is the fraction of $i$ that the solution $x$ removed from the buffer since the last time $i$ was charged.

We call the set of all items of a specific color in the algorithm's buffer a block. Let $\mathcal{B}^{j}=\left\{B_{1}^{j}, B_{2}^{j}, \ldots, B_{m_{j}}^{j}\right\}$ denote the set of blocks in the algorithm's buffer at time $j$ (before removing from the buffer an item at time $j$ ). For $r=1,2, \ldots, m_{j}$, let $f_{r}^{j}$ denote the earliest item in $B_{r}^{j}$. For an item $i$, we denote $t(i)=\min \left\{t: w_{i}^{t} \leq 1-\delta_{1}\right\}$. We assume that the blocks are ordered so that $t\left(f_{1}^{j}\right) \leq t\left(f_{2}^{j}\right) \leq \cdots \leq t\left(f_{m_{j}}^{j}\right)$. We denote by $\Delta_{j}$ the difference in volume between the algorithm's buffer and the fractional buffer at time $j$. Formally,

$$
\Delta_{j}=\frac{1}{2}\left\|\bar{w}^{j}-w^{j}\right\|_{1}=\sum_{r=1}^{m_{j}} \sum_{i \in B_{r}^{j}}\left(1-w_{i}^{j}\right)=\sum_{i \leq j: \bar{w}_{i}^{j}=0} w_{i}^{j}
$$

where $\bar{w}=w(\bar{x})$. For a current time step $j$ and a target time step $t_{q}$, let

$$
I_{q}^{j}=\operatorname{argmax}\left\{|I|: I \subset\left[j, t_{q}\right] \wedge \forall i, i^{\prime} \in I, c(i)=c\left(i^{\prime}\right)\right\}
$$

and put

$$
t_{q}^{j}=\max \left\{j, t_{q}+1-\left|I_{q}^{j}\right|\right\}
$$

This is the earliest time where items from one color that arrive after time $j-1$ can be removed from the buffer consecutively, reaching time $t_{q}$ or later. (Notice that if $t_{q}<j$, then $t_{q}^{j}=j$.) Intuitively, if we reach $t_{q}^{j}$ without using items from $I_{q}^{j}$, then in one more step we can reach our target $t_{q}$. Consider a decision time $j$ (the previous phase ended at time $j-1$ ). We execute the following procedure:

0 . While our buffer contains an item $i$ such that $t(i) \leq j$, we evict color $c(i)$, and we increment $j$ to be the first time step following the one we've reached. When there are no more steps of this case, we execute the first case among $1 \sqrt[4]{4}$ that applies.

1. If there is a color $c$ that we can evict and reach $t_{q}$, we evict one such color, thus ending the phase.
2. If our buffer contains $t_{q}^{j}-j$ items from one or two colors, we evict those colors in two steps, and if we haven't reached $t_{q}$, we also evict the color of $I_{q}^{j}$. We will prove in Claim 3.2 that we reach $t_{q}$, so the phase ends.
3. If $\Delta_{j} \geq \frac{1}{\gamma} \cdot\left(t_{q}^{j}-j\right)$, then we do the following. For $B \in \mathcal{B}^{j}$, put $\hat{d}_{B}^{j}=\frac{1}{|B|} \cdot \sum_{i \in B} d_{i}^{j}$. Define $s_{1}>s_{2}>$ $\cdots>s_{p_{j}}$ inductively as follows. Initially set $s_{1}=m_{j}$. Assuming $s_{p}$ is defined, let $r_{p} \leq s_{p}$ be the largest index for which $\sum_{u=r_{p}}^{s_{p}-1} \sum_{i \in B_{u}^{j}} d_{i}^{j} \leq \delta_{2}\left|B_{s_{p}}^{j}\right|$. If $\sum_{u=r_{p}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j} \geq \delta_{1}$, let $r_{p}^{\prime} \in\left[r_{p}, s_{p}\right]$ be the largest index for which $\sum_{u=r_{p}^{\prime}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j} \geq \delta_{1}$. (See Figure 1 in the appendix.) We evict the color of block $B_{s_{p}}^{j}$, and charge the items in $B_{r_{p}^{\prime}}^{j}, \ldots, B_{s_{p}-1}^{j}$ (i.e., for each charged item $i$, set $\tau_{i, j+1}=j$ ). If $r_{p}^{\prime}>1$, set $s_{p+1}$ to be $r_{p}^{\prime}-1$, else set $p_{j}=p$. Otherwise, if $\sum_{u=r_{p}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j}<\delta_{1}$ and $r_{p}>1$, set $s_{p+1}=\operatorname{argmax}\left\{\left|B_{u}^{j}\right|: u \in\left[r_{p}-1, s_{p}-1\right]\right\}$. Otherwise, if $r_{p}=1$, set $p_{j}=p$. If the entire process removes fewer than $t_{q}^{j}-j$ items that were in our buffer at time $j$, we evict the color of the largest block $B \in \mathcal{B}^{j}$ that remains. We will prove in Claims 3.3 and 3.4 that at this point we can evict the color of $I_{q}^{j}$ and reach $t_{q}$, thus ending the phase.
4. In the remaining case, we evict the color of the largest block $B \in \mathcal{B}^{j}$. We increment $j$ to be the time step following the last output step. Now, we execute the procedure again. We will prove in Claim 3.5 that in a phase we never reach case 4 twice.

This completes the definition of $\bar{x}$.

## 3 Analysis

In this section we prove our main result, Theorem 2.4 . We first give an interpretation of the feasible fractional solution $x$. Consider a color $c$, a sequence $I$ of color $c$ items $i_{1}, i_{2}, \ldots, i_{m}$ and a starting time $j$. Let $M_{I, j}$ denote the matching given by $M_{I, j}\left(i_{s}\right)=j+s$, for all $s=1,2, \ldots, m$. We say that $M_{I, j}$ is a monochromatic sequence matching (MSM) iff the items are a maximal sequence of consecutive items of the same color $c\left(i_{1}\right)$. In other words, $M_{I, j}$ is an MSM iff it satisfies the following conditions: $(i)$ for every $s=1,2, \ldots, m-1$ it holds that $c\left(i_{s}\right)=c\left(i_{s+1}\right)$ and $n\left(i_{s}\right)=i_{s+1} ;(i i) j+s \geq i_{s}$ for every $s=1,2, \ldots, m$; (iii) $j+m<n\left(i_{m}\right)-1$.

Proposition 3.1. For every feasible fractional solution $x$ there is a fractional packing of monochromatic sequence matchings $\lambda=\lambda(x)$ that satisfies the following constraints: (a) for each input item $i, \sum_{I, j: i \in I} \lambda_{I, j}=1$; (b) for each time slot $t, \sum_{I, j: t \in[j+1, j+|I|]} \lambda_{I, j}=1 ;(c) z(x)=\sum_{I, j} \lambda_{I, j}$.

Proof Sketch. We can construct $\lambda$ by the following algorithm. While there exist $i_{1}, j$ such that $x_{i_{1}, j+1}>0$, find such a pair with minimum $j$. Find a maximal sequence $I=\left(i_{1}, i_{2}, \ldots, i_{m}\right)$ of items of color $c\left(i_{1}\right)$ with $x_{i_{s}, j+s}>0$. By constraints (3) (which are maintained through the induction), it must be that $x_{i_{s}, j+s} \geq x_{i_{1}, j+1}$ and $j+m<n\left(i_{m}\right)-1$. Put $\lambda_{I, j}=x_{i_{1}, j+1}$ and subtract $\lambda_{I, j}$ from $x_{i_{s}, j+s}$ for all $s=1,2, \ldots, m$. (Notice that this will not cause constraints (3) to be violated.) Constraints (a), (b) follow from constraints (1], (2) of the LP. Equation (c) follows from the fact that every MSM $M_{I, j}$ that we construct ends at time $j+|I|$ which precedes the arrival of the next item of this color.

We next prove that the algorithm is well-defined.
Claim 3.2. Executing case 2 ends a phase.
Proof. Consider a phase $q$ where we execute case 2 at time $j$. Let $B, B^{\prime} \in \mathcal{B}^{j}$ denote the two blocks with $|B|+\left|B^{\prime}\right| \geq t_{q}^{j}-j$, and let $c, c^{\prime}$ denote their colors. (If the $t_{q}^{j}-j$ items stipulated by case 2 are of a single color,
set $B^{\prime}=\emptyset$ and $c^{\prime}=c$.) Recall that $I_{q}^{j}$ is the set of items that determine $t_{q}^{j}$. Notice that $I_{q}^{j} \cap\left(B \cup B^{\prime}\right)=\emptyset$, because all the items in $B \cup B^{\prime}$ arrived before time $j$, and all the items in $I_{q}^{j}$ arrive at time $j$ or later. Let $b$ denote the number of items in $I_{q}^{j}$ that are removed when we evict the colors $c, c^{\prime}$. Let $I^{\prime}$ denote the set of remaining items from $I_{q}^{j}$. Notice that $b>0$ only if the color of $I_{q}^{j}$ is either $c$ or $c^{\prime}$. When we evict the colors of $B$, $B^{\prime}$, we reach $t_{q}^{j}+b-1$. As the items in $I^{\prime}$ can be removed starting from time $t_{q}^{j}+b$ and ending at time $t_{q}$, evicting the color of $I^{\prime}$ ends the phase.

Claim 3.3. For every $\delta_{1}, \delta_{2}>0$ such that $\delta_{2}>2 \delta_{1}$ there exists $\gamma=\gamma\left(\delta_{1}, \delta_{2}\right)$ such that applying the process in case 3 starting at time $j$ removes at least $t_{q}^{j}-j$ items that were in the buffer at time $j$.

Proof. Let $\Delta_{j}^{F}=\sum_{B \in \mathcal{B}^{j}} \sum_{i \in B} d_{i}^{j}$ be the uncharged portion of items that are removed by $x$, but the algorithm holds at time $j$. We start by showing that $\Delta_{j}^{F} \geq \frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}} \cdot \Delta_{j}$. Each block $B$ that the algorithm removed from the buffer before time $j$ contributes $\sum_{i \in B} w_{i}^{j}$ to $\Delta_{j}$ (and the sum of all those contributions is exactly $\Delta_{j}$ ). Whenever the algorithm removes a block $B$, it charges some of the volume of the items that remain in its buffer and paid for removing $B$. The total volume charged is at most $\delta_{2} \cdot|B|$. (This is trivially true when $B$ is not removed during a case 3 process and does not charge anything.) If all the items that are charged when $B$ is removed are not in the algorithm's buffer at time $j$, the same contribution of $\sum_{i \in B} w_{i}^{j}$ contributes to $\Delta_{j}^{F}$ as well. Now consider the case that the buffer does contain items that were charged for the removal of $B$. Let $j^{\prime}$ be the beginning of the case 3 process in which block $B=B_{s}^{j^{\prime}}$ was removed, and let $B_{p}^{j^{\prime}}$ be one of the blocks that were charged for removing $B$ and its items are still in the algorithm's buffer at time $j$. By the ordering of the blocks in $\mathcal{B}^{j^{\prime}}$, it must be that $t\left(f_{p}^{j^{\prime}}\right) \leq t\left(f_{s}^{j^{\prime}}\right)$. As we can't apply case $0, w_{f_{p}^{j^{\prime}}}^{j} \geq 1-\delta_{1}$. Therefore, the first item $f_{s}^{j^{\prime}}$ of block $B$ also has $w_{f_{s}^{j^{\prime}}}^{j} \geq 1-\delta_{1}$. As every item in block $B$ must have at least the same weight as the first item, we get that $\sum_{i \in B} w_{i}^{j} \geq\left(1-\delta_{1}\right) \cdot|B|$. Block $B$ only charges at most a volume of $\delta_{2} \cdot|B|$ of items that are still in the algorithm's buffer at time $j$. Putting $\sum_{i \in B} w_{i}^{j}=(1-\theta) \cdot|B|$, we get that $B$ contributes to $\Delta_{j}^{F}$ at least

$$
\frac{(1-\theta) \cdot|B|-\delta_{2} \cdot|B|}{(1-\theta) \cdot|B|} \geq \frac{1-\theta-\delta_{2}-\left(\delta_{1}-\theta\right)}{1-\theta-\left(\delta_{1}-\theta\right)}=\frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}}
$$

of the portion it contributes to $\Delta_{j}$.
Going back to the main argument, Let $s_{e(1)}>s_{e(2)}>\cdots>s_{e(\ell)}$ denote the indices of the blocks that we removed during the case 3 process $\left(\left\{s_{e(1)}, \ldots, s_{e(\ell)}\right\} \subseteq\left\{s_{1}, s_{2}, \ldots, s_{p_{j}}\right\}\right.$ ). For each $p \in\{1, \ldots, \ell\}$, by the definition of $r_{e(p)}^{\prime}, \sum_{u=r_{e(p)}^{\prime}}^{s_{e(p)}^{-1}} \sum_{i \in B_{u}^{j}} d_{i}^{j} \leq \delta_{2} \cdot\left|B_{s_{e(p)}}^{j}\right|$, and $\sum_{u=r_{e(p)}^{\prime}}^{s_{e(p)}} \hat{d}_{B_{u}^{j}}^{j} \geq \delta_{1}$. Consider the indices $s_{e(p)+1}, s_{e(p)+2}, \ldots, s_{e(p+1)-1}$. (Those are the indices defined in the process of blocks that weren't removed between removing block $B_{s_{e(p)}}^{j}$ and block $B_{s_{e(p+1)}}^{j}$.) We now show that for each $u \in[e(p)+1, e(p+1)-1]$, $\left|B_{s_{u}}^{j}\right|<\frac{2 \delta_{1}}{\delta_{2}} \cdot\left|B_{s_{u+1}}^{j}\right|$ (the same holds for $u \in[1, e(1)-1]$ and $u \in\left[e(\ell)+1, p_{j}-1\right]$ ):

$$
\begin{aligned}
\delta_{2} \cdot\left|B_{s_{u}}^{j}\right| & <\sum_{g=r_{u}-1}^{s_{u}-1} \sum_{i \in B_{g}^{j}} d_{i}^{j}=\sum_{g=r_{u}-1}^{s_{u}-1} \hat{d}_{B_{g}^{j}}^{j} \cdot\left|B_{g}^{j}\right| \leq \sum_{g=r_{u}-1}^{s_{u}-1} \hat{d}_{B_{g}^{j}}^{j} \cdot\left|B_{s_{u+1}}^{j}\right| \\
& =\left|B_{s_{u+1}}^{j}\right| \cdot\left(\hat{d}_{B_{r_{u}-1}^{j}}^{j}+\sum_{g=r_{u}}^{s_{u}-1} \hat{d}_{B_{g}^{j}}^{j}\right)<2 \delta_{1} \cdot\left|B_{s_{u+1}}^{j}\right| .
\end{aligned}
$$

The first inequality follows from the definition of $r_{u}$. The second inequality follows as $B_{s_{u}}^{j}$ is not removed by the process, therefore, $B_{s_{u+1}}^{j}$ is defined to be the maximal block in $\left\{B_{r_{u}-1}^{j}, B_{r_{u}}^{j}, \ldots, B_{s_{u}-1}^{j}\right\}$. The last inequality follows as $\sum_{u^{\prime}=r_{u}}^{s_{u}-1} \hat{d}_{B_{u^{\prime}}^{j}}^{j}<\delta_{1}$. Furthermore, as the algorithm did not continue to execute case 0 at time $j$,
$d_{i}^{j} \leq 1-w_{i}^{j}<\delta_{1}$ for every $i \in B_{r_{u}-1}^{j}$, and therefore the average over $i \in B_{r_{u}-1}^{j}$ of $d_{i}^{j}$ is also less than $\delta_{1}$. Thus we get $\left|B_{s_{u}}^{j}\right|<\left(\frac{2 \delta_{1}}{\delta_{2}}\right)^{e(p+1)-u} \cdot\left|B_{s_{e(p+1)}}^{j}\right|$. We can now bound the contribution to $\Delta_{j}^{F}$ of the blocks with indices in $\left[s_{e(p+1)}+1, s_{e(p)+1}\right]$ (blocks that weren't removed and weren't charged for removing any block) as follows:

$$
\begin{aligned}
\sum_{g=s_{e(p+1)}+1}^{s_{e(p)+1}} \sum_{i \in B_{g}^{j}} d_{i}^{j} & =\sum_{u=e(p)+1}^{e(p+1)-1} \sum_{g=s_{u+1}+1}^{s_{u}} \sum_{i \in B_{g}^{j}} d_{i}^{j} \leq \sum_{u=e(p)+1}^{e(p+1)-1}\left(\delta_{2}+\delta_{1}\right) \cdot\left|B_{s_{u}}^{j}\right| \\
& \leq\left(\delta_{2}+\delta_{1}\right) \cdot \sum_{u=e(p)+1}^{e(p+1)-1}\left(\frac{2 \delta_{1}}{\delta_{2}}\right)^{e(p+1)-u} \cdot\left|B_{s_{e(p+1)}}^{j}\right| \\
& \leq\left(\delta_{2}+\delta_{1}\right) \cdot\left|B_{s_{e(p+1)}}^{j}\right| \cdot \frac{2 \delta_{1}}{\delta_{2}-2 \delta_{1}}=\frac{2 \delta_{1}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot\left|B_{s_{e(p+1)}}^{j}\right|,
\end{aligned}
$$

Adding the contributions of block $B_{s_{e(p+1)}}^{j}$ and the blocks that were charged for its removal, we get:

$$
\sum_{g=r_{e(p+1)}^{\prime}}^{s_{e(p)+1}} \sum_{i \in B_{g}^{j}} d_{i}^{j} \leq\left(\delta_{2}+\delta_{1}+\frac{2 \delta_{1}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}}\right) \cdot\left|B_{s_{e(p+1)}}^{j}\right|=\frac{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot\left|B_{s_{e(p+1)}}^{j}\right|
$$

For the same reason,

$$
\sum_{g=r_{e(1)}^{\prime}}^{s_{1}=m(j)} \sum_{i \in B_{g}^{j}} d_{i}^{j} \leq \frac{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot\left|B_{s_{e(1)}}^{j}\right|
$$

and

$$
\sum_{g=r_{p_{j}}=1}^{s_{e(\ell)+1}} \sum_{i \in B_{g}^{j}} d_{i}^{j} \leq \frac{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot\left|B_{s_{p_{j}}}^{j}\right|
$$

Therefore, if $p_{j}>e(\ell)$ then

$$
\begin{align*}
\Delta_{j}^{F} & =\sum_{g=1}^{m(j)} \sum_{i \in B_{g}^{j}} d_{i}^{j}=\sum_{g=r_{p_{j}}=1}^{s_{e(\ell)+1}} \sum_{i \in B_{g}^{j}} d_{i}^{j}+\sum_{p=1}^{\ell-1} \sum_{g=r_{e(p+1)}^{\prime}}^{s_{e(p)+1}} \sum_{i \in B_{g}^{j}} d_{i}^{j}+\sum_{g=r_{e(1)}^{\prime}}^{s_{1}=m(j)} \sum_{i \in B_{g}^{j}} d_{i}^{j} \\
& \leq \frac{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot\left(\left|B_{s_{p_{j}}}^{j}\right|+\sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right|\right) . \tag{6}
\end{align*}
$$

Thus,

$$
\begin{align*}
\left|B_{s_{p_{j}}}^{j}\right|+\sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right| & \geq \frac{\delta_{2}-2 \delta_{1}}{\delta_{2}\left(\delta_{2}+\delta_{1}\right)} \cdot \Delta_{j}^{F} \geq \frac{\delta_{2}-2 \delta_{1}}{\delta_{2}\left(\delta_{2}+\delta_{1}\right)} \cdot \frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}} \cdot \Delta_{j} \\
& \geq \frac{\delta_{2}-2 \delta_{1}}{\delta_{2}\left(\delta_{2}+\delta_{1}\right)} \cdot \frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}} \cdot \frac{1}{\gamma} \cdot\left(t_{q}^{j}-j\right) . \tag{7}
\end{align*}
$$

Choosing $\gamma=\frac{\delta_{2}-2 \delta_{1}}{\delta_{2}\left(\delta_{2}+\delta_{1}\right)} \cdot \frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}}$ we get that $\left|B_{S_{p_{j}}}^{j}\right|+\sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right|$ is at least $t_{q}^{j}-j$.
To conclude, by the definition of the process we removed from the buffer at least $\sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right|$ items from the items that were in the buffer at time $j$. Notice that if $p_{j}=e(\ell)$ we get that, similar to Equation (6), $\Delta_{j}^{F} \leq$
$\frac{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}{\delta_{2}-2 \delta_{1}} \cdot \sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right|$, and for the same reason as in Equation $[7], \sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right|>t_{q}^{j}-j$. Otherwise, it must be that the last block $B_{s_{p_{j}}}^{j}$ considered in the process was not removed. Thus, removing the largest block $B \in \mathcal{B}^{j}$ that remained will remove at least $\left|B_{s_{p_{j}}}^{j}\right|$ items, and overall at least $\left|B_{s_{p_{j}}}^{j}\right|+\sum_{p=1}^{\ell}\left|B_{s_{e(p)}}^{j}\right| \geq t_{q}^{j}-j$ items.
Claim 3.4. Executing case 3 ends a phase.
Proof. Consider phase $q$ where we execute case 3 at time $j$. Let $j^{\prime}$ be the time we execute the last step of case 3 , and let $I$ be the set of items from $I_{q}^{j}$ that were removed before time $j^{\prime}$. By Claim 3.3, at least $t_{q}^{j}-j$ removed items were in the buffer at time $j$, therefore at least $t_{q}^{j}-j+|I|$ items were removed overall. Thus, $j^{\prime} \geq t_{q}^{j}+|I|$. We can now evict the color of $I_{q}^{j}$ and reach $t_{q}$, as there are at least $\left|I_{q}^{j}\right|-|I|=t_{q}-t_{q}^{j}+1-|I| \geq t_{q}-j^{\prime}+1$ items of this color that can be removed from the buffer consecutively starting at time $j^{\prime}$.

Let $\phi=\frac{1+\sqrt{5}}{2} \approx 1.618$ denote the golden ratio.
Claim 3.5. Assuming that $\gamma>\frac{1+\phi}{1-\delta_{3}}$, if we've reached case 4 then in the repeated execution of the procedure we execute one of the cases $1-3$

Proof. Suppose that case 4 is executed at time $j$ in phase $q$. We may assume that $t_{q}>j$, otherwise the claim is vacuous. Let

$$
\mathcal{L}=\left\{\left(I, j^{\prime}\right): j^{\prime}<j \wedge j^{\prime}+|I| \geq t_{q} \wedge \lambda_{I, j^{\prime}}>0\right\}
$$

denote the set of monochromatic sequences in the packing $\lambda$ that are matched to an interval containing the entire interval $\left[j, t_{q}\right]$. (See Figure 2 in the appendix.) Notice that by the definition of $t_{q}$ and the fact that $t_{q-1}<j<t_{q}$,

$$
\begin{equation*}
\Lambda=\sum_{\left(I, j^{\prime}\right) \in \mathcal{L}} \lambda_{I, j^{\prime}} \geq 1-\delta_{3} . \tag{8}
\end{equation*}
$$

By the definition of $t_{q}^{j}$, for every $\left(I, j^{\prime}\right) \in \mathcal{L}$, none of the items in $M_{I, j^{\prime}}^{-1}\left(\left[j, t_{q}^{j}-1\right]\right)$ (the items matched by $M_{I, j^{\prime}}$ to the interval $\left[j, t_{q}^{j}-1\right]$ ) arrive at time $j$ or later. As we've reached case 4 , we may conclude that $\max \{|B|: B \in$ $\left.\mathcal{B}^{j}\right\}<t_{q}^{j}-j$ (otherwise case 2 would apply) and $\Delta_{j}<\frac{1}{\gamma} \cdot\left(t_{q}^{j}-j\right)$ (otherwise case 3 would apply). In particular, consider $\left(I, j^{\prime}\right) \in \mathcal{L}$. Let $t_{I, j^{\prime}}$ denote the minimum time $t^{\prime}$ for which $M_{I, j^{\prime}}^{-1}\left(t^{\prime}\right)$ is in the algorithm's buffer. If no such time exists, set $t_{I, j^{\prime}}=t_{q}^{j}$. The items in $M_{I, j^{\prime}}^{-1}\left(\left[j, t_{I, j^{\prime}}-1\right]\right)$ are no longer in the algorithm's buffer at time $j$. Therefore, $\sum_{\left(I, j^{\prime}\right) \in \mathcal{L}} \lambda_{I, j^{\prime}}\left(t_{I, j^{\prime}}-j\right) \leq \Delta_{j}$. Using Equation (8), we conclude that $E\left[t_{I, j^{\prime}}-j\right]<\frac{1}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right)$, where the expectation is taken over $\left(I, j^{\prime}\right) \in \mathcal{L}$ with probability distribution $\operatorname{Pr}\left[\left(I, j^{\prime}\right)\right]=\frac{\lambda_{I, j^{\prime}}}{\Lambda}$. (See Figure 2 in the appendix.) In particular,

$$
\begin{equation*}
\min \left\{t_{I, j^{\prime}}-j:\left(I, j^{\prime}\right) \in \mathcal{L}\right\}<\frac{1}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right), \tag{9}
\end{equation*}
$$

and by Markov's inequality

$$
\begin{equation*}
\operatorname{Pr}\left[t_{I, j^{\prime}}-j<\frac{\phi}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right)\right] \geq 1-\frac{1}{\phi}=\frac{\phi-1}{\phi}=\frac{1}{1+\phi} . \tag{10}
\end{equation*}
$$

Let

$$
\left(I_{\min }, j_{\min }^{\prime}\right)=\operatorname{argmin}\left\{t_{I, j^{\prime}}-j:\left(I, j^{\prime}\right) \in \mathcal{L}\right\} .
$$

Let $B \in \mathcal{B}^{j}$ denote the block containing items from $I_{\text {min }}$ in the algorithm's buffer. Notice that by Equation (9),

$$
|B|>\left(1-\frac{1}{\gamma\left(1-\delta_{3}\right)}\right) \cdot\left(t_{q}^{j}-j\right) .
$$

As we've reached case 4 (and therefore case 2 does not apply), for all other blocks $B^{\prime} \in \mathcal{B}^{j}$,

$$
\left|B^{\prime}\right|<t_{q}^{j}-j-|B| \leq \frac{1}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right)<\left(1-\frac{1}{\gamma\left(1-\delta_{3}\right)}\right) \cdot\left(t_{q}^{j}-j\right)
$$

where the last inequality uses $\gamma>\frac{1+\phi}{1-\delta_{3}}>\frac{2}{1-\delta_{3}}$. Thus, the case 4 step at $j$ must evict the color of $B$. As case 1 did not apply, evicting the color of $B$ does not reach $t_{q}$. Denote

$$
\mathcal{L}^{\prime}=\left\{\left(I, j^{\prime}\right) \in \mathcal{L}: t_{I, j^{\prime}}-j<\frac{\phi}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right)\right\}
$$

For every $\left(I, j^{\prime}\right) \in \mathcal{L}^{\prime}$ it must hold that the portion of $I$ in the algorithm's buffer is $B$. This is because for all other colors $t_{I, j^{\prime}}-j>\left(1-\frac{1}{\gamma\left(1-\delta_{3}\right)}\right)\left(t_{q}^{j}-j\right)$, and as $\gamma>\frac{1+\phi}{1-\delta_{3}}$, we get $t_{I, j^{\prime}}-j>\frac{\phi}{\gamma\left(1-\delta_{3}\right)}\left(t_{q}^{j}-j\right)$. So consider the situation after the step at $j$, where we remove $B$ and possibly additional items of $B$ 's color, and we reach $t^{\prime}<t_{q}$. Consider the set $A$ of the last $t_{q}^{j}-j-|B|$ items that the algorithm removed so far. For all $i \in A$ and for all $\left(I, j^{\prime}\right) \in \mathcal{L}^{\prime}, M_{I, j^{\prime}}(i)>t^{\prime}$. Therefore, by Equation $10, w_{i}^{t^{\prime}+1} \geq \frac{1}{1+\phi}\left(1-\delta_{3}\right)>\frac{1}{\gamma}$. On the other hand, $\bar{w}_{i}^{t^{\prime}+1}=0$. Let $t^{\prime \prime}$ be the point matched by $M_{I_{\min }, j_{\min }^{\prime}}$ to the first item $i^{\prime}$ of $I_{\min }$ that wasn't yet encountered. Notice that for all $\left(I, j^{\prime}\right) \in \mathcal{L}^{\prime}, M_{I, j^{\prime}}\left(i^{\prime}\right) \geq t^{\prime \prime}$. Clearly, $t^{\prime \prime} \geq t_{q}^{t^{\prime}+1}$. Therefore,

$$
\Delta_{t^{\prime}+1} \geq \sum_{i \in A} w_{i}^{t^{\prime}+1}>\frac{1}{\gamma} \cdot\left(t^{\prime \prime}-t^{\prime}-1\right) \geq \frac{1}{\gamma} \cdot\left(t_{q}^{t^{\prime}+1}-t^{\prime}-1\right)
$$

When we execute the procedure again, we first execute case 0. Assuming that we haven't reached $t_{q}$, the following holds. Each removed item moves our current position $t^{\prime}$ by 1 , and we may lose $\frac{1}{\gamma}$ in our estimate of $\Delta_{t^{\prime}+1}$ for each increment of $t^{\prime}$. Each removed item with the color of $B$ moves the target $t^{\prime \prime}$ by 1 , but in those steps we do not lose $\frac{1}{\gamma}$ in our estimate of $\Delta_{t^{\prime}+1}$. (Notice that if we do not reach $t_{q}$, then we haven't yet encountered this item in any of the sequences in $\mathcal{L}^{\prime}$ and the reason for its removal must be other sequences.) Thus, with respect to the new $t^{\prime}$, we still have that $\Delta_{t^{\prime}+1} \geq \frac{1}{\gamma} \cdot\left(t_{q}^{t^{\prime}+1}-t^{\prime}-1\right)$, so if cases 1 and 2 do not apply, then case 3 applies.

We now analyze the charging scheme that is used in case 3. We say that a block $B_{u}^{j} \in \mathcal{B}^{j}$ that is charged at time $j$ pays $\hat{d}_{B_{u}^{j}}^{j}$ (towards evicting the color of $B_{s_{p}}^{j}$ for which $r_{p}^{\prime} \leq u<s_{p}$ ). Denote by $\widehat{\mathcal{B}}^{j}$ the set of blocks that are charged at time $j$.

Lemma 3.6. $\sum_{j} \sum_{B \in \widehat{\mathcal{B}}^{j}} \hat{d}_{B}^{j} \leq 2 \cdot z(x)$.
Proof. Fix $j$ and consider a block $B \in \widehat{\mathcal{B}}^{j}$ which pays $\hat{d}_{B}^{j}$ at time $j$. Notice that

$$
\hat{d}_{B}^{j}=\frac{1}{|B|} \cdot \sum_{i \in B} d_{i}^{j} \leq \max \left\{d_{i}^{j}: i \in B\right\}
$$

Let $i_{B}=\operatorname{argmax}\left\{d_{i}^{j}: i \in B\right\}$. Then, $d_{i_{B}}^{j}$ is simply the sum of $\lambda_{I, j^{\prime}}$ over $\left(I, j^{\prime}\right)$ such that $M_{I, j^{\prime}}\left(i_{B}\right) \in\left(\tau_{i_{B}, j}, j\right]$. So we can think of such $\left(I, j^{\prime}\right)$ as contributing $\lambda_{I, j^{\prime}}$ towards the payment of $\hat{d}_{B}^{j}$. If $j^{\prime}+|I| \leq j$, then $\left(I, j^{\prime}\right)$ will never be "asked" to contribute again. However, if $j^{\prime}+|I|>j$, then $I$ contains items that are not charged at time $j$, and such items may appear in a future block $B^{\prime}$ that pays for removing some block in a future phase. We argue that it must be the case that $j^{\prime}+|I|<t_{q}$. To prove that, assume for contradiction that this is not the case. Notice that at time $j$ the item $i_{B}$ is still in the algorithm's buffer, and $M_{I, j^{\prime}}\left(i_{B}\right) \leq j$. So if the algorithm evicts $c\left(i_{B}\right)$, it must reach $t_{q}$. This contradicts the assumption that at time $j$ the algorithm executes case 3, as case 1 applies. Therefore, if $\left(I, j^{\prime}\right)$ contributes again in a future phase at some time $j^{\prime \prime}$, we have that $j^{\prime}+|I|<t_{q}<j^{\prime \prime}$, so $\left(I, j^{\prime}\right)$ will never contribute a third time.

We are now ready to prove our main result.
Proof of Theorem 2.4 We choose $\delta_{1}, \delta_{2}, \delta_{3} \in(0,1)$ such that $\delta_{2}>2 \delta_{1}$ and $\gamma=\frac{1-\delta_{1}-\delta_{2}}{1-\delta_{1}} \cdot \frac{\delta_{2}-2 \delta_{1}}{\delta_{2}\left(\delta_{2}+\delta_{1}\right)}>\frac{1+\phi}{1-\delta_{3}}$. (For example, we can choose $\delta_{1}=\frac{1}{40}, \delta_{2}=\frac{1}{10}$, and $\delta_{3}=\frac{1}{5}$.)

The number of phases is at most $\left\lceil z(x) / \delta_{3}\right\rceil$. In a phase, cases $1-4$ are executed at most once. The total number of steps due to case 1, case 2, case 4, and the last two steps of case 3 is at most 4 . (The worst case is when the algorithm executes case 4 and then case 2 ). Therefore, the total cost of those steps is at most $\frac{4}{\delta_{3}} \cdot z(x)+4$.

Now consider case 0. When a color $c$ is evicted at time $j$, this is because there's a block $B$ in the algorithm's buffer with $i \in B$ such that $w_{i}^{j} \leq 1-\delta_{1}$. In particular, $w_{f_{B}^{j}}^{j} \leq 1-\delta_{1}$. We remove $f_{B}^{j}$ at time $j$, so for every monochromatic sequence $\left(I, j^{\prime}\right)$ with $M_{I, j^{\prime}}\left(f_{B}^{j}\right) \leq j$, when this step is over we've reached $j^{\prime}+|I|$. Those are the sequences that pay for the drop of at least $\delta_{1}$ by time $j$ in the weight of $f_{B}^{j}$. As we've reached past them, we will never count them again for another step of case 0 . Thus, the total number of such steps is at most $\frac{1}{\delta_{1}} \cdot z(x)$.

The remaining steps are color evictions due to the charging scheme of case 3. Notice that whenever a block $B_{s_{p}} \in \mathcal{B}^{j}$ is removed by this case, we have that $\sum_{u=r_{p}^{\prime}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j} \geq \delta_{1}$. Therefore, the number of such steps is at most $\frac{1}{\delta_{1}} \cdot \sum_{j} \sum_{p \in I_{e}^{j}} \sum_{u=r_{p}^{\prime}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j}$, where $I_{e}^{j}$ denotes the set of indices of blocks whose removal created a charge at time $j$. By Lemma 3.6, $\frac{1}{\delta_{1}} \cdot \sum_{j} \sum_{p \in I_{e}^{j}} \sum_{u=r_{p}^{\prime}}^{s_{p}} \hat{d}_{B_{u}^{j}}^{j} \leq \frac{2}{\delta_{1}} \cdot z(x)$. The total cost of all the cases is $z(\bar{x}) \leq\left(\frac{3}{\delta_{1}}+\frac{4}{\delta_{3}}\right) \cdot z(x)+4$. As $z(x) \geq|C| \geq 1$, we get that the approximation guarantee $\alpha$ satisfies $\alpha \leq \frac{3}{\delta_{1}}+\frac{4}{\delta_{3}}+4$.

## 4 Concluding Remarks

Our methods can be adapted easily to handle some additional constraints, such as incurring a color change cost whenever we've accummulated too many time steps without a color change (a constraint relevant to [10]). Numerical estimates of the best constant that the above analysis gives indicate that it is below 135 , taking $\delta_{1} \approx 0.02763$, $\delta_{2} \approx 0.11416$, and $\delta_{3} \approx 0.18481$. We did not attempt to optimize our analysis, however, it is unlikely that our methods can be pushed to yield a very small constant (such as 2). Substantially improving the approximation guarantee for RBM is an interesting open problem. Also, adapting our methods to deal with more general cost measures appears to be a non-trivial task. In a variant of RBM called the (uniform) $k$-client problem [4], instead of a buffer there are $k$ input sequences. At each time step, the next item from one of the sequences is chosen and moved to the output sequence. (So the choice of which item to remove affects the order of the combined input sequence.) The goal is the same as RBM: to minimize the number of color changes in the output sequence. Adapting our methods to deal with this setting seems to be another fascinating problem.

## References

[1] A. Aboud. Correlation clustering with penalties and approximating the reordering buffer management problem. Master's thesis, Computer Science Department, The Technion - Israel Institute of Technology, January 2008.
[2] A. Adamaszek, A. Czumaj, M. Englert, and H. Räcke. Almost tight bounds for reordering buffer management. In Proc. of the 43rd Ann. ACM Symp. on Theory of Computing, pages 607-616, June 2011.
[3] S. Albers. New results on web caching with request reordering. In Proc. of the 16th ACM Symp. on Parallel Algorithms and Architectures, pages 84-92, 2004.
[4] H. Alborzi, E. Torng, P. Uthaisombut, and S. Wagner. The k-client problem. J. Algorithms, 41(2):115-173, 2001.
[5] Y. Asahiro, K. Kawahara, and E. Miyano. NP-hardness of the sorting buffer problem on the uniform metric. Unpublished, 2010.
[6] N. Avigdor-Elgrabli and Y. Rabani. An improved competitive algorithm for reordering buffer management. In Proc. of the 21st Ann. ACM-SIAM Symp. on Discrete Algorithms, pages 13-21, January 2010.
[7] R. Bar-Yehuda and J. Laserson. Exploiting locality: approximating sorting buffers. J. of Discrete Algorithms, 5(4):729-738, 2007.
[8] D. Blandford and G. Blelloch. Index compression through document reordering. In Data Compression Conference, pages 342-351, 2002.
[9] H.-L. Chan, N. Megow, R. van Stee, and R. Sitters. The sorting buffer problem is NP-hard. CoRR, abs/1009.4355, 2010.
[10] V-D. Cung, A. Nguyen, Y. Khacheni, C. Artigues, C.M. Li, and B. Penz. Société française de Recherche Opérationnelle et Aide à la Décision (ROADEF) Challenge 2005. http://challenge.roadef.org/2005/en/
[11] M. Englert, H. Räcke, and M. Westermann. Reordering buffers for general metric spaces. In Proc. of the 39th Ann. ACM Symp. on Theory of Computing, pages 556-564, 2007.
[12] M. Englert and M. Westermann. Reordering buffer management for non-uniform cost models. In Proc. of the 32nd Ann. Int'l Colloq. on Algorithms, Langauages, and Programming, pages 627-638, 2005.
[13] I. Gamzu and D. Segev. Improved online algorithms for the sorting buffer problem. In Proc. of the 24th Ann. Int'l Symp. on Theoretical Aspects of Computer Science, pages 658-669, 2007.
[14] K. Gutenschwager, S. Spiekermann, and S. Vos. A sequential ordering problem in automotive paint shops. Int'l J. of Production Research, 42(9):1865-1878, 2004.
[15] R. Khandekar and V. Pandit. Online sorting buffers on line. In Proc. of the 23rd Ann. Int'l Symp. on Theoretical Aspects of Computer Science, pages 584-595, 2006.
[16] J. Kohrt and K. Pruhs. Constant approximation algorithm for sorting buffers. In Proc. of the 6th Latin American Symp. on Theoretical Informatics, pages 193-202, Buenos Aires, Argentina, 2004.
[17] J. Krokowski, H. Räcke, C. Sohler, and M. Westermann. Reducing state changes with a pipeline buffer. In Proc. of the 9th Int'l Workshop on Vision, Modeling and Visualization, page 217, 2004.
[18] J. Li and S.M. Meerkov. Production Systems Engineering. Springer, 2009.
[19] H. Räcke, C. Sohler, and M. Westermann. Online scheduling for sorting buffers. In Proc. of the 10th Ann. European Symp. on Algorithms, pages 820-832, 2002.
[20] A. Silberschatz, P. Galvin, and G. Gagne. Operating System Concepts, 8th edition. J. Wiley, 2009.

## Appendix: Figures



Figure 1: The black area indicates the $d_{i}^{j}$-s, the dark grey area indicates the remaining portion of the $x_{i, j}$-s that was previously charged, and the light grey area indicates the $w_{i}^{j}$-s.


Figure 2: The strips of varying tones indicate the packing of MSMs in the fractional solution, and the outlined rectangles indicate the $t_{I, j^{\prime}}-\mathrm{s}$.


[^0]:    ${ }^{8}$ Computer Science Department, Technion-Israel Institute of Technology, Haifa 32000, Israel. Email: noaelg@cs.technion.ac.il.
    ${ }^{\top}$ The Rachel and Selim Benin School of Computer Science and Engineering and the Center of Excellence on Algorithms, The Hebrew University of Jerusalem, Jerusalem 91904, Israel. Email: yrabani@cs.huji.ac.il. Research supported by ISF grants 1109-07 and 856-11 and by BSF grant 2008059.

