# Near Linear Time Approximation Schemes for Uncapacitated and Capacitated b-Matching Problems in Nonbipartite Graphs* 

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#### Abstract

We present the first near optimal approximation schemes for the maximum weighted (uncapacitated or capacitated) $b$-matching problems for non-bipartite graphs that run in time (near) linear in the number of edges. For any $\delta>3 / \sqrt{n}$ the algorithm produces a $(1-\delta)$ approximation in $O\left(m \operatorname{poly}\left(\delta^{-1}, \log n\right)\right)$ time. We provide fractional solutions for the standard linear programming formulations for these problems and subsequently also provide (near) linear time approximation schemes for rounding the fractional solutions. Through these problems as a vehicle, we also present several ideas in the context of solving linear programs approximately using fast primal-dual algorithms. First, even though the dual of these problems have exponentially many variables and an efficient exact computation of dual weights is infeasible, we show that we can efficiently compute and use a sparse approximation of the dual weights using a combination of (i) adding perturbation to the constraints of the polytope and (ii) amplification followed by thresholding of the dual weights. Second, we show that approximation algorithms can be used to reduce the width of the formulation, and faster convergence.


## 1 Introduction

The $b$-matching problem is a fundamental problem with a rich history in combinatorial optimization, see [29, Chapters 31-33]. In this paper we focus on finding near optimal approximation schemes for finding fractional as well as integral solutions for maximum $b$-matching problems in non-bipartite graphs. The algorithms produce a $(1-O(\delta))$ approximations and run in time $O((m+n) \cdot \operatorname{poly}(\log n, 1 / \delta))$ time for $\delta \geq 3 / \sqrt{n}$.

Definition 1. [29, Chapter 31] In the $\mathbf{b}-$ matching problem we are given a weighted (possibly nonbipartite) graph $G=\left(V, E,\left\{w_{i j}\right\},\left\{b_{i}\right\}\right)$ where $w_{i j}$ is the weight of edge $(i, j)$ and $b_{i}$ is the capacity of the vertex $i$. Let $|V|=n$ and $|E|=m$. We assume $b_{i}$ are integers in $[1$, poly $n]$. We can select an edge $(i, j)$ with multiplicity $y_{i j}$ such that $\sum_{j:(i, j) \in E} y_{i j} \leq b_{i}$ for all vertices $i$ and the goal is to maximize $\sum_{(i, j) \in E} w_{i j} y_{i j}$. Let $B=\sum_{i} b_{i}$, and note $B \geq n$.
Definition 2. [29, Chapters 32 \&33] In the Capacitated b-matching problem we have an additional restriction that the multiplicity of an edge $(i, j) \in E$ is at most $c_{i j}$ where $c_{i j}$ are also given in the input (also assumed to be an integer in $[0$, poly $n]$ ). Observe that we can assume $c_{i j} \leq \min \left\{b_{i}, b_{j}\right\}$ without loss of generality. A problem with $c_{i j}=1$ for all $(i, j) \in E$ is also referred to as an "unit capacity" or "simple" b-matching problem in the literature.

[^0]Müller-Hannemann and Schwartz [26] provide an excellent survey of different algorithms for variants of $b$-matching. Approaches that solve regular matching do not extend to $b$-matchings without significant loss of efficiency. We revisit some of the reasons shortly. In the interest of space we summarize the main results for the $b$-matching problem briefly. Gabow [14] gave an $O(n m \log n)$ algorithm for the unweighted $\left(w_{i j}=1\right)$ capacitated problem. For $c_{i j}=1$ this reduces to $O(\min \{\sqrt{B} m, n m \log n\})$. For the weighted uncapacitated case Anstee [4] gave an $O\left(n^{2} m\right)$ algorithm; an $\tilde{O}\left(m^{2}\right)$ algorithm is in [14]. Letchford et al. [23], building on Padberg and Rao [27], gave an $O\left(n^{2} m \log \left(n^{2} / m\right)\right)$ time algorithm for the decision version of the weighted, uncapacitated/capacitated problem. In summary the best exact algorithms to date for the $b$-matching problem in general graphs are super-linear (see [29, Chapter 31]) in the size of the input.

It is known that solving the bipartite relaxation for the weighted $b$-matching problem within a $(1-\delta)$ approximation (for any $\delta>0$ ) will always produce a ( $\frac{2}{3}-\delta$ )-approximation algorithm for general non-bipartite graphs [12, 13]. This approximation is also tight (consider all $b_{i}=1, w_{i j}=1$ for a triangle graph) - no approach which only uses bipartite relaxations will breach the $\frac{2}{3}$ barrier. Distributed algorithms with $O(1)$ or weaker approximation guarantees have been discussed by Koufogiannakis and Young [22]. Mestre [25] provided a $\left(\frac{2}{3}-\delta\right)$ approximation algorithm running in $O\left(m\left(\max _{i} b_{i}\right) \log \frac{1}{\delta}\right)$ time for weighted unit capacity $b$-matching [25]. However a constant factor approximation does not seem to be a natural stopping point.

Given the recent growth in data sets and sizes of the graphs defining instances of matching it is natural to consider approximation algorithms that trade off the quality of the solution versus running time. Typically these algorithms provide an $f$-approximation, that is, for any instance we return a feasible solution whose value is at least $f$ times the value of the true optimum for that instance (maximum version). In particular efficient algorithms which are ( $1-\delta$ )-approximation schemes (for any absolute constant $\delta>0$, independent of $n$ ) and faster than computing the optimum solution are useful in this context. It would be preferable that the running time depended polynomially on $1 / \delta$ (instead of exponential dependence) - even though $\delta$ is assumed constant. It is possible that each vertex has $b_{i}=\sqrt{n}$ and a linear dependence on $B$ is not a near linear time algorithm. This paper provides the first near linear time approximation scheme for $b$-matching.

### 1.1 Existing Approaches and Challenges

We begin with the natural question about similarity and differences vis-a-vis weighted matching, which correspond to $b_{i}=1$ for all vertices $i$. Efficient approximation schemes exist for maximum weighted matching, even for the non-bipartite case, see [9, 10] and references therein. All of these algorithms maintain a feasible matching and repeatedly use augmentation paths - paths between two unmatched vertices such that the alternate edges are matched. In the non-bipartite case, if the two endpoints are the same vertex then this path is known as a "blossom". An efficient search for good augmentation paths, in the weighted case, requires contraction of blossoms. However this approach does not extend to non-bipartite $b$-matching for the case $b_{i}>1$. The augmentation structures needed for $b$-matching are not just blossoms but also blossoms with forests that are attached to the blossom (often known as petals/arms), see the discussion in [26]. Searching over this space of odd cycles with attached forests is significantly more difficult and inefficient. In the language of linear programming (which we discuss in more detail shortly), augmentation paths preserve primal feasibility for the matching problem. In our approach we explicitly maintain a primal infeasible solution (by violating the capacities) except at the last step.

It is known that if we copy each node $b_{i}$ times then the $b$-matching problems reduce to maximum weighted matching. As an example the pairs of edges $(u, v)$ and $(v, w)$ where the vertex capacities are $3,2,3$ as shown, correspond to 8 vertices and 12 edges.


The size of the graph increases significantly under such a transformation - consider a star graph where the central node has $b_{i}=n$ and the leaf nodes have $b_{i}=1$ - replication of that central node will make the number of edges $n^{2}$. If we are seeking near linear running times then transformations such as copying do not help since the number of edges and vertices can increase by polynomial factors. This blowup was known since [14], judicious use of this approach has been used to achieve superlinear time (in $n$ ) optimal algorithms that also depend on $B$, for example as in [15]. However near linear time algorithms have remained elusive.

Linear Programming Formulations. Consider the following definition and linear programming formulation LP1 for the uncapacitated $b$-matching problem.

Definition 3 (Odd Sets and Small Odd Sets). Given a graph $G=(V, E)$, with $|V|=n$ and $|E|=m$, and non-negative integer $b_{i}$ for each $i \in V$, for each $U \subseteq V$ let $\|U\|_{b}=\sum_{i \in U} b_{i}$. Define $\mathcal{O}=\left\{U \mid\|U\|_{b}\right.$ is odd and $\geq 3, U$ has more than one vertex $\}$. Let $\mathcal{O}_{\delta}=\left\{U \mid U \in \mathcal{O} ;\|U\|_{b} \leq 1 / \delta\right\}$.

$$
\begin{array}{rlrl}
\beta^{*} & =\boxed{\mathrm{LP} 1}(\mathbf{b})=\max \sum_{(i, j) \in E} w_{i j} y_{i j} & \\
& \sum_{j:(i, j) \in E} y_{i j} \leq b_{i} & \forall i \in V  \tag{LP1}\\
& \sum_{\substack{(i, j) \in E: i, j \in U}} y_{i j} \leq\left\lfloor\|U\|_{b} / 2\right\rfloor & & \forall U \in \mathcal{O} \\
y_{i j} \geq 0
\end{array} \quad \forall(i, j) \in E .
$$

The constraints of LP1 represent the "b-matching polytope"; any vector in this polytope can be expressed as a convex combination of integral b-matching solutions, see [29, Chapter 31].

The constraints in LP1 correspond to the vertices and odd sets. The variable $y_{i j}$ (which is the same as $y_{j i}$ ) corresponds to the fractional relaxation of the "multiplicity" of the edge $(i, j)$ in the uncapacitated $b$-matching. It is known that the formulation LP1 has an integral optimum solution when $b_{i}$ are integers. The formulation has $m$ variables and $2^{\Omega(n)}$ constraints - but can be solved in polynomial time since the oracle for computing the maximum violated constraint can be implemented in polynomial time using standard techniques [23]. That approach is the "minimum odd-cut" approach of Padberg and Rao [27]. If we only retain the constraints for odd sets $U \in \mathcal{O}_{\delta}$ then a fractional solution of the modified system, when multiplied by $(1-\delta)$, satisfies LP1. That relaxed formulation, still has $n^{1 / \delta}$ constraints which is exponential in $1 / \delta$. Note that an approximate solution of the dual does not immediately provide us a solution for the prima

It may be tempting to postulate that applying existing multiplicative weight algorithms such as [24, 28, 18] and many others (see the surveys [11, 5]) can help provide us approximate solutions to LP1 efficiently. However that is not the case due to several reasons. First, the existing algorithms have to maintain weights for each of the $n^{1 / \delta}$ constraints. Second, even if we are provided an

[^1]approximately feasible fractional solution, no efficient algorithm exists that easily computes the maximum violation of the constraints in LP1. Moreover it is nontrivial to verify that we have already achieved an approximately feasible solution. The only known algorithms for computing the maximum violation (for just the odd-sets) still correspond to the minimum odd-cut problem. Those solutions are at least cubic (see [23]).

Capacitated $b$-Matching The situation is more dire in presence of edge capacities. The capacitated $b$-matching problem has two known solution approaches. In the first one [29, Theorem 32.2 , page 564], the matching polytope is defined by where the set constraints are for every subset $U$ and every subset $F$ of the cut defined by $U$.

$$
\begin{array}{ll}
\sum_{\substack{j:(i, j) \in E \\
y_{i j} \leq c_{i j}}} y_{i j} \leq b_{i} & \forall i \in V \\
\sum_{(i, j) \in E: i, j \in U} y_{i j}+\sum_{(i, j) \in F} y_{i j} \leq\left\lfloor\frac{1}{2}\left(\|U\|_{b}+\sum_{(i, j) \in F} c_{i j}\right)\right\rfloor \\
y_{i j} \geq 0 & \forall(i, j) \in E  \tag{LP2}\\
\forall U \subseteq V, F \subseteq\{(i, j) \mid i \in U, j \notin U\} \\
\text { and }\|U\|_{b}+\sum_{(i, j) \in F} c_{i j} \text { is odd } \\
\forall(i, j) \in E
\end{array}
$$

Expressing the dual of LP2 is already nontrivial, let alone any combinatorial manipulation. The second approach corresponds to compressed representations introduced in [14, see also [29, Theorem 32.4,page 567]. It corresponds to subdividing each edge $e=(i, j)$ to introduce two new vertices $p_{e i}$ and $p_{e j}$ and creating three edges, where $b_{p_{e i}}=c_{i j}$ as shown in the example below. There are no capacities on edges but we are constrained to always saturate the newly created vertices $p_{e i}, p_{e j}$ for every edge $(i, j)$, i.e.,

$$
y_{i p_{e i}}+y_{p_{e i} p_{e j}}=c_{i j} \quad \text { and } \quad y_{p_{e j} j}+y_{p_{e i} p_{e j}}=c_{i j}
$$



Observe that the equality rules out simply scaling the vector $y$ by a constant smaller than 1 . The all-zero vector $\mathbf{0}$ is not even in the polytope! Even though the polytope is convex, the lack of closure under affine transformations makes it unwieldy for most known techniques that produce fast approximate solutions. The transformation creates unusual difficulties for approaches that are not based on linear programming as well, see [21. New ideas are required to address these issues and the development of such is the goal of this paper.

### 1.2 Contributions

The paper combines several novel structural properties of the $b$-matching polytope with novel modifications of the multiplicative weights method, and uses approximation algorithms to efficiently solve the subproblems produced by the said multiplicative weights method. All three of these facets function in tandem, and the overall technical theme of the solution are independently of interest.

Main Results We assume that the edges in the graph $G=(V, E)$ are presented as a read only list $\left\langle\ldots,\left(i, j, w_{i j}\right), \ldots\right\rangle$ in arbitrary order where $w_{i j}$ is the weight of the edge $(i, j)$. The space complexity will be measured in words and we assume that the integers in the input are bounded from above by poly $n$ to avoid bit-complexity issues. We prove the following theorems about $b$-matching.

Theorem 1 (Fractional b-matching). Given any non-bipartite graph, for any $\frac{3}{\sqrt{n}}<\delta \leq 1 / 16$, we find a ( $1-O(\delta)$ )-approximate (to LP1) fractional weighted b-matching using additional "work" space (space excluding the read-only input) $O\left(n\right.$ poly $\left.\left(\delta^{-1}, \ln n\right)\right)$ and making $T=O\left(\delta^{-4}(\ln (1 / \delta)) \ln n\right)$ passes over the list of edges. The running tim $\rrbracket^{2}$ is $O\left(m T+n \operatorname{poly}\left(\delta^{-1}, \ln n\right)\right)$.

Theorem 2 (Integral b-matching). Given a fractional b-matching y for a non-bipartite graph which satisfies the constraints in the standard LP formulation and has weight $W_{0}$, we find an integral b-matching of weight at least $(1-2 \delta) W_{0}$ in $O\left(m^{\prime} \delta^{-3} \ln (1 / \delta)\right)$ time and $O\left(m^{\prime} / \delta^{2}\right)$ space where $m^{\prime}=\left|\left\{(i, j) \mid y_{i j}>0\right\}\right|$.

The computation for the capacitated $b$-matching problem maintains the invariant that edge capacities are never violated at any stage of the algorithm. This yields a new approximation version of the capacitated matching problem where we exceed the vertex capacities but do not exceed the edge capacities at all and (almost) preserve the objective function. We prove:

Theorem 3 (Fractional, Capacitated). Given any weighted non-bipartite graph, for any $\frac{3}{\sqrt{n}}<$ $\delta \leq 1 / 16$, we find a $(1-O(\delta))$-approximate fractional capacitated b-matching using $O(m R / \delta+$ $\left.\min \{B, m\} \operatorname{poly}\left(\delta^{-1}, \ln n\right)\right)$ time, $O\left(\min \{m, B\} \operatorname{poly}\left(\delta^{-1}, \ln n\right)\right)$ additional "work" space with $R=$ $O\left(\delta^{-4}\left(\ln ^{2}(1 / \delta)\right) \ln n\right)$ passes over the list of edges where $B=\sum_{i} b_{i}$. The algorithm returns a solution $\left\{\hat{y}_{i j}\right\}$ such that the subgraph $\hat{E}=\left\{(i, j) \mid(i, j) \in E, \hat{y}_{i j}>0\right\}$ satisfies $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq 16 R \beta^{*, c}$ where $\beta^{*, c}$ is the weight of the integral maximum capacitated $b$-matching.

The restriction on $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j}$ is explicitly used in the next theorem.
Theorem 4 (Integral, Capacitated). Given a feasible fractional solution y to the linear program referred in Theorem 3 for a non-bipartite graph such that the optimum solution is at most $\beta^{*, c}$ and $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq 16 R \beta^{*, c}$ where $\hat{E}=\left\{(i, j) \mid y_{i j}>0\right\}$, we find an integral b-matching of weight at least $(1-\delta) \sum_{(i, j)} w_{i j} y_{i j}-\delta \beta^{*, c}$ in $O\left(m^{\prime} R \delta^{-3} \ln (R / \delta)\right)$ time and $O\left(m^{\prime} / \delta^{2}\right)$ space where $m^{\prime}=|\hat{E}|$ is the number of nontrivial edges (as defined by the linear program) in the fractional solution. As a consequence we have a $(1-O(\delta))$-approximate integral solution.

Technical Themes To prove the Theorems 14 this paper makes novel contributions towards the structure of $b$-matching polytope as well as techniques for speeding up multiplicative weights methods.

Multiplicative Weights Methods. We show that we can use existing constant factor approximation algorithms for $b$-matching to produce a $(1-\delta)$-approximate solution. The approximation factor surfaces in the speed of convergence of the multiplicative weights method used but the final solution produced is a $(1-\delta)$ approximation. This provides fairly straightforward proofs for near linear time $(1-\delta)$ approximation schemes for bipartite graphs using standard multiplicative weights methods. While the results for bipartite case in this paper do not completely dominate existing results (e.g., [1]), they serve as a warmup for non-bipartite graphs. Many of the pieces which are demonstrated with relatively less complexity in the bipartite case (initial solutions, Lagrangians, etc.) are also re-used in the non-bipartite case.

We then use specific structural properties of the $b$-matching polytope (and perturbations, described shortly) to show that the non-bipartite $b$-matching problem can be solved via a sequence of weighted bipartite $b$-matching problems. The overall approach can be viewed as dual thresholding where we only focus on the large weights in the multiplicative weights method (which are candidate

[^2]dual variables) and ignore the remainder. If we modify (perturb) the $b$-matching polytope then the number of constraints with large weights is small. However the choice of these constraints vary from iteration to iteration - and our algorithm differs from the application of standard multiplicative weights techniques in this aspect. Naturally, this requires a proof that the modified approach converges. This is shown in Section 3 and is used to prove Theorem 1 for uncapacitated $b$-matching. The framework extends to capacities helping prove Theorem 3.

Polytope and Perturbations. We investigate the laminarity of the sets corresponding to the unsatisfied constraints in LP1 at the neighborhood of any infeasible primal. A collection of sets $L$ is a laminar family if for any two sets $U, U^{\prime} \in L, U \cap U^{\prime}$ is either $U, U^{\prime}$ or $\emptyset$. We show that if we modify the polytope by introducing a small perturbation, then the constraints corresponding to the small odd sets that are "almost maximally violated" define a laminar family. Since a laminar family has $O(n)$ sets, this provides the small subset of constraints to the modified multiplicative weights method (note that the total number of constraints is $\Omega\left(n^{1 / \delta}\right)$ ). In that sense this approach generalizes the minimum odd-cut approach.

Many algorithms using the minimum odd-cut approach rely on the following fact: the sets corresponding to the nonzero variables of the optimum dual solution of LP1 define a laminar family (see Giles and Pulleyblank [16], Cook [7, Cunningham and Marsh [8], and also Schrijver [29]). However all these techniques rely on the exact optimality of the pair of primal and dual solutions. In fact, such relationships do not exist for arbitrary candidate primal or dual solutions. It is surprising that the maximally violated constraints of the perturbed polytope shows this property. This is shown in Theorem 5 .

Theorem 5. For a graph $G$ with $n$ vertices and any non-negative edge weights $\hat{\mathbf{y}}$ suppose that we are given $\hat{\mathbf{y}}$ satisfying $\hat{y}_{i i}=0$ for all $i$ and $\sum_{j:(i, j) \in E} \hat{y}_{i j} \leq b_{i}$ for all $i$. Define a perturbation of $b_{i}, b_{U}=\left\lfloor\|U\|_{b} / 2\right\rfloor$ as $\tilde{b}_{i}=(1-4 \delta) b_{i}$ and $\tilde{b}_{U}=\left\lfloor\|U\|_{b} / 2\right\rfloor-\frac{\delta^{2}\|U\|_{b}^{2}}{4}$. Let $\hat{\lambda}_{U}=\left(\sum_{(i, j) \in E: i, j \in U} \hat{y}_{i j}\right) / \tilde{b}_{U}$ and $\hat{\lambda}=\max _{U \in \mathcal{O}_{\delta}} \hat{\lambda}_{U}$. If $\delta \leq \frac{1}{16}$ and $\hat{\lambda} \geq 1+3 \delta$, the set $L_{1}=\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{3} ; U \in \mathcal{O}_{\delta}\right\}$ forms a laminar family. Moreover for any $x \geq 2$ we have $\left|\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{x} ; U \in \mathcal{O}_{\delta}\right\}\right| \leq n^{3}+(n / \delta)^{1+\delta^{(x-3) / 2}}$.

In other words, if we were provided an infeasible (with respect to the perturbed polytope) primal solution $\left\{\hat{y}_{i j}\right\}$ then the constraints that are almost as violated as the maximum violated constraint of the perturbed polytope (in ratio of LHS to RHS) correspond to a laminar family. Intuitively, $\sum_{(i, j): i, j \in U} \hat{y}_{i j}=\hat{\lambda}_{U} \tilde{b}_{U}$ and for a fixed $\hat{\lambda}_{U}$, if we could ignore the floor and ceil functions, the right hand side is a concave function of $\|U\|_{b}$. As a result if two such $U_{1}, U_{2}$ intersect at a non-singleton odd set $U_{3} \neq U_{1}, U_{2}$ (the union $U_{4} \neq U_{1}, U_{2}$ is also an odd set) then $\max \left\{\hat{\lambda}_{U_{3}}, \hat{\lambda}_{U_{4}}\right\}$ will exceed $\min \left\{\hat{\lambda}_{U_{1}}, \hat{\lambda}_{U_{2}}\right\}$ by $\delta^{3}$. Of course, the floor and ceil functions, singleton sets cannot be ignored and more details are required, and Theorem 5 is proved in Section 4 However Theorem 5 , does not give us an algorithm. But the laminarity of the "almost maximally violated" constraints allow us to design an algorithm that finds these constraints (small odd sets) without the knowledge of the maximum violation. Since the laminarity guarantees that at most $O(n)$ such sets can be found, we can compute the maximum violated constraint more efficiently than the existing algorithms. This is formalized in Theorem 6

Theorem 6. For a graph $G$ with $n$ vertices and $\left\{\hat{y}_{i j}\right\}$ and the definitions of $\left\{\hat{\lambda}_{U}\right\}$ exactly as in the statement of Theorem 5 and $\delta \in\left(0, \frac{1}{16}\right]$, if $\hat{\lambda} \geq 1+3 \delta$ we can find the set $L_{2}=\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\frac{\delta^{3}}{10} ; U \in\right.$ $\left.\mathcal{O}_{\delta}\right\}$ in $O\left(m^{\prime}+n\right.$ poly $\left.\left\{\delta^{-1}, \log n\right\}\right)$ time using $O\left(n \delta^{-5}\right)$ space where $m^{\prime}=\left|\left\{(i, j) \mid \hat{y}_{i j}>0\right\}\right|$.

The proof of Theorem 6 combines the insights of the minimum odd-cut approach [27] along with the fact that $L_{2} \subseteq L_{1}$ is a laminar family as proved in Theorem 5 .

Roadmap. Theorems 5 and 6 are proved in Sections 4 and 5 respectively. We discuss the bipartite $b$-matching problem in Section 2 to serve as a warmup as well as to develop pieces (such as initial solutions, etc.) that would be required to solve the non-bipartite problem. In particular we make the connection between fast constant factor approximation algorithms and the convergence of the multiplicative weights method. Section 3 which discusses perturbations and thresholding and provides a modified multiplicative weights framework which is likely of interest in other problems where we have a large number of constraints. Theorem 1 follows immediately from the application of the framework and the bipartite relaxation discussed in Section 2. Section 6 proves Theorem 2. Section 7 discusses capacitated $b$-matching.

## 2 Approximations to speed up Multiplicative Weights Method

The goal of this section is to illustrate how multiplicative weights method can be used in the context of $b$-matching. We focus on the bipartite case in this section. The results obtained in this section do not always dominate the best known results for bipartite $b$-matching, see for example [1]. But the main purpose of this section is to provide a simple illustration of the ideas that are required for the non-bipartite case. We use existing multiplicative weights methods (see [5] for a comprehensive review of these) and show how they apply to the bipartite $b$-matching case without any modification. At the end of the section we discuss why existing techniques will not work directly in the non-bipartite case. However the different parts of the overall solution for bipartite graphs will be reused in the non-bipartite context.

From the perspective of algorithms for matching problems, the multiplicative weights method provides an approach different from that of augmentation paths. Instead of maintaining a feasible solution and increasing the value of that feasible solution using augmenting paths, we maintain an infeasible solution of a certain value and reduce the infeasibility. The overall algorithm is iterative, at each point we identify parts of the graph where our solution is infeasible - we construct a new partial solution that reduces the effect of these parts and consider a convex combination of the old and new solutions. However the new partial solution, in itself can be significantly unhelpful for the original problem! In particular the new solution will either be a matching that allows vertex $i$ to have up to $6 b_{i}$ edges instead of the at most $b_{i}$ as specified in the problem, or have $1 / 6$ the desired objective value (which depends both on the weight of the maximum matching as well constraints in the framework). Of course, this deviation also allows us to find the solution efficiently. However, even though each individual solution is not helpful, the average of the solutions is a $(1-O(\delta))$ approximation for the original problem for a small $\delta>3 / \sqrt{n}$.

### 2.1 Existing Multiplicative Weights Methods

Let $\mathbf{A}^{\prime}$ be a non-negative $m \times N$ matrix, and suppose $\mathbf{b}^{\prime} \geq \mathbf{0}$. Suppose that we seek to solve $\mathbf{A}^{\prime} \mathbf{y} \leq \mathbf{b}^{\prime}, \mathbf{y} \in \mathcal{P}^{\prime}$ where $\mathcal{P}^{\prime} \subseteq\{\mathbf{y} \mid \mathbf{y} \geq \mathbf{0}\}$ is convex. The literature on Multiplicative Weights method shows that it suffices to repeatedly average $\mathbf{y}(t)$ corresponding to iteration $t$. In iteration $t$, given a non-negative vector $\mathbf{u}(t)$, the methods ask for an oracle to supply $\mathbf{y}(t)$ such that $\mathbf{u}(t)^{T} \mathbf{A}^{\prime} \mathbf{y}(t) \leq$ $(1+O(\delta)) \mathbf{u}(t)^{T} \mathbf{b}^{\prime}, \mathbf{y}(t) \in \mathcal{P}^{\prime}$ and $\mathbf{A}^{\prime} \mathbf{y}(t) \leq \rho \mathbf{b}^{\prime}$ where $\rho>1$ is the width parameter. The $\mathbf{u}(t)$ are referred to as the Multiplicative Weights, because the vector $\mathbf{u}(t)$ in the expression $\mathbf{u}(t)^{T} \mathbf{A}^{\prime} \mathbf{y}(t)$ implies an assignment weights to the rows of $\mathbf{A}^{\prime}$ which correspond to constraints. The multiplicative weights method states that as long as we have bounded solutions $\mathbf{A}^{\prime} \mathbf{y}(t) \leq \rho \mathbf{b}^{\prime}$, a (weighted) average $\mathbf{y}$ of $\mathbf{y}(t)$ satisfies $\mathbf{A}^{\prime} \mathbf{y} \leq(1+O(\delta)) \mathbf{b}^{\prime}$. We note that many variations of the multiplicative weights method exist but for the purposes of this section we focus on the version in [28]. In that version the average is a predetermined weighted average and the $j$-th entry of $\mathbf{u}(t)$ corresponds to a scaled
exponential of $\left(\mathbf{A}^{\prime} \mathbf{y}^{\prime}\right)_{j} / \mathbf{b}_{j}^{\prime}$ where $\mathbf{y}^{\prime}$ is the corresponding weighted average of $\mathbf{y}(0), \ldots, \mathbf{y}(t-1)$. Intuitively, if the $j$-th constraint is violated more, its weight would be large and the desired $\mathbf{y}(t)$ would prioritize satisfying the $j$-th constraint.

Theorem 7. [28] Starting from an initial solution $\mathbf{y}(0)$ such that $\mathbf{A}^{\prime} \mathbf{y}(0) \leq \rho \mathbf{b}^{\prime}$, after $O\left(\rho\left(\delta^{-2}+\right.\right.$ $\log \rho) \log N$ ) iterations we have a $\mathbf{y} \in \mathcal{P}^{\prime}$ that satisfies $\mathbf{A}^{\prime} \mathbf{y}^{\prime} \leq(1+\delta) \mathbf{b}^{\prime}$.

### 2.2 Boosting Constant Factor Approximations to $(1-\delta)$-approximations

We begin with Theorem 8 and consider its applications.
Theorem 8 (Proved in Section 2.3). Let $f_{1}, f_{2}>0, \mathbf{h} \geq \mathbf{0}$. Let $\widehat{\mathcal{Q}} \subseteq \widehat{\mathcal{P}} \subseteq\{\mathbf{y} \mid \mathbf{y} \geq \mathbf{0}\}$. Suppose $\widehat{\mathcal{P}}, \widehat{\mathcal{Q}}$ are convex and $\mathbf{0} \in \widehat{\mathcal{Q}}$. Suppose we have a subroutine that for any $\mathbf{z}$ (which can be negative) provides $a \mathbf{y} \in \widehat{\mathcal{P}}$ such that $\mathbf{z}^{T} \mathbf{y} \geq(1-\delta / 2) \max \left\{\mathbf{z}^{T} \mathbf{y}^{\prime} \mid \mathbf{y}^{\prime} \in \widehat{\mathcal{Q}}\right\}{ }^{3}$

1. If $\left\{\mathbf{y} \mid \mathbf{w}^{T} \mathbf{y} \geq f_{1}, \mathbf{h}^{T} \mathbf{y} \leq f_{2}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$ is non-empty then using $O\left(\ln \frac{1}{\delta}\right)$ invocations of the subroutine we can find $a \mathbf{y} \in \widehat{\mathcal{P}}$ such that $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) f_{1}$ and $\mathbf{h}^{T} \mathbf{y} \leq f_{2}$.
2. Suppose $\widehat{\mathbf{A}}, \widehat{\mathbf{b}}$ are non-negative and $\mathbf{b} \in R^{N}$, let $\widehat{\beta}=\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$. If $\left\{\mathbf{y} / \lambda_{0} \mid \mathbf{y} \in \widehat{\mathcal{P}}\right\} \subseteq\{\mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\}$ then we can compute $\mathbf{y}$ that satisfies $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta)^{2} \widehat{\beta}$,
$\widehat{\mathbf{A}} \mathbf{y} \leq(1+\delta) \widehat{\mathbf{b}}$ and $\mathbf{y} \in \widehat{\mathcal{P}}$ using $O\left(\lambda_{0}\left(\delta^{-2}+\delta^{-1} \log \lambda_{0}\right)(\log N)(\log 1 / \delta)\right)$ invocations of the subroutine.

Note that if $\left\{\lambda_{0} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}=\widehat{\mathcal{P}}$ for some $\lambda_{0} \geq 1$, then a $\left(1 / \lambda_{0}\right)$-approximate solution to $\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$ can be multiplied by $\lambda_{0}$ to achieve the subroutine mentioned above and therefore using $O\left(\left(\lambda_{0}\left(\delta^{-2}+\log \lambda_{0}\right) \log N+\delta^{-1} \log \lambda_{0}\right) \log (1 / \delta)\right)$ invocations we find a (fractional) $\mathbf{y}$ as described in (2).

Bipartite Uncapacitated $b$-matching. The problem is expressed by linear program LP3. Variable $y_{i j}$ corresponds to the fraction with which $(i, j) \in E$ is present in the solution.

$$
\begin{align*}
& \beta_{b}^{*}=\max \sum_{(i, j) \in E} w_{i j} y_{i j} \\
& \mathcal{Q}: \begin{cases}\sum_{\substack{j:(i, j) \in E \\
y_{i j} \geq 0}} y_{i j} \leq b_{i} & \forall i \in V \\
\forall(i, j) \in E\end{cases} \tag{LP3}
\end{align*}
$$

Observe that negative weight edges can simply be ignored by any approximation algorithm. While many constant factor approximation algorithms for uncapacitated $b$-matching exist, we use Theorem 9 which has no dependence on $B=\sum_{i} b_{i}$.

Theorem 9. [Proved in Section 2.4] For the bipartite uncapacitated b-matching problem we can provide a $1 / 6$ approximation in $O(m \log n)$ time and $O(n)$ space.

We now define $\mathcal{Q}$ as in LP3 and set $\widehat{\mathcal{Q}}=\mathcal{Q},\{\widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}\}=\mathcal{Q}$ and $\widehat{\mathcal{P}}=\{6 \mathbf{y} \mid \mathbf{y} \in \widehat{Q}\}$. We multiply the solution provided by Theorem 9 by a factor 6 and as a consequence of the final part of Theorem 8 we obtain a non-negative (fractional) solution $\left\{y_{i j}\right\}$ that satisfies $\sum_{j:(i, j) \in E} y_{i j} \leq(1+\delta) b_{i}$ for all $i \in V$ corresponding to $\widehat{\mathbf{A}} \mathbf{y} \leq(1+\delta) \widehat{\mathbf{b}}$. Dividing each $y_{i j}$ by $(1+\delta)$ provides us a $\left((1-\delta)^{2} /(1+\delta)\right)$ approximation to the optimum bipartite $b$-matching solution in time $O\left(m \delta^{-2}\left(\log ^{2} n\right)(\log 2 / \delta)\right)$.

[^3]Bipartite Capacitated $b$-matching. The problem is expressed as a linear program in LP4 where $c_{i j}$ are integer capacities on the edge $(i, j) \in E$. Without loss of generality $c_{i j} \leq \min \left\{b_{i}, b_{j}\right\}$.

$$
\begin{align*}
& \beta_{b}^{*, c}=\max \sum_{(i, j) \in E} w_{i j} y_{i j} \\
& \mathcal{Q}^{c}: \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq b_{i} & \forall i \in V \\
y_{i j} \leq c_{i j} & \forall(i, j) \in E \\
y_{i j} \geq 0 & \forall(i, j) \in E\end{cases} \tag{LP4}
\end{align*}
$$

Define $\mathcal{P}^{c}$ as:

$$
\mathcal{P}^{c}: \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq \lambda_{0} b_{i} / 2 & \forall i \in V  \tag{LP5}\\ y_{i j} \leq c_{i j} & \forall(i, j) \in E \\ y_{i j} \geq 0 & \forall(i, j) \in E\end{cases}
$$

Theorem 10. [Proved in Section 2.4] If $c_{i j} \leq \min \left\{b_{i}, b_{j}\right\}$ then given any weight vector $\mathbf{w}$, using Theorem 9 at most $k=O(\log 1 / \delta)$ times we can compute a solution $\mathbf{y}^{\dagger, c} \in \mathcal{P}^{c}$ with $\lambda_{0}=16 \ln \frac{2}{\delta}$ such that $\mathbf{w}^{T} \mathbf{y}^{\dagger, c} \geq 2 \beta_{b}^{*, c} / \lambda_{0}$. If $\hat{E}=\left\{(i, j) \mid y_{i j}^{\dagger, c}>0\right\}$ then $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq 8 k \beta_{b}^{*, c}$.

We cannot use an arbitrary algorithm in lieu of Theorem 10 - because we only relax a part of the constraints. The final property of Theorem 10 is used to guarantee that a fractional solution can be rounded in near linear time (Theorem 4). We define $\widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}$ to be $\left\{\sum_{j:(i, j) \in E} y_{i j} \leq b_{i}, \forall i \in V\right\}$ and let $\widehat{\mathcal{Q}}=\mathcal{Q}^{c}$ and $\widehat{\mathcal{P}}=\mathcal{P}^{c}$ for $\lambda_{0}=16 \ln 2 / \delta$. We apply Theorem 8 to get a solution which satisfies $\sum_{j:(i, j) \in E} y_{i j} \leq(1+\delta) b_{i}$ for all $i$ as well as $y_{i j} \leq c_{i j}$ for all $(i, j) \in E$. An appropriate scaling of the solution provides a $(1-O(\delta))$-approximation.

Theorem 11. We can compute a fractional solution which is a $(1-\delta)$ approximation to the optimum capacitated $b$-matching in $O\left(m \delta^{-2}\left(\log ^{2} n\right)\left(\log ^{2} 1 / \delta\right)\right)$ time in a bipartite graph.

### 2.3 Proof of Theorem 8

Theorem 8. Let $f_{1}, f_{2}>0, \mathbf{h} \geq \mathbf{0}$. Let $\widehat{\mathcal{Q}} \subseteq \widehat{\mathcal{P}} \subseteq\{\mathbf{y} \mid \mathbf{y} \geq \mathbf{0}\}$. Suppose $\widehat{\mathcal{P}}, \widehat{\mathcal{Q}}$ are convex and $\mathbf{0} \in \widehat{\mathcal{Q}}$. Suppose we have a subroutine that for any $\mathbf{z}$ (which can be negative) provides $a \mathbf{y} \in \widehat{\mathcal{P}}$ such that $\mathbf{z}^{T} \mathbf{y} \geq(1-\delta / 2) \max \left\{\mathbf{z}^{T} \mathbf{y}^{\prime} \mid \mathbf{y}^{\prime} \in \widehat{\mathcal{Q}}\right\}$.

1. If $\left\{\mathbf{y} \mid \mathbf{w}^{T} \mathbf{y} \geq f_{1}, \mathbf{h}^{T} \mathbf{y} \leq f_{2}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$ is non-empty then using $O\left(\ln \frac{1}{\delta}\right)$ invocations of the subroutine we can find $a \mathbf{y} \in \widehat{\mathcal{P}}$ such that $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) f_{1}$ and $\mathbf{h}^{T} \mathbf{y} \leq f_{2}$.
2. Suppose $\widehat{\mathbf{A}}, \widehat{\mathbf{b}}$ are non-negative and $\mathbf{b} \in R^{N}$, let $\widehat{\beta}=\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$. If $\left\{\mathbf{y} / \lambda_{0} \mid \mathbf{y} \in \widehat{\mathcal{P}}\right\} \subseteq\{\mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\}$ then we can compute $\mathbf{y}$ that satisfies $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta)^{2} \widehat{\beta}$, $\widehat{\mathbf{A}} \mathbf{y} \leq(1+\delta) \widehat{\mathbf{b}}$ and $\mathbf{y} \in \widehat{\mathcal{P}}$ using $O\left(\lambda_{0}\left(\delta^{-2}+\delta^{-1} \log \lambda_{0}\right)(\log N)(\log 1 / \delta)\right)$ invocations of the subroutine.

Note that if $\left\{\lambda_{0} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}=\widehat{\mathcal{P}}$ for some $\lambda_{0} \geq 1$, then a $\left(1 / \lambda_{0}\right)$-approximate solution to $\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$ can be multiplied by $\lambda_{0}$ to achieve the subroutine mentioned above and therefore using $O\left(\left(\lambda_{0}\left(\delta^{-2}+\log \lambda_{0}\right) \log N+\delta^{-1} \log \lambda_{0}\right) \log (1 / \delta)\right)$ invocations we find a (fractional) $\mathbf{y}$ as described in part (2).

Proof: Define $g(\varrho)=\max \left\{\left(\mathbf{w}^{T}-\varrho \mathbf{h}^{T}\right) \mathbf{y} \mid \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$. Since $\left\{\mathbf{y} \mid \mathbf{w}^{T} \mathbf{y} \geq f_{1}, \mathbf{h}^{T} \mathbf{y} \leq f_{2}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$ is non-empty, $g(\varrho)$ exists and is at least $f_{1}-\varrho f_{2}$. Let $\mathcal{L}(\mathbf{y}, \varrho)=\left(\mathbf{w}^{T}-\varrho \mathbf{h}^{T}\right) \mathbf{y}$ and let $\mathbf{y}^{\varrho}$ be the solution returned by the subroutine for $\mathbf{z}=\mathbf{w}^{T}-\varrho \mathbf{h}^{T}$.

For $\varrho=0$, the returned solution $\mathbf{y}^{0}$ satisfies $\mathcal{L}\left(\mathbf{y}^{0}, 0\right)=\left(\mathbf{w}^{T}-\varrho \mathbf{h}^{T}\right) \mathbf{y}^{0} \geq(1-\delta / 2) g(0)=$ $(1-\delta / 2)\left(f_{1}-\varrho f_{2}\right)$. This implies $\mathbf{w}^{T} \mathbf{y}^{0} \geq(1-\delta / 2) f_{1}$. If $\mathbf{y}^{0}$ also satisfies $\mathbf{h}^{T} \mathbf{y}^{0} \leq f_{2}$, then $\mathbf{y}^{0}$ is our desired solution for the first part of the theorem. We therefore consider the case $\mathbf{h}^{T} \mathbf{y}^{0}>f_{2}$.

Consider $\varrho=f_{1} / f_{2}$ and set $\mathbf{y}^{\varrho}=\mathbf{0}$. Note we do not run the subroutine. Note $\mathbf{y}^{\varrho} \in \widehat{\mathcal{P}}$ and $\mathbf{h}^{T} \mathbf{y}^{\varrho}=0 \leq f_{2}$ and $\mathcal{L}\left(\mathbf{y}^{\varrho}, \varrho\right)=\left(\mathbf{w}^{T}-\varrho \mathbf{h}^{T}\right) \mathbf{y}^{\varrho}=0 \geq(1-\delta / 2)\left(f_{1}-\frac{f_{1}}{f_{2}} f_{2}\right)=(1-\delta / 2)\left(f_{1}-\varrho f_{2}\right)$ Therefore over the endpoints of the interval $\varrho \in\left[0, f_{1} / f_{2}\right]=\left[\varrho^{-}, \varrho^{+}\right]$we have two solutions $\mathbf{y}^{\varrho^{-}}, \mathbf{y}^{\varrho^{+}}$ that satisfy
(1) $\mathcal{L}\left(\mathbf{y}^{\varrho^{-}}, \varrho^{-}\right) \geq(1-\delta / 2)\left(f_{1}-\varrho^{-} f_{2}\right), \mathbf{h}^{T} \mathbf{y}^{\varrho^{-}}>f_{2}$
(2) $\mathcal{L}\left(\mathbf{y}^{\varrho^{+}}, \varrho^{+}\right) \geq(1-\delta / 2)\left(f_{1}-\varrho^{+} f_{2}\right), \mathbf{h}^{T} \mathbf{y}^{\varrho^{+}} \leq f_{2}$

Now consider running the subroutine for $\varrho=\frac{1}{2}\left(\varrho^{-}+\varrho^{+}\right)$. Again based on the subroutine we know that we will obtain a solution $\mathbf{y}^{\varrho}$ which satisfies:

$$
\mathcal{L}\left(\mathbf{y}^{\varrho}, \varrho\right)=\left(\mathbf{w}^{T}-\varrho \mathbf{h}^{T}\right) \mathbf{y}^{\varrho} \geq(1-\delta / 2) g(\varrho) \geq(1-\delta / 2)\left(f_{1}-\varrho f_{2}\right)
$$

If $\mathbf{h}^{T} \mathbf{y}^{\varrho}>f_{2}$ then we focus on $\left[\varrho, \varrho^{+}\right]$. Otherwise we focus on $\left[\varrho^{-}, \varrho\right]$. Observe that we are maintaining the invariants (1) and (2). Now we use binary search to find $\varrho^{+}, \varrho^{-}$such that $0 \leq$ $\varrho^{+}-\varrho^{-} \leq \frac{\delta f_{1}}{2 f_{2}}$. This requires $O\left(\ln \frac{2}{\delta}\right)$ invocations of the subroutine. We take a linear combination $\mathbf{y}=a \mathbf{y}^{\varrho^{+}}+(1-a) \mathbf{y}^{\varrho^{-}}, a \in[0,1]$ such that $\mathbf{h}^{T} \mathbf{y}=f_{2}$. Since $\mathbf{y}^{\varrho+}, \mathbf{y}^{\varrho^{-}} \in \widehat{\mathcal{P}}$, their linear combination $\mathbf{y}$ is also in $\hat{\mathcal{P}}$. Note that

$$
\begin{aligned}
& a \mathcal{L}\left(\mathbf{y}^{\varrho^{+}}, \varrho^{+}\right)+(1-a) \mathcal{L}\left(\mathbf{y}^{\varrho^{-}}, \varrho^{-}\right) \geq(1-\delta / 2) f_{1}-(1-\delta / 2) \varrho^{-} f_{2}-a(1-\delta / 2)\left(\varrho^{+}-\varrho^{-}\right) f_{2} \\
& \geq(1-\delta) f_{1}-\varrho^{-} f_{2}
\end{aligned}
$$

because $a \leq 1, \varrho^{+}-\varrho^{-} \leq \frac{\delta f_{1}}{2 f_{2}}$ and $f_{2} \geq 0$. Thus

$$
\begin{aligned}
\mathbf{w}^{T} \mathbf{y} & =a \mathcal{L}\left(\mathbf{y}^{\varrho^{+}}, \varrho^{+}\right)+(1-a) \mathcal{L}\left(\mathbf{y}^{\varrho^{-}}, \varrho^{-}\right)+a \varrho^{+} \mathbf{h}^{T} \mathbf{y}^{\varrho^{+}}+(1-a) \varrho^{-} \mathbf{h}^{T} \mathbf{y}^{\varrho^{-}} \\
& \geq(1-\delta) f_{1}-\varrho^{-} f_{2}+a\left(\varrho^{+}-\varrho^{-}\right) \mathbf{h}^{T} \mathbf{y}^{\varrho^{+}}+\mathbf{h}^{T}\left(a \varrho^{-} \mathbf{y}^{\varrho^{+}}+(1-a) \varrho^{-} \mathbf{y}^{\varrho^{-}}\right) \\
& \geq(1-\delta) f_{1}-\varrho^{-} f_{2}+\mathbf{h}^{T}\left(a \varrho^{-} \mathbf{y}^{\varrho^{+}}+(1-a) \varrho^{-} \mathbf{y}^{\varrho^{-}}\right) \quad\left(\text { Using } \mathbf{h}^{T} \mathbf{y}^{\varrho^{+}} \geq 0 \text { and } \varrho^{+}-\varrho^{-} \geq 0\right) \\
& \geq(1-\delta) f_{1}-\varrho^{-} f_{2}+\mathbf{h}^{T} \varrho^{-} \mathbf{y} \quad\left(\text { Using } \mathbf{y}=a \mathbf{y}^{\varrho^{+}}+(1-a) \mathbf{y}^{\varrho^{-}}\right) \\
& \geq(1-\delta) f_{1}-\varrho^{-}\left(f_{2}-\mathbf{h}^{T} \mathbf{y}\right)=(1-\delta) f_{1} \quad \text { (Since } \mathbf{h}^{T} \mathbf{y}=f_{2} \text { by construction) }
\end{aligned}
$$

The first part of the theorem follows. Note that $\mathbf{h}^{T} \mathbf{y} \leq f_{2}$ from the case $\mathbf{h}^{T} \mathbf{y}^{0} \leq f_{2}$.
For the second part, observe that setting $\mathbf{z}=\mathbf{w}$ we get a $\mathbf{y}(0) \in \widehat{\mathcal{P}}$ such that $\mathbf{z}^{T} \mathbf{y}(0) \geq(1-\delta) \widehat{\beta}$ using the subroutine. Moreover $\mathbf{z}^{T} \mathbf{y}(0) \leq \lambda_{0} \widehat{\beta}$ since $\mathbf{y}(0) / \lambda_{0} \in\{\mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\}$. This provides an initial solution. Observe that the width is $\lambda_{0}$ by construction. We can now apply Theorem 7 . If $0<\beta \leq \widehat{\beta}$ we get a solution for $\mathbf{y} \in \widehat{\mathcal{P}}$ that satisfies $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) \beta$ and $\mathbf{u}(t)^{T} \widehat{\mathbf{A}} \mathbf{y} \leq \mathbf{u}(t)^{T} \widehat{\mathbf{b}}$ from the first part of the theorem setting $f_{1}=\beta, f_{2}=\mathbf{u}(t)^{T} \widehat{\mathbf{b}}$. If we fail to find a solution to the first part for some $\beta$ then we decrease $\beta$ by a factor of $(1-\delta)$. Observe that we would decrease $\beta$ at most $O\left(\delta^{-1} \log \lambda_{0}\right)$ times and eventually we would reach $(1-\delta) \widehat{\beta} \leq \beta \leq \widehat{\beta}$ since the initial $\beta=\mathbf{w}^{T} \mathbf{y}(0)$ is at most $\lambda_{0} \beta$. Note $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) \beta \geq\left(1-\delta^{2}\right) \widehat{\beta}$.

Observe that the iterations for larger $\beta$ remain valid for a smaller $\beta$. Therefore if we classify the iterations according to (a) decrease of $\beta$ because we did not find a solution for for the first part and
(b) invocations where we succeed in finding a solution for the first part. The number corresponding to (a) is at most $O\left(\delta^{-1} \log \lambda_{0}\right)$ (decreases of $\beta$ ) times $O((\log 1 / \delta)$ ), the multiplier due to the reduction. The number corresponding to (b) cannot be more than $O\left(\lambda_{0}\left(\delta^{-2}+\log \lambda_{0}\right) \log N\right)(\log (1 / \delta))$ because then we would have already gotten a better solution based on Theorem 7 - once again, because the $\mathbf{y}$ found for larger $\beta$ remain valid for a smaller $\beta$. The total number of invocations of the subroutine is $O\left(\left(\lambda_{0}\left(\delta^{-2}+\log \lambda_{0}\right) \log N+\delta^{-1} \log \lambda_{0}\right) \log (1 / \delta)\right)$. The second part of the theorem follows.

For the final remark, a $\lambda_{0}$ approximation implies that we have a feasible solution solution $\mathbf{y}^{\prime}$ satisfying $\widehat{\mathbf{A}} \mathbf{y}^{\prime} \leq \widehat{\mathbf{b}}, \mathbf{y}^{\prime} \in \mathcal{Q}$. The claim follows from the second part.

### 2.4 Proofs of Theorem 9 and 10

In this section we provide primal-dual approximation algorithms for both uncapacitated and capacitated $b$-matching. The capacities $b_{i}, c_{i j}$, for vertices and edges respectively are integral. Each edge $(i, j)$ has weight $w_{i j}$. In the uncapacitated case the edge constraints are not present; one can model that by setting $c_{i j}=\min \left\{b_{i}, b_{j}\right\}$ for every edge $(i, j)$. The formulation LP4 expresses a bipartite relaxation which omits non-bipartite constraints. Therefore $\beta_{b}^{*, c} \geq \beta^{*, c}$ (the maximum capacitated $b$-matching) as well as $\beta_{b}^{*, c} \geq \beta^{*}$ (the maximum uncapacitated $b$-matching, assuming $c_{i j}=\min \left\{b_{i}, b_{j}\right\}$ for every edge $\left.(i, j)\right)$. The system LP6 is the dual of LP4.

$$
\begin{array}{ll|l}
\beta_{b}^{*, c}=\max & \\
\sum_{(i, j) \in E} w_{i j} y_{i j} & & \\
\frac{1}{b_{i}} \sum_{j:(i, j) \in E} y_{i j} \leq 1 \quad \forall i & \boxed{\boxed{L P} 4} & \beta_{b}^{*, c}=\min \sum_{i} p_{i}+\sum_{(i, j) \in E} q_{i j} \\
\frac{1}{c_{i j}} y_{i j} \leq 1 & \forall(i, j) \in E & \\
\frac{p_{i}}{b_{i}}+\frac{p_{j}}{b_{j}}+\frac{q_{i j}}{c_{i j}} \geq w_{i j} \quad \forall(i, j) \in E  \tag{LP6}\\
p_{i}, q_{i j} \geq 0 \quad \forall i, \forall(i, j) \in E
\end{array}
$$

$$
y_{i j} \geq 0 \quad \forall(i, j) \in E
$$

Algorithm 1 satisfies the following invariants; and the next lemma is the core of the proof.
(I1) We maintain a feasible primal solution $\left\{y_{i j}\right\}$.
(I2) If we insert an edge into the solution $y_{i j}=c_{i j}$ (but some copies of this edge can be deleted later).
(I3) Once an edge is processed (ignored or inserted) we ensure that $\frac{p_{i}}{b_{i}}+\frac{p_{j}}{b_{j}}+\frac{q_{i j}}{c_{i j}} \geq w_{i j}$.
(I4) We ensure that $\left\{p_{i}, q_{i j}\right\}$ are non-decreasing and therefore the final $\left\{p_{i}, q_{i j}\right\}$ satisfies the constraints of LP6, and $\sum_{i} p_{i}+\sum_{(i, j)} q_{i j} \geq \beta_{b}^{*, c}$.
(I5) At the end of step 3(e), we have the invariant $p_{i} \geq 2 \sum_{j} w_{i j} y_{i j}$.
Lemma 12. Let $\Delta$ be the decrease in $\sum_{(i, j)} w_{i j} y_{i j}$ in Steps 3(c) and 3(d) due to the deletions before the edge $(i, j)$ is added in Step 3(e). $\Delta \leq w_{i j} c_{i j} / 2$.

```
Algorithm 1 A near linear time algorithm for capacitated \(b\)-matching
    1: We start with all \(p_{i}=0\). Initially the graph is empty and all \(y_{i j}=q_{i j}=0\). In the following \(y_{i j}=y_{j i}\),
        the variables are defined on the edges.
    2: Order the edges \(E\) according to an arbitrary ordering and consider the edges one by one.
    3: for each new edge \(e=(i, j)\) do
```

    (a) If \(\frac{p_{i}}{b_{i}}+\frac{p_{j}}{b_{j}} \geq w_{i j}\) then ignore the edge, otherwise:
    (b) We will be eventually inserting \(c_{i j}\) copies of the edge \((i, j)\). Recall for the uncapacitated case
        \(c_{i j}=\min \left\{b_{i}, b_{j}\right\}\).
    (c) Suppose that \(c_{i j}+\sum_{j} y_{i j}>b_{i}\). In that case we need to delete \(\left(\sum_{j^{\prime}} y_{i j^{\prime}}-b_{i}+c_{i j}\right)\) edges such that
        when we add the \(c_{i j}\) copies of \((i, j)\) the vertex constraint \(\sum_{j^{\prime}} y_{i j^{\prime}} \leq b_{i}\) will be satisfied. Therefore we
        delete \(x_{i}=\max \left\{0, \sum_{j^{\prime}} y_{i j^{\prime}}-b_{i}+c_{i j}\right\}\) edges incident to \(i\) - but we delete the edges with the lowest
        \(w_{i j^{\prime}}\) with \(y_{i j^{\prime}}>0\).
    (d) Likewise we delete the $x_{j}=\max \left\{0, \sum_{i^{\prime}} y_{i^{\prime} j}-b_{j}+c_{i j}\right\}$ edges incident to $j$, with the lowest $w_{i^{\prime} j}$ amongst $y_{i^{\prime} j}>0$.
(e) Set $y_{i j}=c_{i j}$, (if required) increase $p_{i}, p_{j}$ to be at least $2 \sum_{j} w_{i j} y_{i j}, 2 \sum_{i} w_{i j} y_{i j}$ respectively. Set $q_{i j}=w_{i j} c_{i j}$.
4: Output $\left\{(i, j) \mid y_{i j}>0\right\}$ and $\left\{p_{i}\right\},\left\{q_{i j}\right\}$.

Proof: Suppose we deleted edges at $i$ for Step 3(c) and $x_{i}>0$. Note that we retained the heaviest $b_{i}-c_{i j}$ edges and therefore the total retained edges have weight at least $\frac{b_{i}-c_{i j}}{\sum_{j^{\prime}} y_{i j^{\prime}}} \sum_{j^{\prime}} w_{i j^{\prime}} y_{i j^{\prime}}$ which is at least $\frac{b_{i}-c_{i j}}{b_{i}} \sum_{j^{\prime}} w_{i j^{\prime}} y_{i j^{\prime}}$ since $\sum_{j^{\prime}} y_{i j^{\prime}} \leq b_{i}$ because $\left\{y_{i j^{\prime}}\right\}$ are feasible. Thus the total weight deleted at $i$ is at most $\frac{c_{i j}}{b_{i}} \sum_{j^{\prime}} w_{i j^{\prime}} y_{i j^{\prime}}$. But since $2 \sum_{j^{\prime}} w_{i j^{\prime}} y_{i j^{\prime}} \leq p_{i}$ at Step 3(e) in the iteration before $(i, j)$ was considered, the total weight deleted at $i$ is at most $\frac{c_{i j} p_{i}}{2 b_{i}}$. Using the same reasoning at $j$, the the total weight deleted by $(i, j)$ at both $i, j$ is at most $\frac{c_{i j} p_{i}}{2 b_{i}}+\frac{c_{i j} p_{j}}{2 b_{j}}$ which is at most $c_{i j} w_{i j} / 2$ since we are past Step 3(a).

Note that we now immediately have a factor $1 / 10$ approximation for both capacitated and uncapacitated $b$-matching. This because the net direct increase to $\sum_{i} p_{i}+\sum_{(i, j)} q_{i j}$ due to inserting $(i, j)$ is at most $5 w_{i j} c_{i j}$. At each of the endpoints $i, j$ the increase is $2 w_{i j} c_{i j}$ and $q_{i j} \leq c_{i j} w_{i j}$. Combined with Lemma 12 we have a $1 / 10$ approximation because the total increase in $\sum_{i} p_{i}+$ $\sum_{(i, j)} q_{i j}$ due to $(i, j)$ is the direct increase from $(i, j)$ plus the increase due to all edges deleted by $(i, j)$ (and the edges which have been recursively deleted). But using Lemma 12 the total weight of all such recursively deleted edges is at most $w_{i j} c_{i j}$. Therefore $10 \sum_{(i, j)} w_{i j} y_{i j} \geq \sum_{i} p_{i}+\sum_{(i, j)} q_{i j} \geq$ $\beta_{b}^{*, c}$. For the remainder of the paper any absolute constant approximation suffices. However since the approximation factor relates to the speed of convergence, we provide a slightly better analysis, and space complexity.

Theorem 9. For the bipartite uncapacitated b-matching problem we can provide a $1 / 6$ approximation in $O(m \log n)$ time and $O(n)$ space.
Proof: We first observe that $q_{i j}=0$ for every edge $(i, j)$ which already improves the approximation to $1 / 8$. We then prove $6 \sum_{(i, j)} w_{i j} y_{i j} \geq \sum_{i} p_{i}+\sum_{(i, j)} q_{i j} \geq \beta_{b}^{*, c}$. In the uncapacitated this case $c_{i j}=\min \left\{b_{i}, b_{j}\right\}$ and the edge $(i, j)$ is inserted with $y_{i j}=c_{i j}$. Therefore both $p_{i}, p_{j} \geq 2 c_{i j} w_{i j}$ due to Step 3(e). Therefore at least one of $p_{i} / b_{i}, p_{j} / b_{j}$ is $2 w_{i j}$. Thus $\frac{p_{i}}{b_{i}}+\frac{p_{j}}{b_{j}} \geq w_{i j}$ which implies $q_{i j}=0$. This also means that at each insertion at least one vertex has exactly one edge (but possibly multiple copies of it) and therefore the total number of edges in the solution is $O(n)$. We now observe that the increase in Step $3(\mathrm{e})$ of $\sum_{i} p_{i}$ is at most $4 w_{i j} c_{i j}-2 \Delta$ (recall $\Delta$ is defined in

Lemma 12. Suppose that we maintained $6 \sum_{(i, j)} w_{i j} y_{i j} \geq \sum_{i} p_{i}$ before we considered the deletions in Steps 3(c) and 3(d). Then the left hand side increased by $6 w_{i j} c_{i j}-6 \Delta$ but

$$
6 w_{i j} c_{i j}-6 \Delta=\left(4 w_{i j} c_{i j}-2 \Delta\right)+\left(2 w_{i j} c_{i j}-4 \Delta\right)
$$

and $\Delta \leq w_{i j} c_{i j} / 2$. This implies that the increase in $6 \sum_{(i, j)} w_{i j} y_{i j}$ after Step $3(\mathrm{e})$ is more than the increase in $\sum_{i} p_{i}$. Therefore the invariant continues to hold and the theorem follows.

Theorem 13. We can solve the capacitated b-matching problem to an approximation factor $1 / 8$ in time $O(m \log n)$. If $E^{\prime}$ is the set of edges $(i, j)$ such that $y_{i j}>0$ at any point of time in the algorithm then $\sum_{(i, j) \in E^{\prime}} w_{i j} c_{i j} \leq 8 \beta_{b}^{*, c}$.

Proof: Unlike the proof of Theorem 9 we cannot assert $q_{i j}=0$. But observe that if we maintained $8 \sum_{(i, j)} w_{i j} y_{i j} \geq \sum_{i} p_{i}+\sum_{(i, j)} q_{i j}$, then the increase to the left hand side is $8 w_{i j} c_{i j}-8 \Delta$ (again following the definition of $\Delta$ from Lemma 12) and the increase to the right hand side is $4 w_{i j} c_{i j}-$ $2 \Delta+w_{i j} c_{i j}$ (the addition is due to $q_{i j}$ ). But

$$
8 w_{i j} c_{i j}-8 \Delta=4 w_{i j} c_{i j}-2 \Delta+w_{i j} c_{i j}+3\left(w_{i j} c_{i j}-2 \Delta\right) \geq 4 w_{i j} c_{i j}-2 \Delta+w_{i j} c_{i j}
$$

Therefore the invariant continues to hold after Step 3(e). For the second part, observe that $\sum_{(i, j) \in E^{\prime}} w_{i j} c_{i j}=\sum_{(i, j)} q_{i j}$ but $\sum_{(i, j)} q_{i j} \leq 8 \sum_{(i, j)} w_{i j} y_{i j}$ and $\left\{y_{i j}\right\}$ are feasible. Therefore the theorem follows.
We use Theorem 13 to prove Theorem 10 .
Theorem 10. Using Algorithm 1 at most $k \leq 8 \ln \frac{2}{\delta}$ times we get an integral solution that satisfies

$$
\begin{align*}
& \sum_{(i, j) \in E} w_{i j} y_{i j} \geq\left(1-\frac{\delta}{2}\right) \beta_{b}^{*, c} \\
& \sum_{j:(i, j) \in E} y_{i j} \leq\left(8 \ln \frac{2}{\delta}\right) b_{i} \quad \forall i  \tag{LP7}\\
& y_{i j} \leq c_{i j} \quad \forall(i, j) \in E \\
& y_{i j} \geq 0 \quad \forall(i, j) \in E
\end{align*}
$$

Moreover if $\hat{E}=\left\{(i, j) \in E \mid y_{i j}>0\right\}$ then $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq(8 k) \beta_{b}^{*, c}$.
Proof: We reuse the notation $\mathbf{y}(t)$ since we would be using an iterative algorithm. For any $b_{i}, c_{i j} \geq$ 0 we can find a solution $\mathbf{y}(1)$ such that $\sum_{(i, j)} w_{i j} y_{i j}(1)=\tau_{1} \geq \beta_{b}^{*, c} / 8$ using Algorithm 1 and Theorem 13

Define $\hat{\beta}(1)=\beta_{b}^{*, c}$. We now run an iterative procedure where we remove the edges $(i, j)$ corresponding to $y_{i j}(1)>0$ and decrease the corresponding capacities. The decrease in capacities corresponds to modifying LP4 by adding the constraint $y_{i j} \leq \max \left\{0, c_{i j}-y_{i j}(1)\right\}$. Let optimum solution of LP4 on this modified graph be denoted by $\hat{\beta}(2)$. We have

$$
\begin{equation*}
\hat{\beta}(1)-\tau_{1} \leq \hat{\beta}(2) \leq \hat{\beta}(1) \tag{1}
\end{equation*}
$$

Consider the optimum solution of LP4 on the unmodified graph. Let that solution be $\left\{y_{i j}^{*}\right\}$. Consider $y_{i j}^{\prime}=\max \left\{y_{i j}^{*}-y_{i j}(1), 0\right\}$. Then $\left\{y_{i j}^{\prime}\right\}$ is a feasible solution of the modified LP4 and $\sum_{i, j} w_{i j} y_{i j} \geq \hat{\beta}(1)-\tau_{1} . \hat{\beta}(2) \leq \hat{\beta}(1)$ follows from the fact that capacities are decreased and Equation (1) follows.

Now we obtain a solution $\mathbf{y}(2)$ such that $\sum_{(i, j)} w_{i j} y_{i j}(2)=\tau_{2} \geq \hat{\beta}(2) / 8$. We now repeat the process by modifying LP4 to $y_{i j} \leq \max \left\{0, c_{i j}-y_{i j}(1)-y_{i j}(2)\right\}$. Proceeding in this fashion we obtain solutions $\left\{y_{i j}(\ell)\right\}_{\ell=1}^{k}$ where $k \leq\left\lceil 8 \ln \frac{2}{\delta}\right\rceil$ or we have no further edges to pick. Observe, that by construction $\sum_{\ell=1}^{k} y_{i j}(\ell) \leq c_{i j}$ for all $(i, j)$ and therefore the union of these $k$ solutions satisfies $y_{i j} \leq c_{i j}$. Moreover for every $\ell$ we have $\sum_{j} y_{i j}(\ell) \leq b_{i}$ and therefore for the union of these $k$ solutions the vertex constraints hold as described in the statement of Theorem 10.

We now claim that $\sum_{\ell=1}^{k} \sum_{(i, j)} w_{i j} y_{i j}(\ell) \geq\left(1-\left(\frac{7}{8}\right)^{k}\right) \hat{\beta}(1)$ by induction on $k$. The base case follows from $\tau_{1} \geq \hat{\beta}(1) / 8$. In the inductive case, applying the hypothesis on $2, \ldots, k$ we get $\sum_{\ell=2}^{k} \sum_{(i, j)} w_{i j} y_{i j}(\ell) \geq\left(1-\left(\frac{7}{8}\right)^{k-1}\right) \hat{\beta}(2)$. Thus:
$\sum_{\ell=1}^{k} \sum_{(i, j)} w_{i j} y_{i j}(\ell) \geq \tau_{1}+\left(1-\left(\frac{7}{8}\right)^{k-1}\right) \hat{\beta}(2) \geq \tau_{1}+\left(1-\left(\frac{7}{8}\right)^{k-1}\right)\left(\hat{\beta}(1)-\tau_{1}\right)=\hat{\beta}(1)-\hat{\beta}(1)\left(\frac{7}{8}\right)^{k-1}+\tau_{1}\left(\frac{7}{8}\right)^{k-1}$
and the claim follows since $\tau_{1} \geq \hat{\beta}(1) / 8$. The first part of the theorem follows. For the second part, Theorem 13 was applied $k$ times and the result follows.

## 3 Perturbations, Thresholding, and Non-bipartite $b$-matching

We considered the bipartite case in Section 2. We provided an algorithm that produces an $(1-\delta)^{2}$ approximate solution for $\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \widehat{\mathbf{A}} \mathbf{y} \leq \widehat{\mathbf{b}}, \mathbf{y} \in \widehat{\mathcal{Q}}\right\}$, by repeatedly, for any $\mathbf{z}$ finding a solution $\mathbf{y} \in \widehat{\mathcal{P}}$ such that $\mathbf{z}^{T} \mathbf{y} \geq(1-\delta / 2) \max \left\{\mathbf{z}^{T} \mathbf{y}^{\prime} \mid \mathbf{y}^{\prime} \in \widehat{\mathcal{Q}}\right\}$ (we omit the connections between $\widehat{\mathcal{Q}}, \widehat{\mathcal{P}}$ for the moment). However that algorithm relied on the Theorem 7 which computes a multiplicative weight for each constraint/row of matrix $\widehat{\mathbf{A}}$. For the bipartite case, the number of constraints was $n$ for the uncapacitated case and $n+m$ for the capacitated case for a graph with $n$ vertices and $m$ edges. In this section we consider non-bipartite matching - the number of constraints are exponential. The number of constraints can be reduced to $n^{\Omega(1 / \delta)}$ to seek a $(1-\delta)$ approximation, but computing the multiplicative weights for all rows of the constraint matrix is infeasible for a near linear time algorithm. We now provide a framework that bypasses the computation of the weights for all rows in Section 3.1. We then apply the framework to uncapacitated $b$-matching in Section 3.2.

### 3.1 A Dual Thresholding Framework

Suppose that $\mathcal{Q} \subseteq \mathcal{P} \subseteq\{\mathbf{y} \mid \mathbf{y} \geq \mathbf{0}\}$. Suppose further that $\mathcal{P}, \mathcal{Q}$ are convex and $\mathbf{0} \in \widehat{\mathcal{Q}}$. The overall goal in this section to solve $\max \left\{\mathbf{w}^{T} \mathbf{y} \mid \mathbf{A y} \leq \mathbf{b}, \mathbf{y} \in \mathcal{Q}\right\}$, by repeatedly, for any $\mathbf{z}$ finding a solution $\mathbf{y} \in \mathcal{P}$ such that $\mathbf{z}^{T} \mathbf{y} \geq(1-\delta / 2) \max \left\{\mathbf{z}^{T} \mathbf{y}^{\prime} \mid \mathbf{y}^{\prime} \in \mathcal{Q}\right\}$. However, we would like to achieve the reduction by only evaluating the multiplicative weights for the constraints $\mathcal{L}$ which are close to the maximum violated constraint. Note that this set $\mathcal{L}$ would change every iteration. We achieve this by perturbing the constraints and focusing on $\mathbf{A y} \leq \tilde{\mathbf{b}}$.

We present the basic Algorithm 2. The proof of convergence is provided in Theorem 16 . The theorem follows from Lemma 15 which computes the rate of monotonic decrease of a potential function (Definition 4). Lemma 14 demonstrates how the ideas in Section 2 are used as critical pieces of Algorithm 2 .
The remainder of this section uses the notation introduced in Algorithm 2.
Definition 4. Extend $\mathbf{u}$ as $u_{\ell}=\exp \left(\alpha \lambda_{\ell}\right) / \tilde{b}_{\ell}$ for all $\ell$ in Line 11 of Algorithm 2, Define $\Psi=$ $\sum_{\ell} e^{\lambda_{\ell} \alpha}=\mathbf{u}^{T} \mathbf{b}$ which only depends on the current solution.

```
Algorithm 2 A Dual Thresholding Multiplicative Weights Algorithm. \(M \gg m \gg K \geq 1\).
    Let \(\mathcal{P}, \mathcal{Q}\) be convex with \(\mathcal{Q} \subseteq \mathcal{P} \subseteq\{\mathbf{y} \geq \mathbf{0}\}\). Let \(\mathbf{A}\) is nonnegative matrix of dimension \(M \times m\).
    Fix \(\delta \in\left(0, \frac{1}{16}\right]\). Let \(\lambda_{0}, K, f(\delta), \alpha\) be parameters. \(\lambda_{0} \geq 1, f(\delta)<\delta, \alpha \leq \frac{1}{f(\delta)} \ln \left(\frac{M \lambda_{0}}{\delta}\right)\).
    Find an initial solution \(\mathbf{y}_{0} \in \mathcal{P}\) with \(\mathbf{w}^{T} \mathbf{y}=\beta_{0}\) and \(\mathbf{A y}_{0} \leq \lambda_{0} \tilde{\mathbf{b}}\). Set \(\beta=\beta_{0}\).
    Let \(\epsilon=\frac{1}{8}\) (note \(\epsilon \geq \delta\) ) and \(t=0\).
    Start a superphase corresponding to \(\epsilon=\frac{1}{8}\). The algorithm proceeds in superphases corresponding to
    a fixed value of \(\epsilon\). We will be decreasing \(\epsilon\). The algorithm ends when \(\lambda \leq 1+8 \delta\). We will not assume \(\lambda\)
    decreases monotonically.
    while true do
        Define \(\lambda_{\ell}=(\mathbf{A y})_{\ell} / \tilde{b}_{\ell}\) and \(\lambda=\max _{\ell} \lambda_{\ell}\). Find \(\mathcal{L}=\left\{\ell \mid \lambda_{\ell} \geq \lambda-f(\delta)\right\}\), assert \(|\mathcal{L}| \leq K\).
        If \((\lambda \leq 1+8 \delta)\) output \(\mathbf{y}\) which satisfies \(\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) \beta\) and \(\mathbf{A y} \leq(1+8 \delta) \tilde{\mathbf{b}}\) and stop.
        If \(\lambda<1+8 \epsilon\) then declare the current superphase to be over.
        Repeatedly set \(\epsilon \leftarrow \max \{2 \epsilon / 3, \delta\}\) till \(\lambda>1+8 \epsilon\) and start a new superphase corresponding to this
        new \(\epsilon\).
        Define \(\mathbf{u}(\mathcal{L})\) as \(u(\mathcal{L})_{\ell}=\exp \left(\alpha \lambda_{\ell}\right) / \tilde{b}_{\ell}\) if \(\ell \in \mathcal{L}\) and 0 otherwise. Let \(\gamma=\mathbf{u}(\mathcal{L})^{T} \tilde{\mathbf{b}}\).
        Using \(O\left(\ln \frac{2}{\delta}\right)\) invocations of a subroutine that for any \(\mathbf{z}\) finds a \(\mathbf{y} \in \mathcal{P}\) such that \(\mathbf{z}^{T} \mathbf{y} \geq(1-\)
        \(\delta / 2) \max \left\{\mathbf{z}^{T} \mathbf{y} \mid \mathbf{y} \in \mathcal{Q}\right\}\), to find a solution \(\tilde{\mathbf{y}}\) of LP8 , otherwise decrease \(\beta \leftarrow(1-\delta) \beta\).
\[
\begin{equation*}
\mathbf{w}^{T} \tilde{\mathbf{y}} \geq(1-\delta) \beta, \quad \mathbf{u}(\mathcal{L})^{T} \mathbf{A} \tilde{\mathbf{y}} \leq \frac{\gamma}{1-\delta}, \quad \tilde{\mathbf{y}} \in \mathcal{P} \tag{LP8}
\end{equation*}
\]
```

        Set }\mathbf{y}\leftarrow(1-\sigma)\mathbf{y}+\sigma\tilde{\mathbf{y}}\mathrm{ where }\sigma=\epsilon/(4\alpha\mp@subsup{\lambda}{0}{})
    ```
        Set }\mathbf{y}\leftarrow(1-\sigma)\mathbf{y}+\sigma\tilde{\mathbf{y}}\mathrm{ where }\sigma=\epsilon/(4\alpha\mp@subsup{\lambda}{0}{})
    end while
```

Lemma 14. Suppose $\Psi \leq \gamma+\frac{\delta \gamma}{\lambda_{0}}$. If $\tilde{\beta}=\max \left\{\mathbf{w}^{T} \mathbf{y} \left\lvert\, \mathbf{u}(\mathcal{L})^{T} \mathbf{A y} \leq \frac{\gamma}{1-\delta}\right., \mathbf{y} \in \mathcal{Q}\right\}$ exists then we have an algorithm for Line 12 of Algorithm 2 for any $\beta<\tilde{\beta}$. Further the final output of Algorithm 2 satisfies $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta) \min \left\{(1-\delta)^{2} \tilde{\beta}, \beta_{0}\right\}$.

Proof: The assumption implies that $\mathbf{u}(\mathcal{L})^{T} \mathbf{A y} \leq \mathbf{u}^{T} \mathbf{A y} \leq \mathbf{u}^{T} \tilde{\mathbf{b}}=\Psi \leq \gamma /(1-\delta)$ for any $\mathbf{y}$ satisfying Ay $\leq \mathbf{b}$. Part (1) of Theorem 8 applies with $\widehat{\mathcal{P}}=\mathcal{P}$ and $\widehat{\mathcal{Q}}=\mathcal{Q}$ and we succeed in solving LP8. This implies that $\beta$ cannot decrease below $(1-\delta) \tilde{\beta}$; the last decrease of $\beta$ corresponds to a value greater than $\tilde{\beta}$.

For $\alpha=\frac{1}{f(\delta)} \ln \left(\frac{M \lambda_{0}}{\delta}\right)$ we satisfy the precondition of Lemma 14 because

$$
\Psi-\mathbf{u}(\mathcal{L})^{T} \tilde{\mathbf{b}}=\mathbf{u}^{T} \tilde{\mathbf{b}}-\mathbf{u}(\mathcal{L})^{T} \tilde{\mathbf{b}}=\sum_{\ell: \lambda_{\ell}<\lambda-f(\delta)} \exp \left(\alpha \lambda_{\ell}\right) \leq \frac{\delta}{\lambda_{0}} e^{\alpha \lambda} \leq \frac{\delta}{\lambda_{0}} \mathbf{u}(\mathcal{L})^{T} \tilde{\mathbf{b}}=\frac{\delta \gamma}{\lambda_{0}}
$$

However we include the condition in the statements of Lemma 15 and Theorem 16 because in the specific case of $b$-matching we would use a value of $\alpha$ which is better by a factor of $1 / \delta$ - therefore we can use Lemma 15 and Theorem 16 without any change. A smaller value of $\alpha$ will result in faster convergence.

Lemma 15. Suppose $\Psi \leq \gamma+\delta \gamma / \lambda_{0}$. Let $\Psi^{\prime}$ be the new potential corresponding to the new $\mathbf{y}$ computed in Step 13. Then if $\lambda \geq 4$ and $\epsilon=1 / 8$ then $\Psi^{\prime} \leq\left(1-\frac{\lambda}{128 \lambda_{0}}\right) \Psi$ otherwise we have $\Psi^{\prime} \leq\left(1-\frac{\epsilon^{2}}{8 \lambda_{0}}\right) \Psi$.

Proof: Observe that the algorithm maintains the invariant $\lambda \geq 1+8 \epsilon$, even though $\lambda$ may not be monotone. After the update, let the new current solution be denoted by $\mathbf{y}^{\prime \prime}$, i.e., $\mathbf{y}^{\prime \prime}=(1-\sigma) \mathbf{y}+\sigma \tilde{\mathbf{y}}$ where $\tilde{\mathbf{y}}$ is the solution of LP8. Recall $\alpha \sigma=\epsilon /\left(4 \lambda_{0}\right)$. Let

$$
\lambda_{\ell}^{\prime \prime}=\left(\mathbf{A y}^{\prime \prime}\right)_{\ell} / \tilde{b}_{\ell}, \quad \text { and } \quad \tilde{\lambda}_{\ell}=\left(\mathbf{A} \tilde{\mathbf{y}}_{\ell} / \tilde{b}_{\ell} \quad \text { therefore } \lambda_{\ell}^{\prime \prime}=(1-\sigma) \lambda_{\ell}+\sigma \tilde{\lambda}_{\ell} \quad \forall \ell\right.
$$

Observe that $\sum_{\ell} e^{\alpha \lambda_{\ell}} \lambda_{\ell}=\sum_{\ell}\left(\tilde{b}_{\ell} u_{\ell}\right) \frac{(\mathbf{A y})_{\ell}}{\tilde{b}_{\ell}}=\mathbf{u}^{T} \mathbf{A} \mathbf{y}$. Likewise $\sum_{\ell} e^{\alpha \lambda_{\ell}} \tilde{\lambda}_{\ell}=\sum_{\ell}\left(\tilde{b}_{\ell} u_{\ell}\right) \frac{(\mathbf{A} \tilde{\mathbf{y}})_{\ell}}{\tilde{b}_{\ell}}=$ $\mathbf{u}^{T} \mathbf{A} \tilde{y}$.

Since $\tilde{\mathbf{y}} \in \mathcal{P}$ we have $\tilde{\lambda}_{\ell} \leq \lambda_{0}$ from Step 2 of Algorithm 2 . Since we repeatedly take convex combination of the current candidate solution $\mathbf{y}$ with a $\tilde{\mathbf{y}} \in \mathcal{P}$, and the initial solution satisfies $\lambda \leq \lambda_{0}$; we have $\lambda_{\ell} \leq \lambda_{0}$ throughout the algorithm. Since $\lambda_{\ell} \leq \lambda_{0}$ we have all $\left|\alpha \sigma\left(\tilde{\lambda}_{\ell}-\lambda_{\ell}\right)\right| \leq \epsilon / 4$. Now for $|\Delta| \leq \frac{\epsilon}{4} \leq \frac{1}{4}$; we have $e^{a+\Delta} \leq e^{a}(1+\Delta+\epsilon|\Delta| / 2)$. Therefore:

$$
e^{\alpha \lambda_{\ell}^{\prime \prime}} \leq e^{\alpha \lambda_{\ell}}\left(1+\sigma \alpha\left(\tilde{\lambda}_{\ell}-\lambda_{\ell}\right)+\frac{1}{2} \epsilon \sigma \alpha\left(\tilde{\lambda}_{\ell}+\lambda_{\ell}\right)\right)=e^{\alpha \lambda_{\ell}}+\left(1+\frac{\epsilon}{2}\right) \sigma \alpha \tilde{\lambda}_{\ell} e^{\alpha \lambda_{\ell}}-\left(1-\frac{\epsilon}{2}\right) \sigma \alpha \lambda_{\ell} e^{\alpha \lambda_{\ell}}
$$

which implies that

$$
\begin{align*}
\Psi^{\prime} & =\sum_{\ell} e^{\alpha \lambda_{\ell}^{\prime \prime}} \leq \Psi+\left(1+\frac{\epsilon}{2}\right) \sigma \alpha \sum_{\ell} e^{\alpha \lambda_{\ell}} \tilde{\lambda}_{\ell}-\left(1-\frac{\epsilon}{2}\right) \sigma \alpha \sum_{\ell} e^{\alpha \lambda_{\ell}} \lambda_{\ell} \\
& =\Psi+\left(1+\frac{\epsilon}{2}\right) \sigma \alpha \mathbf{u}^{T} \mathbf{A} \tilde{\mathbf{y}}-\left(1-\frac{\epsilon}{2}\right) \sigma \alpha \mathbf{u}^{T} \mathbf{A y} \tag{2}
\end{align*}
$$

$\tilde{\mathbf{y}}$ satisfies LP8 and therefore $\mathbf{u}(\mathcal{L})^{T} \mathbf{A} \tilde{\mathbf{y}} \leq \frac{\gamma}{1-\delta}$, which along with $\tilde{\lambda}_{\ell} \leq \lambda_{0}, \delta \leq 1 / 16$, implies:

$$
\begin{equation*}
\mathbf{u}^{T} \mathbf{A} \tilde{\mathbf{y}}=\mathbf{u}(\mathcal{L})^{T} \mathbf{A} \tilde{\mathbf{y}}+\sum_{\ell: \lambda_{\ell}<\lambda-f(\delta)} \tilde{\lambda}_{\ell} e^{\lambda_{\ell} \alpha} \leq \frac{\gamma}{1-\delta}+\lambda_{0} \sum_{\ell: \lambda_{\ell}<\lambda-f(\delta)} e^{\lambda_{\ell} \alpha} \leq \frac{\gamma}{1-\delta}+\lambda_{0}(\Psi-\gamma) \leq(1+3 \delta) \gamma \tag{3}
\end{equation*}
$$

Finally observe that since $\lambda>1+8 \epsilon, f(\delta) \leq \delta$, and $\gamma=\sum_{\ell: \lambda_{\ell}>\lambda-f(\delta)} \lambda_{\ell} e^{\lambda_{\ell} \alpha}$,

$$
\begin{equation*}
\mathbf{u}^{T} \mathbf{A} \mathbf{y} \geq \mathbf{u}(\mathcal{L})^{T} \mathbf{A} \mathbf{y}=\sum_{\ell: \lambda_{\ell} \geq \lambda-f(\delta)} \lambda_{\ell} e^{\lambda_{\ell} \alpha} \geq(\lambda-f(\delta))\left(\sum_{\ell: \lambda_{\ell} \geq \lambda-f(\delta)} e^{\lambda_{\ell} \alpha}\right)=(\lambda-f(\delta)) \gamma \geq(\lambda-\delta) \gamma \tag{4}
\end{equation*}
$$

Using Equations (2)-(4) we have:

$$
\begin{equation*}
\Psi^{\prime} \leq \Psi-\left(\lambda-1-4 \delta-\frac{(\lambda+4 \delta+1) \epsilon}{2}\right) \gamma \alpha \sigma \tag{5}
\end{equation*}
$$

Note $\delta \leq \epsilon \leq 1 / 8$. If $\lambda \geq 4$ then $\lambda-1-4 \delta-\frac{(\lambda+4 \delta+1) \epsilon}{2} \geq \lambda / 2$. From the statement of the lemma, $\Psi \leq 2 \gamma$. Note that if $\lambda \geq 4$ and $\epsilon=\frac{1}{8}$ then $\frac{\lambda \alpha \sigma}{4}=\frac{\lambda}{128 \lambda_{0}}$. Thus in the case when $\lambda \geq 4$ and $\epsilon=1 / 8$,

$$
\Psi^{\prime} \leq \Psi-\frac{\lambda \alpha \sigma \gamma}{2} \leq \Psi\left(1-\frac{\lambda \alpha \sigma}{4}\right) \leq \Psi\left(1-\frac{\lambda}{128 \lambda_{0}}\right)
$$

Otherwise using $\lambda \geq 1+8 \epsilon$,

$$
\lambda-1-4 \delta-\frac{(\lambda+4 \delta+1) \epsilon}{2} \geq(1+8 \epsilon)\left(1-\frac{\epsilon}{2}\right)-(1+4 \delta)\left(1+\frac{\epsilon}{2}\right) \geq 7 \epsilon-4 \delta-\left(4 \epsilon^{2}+2 \delta \epsilon\right) \geq \epsilon
$$

Combining with Equation (5) we get $\Psi^{\prime} \leq \Psi-\epsilon \sigma \alpha \gamma \leq \Psi\left(1-\frac{\epsilon \alpha \sigma}{2}\right)=\Psi\left(1-\frac{\epsilon^{2}}{8 \lambda_{0}}\right)$.
Lemma 15 proves that $\Psi$ decreases monotonically, even though $\lambda$ may not.
Theorem 16. Suppose $\Psi \leq \gamma+\delta \gamma / \lambda_{0}$. Algorithm 2 converges within $\tau=O\left(\lambda_{0}\left(\frac{\ln (2 K)}{\delta^{2}}+\frac{\alpha}{\delta}+\alpha \ln \lambda_{0}\right)\right)$ invocations of LP8 and provides a solution as described in line 8 .

Proof: Observe that $e^{\alpha \lambda} \leq \gamma \leq K e^{\alpha \lambda}$ since there are at most $K$ constraints in $\mathcal{L}$. Since $\gamma \leq \Psi \leq 2 \gamma$ we know that $e^{\alpha \lambda} \leq \Psi \leq 2 K e^{\alpha \lambda}$. We partition the number of iterations into three parts:
$(\mathcal{C} 1)$ The number of iterations till we observe $\lambda<4$ for the first time.
(C2) The number of iterations after we observe $\lambda<4$ for the first time till $\lambda<2$ for the first time.
(C3) The number of iterations since $\lambda<2$ for the first time.
Observe $(\mathcal{C} 1)$ and $(\mathcal{C} 2)$ correspond to the first superphase during which $\epsilon=\frac{1}{8}$. In case ( $\left.\mathcal{C} 1\right)$, consider the total number of iterations when $4 \leq 2^{j} \leq \lambda \leq 2^{j+1}$. The potential $\Psi$ must be below $2 K e^{\alpha 2^{j+1}}$. If we perform $r$ updates to $\mathbf{y}$ then the potential decreased by at least $\left(1-2^{j-5} / \lambda_{0}\right)^{r}$ but if $r \geq \frac{\lambda_{0}}{2^{j-5}}\left(\ln (2 K)+2^{j+1} \alpha\right)$ then the new potential will be below 1 , which is impossible since the potential must be at least $e^{4 \alpha}$. Therefore the total number of updates corresponding to $4 \leq 2^{j} \leq \lambda \leq 2^{j+1}$ for a fixed $j$ is at most $128 \lambda_{0} 2^{-j} \ln (2 K)+256 \lambda_{0} \alpha$. Summed over all $j \geq 2$ the number of updates in case ( $\mathcal{C} 1$ ) is $O\left(\lambda_{0} \ln (2 K)+\lambda_{0} \alpha \ln \lambda_{0}\right)$.

In case $(\mathcal{C} 2)$, the potential decreases by a factor $\left(1-1 /\left(512 \lambda_{0}\right)\right)$. By the same exact argument as in case $(\mathcal{C} 1)$, if the number of updates exceed $512 \lambda_{0}(\ln (2 K)+4 \alpha)$ then the potential would be below 1 , which again is impossible since the potential must be at least $e^{2 \alpha}$. Therefore the number of updates in this case is $O\left(\lambda_{0}(\ln (2 K)+\alpha)\right)$.
In case ( $\mathcal{C} 3$ ), we partition a superphase into a number of different phases.
Definition 5. A phase starts when a superphase starts, and we remember the $\lambda$ value at the start of a phase. Let the value of $\lambda$ at the start of phase $t$ be $\lambda_{t}$. If at some point of time during phase $t$, we observe $\lambda<(1-\delta) \lambda_{t}$, then we mark the end of phase $t$ and start phase $t+1$ with $\lambda_{t+1}=\lambda$. A phase also ends when $\lambda<1+8 \epsilon$ because the corresponding superphase ends as well.

Note that while $\lambda$ is not monotone, $\lambda_{t}$ are monotone and we will use $\lambda_{t}$ to bound the number of iterations. Since in each phase $\lambda$ decreases by at least $(1-\delta)$ factor the number of phases in the superphase corresponding to $\epsilon$ is $O\left(\log _{\frac{1}{(1-\delta)}} \frac{(1+12 \epsilon)}{(1+8 \epsilon)}\right)=O\left(\frac{4 \epsilon}{\delta}\right)$. In each of these phases (say in phase $t$ ) we have $e^{\alpha(1-\delta) \lambda_{t}} \leq \Psi \leq 2 K e^{\alpha \lambda_{t}}$ and $\Psi$ decreases by a factor $\left(1-\epsilon^{2} /\left(8 \lambda_{0}\right)\right)$. Note that if $\Psi$ decreases by a factor of $2 K e^{\delta \alpha \lambda_{t}}$ then the phase would be over. Therefore the number of updates in a phase is at most $\frac{4 \lambda_{0}}{\epsilon^{2}}(\ln (2 K)+2 \delta \alpha)$ using $\lambda_{t} \leq 2$ - note $\lambda_{t}$ decreases monotonically. Therefore the number of updates in a superphase corresponding to an $\epsilon$ is

$$
\begin{equation*}
\frac{4 \epsilon}{\delta} \frac{4 \lambda_{0}}{\epsilon^{2}}(\ln (2 K)+2 \delta \alpha)=\frac{16 \lambda_{0}}{\delta \epsilon} \ln (2 K)+\frac{48 \alpha \lambda_{0}}{\epsilon} \tag{6}
\end{equation*}
$$

However note that $\epsilon$ decreases by a factor of at least $2 / 3$ (unless it is close to $\delta$ ) and the terms in Equation (6) define a geometric series each and the smallest two values of $\epsilon$ dominate because we may have $\epsilon=1.01 \delta$ followed by $\epsilon=\delta$. Therefore the total number of updates in this case is $O\left(\lambda_{0}\left(\frac{\ln (2 K)}{\delta^{2}}+\frac{\alpha}{\delta}\right)\right)$. Summing up the three cases, the number of updates is $O\left(\lambda_{0}\left(\frac{\ln (2 K)}{\delta^{2}}+\frac{\alpha}{\delta}+\alpha \ln \lambda_{0}\right)\right)$. The bound on the number of non-zero edges follows from multiplying the number of updates by the number of nonzero entries guaranteed by LP8 which is $O\left(n^{\prime}\right)$. This proves Theorem 16 . Note that $\lambda_{t}$ and its monotonicity was used in (C3), even though $\lambda$ need not be monotone.

### 3.2 Applying Algorithm 2 to Uncapacitated $b$-matching

$\mathbf{A y} \leq \tilde{\mathbf{b}}, \mathcal{Q}$ and $\mathcal{P}$ are defined below:

$$
\begin{aligned}
\{\mathbf{A y} \leq \tilde{\mathbf{b}}\} & = \begin{cases}\sum_{i:(i, j) \in E} y_{i j} \leq \tilde{b}_{i} & \forall i \in V \\
\sum_{(i, j) \in E: i, j \in U} y_{i j} \leq \tilde{b}_{U} & \forall U \in \mathcal{O}_{\delta} \quad \text { where } \tilde{b}_{i}=(1-4 \delta) b_{i} \\
\mathcal{Q} & = \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq b_{i} & \forall i \in V \\
y_{i j} \geq 0 & \forall(i, j) \in E\end{cases} \\
\mathcal{P} & = \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq \lambda_{0} b_{i} / 2 & \forall i \in V \\
y_{i j} \geq 0 & \left.\left.\|U\|_{b}\right\rfloor-\frac{\delta^{2}\|U\|_{b}^{2}}{4}\right)\end{cases} \\
\quad \forall(i, j) \in E\end{cases}
\end{aligned}
$$

Note the number of constraints in $\mathbf{A y} \leq \tilde{\mathbf{b}}$ is $M=n^{O(1 / \delta)} \gg m$, the number of edges. We observe that $\left\{\mathbf{y} / \lambda_{0} \mid \mathbf{y} \in P\right\} \subseteq\{\mathbf{y} \mid, \mathbf{A y} \leq \tilde{\mathbf{b}}, \mathcal{Q}\}$ for $\delta \leq 1 / 16$.
Definition 6. Let $\tilde{\beta}=\left\{\mathbf{w}^{T} \mathbf{y} \mid \mathbf{A y} \leq \tilde{\mathbf{b}}, \mathbf{y} \in \mathcal{Q}\right\}$; note $\tilde{\beta}$ exists.
We can now apply Lemma 14 and Theorem 16 and obtain a solution $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta)^{2} \tilde{\beta}$ and Ay $\leq(1+8 \delta)$ b (we ignore $\mathcal{P}$ ). We can extract a $(1-O(\delta))$ approximation to the optimum uncapacitated $b$-matching from $\mathbf{y}$. Set $y_{i j}^{\dagger}=\frac{(1-\delta)}{(1+8 \delta)} y_{i j}$. Since $\tilde{b}_{i} \leq b_{i}$ and $\tilde{b}_{U} \leq\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor$ the constraints corresponding to vertices and $U \in \mathcal{O}_{\delta}$ are satisfied in LP1. For the $U \notin \mathcal{O}_{\delta}$, which have $\|U\|_{b} \geq 1 / \delta$ observe that

$$
\sum_{j:(i, j) \in E} y_{i j}^{\dagger} \leq(1-\delta) b_{i} \quad \Longrightarrow \quad \sum_{(i, j) \in E, i, j \in U} y_{i j}^{\dagger} \leq \frac{1}{2} \sum_{i \in U} \sum_{j:(i, j) \in E} y_{i j}^{\dagger} \leq(1-\delta) \frac{\|U\|_{b}}{2} \leq\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor
$$

At the same time $\mathbf{w}^{T} \mathbf{y} \geq(1-\delta)^{2} \tilde{\beta}$ and thus $\mathbf{w}^{T} \mathbf{y}^{\dagger} \geq \frac{(1-\delta)^{2}}{(1+8 \delta)} \tilde{\beta}$. Note that the same argument also proves that if we consider the optimum solution of $\max \left\{\mathbf{w}^{t} \mathbf{y} \mid \tilde{A y} \leq \tilde{\mathbf{b}}, \mathbf{y} \in \mathcal{Q}\right\}$ and multiply by $(1-\delta)$ then we satisfy the constraints of LP1. Therefore $(1-\delta) \tilde{\beta} \leq \beta^{*}$. We observe $\tilde{\beta} \geq(1-4 \delta) \beta^{*}$. The latter equation follows from Likewise consider the optimum $b$-matching and multiply that solution by $(1-4 \delta)$. That modified solution $\mathbf{y}^{\prime}$ satisfies $\mathbf{A y}{ }^{\prime} \leq \tilde{\mathbf{b}}$ when $\delta \leq 1 / 16$. Thus $\tilde{\beta} \geq$ $(1-4 \delta) \beta^{*}$. Therefore $\mathbf{y}^{\dagger}$ provides a $(1-O(\delta))$-approximation to LP1 (page 3), the uncapacitated $b$-matching LP that characterizes the optimum solution.

We set $\lambda_{0}=12$. The initial solution is a solution of the bipartite relaxation (Theorem 9) multiplied by $\lambda_{0} / 2=6$. This solution value $\beta_{0}$ will be at least $\beta_{b}^{*}$ due to the approximation guarantee. But $\beta_{b}^{*} \geq \tilde{\beta}$, based on the constraints. On the other hand $\beta_{0}$ can be as large as $6 \beta_{b}^{*}$ (the bipartite optimum) which is at most $9 \beta^{*}$ since the gap between bipartite and non-bipartite solution is at most a factor of 1.5 . But $9 \beta^{*} \leq \frac{1}{1-4 \delta} \tilde{\beta} \leq 12 \tilde{\beta}$. This proves the bound on the initial solution. The parameter $\lambda_{0}$ can be improved (e.g., it can be argued that $\tilde{\beta} \leq \beta^{*}$ ), but that only affects the running time by a $O(1)$ factor. Observe that an algorithm for solving LP8 is also provided by Theorem 9 and multiplying the solution by 6 . We now focus on Step (7).

We set $f(\delta)=\delta^{3} / 10$. We show that in Step (7) the sparse set of constraints will be of size at most $K=2 n$, since that collection would define a laminar family. More specifically:

Lemma 17. Let $\frac{3}{\sqrt{n}} \leq \delta \leq \frac{1}{16}$. If $\lambda>1+8 \delta$ then we can find $\mathcal{L}=\left\{U \mid \lambda_{U} \geq \lambda-\delta^{3} / 10 ; U \in \mathcal{O}_{\delta}\right\}$ in $O\left(m+n \operatorname{poly}\left(\delta^{-1}, \ln n\right)\right)$ time. We find $\mathcal{L}$ without knowing $\lambda$ and once $\mathcal{L}$ is known, we know $\lambda$ as well.

Proof: Let $\underline{\lambda}=\max \left\{1, \max _{i}(1-4 \delta) \lambda_{i}\right\}=\max \left\{1, \max _{i} \sum_{j} y_{i j} / b_{i}\right\}$ and $\hat{y}_{i j}=y_{i j} / \underline{\lambda}$. Let $\hat{\lambda}_{U}=$ $\sum_{i, j \in U} \hat{y}_{i j} / \tilde{b}_{U}$ and $\hat{\lambda}=\max _{U \in \mathcal{O}_{\delta}} \hat{\lambda}_{U}$. Observe that $\lambda_{U}=\underline{\lambda} \hat{\lambda}_{U}$ and if $\lambda>\max _{i} \lambda_{i}$ then $\lambda=\underline{\lambda} \hat{\lambda}$. Note $\sum_{j} \hat{y}_{i j} \leq b_{i}$.

Suppose that $\hat{\lambda} \leq 1+3 \delta$ and $\underline{\lambda}=1$. Then for all $U$ we have $\lambda_{U}=\underline{\lambda} \hat{\lambda}_{U} \leq \underline{\lambda} \hat{\lambda} \leq \underline{\lambda}(1+3 \delta)<1+8 \delta$ and $\max _{i} \lambda_{i} \leq 1 /(1-4 \delta) \leq 1+8 \delta$ for $\delta \in\left(0, \frac{1}{16}\right]$. This contradicts the assumption that $\lambda>1+8 \delta$. Therefore, if $\hat{\lambda} \leq 1+3 \delta$ then we must have $\underline{\lambda}>1$. Now consider the vertex $i$ which defined $\underline{\lambda}$; then

$$
\lambda \geq \lambda_{i}=\frac{\underline{\lambda}}{1-4 \delta} \geq(1+4 \delta) \underline{\lambda} \geq(1+3 \delta) \underline{\lambda}+\delta \underline{\lambda}>\hat{\lambda} \underline{\lambda}+\delta
$$

which implies $\lambda-\delta \geq \lambda_{U}$ for every $U$. In this case $\mathcal{L}=\emptyset$ and $\left|\left\{U: \lambda_{U} \geq \lambda-\delta^{x} ; U \in \mathcal{O}_{\delta}\right\}\right|=0$ for $x \geq 2$. Therefore the remaining case is $\hat{\lambda}>1+3 \delta$. But in this case Theorems 5 and 6 apply because we satisfy $\sum_{j} \hat{y}_{i j} \leq b_{i}$. To find $L$, compute $\underline{\lambda}, \hat{y}_{i j}$ and run the algorithm in Theorem 6. We can compute $\lambda=\left\{\underline{\lambda} \hat{\lambda}, \max _{i} \lambda_{i}\right\}$ and return the sets satisfying $\lambda_{U} \geq \lambda-\delta^{3} / 10$.
To compute $\mathbf{z}^{T}=\mathbf{w}^{T}-\mathbf{u}(\mathcal{L})^{T} \mathbf{A}$ in LP8, note that $z_{i j}=w_{i j}-\varrho\left(x_{i}+x_{j}+\sum_{U: i, j \in U} z_{U}\right)$ where $x_{i}$ corresponds to the vertices and $z_{U}$ correspond to the odd set in $\mathbf{u}$. These weights can be computed in $O(1)$ time if we precompute the $\sum_{U \in \mathcal{L}, i, j \in U} z_{U}$ for each pair of vertices $(i, j)$. Note there can be at most $\sum_{s=1}^{O(1 / \delta)} \frac{n}{s} s^{2}=O\left(n \delta^{-2}\right)$ such pairs for any $\mathcal{L}$. If we use $\alpha=\frac{1}{f(\delta)} \ln \left(\frac{M \lambda_{0}}{\delta}\right)$ we get an algorithm which converges in $O\left(\delta^{-5} \log n\right)$ invocations of LP8 and provides a $(1-O(\delta))$-approximation. We show that $\alpha$ can be chosen to be smaller.
Lemma 18. For $\frac{3}{\sqrt{n}} \leq \delta \leq \frac{1}{16}, n \geq \lambda_{0}, \alpha=50 \delta^{-3} \ln n$ and the definition of $\gamma$ in Algorithm 2,

$$
\sum_{i: \lambda_{i} \leq \lambda-\delta^{3} / 10} \tilde{b}_{i} x_{i}+\sum_{\lambda_{U}<\lambda-\delta^{3} / 10} z_{U} \tilde{b}_{U}=\sum_{i: \lambda_{i} \leq \lambda-\delta^{3} / 10} e^{\lambda_{i} \alpha}+\sum_{\lambda_{U}<\lambda-\delta^{3} / 10} e^{\lambda_{U} \alpha} \leq \frac{e^{\lambda \alpha} \delta}{n} \leq \frac{\delta \gamma}{n}<\frac{\delta \gamma}{\lambda_{0}}
$$

Proof: Observe that $e^{\lambda \alpha} \leq \gamma$ since $\lambda=\max _{i} \lambda_{i}$ or $\lambda=\max _{U \in L} \lambda_{U}$. We first focus on $U \in \mathcal{O}_{\delta}$. Observe the $U$ considered in the left hand side of the statement of the inequality in the Lemma can be partitioned into three classes (i) $\lambda_{U} \leq \lambda-\delta^{2}$ (ii) $\lambda-\delta^{\left(x_{0}+3\right) / 2} \leq \lambda_{U} \leq \lambda-\delta^{3} / 10$, where $x_{0}$ is the largest value of $x \geq 2$ such that $\delta^{(x-3) / 2} \geq 2$. Note that $x_{0}<3$ exists $घ^{4}$ given $\delta \leq 1 / 16$, and (iii) $\lambda-\delta^{2} \leq \lambda_{U}<\lambda-\delta^{\left(x_{0}+3\right) / 2}$.

For case (i) observe that the corresponding $e^{\lambda_{U} \alpha} \leq e^{\lambda \alpha-\delta^{2} \alpha} \leq e^{\lambda \alpha} e^{-50 \delta^{-1} \ln n}=e^{\alpha \lambda} / n^{(50 / \delta)}$. There are at most $n^{1 / \delta}$ such sets and therefore $\sum_{U: \lambda_{U} \leq\left(1-\delta^{2}\right) \lambda} e^{\lambda_{U} \alpha} \leq e^{\lambda \alpha} / n^{(49 / \delta)}$.

For case (ii), perform the same transformation as in the first two lines of Lemma 17. The bound on $\lambda_{U}$ corresponds to $\hat{\lambda}_{U} \geq\left(\hat{\lambda}-\delta^{\left(3+x_{0}\right) / 2}\right) / \underline{\lambda} \geq \hat{\lambda}-\delta^{\left(3+x_{0}\right) / 2}$ since $\underline{\lambda} \geq 1$. Using Theorem 5 we know that there are at most $n^{3}+(n / \delta)^{1+\delta^{\left(x_{0}-3\right) / 2}} \leq 2(n / \delta)^{3}$ such sets. For each such set $U, e^{\lambda_{U} \alpha} \leq e^{\lambda \alpha} e^{-\delta^{3} \alpha / 10}=e^{\lambda \alpha} e^{-5 \ln n}=e^{\lambda \alpha} / n^{5}$. Summing up over such $2(n / \delta)^{3}$ sets the total contribution to the left hand side of the inequality in the statement of the lemma is at most $2 e^{\lambda \alpha} \delta^{-3} / n^{2}$.

For case (iii), we partition the interval $\left(\lambda-\delta^{2}, \lambda-\delta^{\left(x_{0}+3\right) / 2}\right.$ ] into subintervals of the form $\left(\lambda-\delta^{x}, \lambda-\delta^{(x+3) / 2}\right]$ for different values of $x$. The last subinterval corresponds to $x=x_{0}$. If we set $x_{1}=2 x_{0}-3$ we have $\left(3+x_{1}\right) / 2=x_{0}$ and thus $x_{1}<x_{0}<3$, which corresponds to the second subinterval. The $j$-th subinterval is defined by $x_{j}$ satisfying $3+x_{j}=2 x_{j-1}$. The number of such subintervals is at most $2+\log \log (1 / \delta)$.

Consider the case $\lambda_{U} \in\left(\lambda-\delta^{x}, \lambda-\delta^{(x+3) / 2}\right]$. Again, performing the transformation as in the first two lines of Lemma 17, we get that $\hat{\lambda}_{U} \geq\left(\hat{\lambda}-\delta^{x}\right) / \underline{\lambda} \geq \hat{\lambda}-\delta^{x}$ (again, $\underline{\lambda} \geq 1$ ). Using Theorem 5
${ }^{4}$ Consider $h=\delta^{(x-3) / 2}$ as $x$ increases from 2 to 3 . The value of $h$ decreases from $\delta^{-1 / 2}$ to 1 .
and $\delta \geq 1 / \sqrt{n}$, the number of odd-sets corresponding to this subinterval is at most

$$
n^{3}+(n / \delta)^{1+\delta^{(x-3) / 2}} \leq 2(n / \delta)^{1+\delta^{(x-3) / 2}} \leq 2 n^{1.5+1.5 \delta^{(x-3) / 2}}
$$

However note that $\lambda_{U} \leq \lambda-\delta^{(3+x) / 2}$ and therefore $e^{\alpha \lambda_{U}}$ is at most

$$
e^{\alpha \lambda} / e^{\alpha \delta^{(x+3) / 2}}=\frac{e^{\alpha \lambda}}{e^{50 \delta^{(x-3) / 2} \ln n}}=\frac{e^{\alpha \lambda}}{n^{50 \delta^{(x-3) / 2}}}
$$

Therefore the total contribution to the left hand side of the inequality in the statement of the lemma for all $U$ such that $\lambda_{U} \in\left(\lambda-\delta^{x}, \lambda-\delta^{(x+3) / 2}\right]$ is at most (since $\left.x \leq x_{0}\right)$ :

$$
\frac{2 e^{\alpha \lambda}}{n^{48.5 \delta^{(x-3) / 2}-1.5}} \leq \frac{2 e^{\alpha \lambda}}{n^{97-1.5}} \leq \frac{e^{\alpha \lambda}}{n^{94}}
$$

For $i \in V$ such that $\lambda_{i} \leq \lambda-\delta^{3} / 10$ the calculation as in the case (iii) applies and $\sum_{i: \lambda_{i} \leq \lambda-\delta^{3} / 10} e^{\lambda_{i} \alpha} \leq$ $2 e^{\lambda \alpha} \delta^{-3} / n^{2}$ as well. Since $\frac{1}{n^{49 / \delta}}+\frac{2+\log \log (1 / \delta)}{n^{94}}+\frac{4 \delta^{-3}}{n^{2}} \leq \frac{5 \delta^{-3}}{n^{2}} \leq \frac{\delta}{n}$; the lemma follows.

We can now conclude Theorem 1 based on the discussion above, Theorems 8,16 and Lemmas 17 18)

Theorem 1. Given any non-bipartite graph, for any $\frac{3}{\sqrt{n}}<\delta \leq 1 / 16$ we find a $(1-O(\delta))$ approximate maximum fractional weighted b-matching using additional "work" space (space excluding the read-only input) $O\left(n\right.$ poly $\left.\left(\delta^{-1}, \ln n\right)\right)$ and making $T=O\left(\delta^{-4}(\ln (1 / \delta)) \ln n\right)$ passes over the list of edges. The running time is $O\left(m T+n \operatorname{poly}\left(\delta^{-1}, \ln n\right)\right)$.

## 4 Proof of Theorem 5

Before proving Theorem 5. recall that $\tilde{b}_{U}=\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor-f\left(\|U\|_{b}\right)$ where $f(\ell)=\frac{\delta^{2} \ell^{2}}{4}$ and $\delta \in\left(0, \frac{1}{16}\right]$. We can verify that $f(\ell)$ is convex, monotonic for $0 \leq \ell \leq 2 / \delta$ and:
$(\mathcal{F} 1)$ : For $3 \leq\|U\|_{b} \leq 2 / \delta-1$ (irrespective of odd or even) we have $\tilde{b}_{U} \geq(1-\delta)\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor$.
$(\mathcal{F} 2):$ For any $\ell_{1}, \ell_{2} ; f\left(\ell_{1}\right)+f\left(\ell_{2}\right)=f\left(\ell_{1}+\ell_{2}-1\right)-\left(2 \ell_{1} \ell_{2}-2 \ell_{1}-2 \ell_{2}+1\right) \frac{\delta^{2}}{4}$.
$(\mathcal{F} 3):$ For integers $\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} \in[3,2 / \delta]$ and $t \geq 0$, such that $\ell_{1}+2 t \leq \ell_{2} \leq \ell_{3} \leq \ell_{4}-2 t$ and $\ell_{1}+\ell_{4}=\ell_{2}+\ell_{3}$, we have $f\left(\ell_{2}\right)+f\left(\ell_{3}\right) \leq f\left(\ell_{1}\right)+f\left(\ell_{4}\right)-2 t^{2} \delta^{2}$.

Theorem 5. For a graph $G$ with $n$ vertices and any non-negative edge weights $\hat{y}_{i j}=\hat{y}_{j i}$ such that $\hat{y}_{i i}=0$ for all $i, \sum_{j} \hat{y}_{i j} \leq b_{i}$ for all $i$, and $\delta \in\left(0, \frac{1}{16}\right]$, define: $\hat{\lambda}_{U}=\frac{\sum_{(i, j): i, j \in U} \hat{y}_{i j}}{\tilde{b}_{U}}$ and $\hat{\lambda}=\max _{U \in \mathcal{O}_{\delta}} \hat{\lambda}_{U}$. If $\hat{\lambda} \geq 1+3 \delta$, the set $L_{1}=\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{3} ; U \in \mathcal{O}_{\delta}\right\}$ defines a laminar family. Moreover for any $x \geq 2$ we have $\left|\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{x} ; U \in \mathcal{O}_{\delta}\right\}\right| \leq n^{3}+(n / \delta)^{1+\delta^{(x-3) / 2}}$.
Proof: Consider two sets $A_{1}, A_{2} \in \mathcal{O}_{\delta}$ such that $\hat{\lambda}_{A_{1}}, \hat{\lambda}_{A_{2}} \geq \hat{\lambda}-\delta^{x}>1+2 \delta$ (since $x \geq 2$ ) and neither $A_{1}-A_{2}, A_{2}-A_{1} \neq \emptyset$. For any set $U$ (with $\|U\|_{b} \geq 1$, even or odd, large or small) define $\hat{Y}_{U}=\sum_{(i, j): i, j \in U} \hat{y}_{i j}$ and $\tilde{b}_{U}$. For $\|U\|_{b}=1$ we have $\hat{Y}_{U}=0$. Let $\hat{\lambda}_{U}=\hat{Y}_{U} / \tilde{b}_{U}$. There are now two cases.

Case I: $\left\|A_{1} \cap A_{2}\right\|_{b}$ is even. Let $D=A_{1} \cap A_{2}$ and $t=\|D\|_{b} / 2$. Let $Q_{1}=\sum_{i \in D} \sum_{j \in A_{1}-A_{2}} \hat{y}_{i j}$ (the cut between $D$ and $A_{1}-A_{2}$ using the edge weights $\hat{y}_{i j}$ ) and $Q_{2}=\sum_{i \in D} \sum_{j \in A_{2}-A_{1}} \hat{y}_{i j}$. Without loss of generality, assume that $Q_{1} \leq Q_{2}$ (otherwise we can switch $A_{1}, A_{2}$ ). Let $C=A_{1}-A_{2}$ and
$A=A_{1}$. Let $2 \ell-1=\|C\|_{b}$ which is odd. From the definitions of $\hat{Y}_{C}, \hat{Y}_{D}$ we have $\hat{Y}_{C}=\hat{Y}_{A}-Q_{1}-\hat{Y}_{D}$ and $\hat{Y}_{D} \leq \frac{1}{2}\left(\sum_{i \in D} \sum_{j} \hat{y}_{i j}-Q_{1}-Q_{2}\right) \leq \frac{\|D\|_{b}}{2}-\frac{Q_{1}+Q_{2}}{2}$. Using $Q_{1} \leq Q_{2}$ we get:

$$
\begin{equation*}
\hat{Y}_{C} \geq \hat{Y}_{A}-\frac{\|D\|_{b}}{2}-\frac{Q_{1}}{2}+\frac{Q_{2}}{2} \geq \hat{Y}_{A}-\frac{\|D\|_{b}}{2}=\hat{Y}_{A}-t . \tag{7}
\end{equation*}
$$

Now $\hat{Y}_{A}=\hat{\lambda}_{A} \tilde{b}_{A}>(1+2 \delta)(1-\delta)\left\lfloor\frac{\|A\|_{b}}{2}\right\rfloor \geq\left\lfloor\frac{\|A\|_{b}}{2}\right\rfloor$ using Condition $\mathcal{F} 1$, and the lower bound on $\hat{\lambda}$. Therefore $\hat{Y}_{A}>t$ and $\hat{Y}_{C}>0$ which means $\|C\|_{b} \geq 3$. Therefore we can refer to $\tilde{b}_{C}, \hat{\lambda}_{C}$. Since $\|D\|_{b}=\|A\|_{b}-\|C\|_{b}$,

$$
\begin{align*}
\tilde{b}_{A}-\tilde{b}_{C} & =\left\lfloor\frac{\|A\|_{b}}{2}\right\rfloor-f\left(\|A\|_{b}\right)-\left\lfloor\frac{\|C\|_{b}}{2}\right\rfloor+f\left(\|C\|_{b}\right)=\frac{\|D\|_{b}}{2}-\left(f\left(\|A\|_{b}\right)-f\left(\|C\|_{b}\right)\right) \\
& =t-t \delta(t+2 \ell-1) \delta \geq(1-\delta) t \tag{8}
\end{align*}
$$

where the last line uses $\frac{1}{\delta} \geq\|A\|_{b} \geq(2 t+2 \ell-1)$ because $A \in \mathcal{O}_{\delta}$. From Equations (7) and (8), and $\hat{Y}_{C}=\hat{\lambda}_{C} \tilde{b}_{C}, \hat{Y}_{A}=\hat{\lambda}_{A} \tilde{b}_{A}$ we get:

$$
\begin{aligned}
\hat{\lambda} \tilde{b}_{C} & \geq \hat{\lambda}_{C} \tilde{b}_{C}=\hat{Y}_{C} \geq \hat{Y}_{A}-t=\hat{\lambda}_{A} \tilde{b}_{A}-t \geq\left(\hat{\lambda}-\delta^{x}\right) \tilde{b}_{A}-t=\hat{\lambda} \tilde{b}_{A}-\delta^{x} \tilde{b}_{A}-t \geq \hat{\lambda}\left(\tilde{b}_{C}+(1-\delta) t\right)-\delta^{x} \tilde{b}_{A}-t \\
& >\hat{\lambda} \tilde{b}_{C}+(1+3 \delta)(1-\delta) t-\delta^{x} \tilde{b}_{A}-t \geq \hat{\lambda} \tilde{b}_{C}+\delta t-\delta^{x} \tilde{b}_{A}
\end{aligned}
$$

Since $\tilde{b}_{A} \leq 1 / \delta$ this implies that $t<\delta^{x-1} \tilde{b}_{A} \leq \delta^{x-2}$ which contradicts $A_{1} \cap A_{2} \neq \emptyset$ for $x \geq 2$.
Case II: $\left\|A_{1} \cap A_{2}\right\|_{b}$ is odd. Let $C=A_{1} \cup A_{2}$, and $D=A_{1} \cap A_{2}$. Let $\left\|A_{1}\right\|_{b}=\ell_{1},\left\|A_{2}\right\|_{b}=\ell_{2}$. We prove

$$
\begin{equation*}
\hat{\lambda}_{C} \leq \hat{\lambda} \tag{9}
\end{equation*}
$$

If $\|C\|_{b} \leq 1 / \delta$ then Equation (9) is true by definition since $\hat{\lambda}$ explicitly optimizes over $\mathcal{O}_{\delta}$ and $C \in \mathcal{O}_{\delta}$. We focus on the case $\|C\|_{b}>1 / \delta$. We extend the definitions $\tilde{b}_{C}=\left\lfloor\frac{\|C\|_{b}}{2}\right\rfloor-f\left(\|C\|_{b}\right)$ and $\hat{\lambda}_{C}=\hat{Y}_{C} / \tilde{b}_{C}$ for all odd subsets with $\|\cdot\|_{b} \leq 2 / \delta$. Now $\hat{Y}_{C} \leq \frac{\|C\|_{b}}{2}$ since $\sum_{j} \hat{y}_{i j} \leq b_{i}$. Note that $\|C\|_{b}=\left\|A_{1}\right\|_{b}+\left\|A_{2}\right\|_{b}-\|D\|_{b}$ and $\|D\|_{b} \geq 1$. Thus $\|C\|_{b} \leq 2 / \delta-1$ and using Condition $\mathcal{F} 1$ :

$$
\tilde{b}_{C} \geq(1-\delta)\left\lfloor\frac{\|C\|_{b}}{2}\right\rfloor=(1-\delta) \frac{\|C\|_{b}}{2}\left(1-\frac{1}{\|C\|_{b}}\right) \geq(1-\delta)^{2} \frac{\|C\|_{b}}{2}
$$

which implies that $\hat{\lambda}_{C} \leq(1-\delta)^{-2} \leq 1+3 \delta<\hat{\lambda}$. Thus Equation (9) holds in this case as well.
Now, $\hat{Y}_{C}+\hat{Y}_{D} \geq \hat{Y}_{A_{1}}+\hat{Y}_{A_{2}}$ and $\left\lfloor\frac{\|C\|_{b}}{2}\right\rfloor+\left\lfloor\frac{\|D\|_{b}}{2}\right\rfloor=\left\lfloor\frac{\left\|A_{1}\right\|_{b}}{2}\right\rfloor+\left\lfloor\frac{\left\|A_{2}\right\|_{b}}{2}\right\rfloor$. Therefore:

$$
\begin{equation*}
\hat{Y}_{C}+\hat{Y}_{D} \geq \hat{Y}_{A_{1}}+\hat{Y}_{A_{2}}=\hat{\lambda}_{A_{1}} \tilde{b}_{A_{1}}+\hat{\lambda}_{A_{2}} \tilde{b}_{A_{2}} \geq\left(\hat{\lambda}-\delta^{x}\right)\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right) \tag{10}
\end{equation*}
$$

If $\|D\|_{b}=1$, then by Condition $\mathcal{F} 3: \quad \tilde{b}_{C}=\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}-\frac{\delta^{2}}{4}\left(2 \ell_{1} \ell_{2}-2 \ell_{1}-2 \ell_{2}+1\right)$, and using Equation (9),

$$
\begin{aligned}
\hat{\lambda} \tilde{b}_{C} & \geq \hat{\lambda}_{C} \tilde{b}_{C}=\hat{Y}_{C} \geq\left(\hat{\lambda}-\delta^{x}\right)\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right) \geq \hat{\lambda}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right)-\delta^{x}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right) \\
& \geq \hat{\lambda}^{2} \tilde{b}_{C}+\frac{\delta^{2} \hat{\lambda}\left(2 \ell_{1} \ell_{2}-2 \ell_{1}-2 \ell_{2}+1\right)}{4}-\delta^{x}\left(\frac{\ell_{1}+\ell_{2}-2}{2}\right)
\end{aligned}
$$

since $\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}} \leq\left(\ell_{1}+\ell_{2}-2\right) / 2$. Therefore we would have a contradiction if

$$
\begin{equation*}
\hat{\lambda}\left(2 \ell_{1} \ell_{2}-2 \ell_{1}-2 \ell_{2}+1\right)-2 \delta^{x-2}\left(\ell_{1}+\ell_{2}-2\right)>0 \tag{11}
\end{equation*}
$$

Observe that for $x \geq 3$ the term $2 \delta^{x-2}\left(\ell_{1}+\ell_{2}-2\right)$ is at most 2 whereas $\left(2 \ell_{1} \ell_{2}-2 \ell_{1}-2 \ell_{2}+1\right) \geq 7$ since $3 \leq \ell_{1}, \ell_{2} \leq \frac{1}{\delta}$. Since $\hat{\lambda}>1$ we have a contradiction for $\|D\|_{b}=1, x \geq 3$.

Now consider $\|D\|_{b} \geq 3$. Without loss of generality, $\left\|A_{2}-D\right\|_{b} \geq\left\|A_{1}-D\right\|_{b}$. Let $\left\|A_{1}-D\right\|_{b}=2 t$. Using Condition $\mathcal{F} 3, \tilde{b}_{C}+\tilde{b}_{D} \leq \tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}-2 t^{2} \delta^{2}$. Note $\hat{\lambda}_{D} \leq \hat{\lambda}$, and from Equation (9) $\hat{\lambda}_{C} \leq \hat{\lambda}$. Therefore $\hat{\lambda}\left(\tilde{b}_{C}+\tilde{b}_{D}\right) \geq \hat{\lambda}_{C} \tilde{b}_{C}+\hat{\lambda}_{D} \tilde{b}_{D}$ and from Equation 10 :

$$
\begin{align*}
\hat{\lambda}\left(\tilde{b}_{C}+\tilde{b}_{D}\right) & \geq \hat{\lambda}_{C} \tilde{b}_{C}+\hat{\lambda}_{D} \tilde{b}_{D}=\hat{Y}_{C}+\hat{Y}_{D} \geq \hat{\lambda}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right)-\delta^{x}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right) \\
& \geq \hat{\lambda}\left(\tilde{b}_{C}+\tilde{b}_{D}\right)+2 t^{2} \delta^{2} \hat{\lambda}-\delta^{x}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right) \tag{12}
\end{align*}
$$

Again, this is infeasible if $x \geq 3$ since $\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}} \leq 2 / \delta$ and $\hat{\lambda} \geq 1$. Therefore for $x \geq 3$, in all cases we arrived at a contradiction to $A_{1} \cap A_{2} \neq \emptyset$. Thus we have proved that $\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{3} ; U \in \mathcal{O}_{\delta}\right\}$ is a laminar family.

We now prove the second part. Consider $L_{\ell}^{\prime}=\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{x} ; U \in \mathcal{O}_{\delta} ;\|U\|_{b}=\ell\right\}$. From Case I, no two distinct sets $A_{1}, A_{2} \in L_{\ell}^{\prime}$ intersect when $\left\|A_{1} \cap A_{2}\right\|_{b}$ is even. From Case II for $\ell \geq 5$, they cannot have $\|D\|_{b}=1$ because $\left(2 \ell^{2}-4 \ell+1\right)-2(2 \ell-2)>0$ for $\ell \geq 5$. Note $\left\|A_{1}-D\right\|_{b}=\left\|A_{2}-D\right\|_{b}$ because $\left\|A_{1}\right\|_{b}=\left\|A_{2}\right\|_{b}=\ell$. Moreover for $t \geq \delta^{(x-3) / 2}$ we would have $2 t^{2} \delta^{2} \hat{\lambda}>\delta^{x}\left(\tilde{b}_{A_{1}}+\tilde{b}_{A_{2}}\right)$ in Equation 12 . Therefore two distinct sets $A_{1}, A_{2} \in L_{\ell}^{\prime}$ which intersect, cannot differ by $\delta^{(x-3) / 2}$ or more elements. This means that $\left|L_{\ell}^{\prime}\right| \leq(n / \delta)^{1+\delta^{(x-3) / 2}}$ for $\ell \geq 5$ - to see this choose a maximal collection of disjoint sets in $L_{\ell}^{\prime}$. This would be at most $n$. Every other set $S$ in $L_{\ell}^{\prime}$ has to intersect one of these sets in the maximal collection. To upper bound the number of such sets $S$ with intersection $t$, we can start from a set in that maximal collection; throw out $t$ elements in $\ell^{t}$ ways and include new elements in $n^{t}$ ways. Note $\ell \leq 1 / \delta$. Thus the number of such sets for a fixed $t$ is $n(n / \delta)^{\delta^{(x-3) / 2}}$. Observe that $t \leq 1 / \delta$ and the bound follows. Finally note $\left|L_{3}^{\prime}\right| \leq n^{3}$. Thus the total number of sets is $n^{3}+(n / \delta)^{1+\delta^{(x-3) / 2}}$. The lemma follows.

## 5 Proof of Theorem 6

An Overview. We combine the insights of the minimum odd-cut approach 27] along with the fact that $L_{2} \subseteq L_{1}$ is a laminar family as proved in Theorem 5 . The algorithm picks the sets based on their sizes. Define $L_{1}(\ell)=\left\{U \mid U \in L_{1},\|U\|_{b}=\ell\right\}$ and $L_{2}(\ell)=\left\{U \mid U \in L_{2},\|U\|_{b}=\ell\right\}$ for $\ell \in[3,1 / \delta]$. Note that $L_{1}(\ell) \supseteq L_{2}(\ell)$. Observe that it suffices to identify $L_{2}(\ell)$ for different values of $\ell$. We construct an unweighted graph $G_{\varphi}(\ell, \tilde{\lambda})$ where $\varphi=O\left(\delta^{-4}\right)$ with a new special node $r(\ell)$ with the following two properties:

Property 1. If $\tilde{\lambda}-\frac{\delta^{3}}{100}<\hat{\lambda} \leq \tilde{\lambda}$, then (i) all sets in $L_{2}(\ell)$ have a cut which is of size at most $\kappa(\ell)$ and (ii) all odd sets of $G_{\varphi}(\ell, \tilde{\lambda})$ which do not contain $r(\ell)$ and have cut of size at most $\kappa(\ell)$ belong to $L_{1}(\ell)$. Here $\kappa(\ell)=\left\lfloor\varphi \lambda\left(1-\delta^{2} \ell^{2} / 2\right)\right\rfloor+\frac{12 \ell}{\delta}+1<2 \varphi$.

Property 2. We show in Lemma 19 that we can extend the algorithm in [27] to efficiently extract a collection $\bar{L}(\ell)$ of maximal odd-sets in $G_{\varphi}(\ell, \tilde{\lambda})$, not containing $r(\ell)$ and cut of size at most $\kappa(\ell)$ - such that any such set which is not chosen must intersect with some set in the collection.

Lemma 19. Given an unweighted graph $G$ with parameter $\kappa(\ell)$ and a special node $r(\ell)$, in time $O(n \operatorname{poly}(\kappa, \log n))$ we can identify a collection $\bar{L}(\ell)$ of odd-sets such that (i) each $U \in \bar{L}(\ell)$ does not contain $r(\ell)$ (ii) each $U \in \bar{L}(\ell)$ defines a cut of size at most $\kappa$ in $G$ and (iii) every other odd set not containing $r(\ell)$ and with a cut less than $\kappa(\ell)$ intersects with a set in $\bar{L}(\ell)$.

The second property follows without much difficulty from the properties of Gomory-Hu trees [17, 19] - trees which represent all pairwise mincuts over a set of nodes. Observe that property 1 implies that we can restrict our attention to only those regions of the graph $G_{\varphi}(\ell, \tilde{\lambda})$ which have
cuts of size at most $O\left(\delta^{-4}\right)$. Therefore if we are given a subset of vertices such that any partition of that vertex set induces a large cut, then either that subset is included entirely within one odd set or excluded completely. This is the notion of a Steiner Mincut which is used to compute the "partial Gomory Hu tree" - where for some $\kappa$ we represent all pairwise min cuts of value at most $\kappa$. Such representations can be computed for unweighted undirected graphs in time $O\left(m+n \kappa^{3} \log ^{2} n\right)$ [20] (see also improvements in [6]). The graph $G_{\varphi}(\ell, \tilde{\lambda})$ is used exactly for this purpose. If we have a maximal collection $\bar{L}(\ell)$ then $\bar{L}(\ell) \subseteq L_{1}(\ell)$ by condition (ii) of Property 1 . Due to Theorem 5 , the intersection of two such sets $U_{1}, U_{2} \in L_{1}(\ell)$ will be either empty or have $\|\cdot\|_{b}=\ell$ by laminarity the latter implies $U_{1}=U_{2}$. Therefore the sets in $L_{1}(\ell)$ are disjoint. Any $U \in L_{2}(\ell)-\bar{L}(\ell)$ has a cut of size at most $\kappa(\ell)$ using condition (i) of Property 1 and therefore must intersect with some set in $\bar{L}(\ell)$. This is impossible because $U \in L_{2}(\ell)$ implies $U \in L_{1}(\ell)$ and $\bar{L}(\ell) \subseteq L_{1}(\ell)$ and we just argued that the sets in $L_{1}(\ell)$ are disjoint! Therefore no such $U$ exists and $L_{2}(\ell) \subseteq \bar{L}(\ell)$. We now have a complete algorithm: we perform a binary search over the estimate $\tilde{\lambda} \in\left[1+3 \delta, \frac{3}{2}+\delta^{2}\right]$, and we can decide if there exists a set $U \in L_{2}(\ell)$ in time $O\left(n \operatorname{poly}\left(\delta^{-1}, \log n\right)\right)$ as we vary $\ell, \tilde{\lambda}$. This gives us $\tilde{\lambda}$. We now find the collection $\bar{L}(\ell)$ for each $\ell$ and compute all $\hat{\lambda}_{U}$ exactly (either remembering the $\hat{y}_{i j}$ of the the edges stored in $G_{\varphi}$ or by another pass over $G$ ). We can now return $\cup_{\ell} L_{2}(\ell)$. We now prove Lemma 19 .

### 5.1 Proof Of Lemma 19

The parameter $\ell$ is not relevant to the proof and is dropped. Algorithm 3 provides the algorithm for this lemma.

Lemma 19. Given an unweighted graph $G$ with parameter $\kappa$ and a special node $r$, in time $O(n \operatorname{poly}(\kappa, \log n))$ we can identify a collection $\bar{L}$ of odd-sets such that (i) each $U \in \bar{L}$ does not contain $r$ (ii) each $U \in \bar{L}$ defines a cut of size at most $\kappa$ in $G$ and (iii) every other odd set not containing $r$ and with a cut less than $\kappa$ intersects with a set in $\bar{L}$.

```
Algorithm 3 Algorithm: Finding a maximal collection of odd-sets
    \(\bar{L} \leftarrow \emptyset\). Initially \(G^{\prime}=G\). The node \(r \in V(G)\).
    repeat
        Assign the \(r\) duplicity \(b_{r}=1\) if \(\sum_{i \in V\left(G^{\prime}\right)} b_{i}\) is odd. Otherwise let \(b_{r}=2\).
        Construct a tree \(\mathcal{T}\) that represents all low \(s-t\) cuts in \(G^{\prime}\) using Theorem 20. The nodes of this tree
        \(\mathcal{T}\) correspond to subsets of vertices of \(V\left(G^{\prime}\right)\).
        Make the vertex set containing \(r\) the root of \(\mathcal{T}\) and orient all edges towards the root. The oriented
        edges represent an edge from a child to a parent. Let \(D(e)\) indicate the set of descendant subsets of
        an edge \(e\) (including the child subset which is the tail of the edge, but not including the parent subset
        which is the head of the edge).
6: Using dynamic programming starting at the leaf, mark every edge as admissible/inadmissible based on the \(\sum_{S \in D(e)} \sum_{i \in S} b_{i}\) over the descendant subsets of that edge being odd/even respectively.
7: Starting from the root \(s\) downwards, pick the edges \(e\) in parallel such that (c1) the weight of \(e\) (corresponding to a cut) is at most \(\kappa\), (c2) \(\sum_{S \in D(e)} \sum_{i \in S} b_{i}\) is odd and (c3) no edge \(e^{\prime}\) on the path from \(e\) to \(r\) satisfies (c1) and (c2). Let the odd-set \(U_{e}\) corresponding to this edge \(e \in \mathcal{T}\) be \(U_{e}=\cup_{S \in D(e)} S\).
8: If the odd-sets found are \(U_{e_{1}}, \ldots, U_{e_{f}}\) then \(\bar{L} \leftarrow \bar{L} \cup\left\{U_{e_{1}}, \ldots, U_{e_{f}}\right\}\). Observe that the sets \(U_{e_{g}}\) are mutually disjoint for \(1 \leq g \leq f\) and do not contain \(r\).
9: \(\quad\) Merge all vertices in \(\bigcup_{g=1}^{\bar{f}} \bar{U}_{e_{g}}\) with \(r\). Observe that for any set \(U\) that does not contain \(r\) and does not intersect with any \(U_{e_{g}}\), the cut \(\operatorname{Cut}(U)\) is unchanged. This defines the new \(G^{\prime}\).
until no new odd set has been found in \(G^{\prime}\)
return \(\bar{L}\).
```

Proof: First, consider the following known theorem and Lemma:

Theorem 20 ( 6,20$]$ ). Given a graph with $n$ nodes and $m$ edges (possibly with parallel edges), in time $O(m)+\tilde{O}\left(n \kappa^{2}\right)$ we can construct a weighted tree $T$ that represents all min $s-t$ cuts in $G^{\prime}$ of value at most $\kappa$. The nodes of this tree are subsets of vertices. The mincut of any pair of vertices that belong to the same subset (the same node in the tree T) is larger than $\kappa$ and for any pair of vertices $i, j$ belonging to different subsets (nodes in the tree $T$ ) the mincut is specified by the partition corresponding to the least weighted edge in the tree $T$ between the two nodes that contain $i$ and $j$ respectively.

Lemma 21 (Implicit in [27]). Suppose that for a graph $G=(V, E), \sum_{i \in V} b_{i}$ is even. For any odd-set $U$ in $G$ with cut $\kappa$, there exists an edge $e$ in the low min s-t cut tree $\mathcal{T}$ such that removing $e$ from the tree results in two connected components of odd sizes and the component $U_{e}$ not containing the root intersects $U$. In addition, the cut between $U_{e}$ and rest of the graph is at most $\kappa$.

Proof:(Of Lemma 21) Observe that the min $u-v$ cut for any $u \in U$ and $v \notin U$ is at most $\kappa$.
We provide an algorithmic proof of the existence - this is not the algorithm to find the odd sets. Let $H_{0}=V(G)$. We will maintain the three invariants that (1) $\left\|H_{z}\right\|_{b}$ is even (2) $H(z)$ defines a connected component in the low min $s$ - $t$ cut tree $\mathcal{T}$ and (3) $H_{z} \cap U \neq \emptyset$ and $H_{z} \cap(V(G)-U) \neq \emptyset$. These hold for $H_{0}$. Staring from $H_{z}$ until we find a desired edge $e$ or find $H_{z+1} \subset H_{z}$ which satisfies the same invariants. This process has to stop eventually and we would have found the desired edge $e$.

Given the invariant, there exists $u \in H_{z} \cap U$ and $v \in H_{z} \cap(V(G)-U)$ such that the min $u$-v cut is at most $\kappa$ and therefore there must exist an edge $e_{z}$ (corresponding to a min $u-v$ cut) within the component $H_{z}$ such that $e_{z}$ separates $u, v$. Let the two connected sub-components of $H_{z}$ defined by the removal of $e_{z}$ be $S_{1}$ and $S_{2}$. If $\left\|S_{1}\right\|_{b},\left\|S_{2}\right\|_{b}$ are both even, then one of them must satisfy condition (3), since $\|U\|_{b}$ is odd.

This process has to stop eventually and we would have found the desired edge $e$. Observe that all the subcomponents of $\mathcal{T}$ created in this manner define even sets until we find $e$. If we add back all the sub-components such that we have the two components corresponding to the two sides of $e$, both of those components must have odd $\|\cdot\|_{b}$. The component not containing $r$ defines $U_{e}$. In addition, the corresponding cut size is less than $\kappa$.
(Continuing with Proof of Lemma 19.) All that remains to be proven is that the loop in Algorithm 3 needs to be run only a few times. Suppose after $t^{\prime}$ repetitions $Q_{t^{\prime}}$ is the maximum collection of disjoint odd-sets which are attached to the remainder of $\mathcal{T}$ with cuts of size at most $\kappa$ and we choose $U_{e_{1}}, \ldots, U_{e_{f}}$ to be added to $\mathcal{L}$ in the $t^{\prime}+1$ st iteration. We first claim that $\left|Q_{t^{\prime}+1}\right| \leq f$. To see this we first map every odd-set in $Q_{t^{\prime}+1}$ to an edge in the tree as specified by the existence proof in Lemma 21. This map need not be constructive - the map is only used for this proof. Note that $\sum_{i} b_{i}$ is even, by construction, in Algorithm 3as required in Lemma 21. Observe that this can be a many to one map; i.e., several sets mapping to the same edge.

Now every edges $e_{1}, \ldots, e_{f}$ chosen in Algorithm 3 satisfy the property for all $j$ : no edge $e^{\prime}$ on the path from the head of $e_{j}$ (recall that the edges are oriented towards the root $r$ ) to $r$ is one of the edges in our map. Because in that case we would have chosen that edge $e^{\prime}$ instead of $e_{j}$.

Therefore the sets in $Q_{t^{\prime}+1}$ could not have mapped to any edges in the path towards $r$. Now, if a set in $Q_{t^{\prime}+1}$ mapped to an edge $e^{\prime}$ which is a descendant of the tail of some $e_{j}$ (again, the edges are oriented towards $r$ ) then this set intersects with our chosen $U_{e_{j}}$ which is not possible.

Therefore any set in $Q_{t^{\prime}+1}$ must have mapped to the same edges in the tree; i.e., $e_{1}, \ldots, e_{f}$. But then the vertex at the head of the edge belongs to the set in $Q_{t^{\prime}+1}$. Therefore there can be at most $f$ such sets. This proves $\left|Q_{t^{\prime}+1}\right| \leq f$.

We next claim that $\left|Q_{t^{\prime}+1}\right| \leq\left|Q_{t^{\prime}}\right|-f$. Consider $Q^{\prime}=Q_{t^{\prime}+1} \cup\left\{U_{e_{1}}, \ldots, U_{e_{f}}\right\} . Q^{\prime}$ is a collection of disjoint odd-sets which define a cut of size $\kappa$ in $G$ after $t^{\prime}$ repetitions. Obviously $\left|Q^{\prime}\right|=\left|Q_{t^{\prime}+1}\right|+f$ and by the definition of $Q_{t^{\prime}},\left|Q^{\prime}\right| \leq\left|Q_{t^{\prime}}\right|$. Therefore, $\left|Q_{t^{\prime}+1}\right| \leq\left|Q_{t^{\prime}}\right|-f$.

Therefore, in the worst case, $\left|Q_{t^{\prime}}\right|$ decreases by a factor $1 / 2$ and therefore in $O(\log n)$ iterations over this loop we would eliminate all odd-sets that define a cut of size $\kappa$ in $G^{\prime}$.

### 5.2 Proof of Theorem 6

Theorem 6. For a graph $G$ with $n$ vertices and any non-negative edge weights $\hat{y}_{i j}=\hat{y}_{j i}$ such that $\hat{y}_{i i}=0$ and $\sum_{j} \hat{y}_{i j} \leq b_{i}$ for all $i$; and $\delta \in\left(0, \frac{1}{16}\right]$, define: $\hat{\lambda}_{U}=\frac{\sum_{(i, j): i, j \in U} \hat{y}_{i j}}{\hat{b}_{U}}$ where $\tilde{b}_{U}=\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor-$ $\frac{\delta^{2}\|U\|_{b}^{2}}{4}$ and $\hat{\lambda}=\max _{U \in \mathcal{O}_{\delta}} \hat{\lambda}_{U}$. If $\hat{\lambda} \geq 1+3 \delta$ we can find the set $L_{2}=\left\{U: \hat{\lambda}_{U} \geq \hat{\lambda}-\delta^{3} / 10 ; U \in \mathcal{O}_{\delta}\right\}$ in $O\left(m^{\prime}+n \operatorname{poly}\left(\delta^{-1}, \log n\right)\right)$ time using $O\left(n \delta^{-5}\right)$ space where $m^{\prime}=\left|\left\{(i, j) \mid \hat{y}_{i j}>0\right\}\right|$.
Proof: We first observe that $L_{2}$ is a laminar family using Theorem 5 and $L_{2} \subseteq L_{1}$. Second, observe that for any $U$ we have $\sum_{(i, j): i, j \in U} \hat{y}_{i j} \leq \frac{1}{2} \sum_{i \in U} \sum_{j} \hat{y}_{i j} \leq \frac{1}{2} \sum_{i \in U} b_{i}=\|U\|_{b} / 2$. Therefore $\hat{\lambda} \leq \frac{3}{2} /\left(1-\frac{\delta^{2}}{4}\right)<\frac{3}{2}+\delta^{2}$. We maintain an estimate $\tilde{\lambda}$ of such that $\tilde{\lambda}-\frac{\delta^{3}}{100}<\hat{\lambda} \leq \tilde{\lambda} \leq \frac{3}{2}+\delta^{2}$. This estimate can be found using binary search (as described below) We now show how to find the sets $U \in L_{2}$ with $\|U\|_{b}=\ell$, denoted by $L_{2}(\ell)$.
Create a graph $G_{\varphi}$ with $p_{i j}=\left\lfloor\varphi \hat{y}_{i j}\right\rfloor$ parallel edges between $i$ and $j$ where $\varphi=50 / \delta^{4}$ (this parameter can be optimized but we omit that in the interest of simplicity). This is an unweighted graph. This graph can be constructed in a single pass over $\{(i, j)\}$. We also "merge" all pairs of vertices $i$ and $j$ if $p_{i j}$ exceeds $2 \varphi$. Moreover delete vertices $i$ with $2 \varphi / \delta$ edges - note that these vertices must have $b_{i} \geq \sum_{j} \hat{y}_{i j} \geq 1 / \delta$ and cannot participate in any odd set in $\mathcal{O}_{\delta}$. This gives us a graph $G_{\varphi}$ with at most $O\left(n \delta^{-5}\right)$ edges.

Now for an odd $\ell \in[3,1 / \delta]$ and $\tilde{\lambda}$, create $G_{\varphi}(\ell, \tilde{\lambda})$ as follows: We begin with $G_{\varphi}$. Let $q_{i}(\ell)=$ $\left\lfloor\varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right) b_{i}\right\rfloor$ for all $i$. Since $q_{i}(\ell)>(1+\delta) \varphi b_{i}>\sum_{j} p_{i j}$ (because $\tilde{\lambda}$ is large) we can add a new node $r(\ell)$ and add $q_{i}(\ell)-\sum_{j} p_{i j}$ edges between $r(\ell)$ and $i$ (for all $\left.i\right)$. This gives us a graph $G_{\varphi}(\ell, \tilde{\lambda})$ of size $O\left(n \delta^{-5}\right)$ edges for all $\ell$. Let $\kappa(\ell)=\left\lfloor\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)\right\rfloor+\frac{12 \ell}{\delta}+1<2 \varphi$. Now:

$$
q_{i}(\ell)-\kappa(\ell) \geq \varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right)-1-\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)-\frac{12 \ell}{\delta}-1=\frac{\varphi \tilde{\lambda} \delta^{2} \ell(\ell-2)}{2}-\frac{12 \ell}{\delta}-2
$$

which is positive for $\varphi=50 / \delta^{4}$ and $\ell \geq 3$. Therefore $q_{i}(\ell)>\kappa(\ell)$.
Define $\operatorname{Cut}(U)$ to be the cut induced by $U$ in $G_{\varphi}(\ell, \tilde{\lambda})$, that is, $\mathcal{C u t}(U)=\sum_{i \in U} q_{i}-2 \sum_{(i, j): i, j \in U} p_{i j}$. We now show that for $\|U\|_{b}>1 / \delta, \operatorname{Cut}(U)>\kappa(\ell)$. For any odd set $U \in \mathcal{O}$ with $\|U\|_{b}>1 / \delta$ :

$$
\begin{aligned}
\operatorname{Cut}(U)-\kappa(\ell) & =\sum_{i \in U} q_{i}(\ell)-2 \sum_{(i, j): i, j \in U} p_{i j}-\kappa(\ell) \\
& \geq \sum_{i \in U}\left(\varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right) b_{i}-1\right)-2 \varphi \sum_{(i, j): i, j \in U} \hat{y}_{i j}-\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)-\frac{12 \ell}{\delta}-1 \\
& \geq \varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right)\|U\|_{b}-|U|-\varphi\|U\|_{b}-\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)-\frac{12 \ell}{\delta}-1 \quad\left(\text { Since } \sum_{(i, j): i, j \in U} \hat{y}_{i j} \leq\|U\|_{b} / 2\right) \\
& \geq \varphi \tilde{\lambda}(1-\delta)\|U\|_{b}-\varphi\|U\|_{b}-\varphi \tilde{\lambda}-\delta^{2} \varphi\|U\|_{b} \quad\left(\text { Since } \ell \leq 1 / \delta \text { and } \delta^{2} \varphi\|U\|_{b}>|U|+\frac{12 \ell}{\delta}+1\right) \\
& =\varphi\left(\tilde{\lambda}(1-\delta)\|U\|_{b}-\tilde{\lambda}-\left(1+\delta^{2}\right)\|U\|_{b}\right) \\
& \geq \varphi\left((1+3 \delta)(1-\delta)\|U\|_{b}-\frac{3}{2}-\delta^{2}-\left(1+\delta^{2}\right)\|U\|_{b}\right) \quad\left(\text { Since } 1+3 \delta \leq \hat{\lambda} \leq \tilde{\lambda} \leq \frac{3}{2}+\delta^{2}\right) \\
& >\varphi\left(2 \delta(1-2 \delta)\|U\|_{b}-\frac{3}{2}-\delta^{2}\right)>\varphi\left(2(1-2 \delta)-\frac{3}{2}-\delta^{2}\right)>0 \quad\left(\text { Since } \delta\|U\|_{b}>1\right)
\end{aligned}
$$

where the last inequality follows $\delta \in\left(0, \frac{1}{16}\right]$. Therefore no odd-set with $\|U\|_{b}>1 / \delta$ satisfies $\operatorname{Cut}(U) \leq \kappa(\ell)$.

We now show Property 1. namely: If $\tilde{\lambda}-\frac{\delta^{3}}{100}<\hat{\lambda} \leq \tilde{\lambda}$, then (i) all sets in $L_{2}(\ell)$ have a cut which is at most $\kappa(\ell)$ and (ii) all odd sets of $G_{\varphi}(\ell, \tilde{\lambda})$ which do not contain $s$ and have cut at most $\kappa(\ell)$ belong to $L_{1}(\ell)$. For part (i) for a set $U \in L_{2}(\ell)$ with $\|U\|_{b}=\ell$, note $|U| \leq\|U\|_{b}=\ell$ and:

$$
\begin{aligned}
\operatorname{Cut}(U) & =\sum_{i \in U} q_{i}-2 \sum_{(i, j): i, j \in U} p_{i j} \leq \sum_{i \in U} \varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right) b_{i}-2 \varphi \sum_{(i, j): i, j, j \in U} \hat{y}_{i j}+|U|^{2} \\
& \leq \varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right)\|U\|_{b}-2 \varphi \hat{\lambda}_{U} \tilde{b}_{U}+\ell^{2} \leq \varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right)\|U\|_{b}-2 \varphi\left(\tilde{\lambda}-\frac{\delta^{3}}{100}-\frac{\delta^{3}}{10}\right) \tilde{b}_{U}+\ell^{2} \\
& =\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)+\frac{11 \delta^{3} \varphi \tilde{b}_{U}}{50}+\ell^{2}=\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)+\frac{11 \tilde{b}_{U}}{\delta}+\ell^{2} \\
& \leq \varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)+\frac{12 \ell}{\delta} \leq \kappa(\ell) \quad\left(\text { since } \tilde{b}_{U}<\|U\|_{b}=\ell \leq 1 / \delta\right)
\end{aligned}
$$

To prove part (ii) if $\operatorname{Cut}(U) \leq \kappa(\ell)$ then:

$$
\begin{aligned}
\sum_{(i, j): i, j \in U} p_{i j} & =\frac{1}{2}\left(\sum_{i \in U} q_{i}-\operatorname{Cut}\left(U^{\prime}\right)\right) \geq \frac{1}{2}\left(\sum_{i \in U}\left(\varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right) b_{i}-1\right)-\kappa(\ell)\right) \\
& \geq \frac{1}{2}\left(\sum_{i \in U}\left(\varphi \tilde{\lambda}\left(1-\delta^{2} \ell\right) b_{i}-1\right)-\varphi \tilde{\lambda}\left(1-\delta^{2} \ell^{2} / 2\right)\right)-\frac{12 \ell}{\delta}-1 \\
& \left.\geq \varphi \tilde{\lambda}\left(\left\lvert\, \frac{\|U\|_{b}}{2}\right.\right\rfloor-\frac{\delta^{2}\|U\|_{b}^{2}}{4}\right)+\frac{\varphi \tilde{\lambda} \delta^{2}}{4}\left(\|U\|_{b}-\ell\right)^{2}-\frac{|U|}{2}-\frac{12 \ell}{\delta}-1 \\
& =\varphi \tilde{\lambda} \tilde{\lambda}_{U}+\frac{\varphi \tilde{\lambda} \delta^{2}}{4}\left(\|U\|_{b}-\ell\right)^{2}-\frac{|U|}{2}-\frac{12 \ell}{\delta}-1
\end{aligned}
$$

But since $\tilde{\lambda} \geq \hat{\lambda} \geq \hat{\lambda}_{U}$ and $\varphi \tilde{b}_{U} \hat{\lambda}_{U}=\varphi \sum_{(i, j): i, j \in U} \hat{y}_{i j} \geq \sum_{(i, j): i, j \in U} p_{i j}$ we have

$$
\begin{equation*}
\varphi \tilde{\lambda} \tilde{b}_{U} \geq \varphi \hat{\lambda}_{U} \tilde{b}_{U} \geq \varphi \tilde{\lambda} \tilde{b}_{U}+\frac{\varphi \tilde{\lambda} \delta^{2}}{4}\left(\|U\|_{b}-\ell\right)^{2}-\frac{|U|}{2}-\frac{12 \ell}{\delta}-1 \tag{13}
\end{equation*}
$$

But that is a contradiction unless $\|U\|_{b}=\ell$, otherwise the quadratic term, $\frac{\varphi \tilde{\delta} \delta^{2}}{4}\left(\|U\|_{b}-\ell\right)^{2} \geq$ $12.5 \delta^{-2}$ is larger than the negative terms which are at most $\frac{1}{2 \delta}+\frac{12}{\delta^{2}}+1$ in the RHS of Equation 13 . Therefore $\operatorname{Cut}(U) \leq \kappa(\ell)$ for an odd-set implies $\|U\|_{b}=\ell$. But then Equation 13 implies (again using $|U| \leq\|U\|_{b}=\ell$ ):

$$
\varphi \hat{\lambda}_{U} \tilde{b}_{U} \geq \varphi \tilde{\lambda} \tilde{b}_{U}-\frac{\ell}{2}-\frac{12 \ell}{\delta}-1
$$

Now $\tilde{b}_{U} \geq \frac{\ell}{3}\left(1-\frac{3 \delta}{4}\right)$ when $\|U\|_{b}=\ell \geq 3$; thus:

$$
\hat{\lambda}_{U} \geq \tilde{\lambda}-\frac{\ell}{2 \varphi \tilde{b}_{U}}-\frac{12 \ell}{\delta \varphi \tilde{b}_{U}}-\frac{1}{\varphi \tilde{b}_{U}} \geq \tilde{\lambda}-\frac{3 \delta^{4}}{100\left(1-\frac{3 \delta}{4}\right)}-\frac{36 \delta^{3}}{50\left(1-\frac{3 \delta}{4}\right)}-\frac{\delta^{4}}{50}>\tilde{\lambda}-\delta^{3} \geq \hat{\lambda}-\delta^{3}
$$

in other words, $\mathcal{C u t}(U) \leq \kappa(\ell)$ for an odd-set implies $U \in L_{1}(\ell)$, as claimed in part(ii).
We now apply Lemma 19 to extract a collection $\bar{L}(\ell)$ of odd-sets in $G_{\varphi}(\ell, \tilde{\lambda})$, not containing $r(\ell)$ and cut at most $\kappa(\ell)$ - such that any such set which is not chosen must intersect with some set in the collection $\bar{L}(\ell)$.

If we have a maximal collection $\bar{L}(\ell)$ then $\bar{L}(\ell) \subseteq L_{1}(\ell)$ by part (ii) of Property 1 Due to Theorem 5. the intersection of two such sets $U_{1}, U_{2} \in L_{1}(\ell)$ will be either empty or of size
$\ell$ by laminarity - the latter implies $U_{1}=U_{2}$. Therefore the sets in $L_{1}(\ell)$ are disjoint. Any $U \in L_{2}(\ell)-\bar{L}(\ell)$ has a cut of size at most $\kappa(\ell)$ using part (i) of Property 1 and therefore must intersect with some set in $\bar{L}(\ell)$. This is impossible because $U \in L_{2}(\ell)$ implies $U \in L_{1}(\ell)$ and $\bar{L}(\ell) \subseteq L_{1}(\ell)$ and we just argued that the sets in $L_{1}(\ell)$ are disjoint. Therefore no such $U$ exists and $L_{2}(\ell) \subseteq \bar{L}(\ell)$. We now have a complete algorithm: we perform a binary search over the estimate $\tilde{\lambda} \in\left[1+3 \delta, \frac{3}{2}+\delta^{2}\right]$, and we can decide if there exists a set $U \in L_{2}(\ell)$ in time $O\left(n \operatorname{poly}\left(\delta^{-1}, \log n\right)\right)$ as we vary $\ell, \tilde{\lambda}$. This gives us $\tilde{\lambda}$. We now find the collections $\bar{L}(\ell)$ for each $\ell$ and compute all $\hat{\lambda}_{U}$ exactly. We can now return $\cup_{\ell} L_{2}(\ell)$. Observe that $G_{\varphi}$ does not need to be constructed more than once; it can be stored and reused. The running time follows from simple counting.

## 6 Rounding Uncapacitated $b$-matchings

Theorem 2 , (Integral b-matching) Given a fractional b-matching y for a non-bipartite graph which satisfies LP1(b) (parametrized over $\mathbf{b})$ where $\left|\left\{(i, j) \mid y_{i j}>0\right\}\right|=m^{\prime}$, we find an integral b-matching of weight at least $(1-2 \delta) \sum_{(i, j)} w_{i j} y_{i j}$ in $O\left(m^{\prime} \delta^{-3} \log (1 / \delta)\right)$ time and $O\left(m^{\prime} / \delta^{2}\right)$ space.

```
Algorithm 4 Rounding a fractional \(b\)-matching
    1: First Phase: (large multiplicities) Let \(t=\lceil 2 / \delta\rceil\) and \(\mathcal{M}^{(0)}=\emptyset\).
    (a) If \(y_{i j} \geq t\) add \(\hat{y}_{i j}^{(0)}=\left\lfloor y_{i j}\right\rfloor-1\) copies of \((i, j)\) to \(\mathcal{M}^{(0)}\).
    (b) Set \(y_{i j}^{(1)}=0\) if \(y_{i j} \geq t\) and \(y_{i j}^{(1)}=y_{i j}\) otherwise.
    (c) Let \(b_{i}^{(1)}=\min \left\{b_{i}-\sum_{j} \hat{y}_{i j}^{(0)},\left\lceil\sum_{j} y_{i j}^{(1)}\right\rceil+1\right\}\).
```

2: Second Phase: (large capacities)
(a) While $\exists i$ s.t. $\sum_{j} y_{i j}^{(1)} \geq 3 t$ do
(i) Order the vertices adjacent to $i$ arbitrarily. Select the prefix $S$ in that order such that the sum is between $t$ and $2 t$ (each edge is at most $t$ from Step 1 b ). Create a new copy $i^{\prime}$ of $i$ with this prefix and $y_{i^{\prime} j}^{(1)}=y_{i j}^{(1)}$ for $j \in S$ and delete the edges from $S$ incident to $i$. Observe that the procedure describes a process where given a set of numbers $q_{1}, \ldots, q_{k}$ such that each $q_{j} \leq 1$ and $\sum_{j} q_{j}=Y \geq 3$; we partition the set of numbers such that each partition $S$ satisfies $1 \leq \sum_{j \in S} q_{j} \leq 2$.
(b) If no copies of $i$ were created then $b_{i}^{(2)}=b_{i}^{(1)}$. For every new $i^{\prime}$ (corresponding to $i$ ) created from the partition $S$ (which may have now become $S^{\prime}$ with subsequent splits), assign $b_{i^{\prime}}^{(2)}=\left\lfloor\sum_{j \in S^{\prime}} y_{i j}^{(1)}\right\rfloor$. Note $b_{i}^{(2)} \leq 3 t$ for all vertices. We now have a vertex set $V^{(2)}$. Set $y_{i j}^{(2)}=(1-\delta) y_{i j}^{(1)}$ for $i, j \in V^{(2)}$.

## 3: Third Phase: Reduction to weighted matching.

(a) For each $i \in V^{(2)}$ with $b_{i}^{(2)}$, create $i(1), i(2), \cdots, i\left(b_{i}^{(2)}\right)$.
(b) For each edge $(i, j)$, create a complete bipartite graph between $i(1), i(2), \cdots$ and $j(1), j(2), \cdots$ with every edge having weight $w_{i j}$. Let this new graph be $G^{(3)}$.
(c) Run any fast approximation for finding a $(1-\epsilon)$-approximate maximum weighted matching in $G^{(3)}$ let this matching be $\mathcal{M}^{(3)}$. Matching $\mathcal{M}^{(3)}$ provides a $b$-matching $\mathcal{M}^{(2)}$ in $G^{(2)}$ of same weight (merge edges). (ii) Matching $\mathcal{M}^{(2)}$ provides a $b$-matching $\mathcal{M}^{(1)}$ in $G^{(1)}$ of same weight (merge vertices).
4: Output: $\mathcal{M}^{(0)} \cup \mathcal{M}^{(1)}$.

As an example of Step $3(\mathrm{~b})$, consider


The algorithm is given in Algorithm 4. We begin with the following lemma:
Lemma 22. (First Phase and the Output Phase) Suppose that all vertex constraints are satisfied and $\sum_{j} y_{i j} \leq b_{i}-1$ for some $i \in V$. Then, for any odd set $U$ that contains $i$, the corresponding odd set constraint is satisfied. The fractional solution $\left\{y_{i j}^{(1)}\right\}$ obtained in the first phase of Algorithm 4 is feasible for LP1 $\left(\mathbf{b}^{(\mathbf{1})}\right)$ - and an integral $\mathcal{M}^{(1)}$ which is a $(1-2 \delta)$-approximation of LP1 $\left(\mathbf{b}^{(\mathbf{1})}\right)$ can be output along with $\mathcal{M}^{(0)}$.
Proof: For any $U \in \mathcal{O}, i \in U, \sum_{i^{\prime}, j \in U} y_{i^{\prime} j} \leq \frac{1}{2} \sum_{i^{\prime} \in U} \sum_{j} y_{i^{\prime} j} \leq \frac{1}{2}\left(\left(\sum_{i^{\prime} \in U} b_{i^{\prime}}\right)-1\right)=\frac{\|U\|_{b}-1}{2}=\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor$. Thus it follows that any vertex which has an edge incident to it in $\mathcal{M}^{(0)}$ cannot be in any violated odd-set in LP1 $\left(\mathbf{b}^{(\mathbf{1})}\right)$. Then any violated odd-set in LP1 $\left(\mathbf{b}^{(\mathbf{1})}\right)$ with respect to $\left\{y_{i j}^{(1)}\right\}$ must also be a violated odd-set in LP1(b); contradicting the fact that we started with a $\left\{y_{i j}\right\}$ is feasible for LP1(b). Now $\mathcal{M}^{(0)} \cup \mathcal{M}^{(1)}$ is feasible since both are integral and we know that $b_{i}^{(1)} \leq b_{i}-\sum_{j} \hat{y}_{i j}^{(0)}$. Observe that $w\left(\mathcal{M}^{(0)}\right) \geq(1-\delta) \sum_{(i, j) \in E} w_{i j}\left(y_{i j}-y_{i j}^{(1)}\right)$ where $w\left(\mathcal{M}^{(0)}\right)=\sum_{(i, j) \in E} \hat{y}_{i j}^{(0)} w_{i j}$. Therefore if $w\left(\mathcal{M}^{(1)}\right) \geq(1-2 \delta) \sum_{(i, j) \in E} w_{i j} y_{i j}^{(1)}$ then $w\left(\mathcal{M}^{(0)}\right)+w\left(\mathcal{M}^{(1)}\right)$ is at least $(1-2 \delta) \sum_{(i, j) \in E} w_{i j} y_{i j}$ as desired.

Lemma 23. (Second Phase) If $\left\{y_{i j}^{(1)}\right\}$ satisfies $\overline{\mathrm{LP} 1}\left(\mathbf{b}^{(\mathbf{1})}\right)$ over $V$, then $\left\{y_{i j}^{(2)}\right\}$ satisfies $\overline{\mathrm{LP} 1}\left(\mathbf{b}^{(\mathbf{2})}\right)$ over $G^{(2)}$ and $\sum_{i, j} w_{i j} y_{i j}^{(2)}=(1-\delta) \sum_{i, j} w_{i j} y_{i j}^{(1)}$.
Proof: Observe that any vertex which participates in any split produces vertices which have (fractionally) at least $t$ edges. After scaling we have (1- $\delta) \sum_{j} y_{i j}^{(1)} \leq \sum_{j} y_{i j}^{(1)}-\delta t \leq \sum_{j} y_{i j}^{(1)}-2 \leq b_{i}^{(2)}-1$ from the definition of $b^{(2)}$ in line (3b) of Algorithm 4 . Therefore the new vertices cannot be in any violated vertex or set constraint; from the first part of Lemma 22 (now applied to LP1 $\left(\mathbf{b}^{(2)}\right)$ instead of LP1(b)). Therefore the Lemma follows.
Finally, observe that any integral $b$-matching in $G^{(2)}$ has an integral matching in $G^{(3)}$ of the same weight and vice versa - moreover given a matching for $G^{(3)}$ the integral $b$-matching for $G^{(2)}$ can be constructed trivially. Also, the number of edges in $G^{(3)}$ is at most $O\left(\delta^{-2} m^{\prime}\right)$ since each vertex in $G^{(2)}$ is split into $O\left(\delta^{-1}\right)$ vertices in $G^{(3)}$. We are guaranteed a maximum $b$-matching in $G^{(2)}$ of weight at least $\sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)}$ since $\left\{y_{i j}^{(2)}\right\}$ satisfies $\mathbb{L P 1}\left(\mathbf{b}^{(2)}\right)$ over $G^{(2)}$. Therefore we are guaranteed a matching of the same weight in $G^{(3)}$. Now, we use the approximation algorithm in [9, 10] which returns a $(1-\delta)$-approximate maximum weighted matching in $G^{(3)}$ in $O\left(m^{\prime} \delta^{-3} \log (1 / \delta)\right)$ time and space. From the $(1-\delta)$-approximate maximum matching we construct a $b$-matching in $G^{(2)}$ of the same weight (and therefore a $b$-matching $\mathcal{M}^{(1)}$ in $G^{(1)}$ of the same weight). Theorem 2 follows.

## 7 The Capacitated $b$-Matching Problem

Definition 2. [29, Chapters 32 83 33] The Capacitated b-matching problem is a b-matching problem where we have an additional restriction that the multiplicity of an edge $(i, j) \in E$ is at
most $c_{i j}$. The vertex and edge capacities $\left\{b_{i}\right\},\left\{c_{i j}\right\}$ are given as input and for this paper are assumed to be integers in $[0$, poly $n]$. Observe that we can assume $c_{i j} \leq \min \left\{b_{i}, b_{j}\right\}$ without loss of generality.
Long and Short Representations: We follow the reduction of the capacitated problem to the uncapacitated problem outlined in [29, Chapter 32], with modifications.

Definition 7. Given a graph $G=(V, E)$ with vertex and edge capacities. Consider subdividing each edge $e=(i, j)$ into $\left(i, p_{i j, i}\right),\left(p_{i j, i}, p_{i j, j}\right),\left(p_{i j, j}, j\right)$ where $p_{i j, i}, p_{i j, j}$ are new additional vertices with capacity $b_{p_{i j, i}}^{c}=b_{p_{i j, j}}^{c}=c_{i j}$. For $i \in V$ set $b_{i}^{c}=b_{i}$. We use the weights denoted by $\mathbf{w}^{c}$ to be $\frac{1}{2} w_{i j}, 0, \frac{1}{2} w_{i j}$ for $\left(i, p_{i j, i}\right),\left(p_{i j, i}, p_{i j, j}\right),\left(p_{i j, j}, j\right)$ respectively. Let the transformation of $G$ be denoted as Long $(G)$; let the vertices and edges of Long $(G)$ be $V^{c}$ and $E^{c}$ respectively. LONG $(G)$ does not have any edge capacities.

For $U^{c} \subseteq V^{c}$ let $\left\|U^{c}\right\|_{b}=\sum_{s \in U^{c}} b_{s}^{c}$ as before. The odd-sets in $\operatorname{LONG}(G)$, i.e. $U^{c}$ such that $\left\|U^{c}\right\|_{b}$ is odd, are denoted by $\mathcal{O}^{c}$ and define $\mathcal{O}_{\delta}^{c}=\left\{U^{c} \in \mathcal{O}^{c},\left\|U^{c}\right\|_{b} \leq 1 / \delta\right\}$.

The above transformation is inspired by the proof of [29, Theorem 32.4, Vol A, page 567] which used the weights $w_{i j}, w_{i j}, w_{i j}$ instead of $\frac{1}{2} w_{i j}, 0, \frac{1}{2} w_{i j}$ for $\left(i, p_{i j, i}\right),\left(p_{i j, i}, p_{i j, j}\right),\left(p_{i j, j}, j\right)$ respectively. In fact [29, Theorem 32.4] computes an optimum solution of value $\beta^{*, c}+\sum_{(i, j) \in E} w_{i j} c_{i j}$. However an approximation of $\beta^{*, c}+\sum_{(i, j) \in E} w_{i j} c_{i j}$ need not provide an approximation of $\beta^{*, c}$ because $\sum_{(i, j) \in E} w_{i j} c_{i j}$ can be significantly larger. We will eventually use the algorithm in [29, Theorem 32.4] to find an integral solution in Section 7.2. We need to bound $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j}$ where $\hat{E}$ is the edgeset in our candidate fractional solution. An example of the transformation is as follows (the edges only have weight in the new graph).


Notation: We will use $i, j$ to denote vertices (and edges) in the original graph $G$ and use $s, r, u, v$ to denote vertices (and edges) in Long $(G)$. We will use the superscript such as $y^{c}, U^{c}$ to indicate variables, subsets in Long $(G)$ to distinguish them from $G$. However we can switch between $G$ and Long $(G)$ as described next.

Definition 8. Let $\lambda_{0}^{c}$ be a parameter which is determined later. Define:

$$
\operatorname{LoNG}\left(\mathcal{Q}^{c}\right):\left\{\begin{array}{ll}
\sum_{r:(s, r) \in E^{c}} y_{s r}^{c} \leq b_{s}^{c} & \forall s \in V^{c} \\
y_{i_{i j, i}}^{c}+y_{p_{i j,}, p_{i j, j}}^{c}=c_{i j} & \forall(i, j) \in E \\
y_{p_{j, i}, p_{i j, j}}^{c}+y_{p_{i, j}, j}^{c}=c_{i j} & \forall(i, j) \in E \\
y_{s r}^{c} \geq 0 & \forall(s, r) \in E^{c}
\end{array} \quad \text { and } \quad \mathcal{Q}^{c}: \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq b_{i} & \forall i \in V \\
y_{i j} \leq c_{i j} & \forall(i, j) \in E \\
y_{i j} \geq 0 & \forall(i, j) \in E\end{cases}\right.
$$

And likewise:
$\operatorname{LoNG}\left(\mathcal{P}^{c}\right):\left\{\begin{array}{ll}\sum_{r:(s, r) \in E_{L}} y_{s r}^{c} \leq \lambda_{0}^{c} b_{s}^{c} / 2 & \forall s \in V^{c} \\ y_{p_{i j, i}}^{c}+y_{p_{i j, j}, p_{i j, j}=c_{i j}}^{c} & \forall(i, j) \in E \\ y_{p_{j, i}, p_{i j, j}}^{c}+y_{p_{i j, j}}^{c}=c_{i j} & \forall(i, j) \in E \\ y_{s r}^{c} \geq 0 & \forall(s, r) \in E^{c}\end{array} \quad\right.$ and $\quad \mathcal{P}^{c}: \begin{cases}\sum_{j:(i, j) \in E} y_{i j} \leq \lambda_{0}^{c} b_{i} / 2 & \forall i \in V \\ y_{i j} \leq c_{i j} & \forall(i, j) \in E \\ y_{i j} \geq 0 & \forall(i, j) \in E\end{cases}$
Given $\mathbf{y}^{c} \in \operatorname{Long}\left(\mathcal{P}^{c}\right)$ define $\operatorname{Short}\left(\mathbf{y}^{c}\right)$ as $y_{i j} \leftarrow y_{i, p_{i j, i}}^{c}\left(=y_{p_{i, j, j}}^{c}\right)$. Observe that $\operatorname{Short}\left(\mathbf{y}^{c}\right) \in$ $\mathcal{P}^{c}$. Likewise given a $\mathbf{y} \in \mathcal{P}^{c}$ define LONG (y) as $y_{i, p_{i, i}}^{c}, y_{p_{i j, j}, j}^{c} \leftarrow y_{i j}$ and $y_{p_{i j, i}, p_{i j, j}} \leftarrow\left(c_{i j}-\right.$ $\left.y_{i j}\right)$. Observe that $\operatorname{Long}(\mathbf{y}) \in \operatorname{LONG}\left(\mathcal{P}^{c}\right)$. Moreover $\operatorname{LoNG}(\cdot), \operatorname{Short}(\cdot)$ are inverse operations;
$\operatorname{Short}\left(\mathbf{y}^{c}\right)=\mathbf{y}$ iff $\operatorname{Long}(\mathbf{y})=\mathbf{y}^{c}$ and define bijections between $\operatorname{Long}\left(\mathcal{P}^{c}\right), \mathcal{P}^{c}$ and between Long ( $\mathcal{Q}^{c}$ ), $\mathcal{Q}^{c}$.

Moreover for any $\mathbf{y}^{c} \in \operatorname{Long}\left(\mathcal{P}^{c}\right)$ (therefore also Long ( $\left.\mathcal{Q}^{c}\right)$ ) we have $\mathbf{w}^{T} \operatorname{Short}\left(\mathbf{y}^{c}\right)=\left(\mathbf{w}^{c}\right)^{T} \mathbf{y}^{c}$. Similarly for any $\mathbf{y} \in \mathcal{P}^{c}$ (therefore also $\left.\mathcal{Q}^{c}\right)\left(\mathbf{w}^{c}\right)^{T} \operatorname{LONG}(\mathbf{y})=\mathbf{w}^{T} \mathbf{y}$.
The next theorem provides the linear program we will use for capacitated $b$-matching.
Theorem 24. The maximum integral weighted capacitated b-matching problem is expressed by the following linear programming relaxation on LONG $(G)$.

$$
\begin{align*}
& \beta^{*, c}=\max \sum_{(s, r) \in E^{c}} w_{s r}^{c} y_{s r}^{c} \\
& \left\{\mathbf{A}^{c} \mathbf{y}^{c} \leq \mathbf{b}^{c}\right\}= \begin{cases}\sum_{r:(s, r) \in E^{c}} y_{s r}^{c} \leq b_{s}^{c} & \forall s \in V^{c} \\
\sum_{(s, r) \in E^{c}: s, r \in U} y_{s r}^{c} \leq\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor & \forall U^{c} \in \mathcal{O}_{\delta}^{c}\end{cases} \\
& \sum_{(s, r) \in E^{c}: s, r \in U} y_{s r}^{c} \leq\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor \quad \forall U^{c} \in \mathcal{O}^{c}-\mathcal{O}_{\delta}^{c}  \tag{LP9}\\
& \operatorname{LONG}\left(\mathcal{Q}^{c}\right)= \begin{cases}\sum_{r:(s, r) \in E^{c}} y_{s r}^{c} \leq b_{s}^{c} & \forall s \in V^{c} \\
y_{i p_{j, i}}^{c}+y_{p_{i j}, i p_{i, j}}^{c}=c_{i j} & \forall(i, j) \in E \\
y_{p_{i, j, i}}^{c} p_{i j, j}+y_{p_{i j, j} j}^{c}=c_{i j} & \forall(i, j) \in E \\
y_{s r}^{c} \geq 0 & \forall(s, r) \in E^{c}\end{cases}
\end{align*}
$$

The final solution is given by $\mathbf{y} \leftarrow \operatorname{Short}\left(\mathbf{y}^{c}\right)$. Some of the constraints are redundant by design.
Proof: Given an integral feasible solution $\mathbf{y}$ for capacitated $b$-matching, the constraints $\left\{\mathbf{A}^{c} \operatorname{LONG}(\mathbf{y}) \leq\right.$ $\left.\mathbf{b}^{c}\right\}$ hold because LONG $(\mathbf{y})$ defines an integral uncapacitated $b$-matching over LONG $(G)$. The new constraints Long ( $\mathcal{Q}^{c}$ ) are satisfied since $\mathbf{y}$ is feasible, i.e., $\mathbf{y} \leq \mathbf{c}$. Note that the objective function value does not change as a consequence of Definition 7. This proves that $\beta^{*, c}$ is an upper bound on the maximum capacitated integral $b$-matching.

In the reverse direction, given a fractional solution $\mathbf{y}^{c}$ with objective value $\beta^{*, c}$, observe that $\mathbf{y}^{c}$ satisfies the conditions of being in the uncapacitated $b$-matching polytope of LONG $(G)$ (recall these constraints are in LP1). Therefore $\mathbf{y}^{c}$ can be expressed as a convex combination of integral uncapacitated $b$-matchings over $\operatorname{Long}(G)$. Since $\mathbf{y}^{c}$ satisfies that the vertex capacities in $V^{c}-V$ as an equality (see LONG $\left(\mathcal{Q}^{c}\right)$ ) - every integral uncapacitated $b$-matching in the decomposition of $\mathbf{y}^{c}$ must satisfy the vertex capacities $V^{c}-V$ as equality. Therefore there exists at least one integral uncapacitated $b$-matching $\tilde{\mathbf{y}}^{c}$ in the decomposition of $\mathbf{y}^{c}$ which has objective value at least $\beta^{*, c}$ and satisfies the vertex capacities for $V^{c}-V$ as equality. Now $\operatorname{Short}\left(\tilde{\mathbf{y}}^{c}\right)$ is an integral capacitated $b$-matching in $G$ of weight at least $\beta^{*, c}$.
Approximate Satisfiability. Since we will not be satisfy the constraints LP9 exactly the next lemma provides an ability to scale solutions.
Lemma 25. Let $q$ be an arbitrary integer and let $\zeta \geq 1$. Suppose that we have a $\mathbf{y}^{c} \in \operatorname{LoNG}\left(\mathcal{P}^{c}\right)$ which for all $U^{c} \subseteq V^{c}$ in $\operatorname{LONG}(G)$ with $\left\|U^{c}\right\|_{b} \leq q$ satisfies

$$
\zeta\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor \geq \sum_{(s, r) \in E^{c}, s, r \in U^{c}} y_{s r}^{c}
$$

then $\hat{\mathbf{y}}^{c}=\operatorname{LONG}\left(\frac{1}{\zeta} \operatorname{SHORT}\left(\mathbf{y}^{c}\right)\right)$ satisfies for all $U^{c} \subseteq V^{c}$ in $\operatorname{LONG}(G)$ with $\left\|U^{c}\right\|_{b} \leq q$,

$$
\begin{equation*}
\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor \geq \sum_{(s, r) \in E^{c}, s, r \in U^{c}} \hat{y}_{s r}^{c} \tag{14}
\end{equation*}
$$

```
Algorithm 5 An approximation scheme for capacitated \(b\)-matching .
    Define Long \(\left(\mathcal{Q}^{c}\right), \operatorname{Long}\left(\mathcal{P}^{c}\right)\) as in Definition 8 Define \(\mathbf{A}^{c} \mathbf{y}^{c} \leq \widetilde{\mathbf{b}^{c}}\) as:
\[
\left\{\mathbf{A}^{c} \mathbf{y}^{c} \leq \widetilde{\mathbf{b}^{c}}\right\}=\left\{\begin{array}{lll}
\sum_{r:(s, r) \in E E^{c}} y_{s r}^{c} \leq \tilde{b}_{s}^{c} & \forall s \in V^{c}, & \text { where } \tilde{b}_{s}^{c}=(1-4 \delta) b_{s}^{c}  \tag{LP10}\\
\sum_{(s, r) \in E^{c}: s, r \in U} y_{s r}^{c} \leq \tilde{b}_{U} & \forall U^{c} \in \mathcal{O}_{\delta}^{c} & \text { where } \tilde{b}_{U}^{c}=\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor-\frac{\delta^{2}\left\|U^{c}\right\|_{b}^{2}}{4}
\end{array}\right.
\]
Fix \(\delta \in\left(\frac{1}{\sqrt{5 n}}, \frac{1}{16}\right]\). Let \(\lambda_{0}^{c}=16 \ln \frac{2}{\delta}\). Let \(\alpha=50 \delta^{-3} \ln (2 m+n)\).
Find a solution \(\mathbf{y}^{c} \in \operatorname{Long}\left(\mathcal{P}^{c}\right)\) where \(\beta_{0}=\left(\mathbf{w}^{c}\right)^{T} \mathbf{y}^{c}\) and \(\mathbf{A}^{c} \mathbf{y}^{c} \leq \lambda_{0} \widetilde{\mathbf{b}^{c}}\).
Let \(\epsilon=\frac{1}{8}\) (note \(\epsilon \geq \delta\) ) and \(t=0\).
while true do
Define \(\lambda=\max \left\{\max _{i} \lambda_{i}, \max _{U^{c} \in \mathcal{O}_{\delta}^{c}} \lambda_{U^{c}}\right\}\) where \(\begin{cases}\lambda_{s}=\sum_{r:(s, r) \in E^{c}} y_{s r}^{c} / b_{s}^{c} & \forall s \in V^{c} \\ \lambda_{U^{c}}=\sum_{(s, r) \in E^{c: s}, r \in U^{c}} y_{s r}^{c} / \tilde{b}_{U}^{c} & \forall U^{c} \in \mathcal{O}_{\delta}^{c}\end{cases}\)
Find a collection of odd sets \(L^{c}=\left\{U^{c} \mid U^{c} \in \mathcal{O}_{\delta}^{c}, \lambda_{U^{c}} \geq \lambda-\frac{\delta^{3}}{10}\right\}\) (without computing all \(\lambda_{U^{c}}\) ).
If \((\lambda \leq 1+8 \delta)\) output \(\frac{(1-\delta)}{(1+8 \delta)} \operatorname{SHort}\left(\mathbf{y}^{c}\right)\) and stop.
If \(\lambda<1+8 \epsilon\) then a new superphase starts; repeatedly set \(\epsilon \leftarrow \max \{2 \epsilon / 3, \delta\}\) till \(\lambda \geq 1+8 \epsilon\).
Set \(\left\{\begin{array}{l}x_{s}=\exp \left(\alpha \lambda_{s}\right) / \tilde{b}_{s}^{c} \text { if } \lambda_{s}>\lambda-\delta^{3} / 10 \text { and } 0 \text { otherwise } \\ z_{U^{c}}=\exp \left(\alpha \lambda_{\left.U^{c}\right)}\right) / \tilde{b}_{U}^{c} \text { if } \lambda_{U^{c}}>\lambda-\delta^{3} / 10 \text { and } 0 \text { otherwise }\end{array}\right.\). Let \(\gamma^{c}=\sum_{s} x_{s} \tilde{b}_{s}^{c}+\sum_{U^{c} \in \mathcal{O}_{\delta}^{c}} z_{U^{c}} \tilde{b}_{U}^{c}\).
Define \(\eta_{s r}=\left(x_{s}^{c}+x_{r}^{c}+\sum_{U^{c} \in \mathcal{L}^{c} ; s, r \in U^{c}} z_{U^{c}}\right)\). Find a solution \(\widetilde{\mathbf{y}^{c}}\) of LP11 otherwise decrease \(\beta \leftarrow(1-\delta) \beta\).
\[
\begin{equation*}
\left\{\sum_{(s, r) \in E^{c}} w_{s r}^{c} \widetilde{y}^{c}{ }_{s r} \geq(1-\delta) \beta, \quad \sum_{(s, r) \in E^{c}} \widetilde{y}_{s r}^{c} \eta_{s r} \leq \frac{\gamma^{c}}{1-\delta}, \quad \widetilde{\mathbf{y}^{c}} \in \operatorname{LoNG}\left(\mathcal{P}^{c}\right)\right\} \tag{LP11}
\end{equation*}
\]
12: \(\quad\) Set \(\mathbf{y}^{c} \leftarrow(1-\sigma) \mathbf{y}^{c}+\sigma \widetilde{\mathbf{y}^{c}}\) where \(\sigma=\epsilon /\left(4 \alpha \lambda_{0}^{c}\right)\). end while
```

Proof: Suppose not. Consider the subset $U^{c}$ with the smallest $\left\|U^{c}\right\|_{b}$ which violates the assertion 14 . Observe that $U^{c}$ cannot contain both $p_{i j, i}, p_{i j, j}$ for any edge $(i, j) \in E$ (in the original $G$ ). Because in that case $U^{c}-\left\{p_{i j, i}, p_{i j, j}\right\}$ will be a smaller set which violates the assertion - since the LHS of Equation 14 will decrease by $c_{i j}$ as well as the RHS! But if $U^{c}$ does not contain both $p_{i j, i}, p_{i j, j}$ for any edge $(i, j) \in E$ (in $G$ ) then

$$
\sum_{(s, r) \in E^{c}, s, r \in U^{c}} \hat{y}_{s r}^{c} \leq \frac{1}{\zeta} \sum_{(s, r) \in E^{c}, s, r \in U^{c}} y_{s r}^{c} \leq\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor
$$

which is a contradiction. The lemma follows.
Therefore the scaling operation still succeeds (on Short $\left(\mathbf{y}^{c}\right)$ ) but its proof is more global compared to the proof in the uncapacitated case. Here we are proving the statement for all subsets of a certain size simultaneously, whereas in the uncapacitated case the proof of feasibility of $U^{c}$ followed from the bound of $\sum_{(i, j) \in E, i, j \in U^{c}} y_{i j}$ for that particular subset $U^{c}$ itself.

### 7.1 Algorithm for Capacitated $b$-Matching

The algorithm is provided in Algorithm 5
Note that Step 7 follows from Lemma 17 . Moreover if we adjust $\alpha$ for the number of vertices, Lemma 18 also follows. Note that $\lambda^{*}=\min \left\{\lambda \mid \mathbf{y}^{c} \in \operatorname{LoNG}\left(\mathcal{Q}^{c}\right), \mathbf{A}^{c} \mathbf{y}^{c} \leq \lambda \widetilde{\mathbf{b}^{c}}\right\}$ is not 1. In Lemma 27 we show that $\lambda^{*} \leq 1 /(1-4 \delta)$ and moreover we can always find a solution of LP11 for
$\beta \leq(1-4 \delta) \beta^{*, c}$. However the choice of $\mathbf{A}^{c} \mathbf{y}^{c} \leq \widetilde{\mathbf{b}^{c}}$ implied that we can reuse Lemma 17 and 18 without any modification.

Before discussing the algorithm for LP11 we argue that the returned solution returned in Line 8 of Algorithm 5 is a feasible capacitated $b$-matching. We apply Lemma 25 with $\zeta=(1+8 \delta)$ and $q=1 / \delta$. Consider $\mathbf{y}^{c, \dagger}=\operatorname{Long}\left(\frac{1}{1+8 \delta} \operatorname{Short}\left(\mathbf{y}^{c}\right)\right)$. Since $\mathbf{A}^{c} \mathbf{y}^{c} \leq(1+8 \delta) \tilde{\mathbf{b}}^{c} \leq(1+8 \delta) \mathbf{b}^{c}$. Note that this operation will imply that all the vertex constraints in $V_{c}$ are satisfied as well as constraints corresponding to all $U^{c} \in \mathcal{O}_{\delta}^{c}$. For the odd subsets $U^{c}$ with $\left\|U^{c}\right\|_{b} \geq 1 / \delta$, since the vertex constraints are satisfied we have:

$$
\sum_{(s, r) \in E^{c}, s, r \in U^{c}} y_{s r}^{c, t} \leq \frac{\left\|U^{c}\right\|_{b}}{2} \leq \frac{1}{(1-\delta)}\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor
$$

the violation is at most $\zeta=\frac{1}{(1-\delta)}$ for any odd set. We now apply the lemma again with $\zeta=\frac{1}{(1-\delta)}$ for all odd sets, i.e., $q=\infty$. The result of the two operations compose and correspond to the output in Line 8 .

Solving LP11. We now focus on the algorithm for solving LP11. Before providing the algorithm we prove Lemma 26 which proves structural properties of the weights resulting from the dual thresholding.
Lemma 26. Suppose that $\lambda>1+8 \delta$ (otherwise the algorithm has stopped) and the current candidate solution in Algorithm 5 is $\mathbf{y}^{c}$.
(a) $x_{s}=0$ for any $s \in V^{c}-V$ (the new vertices that are introduced).
(b) Suppose $U^{c} \in \mathcal{O}_{\delta}^{c}$ contains $p_{i j, i}, p_{i j, j}$ for some edge $(i, j) \in E$ (of $G$ ). If neither $i, j \notin U$, $z_{U^{c}}=0$.
(c) If for some edge $(i, j) \in E$ we have $y_{i p_{i j, i}}^{c}=y_{p_{i j, j}}^{c}=0$, then neither $p_{i j, i}, p_{i j, j}$ belong to an add set $U^{c} \in \mathcal{O}_{\delta}^{c}$ with $z_{U^{c}}>0$. As a consequence, We can compute $\mathcal{L}^{c}$ in time $O\left(m^{\prime} \operatorname{poly}\left\{\delta^{-1}, \log n\right\}\right)$ where $m^{\prime}=\left|\left\{(i, j) \mid y_{p_{i j, j}}^{c} \neq 0\right\}\right|$ because the other edges cannot define any odd set in $\mathcal{L}_{c}$.
(d) Let $\operatorname{Short}(\boldsymbol{\eta})_{i j}=\eta_{i p_{i j, i}}+\eta_{p_{i j, j}}-\eta_{p_{i j, i} p_{i j, j}}$. Then $\operatorname{Short}(\boldsymbol{\eta})_{i j} \geq 0$ for every $(i, j) \in E$.
(e) Let $\operatorname{Shift}(\boldsymbol{\eta})=\sum_{(i, j) \in E} c_{i j} \eta_{p_{i j, i}, p_{i j, j}}$ then $\frac{\gamma^{c}}{(1-\delta)} \geq \operatorname{Shift}(\boldsymbol{\eta})$.

Proof: Part (a) follows from the fact that $\lambda_{s}=\frac{1}{1-4 \delta}<\lambda-\delta^{3} / 10$.
For part (b) suppose that $p_{i j, i}, p_{i j, j} \in U^{c}, z_{U^{c}} \neq 0$ for some $U^{c} \in \mathcal{O}_{\delta}^{c}$. Note $\left\|U^{c}\right\|_{b} \leq 1 / \delta$ and thus:

$$
\begin{equation*}
\sum_{(s, r) \in E^{c}: s, r \in U^{c}} y_{c}^{c} \geq\left(\lambda-\frac{\delta^{3}}{10}\right) \tilde{b}_{U}^{c}=\left(\lambda-\frac{\delta^{3}}{10}\right)\left(\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor-\frac{\delta^{2}\left\|U^{c}\right\|_{b, c}^{2}}{4}\right) \geq\left(\lambda-\frac{\delta^{3}}{10}\right) \frac{\left\|U^{c}\right\|_{b}-1}{2}-\lambda \frac{\delta^{2}\|U\|_{b, c}^{2}}{4} \tag{15}
\end{equation*}
$$

Consider $U_{1}^{c}=U^{c}-\left\{p_{i j, j}, p_{i j, i}\right\}$. Let $\left\|U_{1}^{c}\right\|_{b}=\ell$. Note since $\left\|U^{c}\right\|_{b}$ is odd, $\ell \geq 1$ and $\ell$ is odd. $\|U\|_{b}=\ell+2 c_{i j} \geq 3$. Since $y_{p_{i j, i} p_{i j, j}} \leq c_{i j}$ and $i, j \notin U^{c}$,

$$
\begin{equation*}
\sum_{(s, r) \in E^{c}: s, r \in U_{1}^{c}} y_{s r}^{c}=\sum_{(s, r) \in E^{c}: s, r \in U} y_{s r}^{c}-y_{p_{i j, i}, p_{i j, j}} \geq \sum_{(s, r) \in E^{c}: s, r \in U^{c}} y_{s r}^{c}-c_{i j} \tag{16}
\end{equation*}
$$

If $U_{1}^{c}$ is a singleton node then the LHS of Equation (16) is 0 . Using Equation (15) and $3 \leq\|U\|_{b, c} \leq$ $1 / \delta$,

$$
0=\sum_{(s, r) \in E^{c}: s, r \in U^{c}} y_{s r}^{c}-y_{p_{i j, i}, p_{i j, j}} \geq\left(\lambda-\frac{\delta^{3}}{10}\right) \tilde{b}_{U}^{c}-c_{i j} \geq\left(\lambda-\frac{\delta^{3}}{10}\right)(1-\delta)\left\lfloor\frac{\ell+2 c_{i j}}{2}\right\rfloor-c_{i j} \geq c_{i j}\left(\left(\lambda-\frac{\delta^{3}}{10}\right)(1-\delta)-1\right)
$$

which is impossible for $\lambda>1+8 \delta$. If $U_{1}^{c}$ is an odd set, then it is in $\mathcal{O}_{\delta}^{c}$ and thus

$$
\begin{equation*}
\lambda\left(\frac{\ell-1}{2}-\frac{\delta^{2} \ell^{2}}{4}\right)=\lambda \tilde{b}_{U_{1}^{c}} \geq\left(\lambda-\frac{\delta^{3}}{10}\right) \frac{\ell+2 c_{i j}-1}{2}-\lambda \frac{\delta^{2}\left(\ell+2 c_{i j}\right)^{2}}{4}-c_{i j} \tag{17}
\end{equation*}
$$

but Equation 17 rearranges to

$$
\frac{\delta^{3}}{10}\left(\frac{\ell-1}{2}\right)+\lambda c_{i j} \delta^{2}\left(c_{i j}+\ell\right) \geq\left(\lambda-\frac{\delta^{3}}{10}-1\right) c_{i j}
$$

which in turn (if we divide by $c_{i j}$ and use $\ell+2 c_{i j} \leq 1 / \delta$ ) implies $1+\frac{\delta^{3}}{10}+\frac{\delta^{2}}{20} \geq(1-\delta) \lambda$ which is impossible for $\lambda>1+8 \delta$. Part (b) of the Lemma follows.

For part (c), suppose for contradiction, $p_{i j, i} \in U^{c} y_{p_{i j, i} p_{i j, j}}^{c}=c_{i j}$ and $z_{U^{c}}>0$. Observe Equation (15) applies because $z_{U^{c}}>0$. If $p_{i j, j} \in U^{c}$ then we consider $U_{1}^{c}=U^{c}-\left\{p_{i j, i}, p_{i j, j}\right\}$ and in part (b). Equation (16) of part (b) holds irrespective of $i, j \in U^{c}$ because neither $p_{i j, i}, p_{i j, j}$ have nonzero edges in $\mathbf{y}^{c}$ to any other vertex. The remainder of part (b) applies as well and we have a contradiction.

Therefore we need to only consider the case $p_{i j, j} \notin U^{c}$. But then consider $U_{2}^{c}=U^{c}-\left\{p_{i j, i}\right\}$. In this case, since $p_{i j, i}$ has no non-zero edge to any vertex in $U_{2}^{c}$ :

$$
\begin{equation*}
\sum_{(s, r) \in E^{c}: s, r \in U_{2}^{c}} y_{s r}^{c}=\sum_{(s, r) \in E^{c}: s, r \in U^{c}} y_{s r}^{c}>0 \tag{18}
\end{equation*}
$$

Again let $\left\|U_{2}^{c}\right\|_{b}=\ell$, thus $\left\|U^{c}\right\|_{b}=\ell+c_{i j}$. Note $\left\|U^{c}\right\|_{b} \leq 1 / \delta$. Now

$$
\sum_{(s, r) \in E^{c}: s, r \in U_{2}^{c}} y_{s r}^{c} \leq \frac{1}{2} \sum_{s \in U_{2}^{c}} \sum_{r:(s, r) \in E^{c}} y_{s r}^{c} \leq \frac{1}{2} \sum_{s \in U_{2}^{c}} \lambda \tilde{b}^{c}{ }_{s}=\frac{(1-4 \delta) \ell}{2} \lambda
$$

Combining the above with Equations 18 and (first part of) 15

$$
\frac{(1-4 \delta) \ell}{2} \lambda \geq\left(\lambda-\frac{\delta^{3}}{10}\right) \tilde{b}_{U^{c}} \geq\left(\lambda-\frac{\delta^{3}}{10}\right)(1-\delta)\left\lfloor\frac{\left\|U^{c}\right\|_{b}}{2}\right\rfloor=\left(\lambda-\frac{\delta^{3}}{10}\right)(1-\delta)\left\lfloor\frac{\ell+c_{i j}}{2}\right\rfloor
$$

and since $c_{i j} \geq 1$, the above implies

$$
\frac{(1-4 \delta) \ell}{2} \lambda \geq\left(\lambda-\frac{\delta^{3}}{10}\right)(1-\delta) \frac{\ell}{2} \quad \Longrightarrow \quad(1-\delta) \frac{\delta^{3}}{10} \geq 3 \delta \lambda
$$

which is not possible for $\lambda>1+8 \delta$. Part (c) follows.
For part (d) observe that:

$$
\begin{aligned}
\operatorname{Short}(\boldsymbol{\eta})_{i j}= & \left(x_{i}+x_{p_{i j, i}}+\sum_{U^{c} ; i, p_{i j, i} \in U^{c}} z_{U^{c}}\right)+\left(x_{j}+x_{p_{i j, j}}+\sum_{U^{c} ; j, p_{i j, j} \in U} z_{U^{c}}\right) \\
& -\left(x_{p_{i j, i}}+x_{p_{i j, j}}+\sum_{U^{c} ; p_{i j, i}, p_{i j, j} \in U^{c}} z_{U^{c}}\right) \\
= & x_{i}+x_{j}+\sum_{U^{c} ; i, p_{i j, i} \in U^{c}} z_{U^{c}}+\sum_{U^{c} ; j, p_{i j, j} \in U^{c}} z_{U^{c}}-\sum_{U^{c} ; p_{i j, i, p, p_{i j, j} \in U^{c}}} z_{U^{c}}
\end{aligned}
$$

but then $\operatorname{Short}(\boldsymbol{\eta})_{i j}$ can be negative only if there exists a set $U^{c} \in L^{c}$ such that $p_{i j, i}, p_{i j, j} \in U^{c}$, neither $i, j \notin U^{c}$ and $z_{U^{c}}>0$. The first part of Lemma rules out that possibility.

Finally for part (d) observe that there exists a solution $y_{p_{i j, i} p_{i j, j}}^{c}=c_{i j}$ and $y_{i, p_{i j, i}}^{c}=y_{i, p_{i j, j}}^{c}=0$. This solution corresponds to not picking any edges in the original graph $G$. This solution belongs to $\mathcal{Q}^{c}$ and satisfies $\mathbf{A}^{c} \mathbf{y}^{c} \leq \widetilde{\mathbf{b}^{c}}$. Therefore for this solution, for every odd set $U^{c} \in \mathcal{O}_{\delta}^{c}$ :

$$
\begin{equation*}
\sum_{(s, r) \in E^{c}: s, r \in U} y_{s r}^{c} \leq\left\lfloor\frac{\|U\|_{b}}{2}\right\rfloor \leq \frac{1}{(1-\delta)} \tilde{b}_{U^{c}} \tag{19}
\end{equation*}
$$

Observe that $y_{s r}^{c} \neq 0$ only for $(s, r)=\left(p_{i j, i}, p_{i j, j}\right)$ for every edge $(i, j) \in E$; and that $y_{s r}^{c}=c_{i j}$. Therefore multiplying Equation 19 by $z_{U^{c}} \geq 0$ and summing over all $U^{c} \in \mathcal{O}_{\delta}^{c}$ we get:

$$
\begin{aligned}
& \sum_{U^{c} \in \mathcal{O}_{\delta}^{c}} z_{U}\left(\sum_{(i, j) \in E: p_{i j, j}, p_{i, i} \in U^{c}} c_{i j}\right) \leq \frac{1}{(1-\delta)} \sum_{U \in \mathcal{O}_{\delta}^{c}} z_{U^{c}} \tilde{b}_{U^{c}} \leq \frac{1}{(1-\delta)} \gamma^{c} \Longrightarrow \\
& \frac{1}{(1-\delta)} \gamma^{c} \geq \sum_{(i, j) \in E} c_{i j}\left(\sum_{U^{c} \in \mathcal{O}_{\delta}^{c}, p_{i, j}, p_{i j, i} \in U^{c}} z_{U^{c}}\right)=\sum_{(i, j) \in E} c_{i j}\left(\eta_{p_{i j, j}, p_{i j, i}}-x_{p_{i j, i}}-x_{p_{i j, j}}\right)=\sum_{(i, j) \in E} c_{i j} \eta_{p_{i j, j}, p_{i j, i}}
\end{aligned}
$$

where the last part follows from $x_{s}=0$ for any $s \in V^{c} \backslash V$ (since $\left.\lambda_{s}=1 /(1-4 \delta)<\lambda-\delta^{3} / 10\right)$. The conclusion (c) follows from the definition of $\operatorname{Shift}(\boldsymbol{\eta})$.
We now provide a solution for LP11, but notice that the solution is only provided for $\beta \leq(1-$ $4 \delta) \widetilde{\beta^{c}}$. This reduces the approximation ratio but the overall approximation remains a $(1-O(\delta))$ approximation.

Lemma 27. Recall LP11 in Algorithm 5.

$$
\left\{\sum_{(s, r) \in E^{c}} w_{s r}^{c} \tilde{y}_{s r}^{c} \geq(1-\delta) \beta \quad \sum_{(s, r) \in E^{c}} \widetilde{\mathbf{y}}_{s r}^{c} \eta_{s r} \leq \frac{\gamma^{c}}{1-\delta} \quad \tilde{\mathbf{y}}^{c} \in \operatorname{LoNG}\left(\mathcal{P}^{c}\right)\right\}
$$

where $\eta_{\text {sr }}$ are as defined in Step 10 . A solution of LP11 is always found for $\beta \leq(1-4 \delta) \widetilde{\beta^{c}}$. The solution requires at most $\ell=O\left(\ln ^{2} \frac{1}{\delta}\right)$ invocations of Theorem 13 and returns a solution $\hat{\mathbf{y}}^{c}$ such that the subgraph (in $G) \hat{E}=\left\{(i, j) \mid(i, j) \in E, \operatorname{SHORT}\left(\hat{y}^{c}\right)_{i j}>0\right\}$ satisfies $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq(16 \ell) \beta^{*, c}$. Recall that $\beta^{*, c}$ is the weight of the optimum capacitated b-matching.

Proof: First observe that for any $H_{1}, H_{2}$ and $\mathbf{y}=\operatorname{SHORT}\left(\mathbf{y}^{c}\right)$ (equivalently $\mathbf{y}^{c}=\operatorname{LoNG}(\mathbf{y})$ ),

$$
\begin{align*}
& \begin{array}{l}
\sum_{(s, r) \in E_{L}} w_{s r}^{c} y_{s r}^{c}=H_{1} \\
\sum_{r:(s, r) \in E_{L}} \eta_{s r} y_{s r}^{c}=H_{2}
\end{array} \\
& \left.\mathbf{y}^{c} \in \operatorname{LONG}\left(\mathcal{P}^{c}\right) \quad \text { (resp. Long }\left(\mathcal{Q}^{c}\right)\right) \\
& \text { Suppose that we can provide a solution for the system: } \\
& \sum_{(i, j) \in E} w_{i j} y_{i j} \geq(1-\delta) \beta \quad \sum_{(i, j) \in E} \operatorname{ShORT}(\boldsymbol{\eta})_{i j} y_{i j} \leq \frac{\gamma^{c}}{(1-\delta)}-\operatorname{ShiFT}(\boldsymbol{\eta}), \quad \mathbf{y} \in \mathcal{P}^{c} \tag{LP12}
\end{align*}
$$

for any $\beta \geq(1-4 \delta) \beta^{*}$, then we have proved the lemma by considering LONG (y).
Consider the optimum capacitated $b$-matching $\mathbf{y}^{c, *}$, and let $\mathbf{y}^{c, \dagger}=\operatorname{LONG}\left((1-4 \delta) \operatorname{Short}\left(\mathbf{y}^{*, c}\right)\right)$. Observe $\mathbf{y}^{c, \dagger} \in \operatorname{Long}\left(\mathcal{Q}_{c}\right)$. Note that for $i \in V$,

$$
\sum_{r} y_{i r}^{c, *} \leq b_{i} \quad \Longrightarrow \sum_{r} y_{i r}^{c, \dagger} \leq(1-4 \delta) b_{i}=\tilde{b}_{i}
$$

We argue that for any $U^{c} \in \mathcal{O}_{\delta}^{c}$ such that if $z_{U^{c}}>0$,

$$
\begin{equation*}
\sum_{(s, r): s, r \in U^{c}} y_{s r}^{c, *} \geq \sum_{(s, r): s, r \in U^{c}} y_{s r}^{c, t} \tag{20}
\end{equation*}
$$

If for every edge $(i, j) \in E$ both $p_{i j, i}, p_{i j, j}$ are not present in $U^{c}$ then Equation 20) follows immediately because in the transformation of $\mathbf{y}^{c, *}$ to $\mathbf{y}^{c, \dagger}$ the only $y^{c}$ values that increase correspond to $y_{p_{i j, i}, p_{i j, j}}^{c}$ for some edge $(i, j) \in E$. On the other hand, suppose that for some $(i, j) \in E$ both $p_{i j, i}, p_{i j, j}$ are present then using Lemma 26, either $i$ or $j \in U^{c}$. Without loss of generality, suppose $i \in U^{c}$. But then the increase in $y_{p_{i j, i}, p_{i j, j}}^{c}$ cancels out the the decrease in $y_{i p_{i j, i}}^{c}$. Therefore Equation (20) follows.

Note $\sum_{r:(s, r) \in E^{c}} w_{r s}^{c} y_{s r}^{c, \dagger}=(1-4 \delta) \beta^{*, c}$ (discussion following Algorithm 5) and $x_{s}=0$ for $s \in$ $V^{c} \backslash V$ (Lemma 26, part(a)). Omitting the implied $U^{c} \in \mathcal{O}_{\delta}^{c}$ for notational simplicity in the sum below, we get:

$$
\begin{aligned}
\sum_{r:(s, r) \in E_{c}} \eta_{s r} y_{s r}^{c, \dagger} & =\sum_{r:(s, r) \in E^{c}}\left(x_{s}+x_{r}+\sum_{U^{c} \in \mathcal{L}^{c} ; s, r \in U} z_{U^{c}}\right) y_{s r}^{c, \dagger}=\sum_{s} x_{s}\left(\sum_{r} y_{s r}^{c, \dagger}\right)+\sum_{U^{c}: z_{U C}>0} z_{U^{c}}\left(\sum_{(s, r): s, r \in U^{c}} y_{s r}^{c, t}\right) \\
& =\sum_{i: x_{i}>0} x_{i}\left(\sum_{r} y_{i r}^{c, \dagger}\right)+\sum_{U^{c}: z_{U^{c}}>0} z_{U^{c}}\left(\sum_{(s, r): s, r \in U^{c}} y_{s r}^{c, \dagger}\right) \leq \sum_{s: x_{s}>0} x_{s} \tilde{b}_{s}+\sum_{U_{:}: z_{U C}>0} z_{U^{c}} \frac{\tilde{b}_{U^{c}}}{(1-\delta)} \\
& \leq \frac{1}{(1-\delta)}\left(\sum_{s: x_{s}>0} x_{s} \tilde{b}_{s}+\sum_{U^{c}: z_{U} c} z_{U^{c}} \tilde{b}_{U^{c}}\right)=\frac{\gamma^{c}}{(1-\delta)}
\end{aligned}
$$

Therefore there exists a solution for

$$
\left\{\sum_{(s, r) \in E^{c}} w_{s r}^{c} y_{s r}^{c, \dagger}=(1-4 \delta) \beta^{*, c} \quad \sum_{r:(s, r) \in E^{c}} \eta_{s r} y_{s r}^{c} \leq \frac{\gamma^{c}}{(1-\delta)} \quad \mathbf{y}^{c, \dagger} \in \operatorname{LONG}\left(\mathcal{Q}^{c}\right)\right\}
$$

which, by the observation made in this proof, implies that for $\beta \leq(1-4 \delta) \beta^{*, c}$ there exists a solution for

$$
\left\{\sum_{(i, j) \in E} w_{i j} y_{i j} \geq \beta, \quad \sum_{(i, j) \in E} \operatorname{SHORT}(\boldsymbol{\eta})_{i j} y_{i j}=\frac{\gamma^{c}}{(1-\delta)}-\operatorname{SHIFT}(\boldsymbol{\eta}), \quad \mathbf{y} \in \mathcal{Q}^{c}\right\}
$$

We can now apply Theorem 8 with $f_{1}=\beta>0, f_{2}=\frac{\gamma^{c}}{(1-\delta)}-\operatorname{Shift}(\boldsymbol{\eta})$ (by Lemma $26, f_{2} \geq 0$ ) and $\mathcal{P}_{1}=\mathcal{Q}_{c}$ and $\mathcal{P}_{2}=\mathcal{P}^{c}$. Note that $\operatorname{Short}(\boldsymbol{\eta}) \geq \mathbf{0}$ by Lemma 26 . Finally $\mathbf{0} \in \mathcal{Q}^{c} \subseteq \mathcal{P}^{c}$. and the algorithm desired by Theorem 8 is provided by Theorem 10. Therefore we have a solution of LP12. The number of iterations in Theorem 8 is $O(\ln (2 / \delta))$ each of which invokes Theorem 10. Theorem 10 involves Theorem 13 repeatedly. The bound on $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j}$ follows from the fact that we average solutions of Theorem 9 for which $\sum_{(i, j): y_{i j}>0} w_{i j} c_{i j} \leq 8 \beta_{b}^{*, c}$ (the bipartite maximum) which can be bounded by $16 \beta^{*, c}$.
We can now conclude Theorem 3 .
Theorem 3. Given any non-bipartite graph, for any $\frac{3}{\sqrt{n}}<\delta \leq 1 / 16$, we find a $(1-O(\delta))$ approximate fractional solution to LP9 using $O\left(m R / \delta+\min \{B, m\}\right.$ poly $\left.\left\{\delta^{-1}, \ln n\right\}\right)$ time, additional "work" space $O\left(\min \{m, B\}\right.$ poly $\left.\left\{\delta^{-1}, \ln n\right\}\right)$ making $R=O\left(\delta^{-4}\left(\ln ^{2}(1 / \delta)\right) \ln n\right)$ passes over the list of edges where $B=\sum_{i} b_{i}$. The algorithm returns a solution $\left\{\hat{y}_{i j}\right\}=\operatorname{SHORT}\left(\mathbf{y}^{c}\right)$ such that the subgraph $\hat{E}=\left\{(i, j) \mid(i, j) \in E, \hat{y}_{i j}>0\right\}$ satisfies $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq 16 R \beta^{*, c}$ where $\beta^{*, c}$ is the weight of the optimum integral capacitated $b$-matching.

### 7.2 Rounding Capacitated b-Matchings

We prove Theorem 4 based on Algorithm 6.

```
Algorithm 6 Rounding capacitated \(b\)-matchings
    First Phase: Removing edges with large multiplicities (no change from Algorithm 4 except tracking
    edge capacities). Let \(t=\lceil 2 / \delta\rceil\) and \(\mathcal{M}_{c}^{(0)}=\emptyset\).
```

    (a) If \(y_{i j} \geq t\) add \(\hat{y}_{i j}^{(0)}=\left\lfloor y_{i j}\right\rfloor-1\) copies of \((i, j)\) to \(\mathcal{M}_{c}^{(0)}\).
    (b) Set \(y_{i j}^{(1)}=\left\{\begin{array}{ll}0 & \text { if } y_{i j} \geq t \\ y_{i j} & \text { otherwise }\end{array} . \quad\right.\) Set \(b_{i}^{(1)}=\min \left\{b_{i}-\sum_{j} \hat{y}_{i j}^{(0)},\left\lceil\sum_{j} y_{i j}^{(1)}\right\rceil+1\right\}\) and \(c_{i j}^{(1)}=\)
        \(\min \left\{c_{i j},\left\lceil y_{i j}^{(1)}\right\rceil+1\right\}\). This describes the graph \(G_{c}^{(1)}=\left(V, E^{(1)}\right)\). Note \(c_{i j}^{(1)} \leq t+1\).
    2: Second Phase: Subdividing vertices with large multiplicities. (no change from Algorithm 4 except tracking edge capacities). We set $c_{i^{\prime} j^{\prime}}^{(2)}=c_{i j}^{(1)}$ where the edge $(i, j)$ got assigned to $i^{\prime}$ and $j^{\prime}$ which are copies of $i$ and $j$ respectively. This defines $G_{c}^{(2)}=\left(V^{(2)}, E^{(2)}\right)$. Note only vertices are split, - the edges are not split, even though they can be assigned to a copy of an original vertex, i.e., $\left|E^{(1)}\right|=\left|E^{(2)}\right|$. Let $\mathcal{W}=\sum_{(i, j) \in E^{(2)}} c_{i j}^{(2)} w_{i j}$.

3: Third Phase: Reducing the problem to a weighted matching on small graph. (different from Algorithm 4). Given $G_{c}^{(2)}$, define $G_{c}^{(3)}$ as follows:
(a) For each $i \in V^{(2)}$ with $b_{i}^{(2)}$, create $i(1), i(2), \cdots, i\left(b_{i}^{(2)}\right)$. For each edge $e=(i, j)$, we create $2 c_{i j}^{(2)}$ vertices $p_{e i, 1}, p_{e i, 2}, \cdots, p_{e i, c_{i j}^{(2)}}, p_{e j, 1}, p_{e j, 2}, \cdots, p_{e j, c_{i j}^{(2)}}$.
(b) Add edges $\left(p_{e i, \ell}, p_{e j, \ell}\right)$ with edge weight $w_{i j}$. Add a complete bipartite graph between $i_{1}, i_{2}, \cdots$ and $p_{e i, 1}, p_{e i, 2}, \cdots$ with edge weight $w_{i j}$.
(c) Run any fast approximation for finding a $\left(1-\frac{\delta}{32 R}\right)$-approximate maximum weighted matching in $G^{(3)_{c}}$. Let this matching be $\mathcal{M}_{c}^{(3) a}$ of weight $W$.
(d) Observe that given any integral matching in $G^{(3)_{c}}$, we can construct a matching of same or greater weight such that every one of the vertices $p_{e i, \ell}, p_{e j, \ell}$ (for all $e=(i, j$ ), $\ell$ ) are matched - if for some $e, \ell$ neither $p_{e i, \ell}, p_{e j, \ell}$ are matched then we can match them, if only one of the pair is matched then we delete the matching edge incident to the other one in the pair and add the matching edge between $p_{e i, \ell}, p_{e j, \ell}$ which is of the same weight. Applying this procedure to $\mathcal{M}_{c}^{(3) a}$ we get $\mathcal{M}_{c}^{(3) b}$ of weight at least $W$.

4: We now merge all the vertices $i(\ell)$ to $i, p_{e i, \ell}$ to $p_{e i}$ and $p_{e j, \ell}$ to $p_{e j}$ for all $e=(i, j), \ell$. Observe $G_{c}^{(3)}$ reduces to Long $\left(G_{c}^{(2)}\right)$ with different edge weights, i.e., for an edge $e=(i, j)$ of weight $w_{i j}$ in the original graph we have the weights of $\left(i, p_{e i}\right),\left(p_{e i} p_{e j}\right)$ and $\left(p_{e j}, j\right)$ are all $w_{i j}$ instead of $\frac{1}{2} w_{i j}, 0, \frac{1}{2} w_{i j}$ as in the definition of LONG $\left(G_{c}^{(2)}\right)$.
However if we merge all the corresponding edges of $\mathcal{M}_{c}^{(3) b}$ then we get a matching $\mathcal{M}^{(3)}$ such that the vertices $p_{e i}$ and $p_{e j}$ are matched to capacity $c_{i j}$ for every edge $e=(i, j)$. Note that $\mathcal{M}_{c}^{(3)}$ has weight at least $W$. $\mathcal{M}^{(3)}$ provides a $b$-matching $\mathcal{M}_{c}^{(2)}$ in $G_{c}^{(2)}$ of weight at least $W-\mathcal{W}$, where we set $y_{i j}^{\dagger}=y_{i p_{e i}}^{\dagger}$. $\mathcal{M}_{c}^{(2)}$ provides a $b$-matching $\mathcal{M}_{c}^{(1)}$ in $G_{c}^{(1)}$ of same weight (merge vertices).
5: Output $\mathcal{M}_{c}^{(0)} \cup \mathcal{M}_{c}^{(1)}$.

For example, in Step 3(b)

which in turn reduces to


Lemma 28. $y_{i j}^{(1)}$ is a feasible fractional capacitated b-matching in $G_{c}^{(1)}$.
Proof: Consider Long $\left(\mathbf{y}^{(\mathbf{1})}\right)$ and Long $\left(G_{c}^{(1)}\right)$ with the new capacities $b_{i}^{(1)}, c_{i j}^{(1)}$ for the vertices and edges in $G_{c}^{(1)}$. The only vertices whose capacities were affected in $\operatorname{Long}\left(G_{c}^{(1)}\right)$ are the following vertices: (i) the corresponding vertex in $G$ has an edge incident to it in $\mathcal{M}_{c}^{(1)}$ and (ii) the corresponding edge $(i, j) \in G$ had $c_{i j}>\left\lceil y_{i j}^{(1)}\right\rceil+1$. In both cases the difference between the sum of the new edge multiplicities and the new capacities (the slack) is at least 1 and the first part of Lemma 22 tells us that these vertices in LONG $\left(G_{c}^{(1)}\right)$ cannot be part of a violated odd-set in Long $\left(G_{c}^{(1)}\right)$. Therefore $\mathbf{y}^{(1)}$ is a feasible fractional (uncapacitated) $b$-matching. The lemma follows from Theorem 24.
Therefore the remaining task is to find a $(1-\delta)$ approximate rounding of the fractional solution $y_{i j}^{(1)}$ on $G_{c}^{(1)}=\left(V, E^{(1)}\right)$ with vertex and edge capacities $\left\{b_{i j}^{(1)}\right\}$ and $\left\{c_{i j}^{(1)}\right\}$ respectively.
Lemma 29. Let $\mathcal{W}=\sum_{(i, j) \in E^{(2)}} c_{i j}^{(2)} w_{i j}$. Then $\mathcal{W} \leq 16 R \beta^{*, c}$.
Proof: Observe that $\left|E^{(2)}\right|=\left|E^{(1)}\right|$ and $E^{(1)} \subseteq \hat{E}$ as defined in the statement of Theorem 4 Moreover $c_{i^{\prime} j^{\prime}}^{(2)}=c_{i j}^{(1)} \leq c_{i j}$. Therefore:

$$
\mathcal{W}=\sum_{\left(i^{\prime}, j^{\prime}\right) \in E^{(2)}} c_{i^{\prime} j^{\prime}}^{(2)} w_{i^{\prime} j^{\prime}}=\sum_{(i, j) \in E^{(1)}} c_{i j}^{(1)} w_{i j} \leq \sum_{(i, j) \in E^{(1)}} c_{i j} w_{i j} \leq \sum_{(i, j) \in \hat{E}} c_{i j} w_{i j} \leq 16 R \beta^{*, c}
$$

Lemma 30. Algorithm 6 outputs a capacitated $b$-matching of weight at least $(1-\delta) \sum_{(i, j) \in E} w_{i j} y_{i j}-$ $\delta \beta^{*}$.

Proof: Let the weight of the maximum matching of this graph $G_{c}^{(3)}$ be $w\left(\mathcal{M}^{*}\right)$. Then

$$
2 \mathcal{W} \geq w\left(\mathcal{M}^{*}\right) \geq \sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)}+\mathcal{W}
$$

since each edge $(i, j) \in G_{c}^{(2)}$ can contribute at most $2 c_{i j}^{(2)} w_{i j}$ to $w\left(\mathcal{M}^{*}\right)$.
Suppose that we find a $\left(1-\frac{\delta}{32 R}\right)$-approximate maximum matching in $G_{c}^{(3)}$, using the algorithm in [9, 10] which takes time $\left|E\left(G_{c}^{(3)}\right)\right|$ times $O\left(\frac{R}{\delta} \log (R / \delta)\right)$ which is $O\left(m^{\prime} R \delta^{-3} \log (R / \delta)\right)$. This gives us a matching of weight at least $W$ where $W \geq w\left(\mathcal{M}^{*}\right)-\frac{\delta}{32 R} w\left(\mathcal{M}^{*}\right)$ which corresponds to a
capacitated $b$-matching in $G_{c}^{(2)}$ with weight at least $w\left(\mathcal{M}^{*}\right)-\frac{\delta}{32 R} w\left(\mathcal{M}^{*}\right)-\mathcal{W}$. Now

$$
\begin{aligned}
& w\left(\mathcal{M}^{*}\right)-\frac{\delta}{32 R} w\left(\mathcal{M}^{*}\right)-\mathcal{W} \geq \sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)}+\mathcal{W}-\frac{\delta}{32 R} w(\mathcal{M} *)-\mathcal{W} \\
& =\sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)}-\frac{\delta w\left(\mathcal{M}^{*}\right)}{32 R} \geq \sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)}-\delta \beta^{*, c}
\end{aligned}
$$

Since the second phase is exactly the same as in the uncapacitated case in Section 6, we have

$$
\sum_{(i, j) \in E^{(2)}} w_{i j} y_{i j}^{(2)} \geq(1-\delta) \sum_{(i, j) \in E^{(1)}} w_{i j} y_{i j}^{(1)}
$$

Thus we get a matching $\mathcal{M}_{c}^{(1)}$ in $G_{c}^{(1)}$ of weight $w\left(\mathcal{M}_{c}^{(1)}\right) \geq(1-\delta) \sum_{(i, j) \in E^{(1)}} w_{i j} y_{i j}^{(1)}-\delta \beta^{*, c}$. Observe that $w\left(\mathcal{M}_{c}^{(0)}\right) \geq(1-\delta) \sum_{(i, j) \in E} w_{i j}\left(y_{i j}-y_{i j}^{(1)}\right)$ where $w\left(\mathcal{M}_{c}^{(0)}\right)=\sum_{(i, j) \in E} y_{i j}^{(0)} w_{i j}$. Then $w\left(\mathcal{M}_{c}^{(0)}\right)+w\left(\mathcal{M}_{c}^{(1)}\right)$ is at least $(1-\delta) \sum_{(i, j) \in E} w_{i j} y_{i j}-\delta \beta^{*}$ as desired. This proves Lemma 30 . Therefore we can conclude Theorem 4.

Theorem 4. Given a fractional capacitated b-matching $\mathbf{y}^{c}$ which is feasible for LPG. Let $\mathbf{y}=\operatorname{SHORT}\left(\mathbf{y}^{c}\right)$ and $\hat{E}=\left\{(i, j) \mid y_{i j}>0\right\}$. Further suppose we are promised that $\sum_{(i, j) \in \hat{E}} w_{i j} c_{i j} \leq$ $16 R \beta^{*, c}$. We find an integral b-matching of weight at least $(1-\delta) \sum_{(i, j)} w_{i j} y_{i j}-\delta \beta^{*, c}$ in $O\left(m^{\prime} R \delta^{-3} \ln (R / \delta)\right)$ time and $O\left(m^{\prime} / \delta^{2}\right)$ space where $m^{\prime}=|\hat{E}|$ is the number of nontrivial edges (as defined by the linear program) in the fractional solution. As a consequence we have a $(1-O(\delta))$-approximate integral solution.

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[^0]:    *A previous extended abstract of this paper appeared in SODA 2014 [2].
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[^1]:    ${ }^{1}$ In subsequent work, in manuscript [3], we show that we can solve the dual to identify the subgraph containing the maximum uncapacitated $b$-matching; but that manuscript uses the results in this paper to construct an actual feasible primal solution on that subgraph. Further the methods of [3] do not apply to the dual of the capacitated $b$-matching problem.

[^2]:    ${ }^{2}$ The exact exponent of $\delta, \log n$ in the poly () term depends on [20, 6] and we omit further discussion in this paper.

[^3]:    ${ }^{3}$ While it may be appealing to discuss closure of $\widehat{\mathcal{Q}}$, note the $(1-\delta / 2)$ factor and therefore lim sup suffices.

